GEOMETRY OF CHARACTERISTIC VARIETIES

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0. Introduction

In this paper we give a quick presentation of the characteristic variety of a complex analytic linear holonomic differential system. The fact that we can view a complex analytic linear holonomic differential system on a complex manifold as a holonomic $\mathcal{D}_X$-module, allows us to present the characteristic variety in an algebro-geometric way as we do here.

We do not define what is a sheaf; for this we refer to the famous book of R. Godement.[2] We also do not define properly $\mathcal{D}_X$-modules although one can find in the lectures of F. Castro the definition of left modules on the Weyl algebra of operators on $\mathbb{C}^n$ with polynomial coefficients which give a local version of $\mathcal{D}_X$-modules.

Also we consider categories, complexes in abelian categories, derived functors and hypercohomologies[1] which are the natural language to be used here. Of course, we cannot define all these notions. One should view these notes as a provocation rather than a self-contained exposition. We hope they will encourage the reader to learn more in the subject.

1. Whitney Conditions

In Ref. 19, §19 p. 540, H. Whitney introduced Whitney conditions. The general idea is to find conditions for the attachment of a non singular analytic space having an analytic closure along a non singular part of its boundary which ensure that the closure is "locally topologically trivial" along the boundary, that is, locally topologically a product of the boundary by a "transverse slice". Whitney's approach can be deemed to be based on the fundamental fact, which he discovered, that any analytic set is, at each of its points, "asymptotically a cone". More precisely, given any closed complex analytic subspace $\mathcal{X} \subset U \subset \mathbb{C}^N$ and a point of X which we may take as the origin 0, given any sequence of non singular points $x_n \in X$ we may consider the tangent spaces $T_{\mathcal{X},x_n}$ of $\mathcal{X}$ and the secant lines $0x_n$. They respectively define points in the Grassmanian $\mathrm{G}(N,d)$ and the projective space $\mathbb{P}^{N-1}$, which are compact. Therefore, possibly after taking a subsequence we may assume the limits $T = \lim_n T_{\mathcal{X},x_n}$ and $\ell = \lim_n 0x_n$ exist. Then we have $\ell \subset T$, which ensures that $\mathcal{X}$ is locally "cone-like" with respect to the "vertex" 0.

Whitney's idea may have been that the topological triviality of the closed analytic space $\mathcal{X}$ along a part $\mathcal{Y}$ would be ensured by the condition that $\mathcal{X}$ should be "cone-like" along the "vertex" $\mathcal{Y}$, which is a way to ensure that locally $\mathcal{X}$ is transversal to the boundaries of small tubes around $\mathcal{Y}$. It gives this:
Let $\mathcal{X}$ be a complex analytic subset of $\mathbb{C}^N$ and $\mathcal{Y}$ be a complex analytic subset of $\mathcal{X}$. One says that $\mathcal{X}$ satisfies the Whitney condition along $\mathcal{Y}$ at the point $y \in \mathcal{Y}$, if:

(1) the point y is non-singular in $\mathcal{Y}$;
(2) for any sequence $(x_n)_{n\in\mathbb{N}}$ of non-singular points of $\mathcal{X}$ which converges to y such that the tangent spaces $T_{x_n}(\mathcal{X})$ have a limit T and, for any sequence $(y_n)_{n\in\mathbb{N}}$ of points of $\mathcal{Y}$ which converges to y, such that the lines $y_n x_n$ have a limit ℓ, we have $\ell \subset T$.

Examples. Consider the complex algebraic subset $\mathcal{X}$ of $\mathbb{C}^3$ defined by the equation:

$$X^2 - YZ^2 = 0.$$

The line $L = \{X = Y = 0\}$ is contained in $\mathcal{X}$. The surface $\mathcal{X}$ satisfies the Whitney condition along L at any point $y \in L \setminus \{0\}$, but not at the point $\{0\}$, because, for the sequence $(0, y_n, 0)$ of non-singular points of $\mathcal{X}$, the limit of tangent spaces is the plane $T = \{Z = 0\}$, and for the points $(0, 0, y_n)$ the limit ℓ is the line $\{X = 0, Y = -Z\}$ which is not contained in T.

Consider the complex algebraic subset $\mathcal{X}$ of $\mathbb{C}^3$ defined by the equation (see Fig. 2):

$$X^2 - Y^3 - Z^2Y^2 = 0.$$

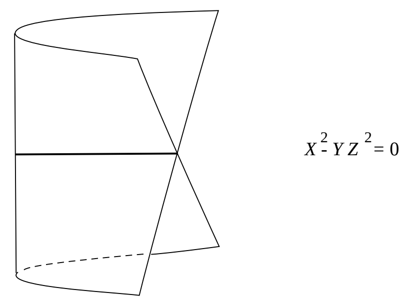

Fig. 1.

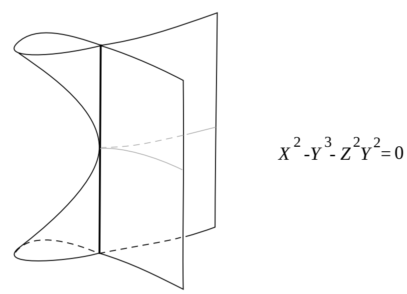

Fig. 2.

The surface $\mathcal{X}$ satisfies Whitney condition along $L = \{X = Y = 0\}$ at any point $y \in L \setminus \{0\}$, but not at $\{0\}$, because, one may consider the sequence of non-singular points $(0, z_n^2, z_n)$ of $\mathcal{X}$ and $(0, 0, z_n)$ of L as z_n tends to 0. The limit ℓ is the line which contains $(0, 0, 1)$, the limit T is the plane orthogonal to $(0, 0, 1)$.

2. Stratifications

Let $\mathcal{X}$ be a complex analytic subset of $\mathbb{C}^N$. A partition $\mathcal{S} = (X_\alpha)_{\alpha \in A}$ is a stratification of $\mathcal{X}$, if:

(1) The closure $\overline{X}_\alpha$ of X_α in $\mathcal{X}$ and $\overline{X}_\alpha \setminus X_\alpha$ are complex analytic subspaces of $\mathcal{X}$;
(2) The family (X_α) is locally finite;
(3) Each X_α is a complex analytic manifold;

(4) If, for a pair (α, β), we have $X_\alpha \cap \overline{X}_\beta \neq \emptyset$, then $X_\alpha \subset \overline{X}_\beta$.

The subsets X_α are called the strata of the stratification $\mathcal{S}$. The condition (4) is called the frontier condition (see e.g. Ref. 13, Définition (1.2.3)).

Examples. Consider the complex algebraic subset of $\mathbb{C}^3$ defined by the equation:

$$X^2 - Y^3 - Z^2Y^2 = 0.$$

One can consider this surface as a deformation of complex plane algebraic curves parametrized by Z. The singular points of these curves are on the Z-axis which if given by $X = Y = 0$. For $Z = 0$, one has a cusp $X^2 - Y^3 = 0$. For $Z \neq 0$, one has a strophoıd with a singular point at the origin.

One has a stratification by considering the strata given by the non-singular points, which are the points of the surface outside of the line $X = Y = 0$ on the surface, and by the singular points which are the points of the line $X = Y = 0$.

Now consider the complex algebraic subset of $\mathbb{C}^3$ defined by the equation (see Fig. 3):

$$X^2 - Y^2Z^3 = 0.$$

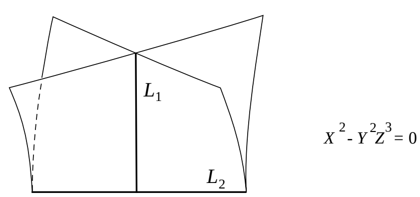

Fig. 3.

It is a surface whose singular points lie on two lines $L_1 = \{X = Y = 0\}$ and $L_2 = \{X = Z = 0\}$. We can define a partition of this surface by considering $\mathcal{S}_0 := \mathcal{X}^0$, the subset of non-singular points of the surface, the punctured line $\mathcal{S}_1 = L_1 \setminus \{0\}$ and the line $\mathcal{S}_2 = L_2$.

This partition does not define a stratification, because it does not satisfy the frontier condition. However, one can consider instead the partition given by $\mathcal{S}_0$, $\mathcal{S}_1$, the punctured line $\mathcal{S}_2' = L_2 \setminus \{0\}$ and the origin $\{0\}$. This is a stratification.

More generally one can prove that, by choosing a “refinement” of a partition satisfying (1), (2) and (3), one obtains the frontier condition (see Ref. 19).

A *complex analytic partition* $\mathcal{S} = (X_\alpha)_{\alpha\in A}$ of a complex analytic space $\mathcal{X}$ is a partition of $\mathcal{X}$ which satisfies (1), (2) and (3) above.

Then, a complex analytic partition $S' = (X'_\alpha)_{\alpha\in A'}$ of $\mathcal{X}$ is finer than the partition $\mathcal{S} = (X_\alpha)_{\alpha\in A}$ of $\mathcal{X}$, if any stratum X_α of $\mathcal{S}$ is the union of strata of $\mathcal{S}'$. One may quote a proposition of H. Whitney in the following way:

Proposition 2.1. *For any complex analytic partition $\mathcal{S} = (X_\alpha)_{\alpha\in A}$ of $\mathcal{X}$, there is a finer complex analytic partition which satisfies the frontier condition.*

Remark also that there is a coarsest finer partition with connected strata; this is obvious by taking the connected components of strata. In stratification theory one often assumes that the strata are connected.

3. Constructible Sheaves

All the sheaves that we consider are sheaves of complex vector spaces.

First, let us define local systems.

Definition 3.1. Let A be a topological space. A sheaf $\mathcal{F}$ on A is called a local system on A if it is locally isomorphic to a constant sheaf.

For instance, a constant sheaf is a local system. In fact, when A is an arcwise connected space, a local system $\mathcal{F}$ on A defines a homomorphism $\pi_1(A, a) \overset{\rho_\mathcal{F}}{\to} Aut_\mathbb{C}\mathcal{F}_a$. This homomorphism is defined in the following way: let γ be a loop at a; one can extend a section of $\mathcal{F}_a$ by continuity along γ, since there is a neighbourhood of a on which $\mathcal{F}$ is constant; because we can cover the image of γ by a finite number of open sets over which $\mathcal{F}$ is constant we define a map of $\mathcal{F}_a$ into itself determined by γ; one can show that this map depends only on the homotopy class of γ and is a complex linear automorphism of $\mathcal{F}_a$.

The correspondence $\mathcal{F} \mapsto \rho_\mathcal{F}$ defines an equivalence of category between the category of local systems on A and the category of representations of the fundamental group $\pi_1(A, a)$ in finite dimensional vector spaces.

Example: In the case A is the circle $\mathbb{S}^1$, the fundamental group $\pi_1(A,a)$ is the group of relative integers $\mathbb{Z}$. Local systems of rank one are given by maps $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^k$, with $k \in \mathbb{Z}$.

Now, we have an important concept which, in some sense, generalises complex analytic partitions:

Definition 3.2. A sheaf L over a complex analytic set $\mathcal{X}$ is constructible if there is a complex analytic partition $\mathcal{S} = (X_\alpha)_{\alpha \in A}$ of $\mathcal{X}$, such that the restriction of L over each X_α is a local system.

Examples: a) Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper algebraic map between two algebraic varieties, the sheaf on $\mathcal{Y}$ whose stalks are the k-th cohomology $\mathrm{H}^k(f^{-1}(y), \mathbb{C})$ of the fibers $f^{-1}(y)$ of f is a constructible sheaf.

b) The cohomologies of the solutions $\mathrm{RHom}_D(M,O)$ of a holonomic D-module M over a complex space $\mathbb{C}^N$ are constructible sheaves (Kashiwara's constructibility Theorem).

c) Let $\mathcal{Z}$ be an algebraic subvariety of $\mathcal{X}$ and $i : \mathcal{Z} \to \mathcal{X}$ be the inclusion. Let $z \in \mathcal{Z}$. The local cohomology $(\mathrm{R}^k i^!)(\mathbb{C})_z \simeq \mathrm{H}^k_{\mathcal{Z} \cap B_z}(\mathcal{X} \cap B_z, \mathbb{C})$, where B_z is a good neighbourhood of z in $\mathcal{X}$, defines a constructible sheaf on $\mathcal{Z}$.

These examples are not easy to prove. The first and third ones are difficult theorems on the topology of algebraic maps, the second one is a basic theorem of the theory of $\mathcal{D}$-modules also difficult to prove.

4. Whitney Stratifications

Let $\mathcal{S} = (X_\alpha)_{\alpha \in A}$ be a stratification of a complex analytic set $\mathcal{X}$. We say that $\mathcal{S}$ is a Whitney stratification of $\mathcal{X}$ if, for any pair (X_α, X_β) of strata, such that $X_\alpha \subset \overline{X}_\beta$, the complex analytic set $\overline{X}_\beta$ satisfies Whitney condition along X_α at any point of X_α.

As we announced above, the interest for Whitney stratifications comes from the fact that they imply local topological triviality. Namely, a theorem of J. Mather (see Ref. 14) and R. Thom[18] gives:

Theorem 4.1. *Let $\mathcal{S} = (X_\alpha)_{\alpha \in A}$ be a Whitney stratification of a complex analytic subset $\mathcal{X}$ of $\mathbb{C}^N$. For any point $x \in X_\alpha$ there is an open neighbourhood U of x in $\mathbb{C}^N$ such that $\mathcal{X} \cap U$ is homeomorphic to $(X_\alpha \cap U) \times (N_\alpha \cap U)$ where N_α is a slice of X_α at x in $\mathcal{X}$ by a transversal affine space in $\mathbb{C}^N$.*

This theorem shows that locally on $\mathcal{X}$, along any strata, the analytic set is topologically a product.

A theorem of H. Whitney (Ref. 19) gives that:

Theorem 4.2. *Let $\mathcal{S} = (X_\alpha)_{\alpha \in A}$ be a stratification of a complex analytic set $\mathcal{X}$. There is a stratification of $\mathcal{X}$ which is finer than $\mathcal{S}$ and is a Whitney stratification for $\mathcal{X}$.*

As a consequence on a singular compact space one can observe only finitely many different topological types of embedded germs of complex spaces, while in general there are continua of different analytic types.

Examples. In the examples given before the stratification of:

$$X^2 - YZ^2 = 0$$

given by the non-singular part $\mathcal{X} \setminus \mathcal{Y}$ of $\mathcal{X}$, $\mathcal{Y} \setminus \{0\}$ and $\{0\}$ gives a Whitney stratification of $\mathcal{X}$ which is finer than $\mathcal{X} \setminus \mathcal{Y}$ and $\mathcal{Y}$.

We have the same for:

$$X^2 - Y^3 - Z^2Y^2 = 0.$$

For the example given by:

$$X^2 - Y^2Z^3 = 0,$$

a Whitney stratification is given by $\mathcal{X} \setminus L_1 \cup L_2$, $L_1 \setminus \{0\}$, $L_2 \setminus \{0\}$ and $\{0\}$.

A theorem of B. Teissier shows that a Whitney stratification can be characterized algebraically and is very useful to know if a stratification is a Whitney stratification. In order to give this criterion, we need to define Polar Varieties.

Let $\mathcal{X}$ be a complex analytic subset of $\mathbb{C}^N$. Let x be a point of $\mathcal{X}$. Assume for simplicity that $\mathcal{X}$ is equidimensional at x. Consider affine projections of $\mathcal{X}$ into $\mathbb{C}^{k+1}$, for $1 \leq k \leq dim_x(\mathcal{X})$. Then, one can prove:

Theorem 4.3. *There is a non-empty Zariski open set Ω_k in the space of projections of $\mathbb{C}^N$ into $\mathbb{C}^{k+1}$, such that for any $p \in \Omega$, there is an open neighbourhood U, such that either the critical locus of the the restriction of p to the non-singular part of $\mathcal{X} \cap U$ is empty or the closure of the critical locus of the restriction of p to the non-singular part of $\mathcal{X} \cap U$ is reduced and has dimension k at x and its multiplicity at x is an integer which is independent of $p \in \Omega_k$.*

In the case where the critical locus of the restriction of p to the non-singular part of $\mathcal{X} \cap U$ is not empty for $p \in \Omega_k$, the closure of the critical locus in $\mathcal{X} \cap U$ is called "the" polar variety $P_k(\mathcal{X}, x, p)$ of $\mathcal{X}$ at x of dimension k, defined by p and the multiplicity $m(P_k(\mathcal{X}, x, p))$ is called the k-th polar multiplicity of $\mathcal{X}$ at the point x. For $P_k(\mathcal{X}, x, p)$ this is an abuse of language since only the "equisingularity type" of $P_k(\mathcal{X}, x, p)$ is well-defined for $p \in \Omega_k$, but the theorem just stated shows that there is no abuse as far as the multiplicity is concerned. When the critical locus of the restriction of p is empty for $p \in \Omega_k$, we say that $m_k(\mathcal{X}, x) = 0$.

Then, the criterion of B. Teissier is the following (Ref. 17):

Theorem 4.4. *Let $\mathcal{X}$ be a complex analytic set. Let $\mathcal{S} = (X_\alpha)_{\alpha \in A}$ be a stratification of $\mathcal{X}$ with connected strata. Suppose that for any pair (X_α, X_β) such that $X_\alpha \subset \overline{X}_\beta$, the multiplicities $m(P_k(\overline{X}_\beta), y)$ are constant for $y \in X_\alpha$, for $1 \leq k \leq dim_y(\overline{X}_\beta)$, then, the stratification $\mathcal{S}$ is a Whitney stratification. Conversely, any Whitney stratification with connected strata has this property.*

Beware that this theorem is true with stratifications: for instance the frontier condition is important. One can consider the case of the surface defined by:

$$X^2 - Y^2 Z^3 = 0.$$

Examples. Let $\mathcal{X}$ be a surface, i.e. a complex analytic set of dimension 2. If, $x \in \mathcal{X}$ is a non-singular point, the 2-nd polar variety at x is $\mathcal{X}$ itself and, by definition, its multiplicy is 1. At x the 1-st polar variety is empty, so $m_1 = 0$. If $x \in \mathcal{X}$ is singular, again the 2-nd polar variety at x is $\mathcal{X}$ itself and $m_2 = m_x(X) > 1$. For almost all singular points, except a finite number locally, the 1-st polar curve is empty, so $m_1 = 0$. So, a Whitney stratification of $\mathcal{X}$ is given by the non-singular part $\mathcal{X}^0$ of $\mathcal{X}$, the nonsingular part of the singular locus minus the points where the polar curve is not empty, the points where the polar curve is not empty and finally the singular points of the singular locus.

For the surface given by:

$$X^2 - Y^3 - Z^2 Y^2 = 0$$

the stratification given by $\mathcal{X} \setminus \mathcal{Y}$, $\mathcal{Y} \setminus \{0\}$ and $\{0\}$ is a Whitney stratification.

A consequence of this theorem is the existence of a minimal Whitney stratification refining a given one (see Ref. 17):

Theorem 4.5. *Let $\mathcal{X}$ be a complex analytic set. Let $\mathcal{S} = (X_\alpha)_{\alpha \in A}$ be a complex analytic partition of $\mathcal{X}$. Then there is a unique coarsest refinement of $\mathcal{S}$ which is a Whitney stratification of $\mathcal{X}$ with connected strata.*

If one takes as partition the non-singular part of $\mathcal{X}$, the non-singular part of the singular locus, and so on, one sees that every complex-analytic space has a unique "minimal" Whitney stratification in the sense that any other Whitney stratification is a refinement of it.

5. Milnor Fibrations

Let $f : U \subset \mathbb{C}^N \to \mathbb{C}$ be a complex analytic function defined on a neighbourhood of the origin 0 in $\mathbb{C}^N$ such that the image $f(0)$ of the origin 0 by f is 0.

One can prove:

Theorem 5.1. *There is $\varepsilon_0 > 0$, such that, for any ε such that $0 < \varepsilon < \varepsilon_0$, there is $\eta(\epsilon) > 0$, such that for any η such that $0 < \eta < \eta(\varepsilon)$, the map*

$$\varphi_{\varepsilon,\eta} : \mathbb{B}_\varepsilon \cap f^{-1}(\mathbb{S})_\eta \to \mathbb{S}_\eta,$$

induced by f over the circle of radius η centered at the origin 0 in the complex plane $\mathbb{C}$, is a locally trivial smooth fibration.

See Ref. 16 for the case where f has an isolated critical point at 0 and Ref. 5 for the general case. We call the fibration given by the theorem the Milnor fibration of f at 0.

When f has a critical point at 0, the fibers of $\varphi_{\varepsilon,\eta}$ have a non-trivial homotopy. Since the "fiber" at 0, i.e., $\mathbb{B}_\varepsilon \cap f^{-1}(0)$ is contractible by Ref. 16, one usually calls *vanishing cycles* the cycles of a fiber of $\varphi_{\varepsilon,\eta}$, i.e. the elements of the homology $\mathrm{H}_*(\mathbb{B}_\varepsilon \cap f^{-1}(t), \mathbb{C})$ for $t \in \mathbb{S}_\eta$.

One calls *neighbouring cycles* the elements of the cohomology $\mathrm{H}^*(\mathbb{B}_\varepsilon \cap f^{-1}(t), \mathbb{C})$. The theorem above shows that these definitions do not depend on $t \in \mathbb{S}_\eta$. One can prove that it does not depend on ε, η chosen conveniently.[13]

One may observe that the complex cohomology $\mathrm{H}^*(\mathbb{B}_\varepsilon \cap f^{-1}(t), \mathbb{C})$ is the sheaf cohomology of $\mathbb{B}_\varepsilon \cap f^{-1}(t)$ with coefficients in the constant sheaf $\mathbb{C}$.

We also have on any analytic set a theorem similar to the one above (see Ref. 9):

Theorem 5.2. *Let $f : \mathcal{X} \to \mathbb{C}$ be a complex analytic function on a complex analytic subset of $\mathbb{C}^N$. Suppose that $0 \in \mathcal{X}$ and $f(0) = 0$. There is $\varepsilon_0 > 0$, such that, for any ε such that $0 < \varepsilon < \varepsilon_0$, there is $\eta(\epsilon) > 0$, such that for any η such that $0 < \eta < \eta(\varepsilon)$, the map*

$$\varphi_{\varepsilon,\eta} : \mathbb{B}_\varepsilon \cap \mathcal{X} \cap f^{-1}(\mathbb{S})_\eta \to \mathbb{S}_\eta,$$

induced by f over the circle of radius η centered at the origin 0 in the complex plane $\mathbb{C}$, is a locally trivial topological fibration.

In particular, this theorem shows that the homology

$$\mathrm{H}_*(\mathbb{B}_\varepsilon \cap \mathcal{X} \cap f^{-1}(t))$$

or the cohomology

$$\mathrm{H}^*(\mathbb{B}_\varepsilon \cap \mathcal{X} \cap f^{-1}(t), \mathbb{C})$$

does not depend on $t \in \mathbb{S}_\eta$ and on ε, η chosen appropriately.

6. Local Constructible Sheaves and Whitney Conditions

Since the constant sheaf $\mathbb{C}$ is a constructible sheaf, one may also consider the neighbouring cycles of a constructible sheaf $\mathcal{L}$ along the function $f : \mathcal{X} \to \mathbb{C}$ as the sheaf cohomology $\mathrm{H}^*(\mathbb{B}_\varepsilon \cap X \cap f^{-1}(t), \mathcal{L})$.

Let p be a projection $\mathbb{C}^N$ into $\mathbb{C}^{k+1}$ which defines a k-th polar variety $P_k(\mathcal{X}, x, p)$ of X at a point $x \in \mathcal{X}$. Then, one has neighbourhood U and V of x and $p(x)$ in $\mathcal{X}$ and $\mathbb{C}^{k+1}$, such that p induces a map $\pi : U \to V$. One can show that $\pi(P_k(\mathcal{X}, x, p) \cap U)$ is a complex analytic subset of V and π induces a locally trivial topological fibration of $\pi^{-1}(V \setminus \pi(P_k(\mathcal{X}, x, p) \cap U))$ over $V \setminus \pi(P_k(\mathcal{X}, x, p) \cap U)$.

The sheaf $(\mathrm{R}^\ell \pi_*)(\mathbb{C}_U)$, whose fiber at $y \in V$ is the ℓ-th cohomology of $\pi^{-1}(y)$, is a constructible sheaf. The Euler characteristic $\chi_k(\mathcal{X}, x)$ of the general fiber of π is called the k-th vanishing Euler characteristic of $\mathcal{X}$ at x. At x, one has $\dim_x(\mathcal{X})$ Euler characteristics $\chi(\mathcal{X}, x) := (\chi_1(\mathcal{X}, x), \ldots, \chi_{dim \mathcal{X}}(\mathcal{X}, x))$.

For simplicity assume that $\mathcal{X}$ is equidimensional. Then, one has a characterization of a Whitney stratification by a result of Lê and Teissier (Ref. 13, Théorème (5.3.1)) similar to the one of Teissier given above:

Theorem 6.1. *Let $\mathcal{X}$ be an equidimensional complex analytic set. Let $\mathcal{S} = (X_\alpha)_{\alpha \in A}$ be a stratification of $\mathcal{X}$. Suppose that, for any pair (X_α, X_β), such that $X_\alpha \subset \overline{X}_\beta$, the Euler characteristics $(\chi_1(X_\beta, y), \ldots, \chi_{\dim_y X_\beta}(X_\beta, y))$ are constant for $y \in X_\alpha$, then, the stratification $\mathcal{S}$ is a Whitney stratification.*

7. Neighbouring Cycles

We saw in Section 5 shows that any analytic functions on an open set defines locally a locally trivial fibration on a circle $\mathbb{S}$. The last theorem of Section 5 shows that this extends to functions on any complex analytic sets.

Let $f : \mathcal{X} \to \mathbb{C}$ be a complex analytic function on a complex analytic subset of $\mathbb{C}^N$. On the fiber $f^{-1}(f(x))$ of the function f through any point

x of $\mathcal{X}$ one can define a sheaf $\mathrm{R}^k(\psi_{f-f(x)})(\mathbb{C})$, with $\mathrm{R}^k(\psi_{f-f(x)})(\mathbb{C})_y \simeq \mathrm{H}^k(F_y, \mathbb{C})$ where F_y is a fiber of the Milnor fibration defined above at $y \in f^{-1}(f(x))$ by f.

This sheaf $\mathrm{R}^k(\psi_{f-f(x)})(\mathbb{C})$ is a constructible sheaf on the fiber of f over $f(x)$. One calls it the *sheaf of k-th neighbouring cycles of f at x.*

Similarly, for any constructible sheaf $\mathcal{L}$, one can define the sheaf of neighbouring cycles $\mathrm{R}^k(\psi_{f-f(x)})(\mathcal{L})$ of f at x, where $\mathrm{R}^k(\psi_{f-f(x)})(\mathcal{L})_y \simeq \mathrm{H}^k(F_y, \mathcal{L})$.

When f is defined on $\mathbb{C}^{n+1}$ and has an isolated singular point at x, the sheaf $\mathrm{R}^k(\psi_{f-f(x)})(\mathbb{C})$ is non-zero when $k = n$ or 0. In this case, $\mathrm{R}^0(\psi_{f-f(x)})(\mathbb{C})$ is the constant sheaf on $f^{-1}(x)$ and $\mathrm{R}^n(\psi_{f-f(x)})(\mathbb{C})$ is a sheaf whose value at x is $\mathbb{C}^\mu$ and which is zero on a neighbourhood of x outside $\{x\}$. In this special case, when $n \geq 1$, one also call $\mathrm{R}^n(\psi_{f-f(x)})(\mathbb{C})$ the sheaf of vanishing cycles of f at x.

When the complex analytic space $\mathcal{X}$ is a Milnor space (see Ref. 10) and the function $f : \mathcal{X} \to \mathbb{C}$ has isolated singularities, i.e. there is a Whitney stratification of $\mathcal{X}$, such that the restriction of f to the strata has maximal rank except at isolated points, one can define the same type of sheaf over the fiber above the image of a singularity. For instance, complete intersection spaces, e.g., hypersurfaces, are Milnor spaces.

Because of the fibration theorem, neighbouring cycles and vanishing cycles at a point y of $f^{-1}(f(x))$ are endowed with the monodromy of the fibration. We have the important theorem (see e.g. Ref. 8, Theorem I p. 89, or Ref. 4):

Theorem 7.1. *The monodromy automorphism of neighbouring (or vanishing) cycles is a quasi-unipotent automorphism, i.e. its eigenvalues are roots of unity.*

8. Constructible Complexes

Constructible sheaves over a complex analytic set $\mathcal{X}$ make a category where objects are constructible sheaves over $\mathcal{X}$, morphisms are morphisms of sheaves. Unit morphisms are identities of constructible sheaves and the composition is the composition of sheaf morphisms.

One can notice that, for each morphism of constructible sheaves over $\mathcal{X}$, one can define the kernel, the image and the cokernel of the morphism.

A complex of sheaves of complex vector spaces over $\mathcal{X}$ is a sequence of morphisms $(\varphi_n)_{n\in\mathbb{Z}}$, say $\phi_n \colon E_n \to E_{n+1}$, such that, for every $n \in \mathbb{Z}$, $\varphi_n \circ \varphi_{n-1} = 0$.

Complexes of sheaves of complex vector spaces over $\mathcal{X}$ make a category whose objects are complexes of sheaves of complex vector spaces over $\mathcal{X}$ and morphisms from $(\varphi_n)_{n\in\mathbb{Z}}$ into $(\psi_n)_{n\in\mathbb{Z}}$ are families $(h_n)_{n\in\mathbb{Z}}$ of sheaf morphisms such that, for any $n \in \mathbb{Z}$,

$$h_{n+1} \circ \varphi_n = \psi_n \circ h_n.$$

In practice we are interested in bounded complexes $(\varphi_n)_{n\in\mathbb{Z}}$ of sheaves of complex vector spaces over $\mathcal{X}$ such that for all $n \in \mathbb{Z}$ except a finite number φ_n is the trivial morphism from the null-sheaf 0 into itself. These complexes make also a full subcategory $D^b(\mathcal{X})$ of the preceding one. Now let $(\varphi_n)_{n\in\mathbb{Z}} \in D^b(\mathcal{X})$, since $\varphi_n \circ \varphi_{n-1} = 0$, the image of φ_{n-1} is a subsheaf of the kernel of φ_n. The quotient of the kernel sheaf of φ_n by the image sheaf of φ_{n-1} is by definition the n-th cohomology of $(\varphi_n)_{n\in\mathbb{Z}}$.

We say that the complex $(\varphi_n)_{n\in\mathbb{Z}}$ is constructible over $\mathcal{X}$, if it is bounded and, for any $n \in \mathbb{Z}$, the n-th cohomology of $(\varphi_n)_{n\in\mathbb{Z}}$ is a constructible sheaf over $\mathcal{X}$.

Constructible complexes over make a full subcategory $D^b_c(\mathcal{X})$ of $D^b(\mathcal{X})$.

Given a constructible complex $\mathbb{K}$ over $\mathcal{X}$, one can consider the hypercohomology $\mathbb{H}^*(\mathcal{X}, \mathbb{K})$ of $\mathbb{K}$ (see Ref. 1, Chap. XVII).

Using the notations of 7, one can consider the neighbouring cycles $\mathrm{R}^k(\psi_{f-f(x)})(\mathbb{K})$ of $\mathbb{K}$ along f at x, where

$$\mathrm{R}^k(\psi_{f-f(x)})(\mathbb{K})_y \simeq \mathbb{H}^k(F_y, \mathbb{K}).$$

9. Vanishing Cycles

Let X be a complex analytic space and $\mathbb{K}$ a constructible complex over $\mathcal{X}$. Let x be a point of $\mathcal{X}$ and $f : U \to \mathbb{C}$ be a complex analytic function defined on an open neighbourhood U of x in $\mathcal{X}$. We have defined the k-th neighbouring cycles of $\mathbb{K}$ along $f - f(x)$ at x as the sheaf $\mathrm{R}^k(\psi_{f-f(x)})(\mathbb{K})$, where $\mathrm{R}^k(\psi_{f-f(x)})(\mathbb{K})_y \simeq \mathbb{H}^k(F_y, \mathbb{K})$, over the space $f^{-1}(f(x))$.

It is convenient to define the sheaf $\mathrm{R}^k(\psi_{f-f(x)})(\mathbb{K})$ as the k-th cohomology of a bounded complex $\mathbf{R}(\psi_{f-f(x)})(\mathbb{K})$. The definition of the complex $\mathbf{R}(\psi_{f-f(x)})(\mathbb{K})$ uses derived categories and is rather abstract. Then, there is a natural morphism from the restriction $\mathbb{K}|f^{-1}(f(x))$ to $\mathbf{R}(\psi_{f-f(x)})(\mathbb{K})$. There is a natural triangle in the appropriate derived category:

$$\begin{tikzcd}
\mathbb{K}|f^{-1}(f(x)) \arrow[rr] & & \mathbf{R}(\psi_{f-f(x)})(\mathbb{K}) \arrow[dl] \\
& \mathbf{R}(\phi_{f-f(x)})(\mathbb{K}) \arrow[ul, "+1"] &
\end{tikzcd}$$

The complex $\mathbf{R}(\phi_{f-f(x)})(\mathbb{K})$ is called the *complex of vanishing cycles of* $\mathbb{K}$ *along* $f - f(x)$. The cohomology gives a long exact sequence:

$$\begin{aligned}\to \mathrm{H}^k(\mathbb{K}|f^{-1}(f(x))) \to \mathrm{R}^k(\psi_{f-f(x)})(\mathbb{K}) \to \mathrm{R}^k(\phi_{f-f(x)})(\mathbb{K})\\ \to \mathrm{H}^{k+1}(\mathbb{K}|f^{-1}(f(x))) \to\end{aligned}$$

The sheaf $\mathrm{R}^k(\phi_{f-f(x)})(\mathbb{K})$ is the sheaf of k-th vanishing cycles of $\mathbb{K}$ along f.

In special cases these sheaves are easy to interpret. Let $\mathcal{X} = \mathbb{C}^{n+1}$ and $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be complex analytic function with an isolated critical point at 0. Let U be an open neighbourhood of 0 in $\mathbb{C}^{n+1}$ where f has the only critical point 0. Let the complex $\mathbb{K}$ be the complex having one term in degree 0 equal to the constant sheaf $\mathbb{C}$ over $\mathbb{C}^{n+1}$ and the sheaf 0 in other degrees. The results of J. Milnor in Ref. 16 show that:

$$\mathrm{R}^k(\psi_{f-f(0)})(\mathbb{K}|U) = \begin{cases}\mathbb{C} \ \text{ if k=0}\\ 0 \text{ if } k \neq 0, n\\ \mathbb{C}^\mu \text{ at 0 if } k = n \text{ and 0 at } x \neq 0\end{cases}$$

Therefore if $n \geq 2$, the complex $\mathbf{R}(\psi_{f-f(0)})(\mathbb{K}|U)$ is the complex with the constant sheaf $\mathbb{C}|U$ in degree 0, the sheaf with one non-trivial stalk $\mathbb{C}^\mu$ at 0 in degree 2, and all morphisms are zero.

The complex $\mathbf{R}(\phi_{f-f(0)})(\mathbb{K}|U)$ has only one term in degree 2 which is the the sheaf with one non-trivial stalk $\mathbb{C}^\mu$ at 0, all the other terms in degree $\neq 2$ being 0.

One can observe that in this special case of isolated critical point the complex of vanishing cycles of the complex $\mathbb{K}|U$ along f consists of a sheaf non-trivial in the degree equal to the dimension of $f^{-1}(0)$ which has only one non-trivial stalk over the isolated critical point 0.

It can be proved that it is true for any space $\mathcal{X}$ satisfying the Milnor condition (see Ref. 10, §5) for functions having isolated singularities in the general sense of (Ref. 10, §1) and for any complex $\mathbb{K}_X$ equal to the constant sheaf $\mathbb{C}_\mathcal{X}$ over $\mathcal{X}$ in degree 0 and to the trivial sheaf 0 in other degrees.

The support of a constructible sheaf $\mathcal{L}$ on the complex analytic space $\mathcal{X}$ is the complex analytic subspace $\mathcal{Y}$ closure of the set of points x where $\mathcal{L}_x \neq 0$.

One says that a constructible complex $\mathbb{K}$ on $\mathbb{C}^N$ satisfies the support condition if the codimension of the support of its i-th cohomology $\mathrm{H}^i(\mathbb{K})$ is $\geq i$.

The Verdier dual of a constructible complex $\mathbb{K}$ on $\mathbb{C}^N$ is the derived complex $\mathrm{RHom}_{\mathbb{C}_{\mathbb{C}^N}}(\mathbb{K}, \mathbb{C}_{\mathbb{C}^N})$. Beware that the Verdier dual of a complex on

$\mathbb{C}^N$ is not the complex of duals. On the other hand, there is also a notion of duality for a constructible complex on a complex analytic space. However in this general case the construction is not as "simple" as the Verdier dual. The construction of the Verdier dual is usually difficult. Properties of the Verdier dual can be found in Ref. 3. One can prove that the Verdier dual of a constructible complex is also a constructible complex.

One says that the constructible complex $\mathbb{K}$ on $\mathbb{C}^N$ satisfies the cosupport condition, if the Verdier dual $\check{\mathbb{K}}$ of $\mathbb{K}$ satisfies the support condition.

A constructible complex $\mathbb{K}$ on $\mathbb{C}^N$ is *perverse*, if it satisfies the support condition and the cosupport condition. We shall call perverse sheaf a perverse constructible complex.

One can prove that more generally, if $\mathbb{K}$ is a perverse sheaf on $\mathbb{C}^N$, the complex of vanishing cycles along a function f which has an isolated critical point in the general sense of Ref. 10, §1 at 0 for all the restrictions of f to the closures of the strata on which all the cohomologies $\mathrm{H}^k(\mathbb{K})$ are locally constant, in an open neighbourhood U of 0, consists in the degree equal to the dimension n of $f^{-1}(0)$, i.e. for $\mathrm{R}^n(\psi_{f-f(0)})(\mathbb{K})|U$, of a sheaf which has only one non-trivial stalk over the isolated critical point 0 and 0 in the other degrees:

$$\mathrm{R}^n(\phi_{f-f(0)})(\mathbb{K})_0 = \mathbb{C}^k$$

$$\mathrm{R}^n(\phi_{f-f(0)})(\mathbb{K})_y = 0, \text{ for } y \neq 0$$

One has the following result due to P. Deligne (see e.g. Ref. 11):

Theorem 9.1. *Let $\mathbb{K}$ be a perverse sheaf on $\mathbb{C}^N$. Let $x \in \mathbb{C}^N$. For almost all linear fonction ℓ of $\mathbb{C}^N$, the sheaf of vanishing cycles of $\mathbb{K}$ along ℓ in an open neighbourhood of x is either zero or a complex which is zero in all degrees except in degree equal to the dimension of $f^{-1}(f(x)$, where it has a non-trivial stalk only at x.*

We can define:

Definition 9.1. Let $\mathbb{K}$ be a perverse sheaf on $\mathbb{C}^N$. The subvariety $\mathcal{V}(\mathbb{K})$ of the cotangent bundle $T^*(\mathbb{C}^N)$ of $\mathbb{C}^N$ is the characteristic variety of $\mathbb{K}$ if, a point $(x, \ell) \in \mathcal{V}(\mathbb{K})$ if and only if it belongs to the closure of the points (y, l) for which there is a neighbourhood U where $\mathrm{R}^k(\phi_{l-l(y)})(\mathbb{K})|U = 0$ for $k \neq n = \dim l^{-1}(0)$ and $\mathrm{R}^n(\phi_{l-l(y)})(\mathbb{K})|U$ is a non-trivial skyscraper sheaf with a non-zero fiber at y.

10. Holonomic $\mathcal{D}$-Modules

Let $\mathcal{X}$ a complex analytic manifold of complex dimension n. On $\mathcal{X}$ consider the sheaf $\mathcal{D}$ of complex analytic differential operators. Notice that this sheaf is filtered by the sheaves $\mathcal{D}(k)$ of differential operators of order $\leq k$.

We consider sheaves on $\mathcal{X}$ which are coherent left $\mathcal{D}$-modules. We shall call these sheaves simply $\mathcal{D}$-modules.

For instance, if $\mathcal{X}$ is a Riemann surface, in this meeting L. Narvaez considered $\mathcal{D}$-modules over a complex analytic manifold of dimension 1.

Since the sheaf $\mathcal{D}$ itself is coherent, a $\mathcal{D}$-module $\mathcal{M}$ is locally of finite presentation, i.e. for any $x \in \mathcal{X}$, there is an open neighbourhood U_x over which one has an exact sequence:

$$(\mathcal{D})|{U_x}^p \stackrel{\psi}{\to} (\mathcal{D}|U_x)^q \stackrel{\varphi}{\to} \mathcal{M}|U_x \to 0.$$

One can notice that $\varphi((\mathcal{D}(k)|U_x)^q) = \mathcal{M}(k)$ defines a *good filtration* of $\mathcal{M}$ (see the lectures of F. Castro).

The complex analytic space Specan $\oplus_{k\geq 0} (\mathcal{D}(k)|U_x)/(\mathcal{D}(k-1)|U_x)$ corresponding to the commutative graded ring $\mathrm{gr}(\mathcal{D}|U_x)$associated to the filtration of $\mathcal{D}|U_x$ by the $\mathcal{D}(k)|U_x$ is the cotangent space of U_x. The support of the $\mathrm{gr}(\mathcal{D}|U_x)$-module $\oplus_{k\geq 0}\mathcal{M}|U_x(k)/\mathcal{M}|U_x(k-1)$ is the characteristic variety $Ch(\mathcal{M}|U_x)$ of the $\mathcal{D}|U_x$-modules $\mathcal{M}|U_x$. One can show that this definition does not depend on the local finite presentation of $\mathcal{M}$.

We shall say that the $\mathcal{D}$-module $\mathcal{M}$ is *holonomic* if the dimension of the characteristic variety $Ch(\mathcal{M})$ is equal to the dimension n of the manifold $\mathcal{X}$.

As it is done in Ref. 7, we obtain the following relation between the characteristic variety of a holonomic $\mathcal{D}$-module and the topology (see Proposition 10.6.5 of Ref. 7):

Theorem 10.1. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module on a complex analytic manifold X. There is a Whitney stratification $(X_\alpha)_{\alpha\in A}$ of X, such that the characterisitc variety of $\mathcal{M}$ is contained in the union of the conormal bundles $T^*_{X_\alpha}X$ of the strata X_α in X.*

Recall that if S is a submanifold of X, the conormal $T^*_S X$ of S in X is the bundle of $(s,\ell) \in T^*X$, such that ℓ is a linear form which vanishes on $T_s(S)$. It can be seen that, when X is a manifold and S is submanifold, the conormal bundle $T^*_S X$ of S in X is a Lagrangean submanifold of T^*X.

In fact we can obtain a more precise result.

As it is done for $n = 1$ in the lectures of L. Narvaez, we define the analytic solutions of $\mathcal{D}$-module $\mathcal{M}$ to be the complex $\mathrm{RHom}_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$. This is the stable (derived) version of the "naive" definition $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$.

M. Kashiwara proved in Ref. 6 that:

Theorem 10.2. *If the $\mathcal{D}$-module $\mathcal{M}$ is holonomic, the analytic solutions $\mathrm{RHom}_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$ of $\mathcal{M}$ is a constructible complex which satisfies the support condition.*

A theorem of Z. Mebkhout (see Ref. 15) shows that:

Theorem 10.3. *The Verdier dual of the analytic solutions $\mathrm{RHom}_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$ of the holonomic $\mathcal{D}$-module $\mathcal{M}$ is the analytic solutions of a holonomic $\mathcal{D}$-module, dual of $\mathcal{M}$.*

It follows that:

Corollary 10.1. *The analytic solutions $\mathrm{RHom}_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$ of the holonomic $\mathcal{D}$-module $\mathcal{M}$ is a perverse sheaf.*

From this result and the ones of the preceding section we can state a result of D. T. Lê and Z. Mebkhout:[12]

Theorem 10.4. *The characteristic variety of an holonomic $\mathcal{D}$-module $\mathcal{M}$ is the characteristic variety of the perverse sheaf of its analytic solutions.*

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