Valuations on complete local rings

Bernard Teissier

Institut Mathématique de Jussieu-Paris Rive Gauche

Special Session on Advances in Valuation Theory

AMS Sectional Meeting at Rutgers University

November 14, 2015

Framework (1)

• A noetherian equicharacteristic local domain R dominated by a valuation ring R_ν of its field of fractions.
• We denote by Φ the value group of the valuation ν determined by R_ν, and by $\Gamma = \nu(R \setminus \{0\})$ the semigroup of values of ν on R.
• Since R is noetherian, Γ is well ordered and so it has a natural minimal system of generators

$$\Gamma = \langle \gamma_1, \ldots, \gamma_i, \ldots, \gamma_\omega, \gamma_{\omega+1}, \ldots \rangle$$

indexed by an ordinal $I \leq \omega^h$, where h is the rank, or height, of the valuation. In fact the ordinal of Γ is $\leq \omega^h$.
• We shall most of the time assume that the valuation is *rational* in the sense that the residue fields extension $k = R/m \subset R_\nu/m_\nu = k_\nu$ is trivial.

Framework (2)

- For any subring R of R_ν we can define the valuation ideals

$$\mathcal{P}_\phi(R) = \{x \in R | \nu(x) \geq \phi\} \text{ and } \mathcal{P}^+_\phi(R) = \{x \in R | \nu(x) > \phi\}.$$

If R is noetherian and so the semigroup of values is well ordered, and $\phi \in \Gamma$, then $\mathcal{P}^+_\phi(R) = \mathcal{P}_{\phi^+}(R)$, where ϕ^+ is the successor of ϕ in Γ. If $\phi \notin \Gamma$, then $\mathcal{P}^+_\phi(R) = \mathcal{P}_\phi(R)$.

- For any subring R of R_ν, define the associated graded ring

$$\mathrm{gr}_\nu R = \bigoplus_{\phi \in \Phi} \mathcal{P}^+_\phi(R)/\mathcal{P}^+_\phi(R)(R).$$

It is not noetherian in general, even if R is.

- $\mathrm{gr}_\nu R_\nu$ is isomorphic to the semigroup algebra $k_\nu[t^{\Phi_{\geq 0}}]$.

In the case of a rational valuation of a noetherian local domain...

• By a theorem of Piltant, the Krull dimension of $\mathrm{gr}_\nu R$ is the rational rank of the group Φ. So Abhyankar's inequality reads:

$$\dim \mathrm{gr}_\nu R \leq \dim R.$$

• Since $k = k_\nu$, the graded algebra $\mathrm{gr}_\nu R$ shares with $\mathrm{gr}_\nu R_\nu$ the property that each nonzero homogeneous component is a one-dimensional vector space over k.
In fact $\mathrm{gr}_\nu R$ is the graded k-subalgebra of $\mathrm{gr}_\nu R_\nu$ generated by the elements whose degree is in the semigroup Γ.
• $\mathrm{gr}_\nu R$ is isomorphic to the semigroup algebra $k[t^\Gamma]$.

If R is noetherian, a minimal system of generators of $\mathrm{gr}_\nu R$ exists and is in bijection with the elements of a minimal system of generators of Γ.

And this is where valuation theory meets toric geometry:

Any graded k-algebra whose nonzero homogeneous components are 1-dimensional k-vector spaces is a quotient of a polynomial algebra over k by a binomial ideal:

$$\Pi\colon k[(U_i)_{i\in I}] \to \mathrm{gr}_\nu R,$$

where the U_i are in bijection with a system of generators of the k-algebra and the kernel is generated by binomials $(U^{m^\ell} - \lambda_\ell U^{n^\ell})_{\ell\in L}$.

Giving U_i the weight γ_i makes Π a graded map of degree zero.

Examples (1)

• Let Φ be $\mathbf{Z}^2$ with the lexicographic order. Then $\Phi_{\geq 0}$ is not well ordered and has no minimal set of generators in the sense given above. However, it is generated, for any $a \in \mathbf{Z}$, by the elements $((0,1),(1,-j)_{j\geq a})$ which are related by:

$$(1,-j) = (1,-(j+1)) + (0,1),$$

and we have:

$$k[t^{\mathbf{Z}^2_{\geq 0}}] = k[W,(U_j)_{j\in\mathbf{N}}]/((U_j - WU_{j+1})_{j\in\mathbf{N}}).$$

Examples (2)

• Assume that k is algebraically closed and $R = k[[u_1, u_2]]/(f)$ with $f \in (u_1, u_2)k[[u_1, u_2]]$ irreducible. The valuation ring is the normalization $k[[t]]$ of R. The semigroup $\Gamma \subset \mathbf{N}$ is finitely generated as are all numerical semigroups, say $\Gamma = \langle \gamma_1, \ldots, \gamma_t \rangle$. There are relations $n_i\gamma_i \in \langle \gamma_1, \ldots, \gamma_{i-1} \rangle$ which generate all relations between the γ_i, and inequalities $n_i\gamma_i < \gamma_{i+1}$. Writing the relations as $n_i\gamma_i = \ell_1^{(i)}\gamma_1 + \cdots + \ell_{i-1}^{(i)}\gamma_{i-1}$, with $\ell_j^{(i)} \in \mathbf{N}$, we see that:

$$\mathrm{gr}_\nu R = k[U_1, \ldots, U_t]/(U_2^{n_2} - \lambda_2 U_1^{\ell_1^{(2)}}, \ldots, U_t^{n_t} - \lambda_t U_1^{\ell_1^{(t)}} \ldots U_{t-1}^{\ell_{t-1}^{(t)}}), \ \ \lambda_i \in k^*.$$

If k is of characteristic zero, by a result of Zariski, the datum of the γ_i is equivalent to the datum of the Puiseux characteristic exponents of the curve. In particular they are equal in numbers. In positive characteristic, even if there is a Puiseux expansion, this is no longer true.

Examples (3)

• Example (Campillo): k of characteristic p, the curve given parametrically by

$$u_1 = t^{p^3}, u_2 = t^{p^3+p^2} + t^{p^3+p^2+p+1}$$

has semigroup

$$\Gamma = \langle p^3,\ p^3+p^2,\ p^4+p^3+p^2+p,\ p^5+p^4+p^3+p^2+p+1\rangle.$$

We see three exponents but the semigroup has four generators. The relation of this phenomenon with Kedlaya's description of the algebraic closure of $k((t))$ should be clarified.

Examples (4)

• A typical example, due to Zariski, of a valuation of $R = k[u_1, u_2]_{(u_1,u_2)}$ for which Abhyankar's inequality is strict has a semigroup which is an infinite version of the curve case:
$\Gamma = \langle \gamma_1, \ldots, \gamma_i, \ldots \rangle$ where the $\gamma_i \in \mathbf{Q}_+$ have relations generated by

$$n_i\gamma_i = \ell_1^{(i)}\gamma_1 + \cdots + \ell_{i-1}^{(i)}\gamma_{i-1}$$

and satisfy the inequalities

$$n_i\gamma_i < \gamma_{i+1}.$$

Then,

$$\mathrm{gr}_\nu R = k[(U_i)_{i\in\mathbf{N}}]/(U_i^{n_i} - \lambda_i U_1^{\ell_1^{(i)}} \ldots U_{i-1}^{\ell_{i-1}^{(i)}})_{i\geq 2}, \;\; \lambda_i \in k^*.$$

By Piltant's theorem the dimension of $\mathrm{gr}_\nu R$ is one while R is of dimension two.

Examples (5): And where is the valuation?

Let us deform the equations defining $\mathrm{gr}_\nu R$ as follows: each one is transformed into

$$u_i^{n_i} - \lambda_i u_1^{\ell_1^{(i)}} \ldots u_{i-1}^{\ell_{i-1}^{(i)}} - u_{i+1} = 0.$$

These polynomials generate an ideal I of $k[(u_i)_{i\in\mathbf{N}}]$ and a direct argument of elimination shows that $k[(u_i)_{i\in\mathbf{N}}]/I = k[u_1, u_2]$.

The inequalities $n_i\gamma_i < \gamma_{i+1}$ mean that we have added to the initial binomial which is homogeneous of degree $n_i\gamma_i$ an element of strictly higher weight.

Given a polynomial $P(u_1, u_2)$, iteratively substituting, whenever possible, $\lambda_i u_1^{\ell_1^{(i)}} \ldots u_{i-1}^{\ell_{i-1}^{(i)}} + u_{i+1}$ for each occurrence of $u_i^{n_i}$ leads in a finite number of steps to a polynomial $\tilde{P}(u_1, u_2, \ldots, u_k)$ where no more substitution is possible and *the map which associates to* $P(u_1, u_2)$ *the weight of* $\tilde{P}(u_1, u_2, \ldots, u_k)$ *defines a valuation on* $R = k[u_1, u_2]_{(u_1,u_2)}$*, whose associated graded ring is our* $\mathrm{gr}_\nu R$.

Deforming binomial ideals (1)

For this lecture, a weight on $k[(u_i)_{i\in I}]$ or $k[[(u_i)_{i\in I}]]$ will be a morphism of semigroups $w\colon M(I)\to \Phi_{\geq 0}$, where $M(I)$ denotes the semigroup of monomials in the u_i, which attributes to each variable u_i a weight $w(u_i)=\gamma_i\in \Phi_{\geq 0}$.

• A weight is compatible with a binomial ideal (an ideal generated by binomials) if each generating binomial is homogeneous.

• The weight of a polynomial, or a series, is the minimum weight of its terms.

Deforming binomial ideals (2)

Given a weight which is compatible with it, an *overweight deformation* of a binomial is an expression

$$F = u^m - \lambda_{mn} u^n + \sum_{w(u^p) > w(u^m)} c_p u^p, \; c_p \in k.$$

If we have any number of binomials and a compatible weight we do the same and consider the deformations

$$F_\ell = u^{m^\ell} - \lambda_\ell u^{n^\ell} + \sum_{w(u^p) > w(u^{m^\ell})} c_p^{(\ell)} u^p, \; c_p^{(\ell)} \in k.$$

One has to add the condition that *the initial binomials of the F_ℓ generate the ideal F_0 of initial forms of the elements of the ideal F generated by the F_ℓ.*

Deforming binomial ideals in power series rings

Let us consider an overweight deformation (F_ℓ) of a prime binomial ideal F_0 in the power series ring $k[[u_1, \ldots, u_N]]$, and the map

$$\pi\colon k[[u_1, \ldots, u_N]] \to R = k[[u_1, \ldots, u_N]]/(F_1, \ldots, F_s).$$

Proposition

1) The map which associates to $x \in R$ the *maximum* of the weights of the elements of $\pi^{-1}(x)$ is well defined and is a valuation ν on R.
2) The associated graded ring of $k[[u_1, \ldots, u_N]]$ with respect to the weight filtration is $k[U_1, \ldots, U_N]$ and the map

$$\Pi\colon k[U_1, \ldots, U_N] \to k[U_1, \ldots, U_N]/F_0 = \mathrm{gr}_\nu R$$

is the associated graded map of π with respect to the weight and valuation filtrations.
3) Given $\tilde{x} \in \pi^{-1}(x)$, we have $w(\tilde{x}) = \nu(x)$ if and only if $\mathrm{in}_w(\tilde{x}) \notin F_0$.

An example

Let $F_1 = u_2^2 - u_1^3 - u_3, F_2 = u_3^2 - u_1^5 u_2$ be an overweight deformation of the binomial ideal $F_0 = (u_2^2 - u_1^3, F_2 = u_3^2 - u_1^5 u_2)$, with u_1 of weight 4, u_2 of weight 6 and u_3 of weight 13.
The element $x \in k[[u_1, u_2, u_3]]/(F_1, F_2)$ which is the image of the element $u_2^2 - u_1^3$, of weight 12, has another counterimage of weight 13, namely u_3, which gives the maximum since it is not in the binomial ideal.
The binomials $U_2^2 - U_1^3, U_3^2 - U_1^5 U_2$ generate the kernel of the surjective map

$$\Pi \colon k[U_1, U_2, U_3] \to k[t^\Gamma] = k[t^4, t^6, t^{13}], \quad U_i \mapsto t^{\gamma_i}.$$

The equations $F_1 = F_2 = 0$ define a curve C in $\mathbf{A}^3(k)$ which is isomorphic to the plane curve $(u_2^2 - u_1^3)^2 - u_1^5 u_2 = 0$, whose semigroup is $\langle 4, 6, 13\rangle$. In $\mathbf{A}^3(k)$ our curve is an overweight deformation of a monomial curve which corresponds to its associated graded ring. We know how to make embedded resolutions of singularities of affine toric varieties in any characteristic. The point is that some of the birational toric maps $Z(\Sigma) \to \mathbf{A}^3(k)$ which resolve the monomial curve will also resolve the curve C because of the overweight condition (in this particular case, they all do).

Some philosophy of filtrations

So it seems that we should study more closely the relation between R and its associated graded ring $\mathrm{gr}_\nu R$.

There is a general principle of commutative algebra: if our local ring contains a field of representatives, a filtration such as the filtration by the valuation ideals $\mathcal{P}_\phi$, determines a faithfully flat (over $k[t^{\Phi_{\geq 0}}]$) specialization of R to the associated graded ring, in our case $\mathrm{gr}_\nu R$.

The classical Cohen theorem concerning the m-adic filtration of a noetherian local ring shows us that unless our local ring is complete we cannot hope for a usable description, for example by equations, of the specialization of R to its associated graded ring. But if the local domain R is complete and equicharacteristic, it has a field of representatives $k \subset R$ and in this case the formal space X corresponding to R embeds in an affine space over k of the same dimension as the embedding dimension of $\mathrm{gr}_m R$, which is that of the Zariski tangent space. Then we can write equations for the specialization of R to $\mathrm{gr}_m R$.

We shall see that we can write equations for R which describe the specialization of R to $\mathrm{gr}_\nu R$, but for that we first must embed X in the same space where $\mathrm{Spec}\,\mathrm{gr}_\nu R$ lives:

The power series ring adapted to Γ

Let $(u_i)_{i\in I}$ be variables indexed by the elements of the minimal system of generators $(\gamma_i)_{i\in I}$ of the semigroup Γ of the valuation ν on R. Give each u_i the weight $w(u_i) = \gamma_i$ and let us consider the set of power series $\sum_{e\in E} d_e u^e$ where $(u^e)_{e\in E}$ is any set of monomials in the variables u_i and $d_e \in k$.

By a theorem of Campillo-Galindo, the semigroup Γ being well ordered is combinatorially finite, which means that for any $\phi \in \Gamma$ the number of different ways of writing ϕ as a sum of elements of Γ is finite. This is equivalent to the fact that the set of exponents e such that $w(u^e) = \phi$ is finite: for any given series the map $w\colon E \to \Gamma,\ e \mapsto w(u^e)$ has finite fibers. Each of these fibers is a finite set of monomials in variables indexed by a totally ordered set, and so can be given the lexicographical order and order-embedded into an interval $1 \leq i \leq n$ of $\mathbf{N}$.

This defines an injection of the set E into $\Gamma \times \mathbf{N}$ equipped with the lexicographical order and thus induces a total order on E, for which it is well ordered. When E is the set of all monomials, this gives a total monomial order.
The combinatorial finiteness implies that we can multiply two series so that this set of series is a k-algebra, which we denote by

$$k\widehat{[(u_i)_{i\in I}]}$$

Since the weights of the elements of a series form a well ordered set and only a finite number of terms of the series have minimum weight, the associated graded ring of $k\widehat{[(u_i)_{i\in I}]}$ with respect to the filtration by weights is the polynomial ring $k[(U_i)_{i\in I}]$.

Facts about $k\widehat{[(u_i)_{i\in I}]}$

• In $k\widehat{[(u_i)_{i\in I}]}$, a series without constant term can be substituted for a variable in another series.

• In particular, if $h(u) \in k\widehat{[(u_i)_{i\in I}]}$ has no constant term, $\sum_{a=0}^{\infty} h(u)^a \in k\widehat{[(u_i)_{i\in I}]}$ so that $k\widehat{[(u_i)_{i\in I}]}$ is a local k-algebra.

• $k\widehat{[(u_i)_{i\in I}]}$ is spherically complete with respect to the "ultrametric" $w(x-y)$. The balls are of the form $\mathbf{B}(x,\gamma) = \{h | w(h-x) \geq \gamma\}$. If $\Phi \subset \mathbf{R}$ the ultrametric is of course $||x-y|| = e^{-w(x-y)}$).

All this is rather easy to prove when the valuation is of rank one.

The natural embedding of the space corresponding to $k[[t^\Gamma]]$

Since Γ is well ordered, the algebra $k[[t^\Gamma]]$ is naturally a subalgebra of the Hahn algebra $k[[t^{\Phi_{\geq 0}}]]$.
There is a natural continuous surjection of topological k-algebras

$$k\widehat{[(u_i)_{i\in I}]} \to k[[t^\Gamma]], \quad u_i \mapsto t^{\gamma_i},$$

which gives the "natural" embedding of the space corresponding to $k[[t^\Gamma]]$ into a non singular space. Indeed, $k\widehat{[(u_i)_{i\in I}]}$ is formally smooth over k.

The valuative Cohen theorem

Assuming that the local noetherian equicharacteristic domain R is complete, with a rational valuation ν, and fixing a field of representatives $k \subset R$, there exist choices of representatives $\xi_i \in R$ of the $\overline{\xi_i}$ generating the k-algebra $\mathrm{gr}_\nu R$ such that the surjective map of k-algebras $k[(U_i)_{i\in I}] \to \mathrm{gr}_\nu R,\ U_i \mapsto \overline{\xi_i}$, is the associated graded map of a continuous surjective map

$$k\widehat{[(u_i)_{i\in I}]} \to R,\ u_i \mapsto \xi_i,$$

of topological k-algebras, with respect to the weight and valuation filtrations respectively. The kernel of this map is generated up to closure by overweight deformations of binomials generating the kernel of $k[(U_i)_{i\in I}] \to \mathrm{gr}_\nu R$, $U_i \mapsto \overline{\xi}_i$.

• If ν is of rank one or if Γ is finitely generated, any choice of representatives ξ_i is permitted.

The proof in rank one is an application of Chevalley's theorem. The proof in general is by induction on the rank and more complicated.

This can be seen as a "comfortable" embedding for the singularity represented by R, but it is in general not finite dimensional. So we must try to extract from it a finite dimensional comfortable embedding, using the fact that R is noetherian.

The good thing is that we have equations for R of the form

$$F_\ell = u^{m^\ell} - \lambda_\ell u^{n^\ell} + \sum_{w(u^p) > w(u^{m^\ell})} c_p^{(\ell)} u^p,$$

and the bad thing is that there are infinitely many variables and equations in general.

But there is a case where all is well:

As a consequence, whenever R is complete and equicharacteristic, if the valuation ν is rational and the semigroup Γ is finitely generated, the ring R is an overweight deformation of the binomial ideal defining its associated graded ring.

Indeed, in this case, the ring $k\widehat{[(u_i)_{i\in I}]}$ is an ordinary power series ring $k[[u_1, \ldots, u_N]]$, except that the variables have weights $\gamma_i \in \Gamma$.

The valuative Cohen theorem is a way of asserting that this is also true when Γ is not finitely generated.

So we are strongly motivated to reduce the study of rational valuations on R to the study of valuations on complete local rings.

Conjecture: If the local equicharacteristic noetherian domain R is excellent, given a rational valuation on R, there exists a prime ideal $H \subset \hat{R}$ such that $H \cap R = (0)$ and an extension $\hat{\nu}_-$ of ν to $\hat{R}/H$ such that the value groups of ν and $\hat{\nu}_-$ are the same. One can even, as suggested by Spivakovsky, hope that after some birational ν-extension of R, the semigroups of R and $\hat{R}/H$ are equal.

This stronger result is known in two cases: the rank of ν is one, or ν is Abhyankar, i.e., we have equality in Abhyankar's inequality.

Let us see how this works for Abhyankar valuations. We can reduce to the study of Abhyankar valuations on complete equicharacteristic noetherian local rings. Let us first study the case where the graded algebra is finitely generated. This implies that the valuation is Abhyankar.

The case of a rational valuation of a complete noetherian domain with an algebraically closed residue field with a finitely generated semigroup

In this case, we have:

$$F_1 = u^{m^1} - \lambda_1 u^{n^1} + \Sigma_{w(p)>w(m^1)} c_p^{(1)} u^p$$
$$F_2 = u^{m^2} - \lambda_2 u^{n^2} + \Sigma_{w(p)>w(m^2)} c_p^{(2)} u^p$$
$$.....$$
$$F_\ell = u^{m^\ell} - \lambda_\ell u^{n^\ell} + \Sigma_{w(p)>w(m^\ell)} c_p^{(\ell)} u^p$$
$$.....$$
$$F_s = u^{m^s} - \lambda_s u^{n^s} + \Sigma_{w(p)>w(m^s)} c_p^{(s)} u^p$$

in $k[[u_1, \ldots, u_N]]$ such that, with respect to the monomial order determined by w, they form a standard basis for the ideal which they generate: their initial forms generate the ideal of initial forms of elements of that ideal.

Toric maps (1)

Denoting by X the formal subspace of $\mathbf{A}^N(k)$ defined by the ideal $F = (F_1, \ldots, F_s)$ of $k[[u_1, \ldots, u_N]]$, we want tu study the effect on X of proper birational toric maps $\pi(\Sigma)\colon Z(\Sigma) \to \mathbf{A}^N(k)$. Such a map is determined by a regular fan Σ refining the fan Σ_0 consisting of the rational convex cone $\sigma_0 = \check{\mathbf{R}}^N_{\geq 0}$ and its faces. Let us just say that $Z(\Sigma)$ is a non singular toric variety which is covered by charts $Z(\sigma)$. Each $Z(\sigma)$ is associated to a regular cone $\sigma \in \Sigma$ generated by integral vectors: $\sigma = \langle a^1, \ldots, a^N \rangle$ where the a^i are a basis of the integral lattice N of $\check{\mathbf{R}}^N$.

Toric maps (2)

Let $\sigma \subset \check{\mathbf{R}}^N_{\geq 0}$ be such an inclusion of regular cones, with

$$\sigma = \langle a^1, \ldots, a^N \rangle, \; a^i \in \check{\mathbf{Z}}^N, \; \det(a^i_j) = \pm 1.$$

It corresponds to the map of affine spaces $Z(\sigma) = \mathbf{A}^N_\sigma \to \mathbf{A}^N$ given in coordinates by:

$$\begin{array}{ll} u_1 = & y_1^{a_1^1} y_2^{a_1^2} \ldots y_N^{a_1^N} \\ u_2 = & y_1^{a_2^1} y_2^{a_2^2} \ldots y_N^{a_2^N} \\ . & \qquad\qquad . \\ . & \qquad\qquad . \\ . & \qquad\qquad . \\ u_N = & y_1^{a_N^1} y_2^{a_N^2} \ldots y_N^{a_N^N} \end{array}$$

NOTE: the divisors which are contracted (exceptional divisors) are exactly those $y_i = 0$ such that the vector a^i is not a vector of the canonical basis of $\mathbf{Z}^N$.

In the toric jargon, the complement of the union of coordinate hyperplanes is called "the torus" and the condition that the vectors a^i generate the integral lattice, which is equivalent to $\det(a^i_j) = \pm 1$, is equivalent to saying that our monomial map $\mathbf{A}^N_\sigma \to \mathbf{A}^N$ induces an isomorphism of the tori of the two affine spaces.

Resolution of singularities of normal toric varieties

Regular refinements of rational cones, Mumford et al., 1973

Given any finite system of rational convex cones in $\check{\mathbf{R}}^N_{\geq 0}$, there exist regular fans filling $\check{\mathbf{R}}^N_{\geq 0}$ and compatible with those cones.

Here compatibility means that each cone of the fan meets the given rational cones along a face.
This result implies that a normal toric variety, which is given combinatorially by a fan of not necessarily regular strictly convex rational cones, has a (non-embedded and non-canonical) resolution of singularities corresponding to a refinement of the given fan into a regular fan.

Embedded resolution of singularities of toric varieties (1)

(Pedro González Pérez and B.T., 2002)

Resolution of binomial ideals

The effect of a monomial map on a binomial is very easy to compute:
If the monomial map is given by

$$
\begin{aligned}
u_1 &= y_1^{a_1^1} y_2^{a_1^2} \ldots y_N^{a_1^N} \\
u_2 &= y_1^{a_2^1} y_2^{a_2^2} \ldots y_N^{a_2^N} \\
&\;\;\vdots \\
u_N &= y_1^{a_N^1} y_2^{a_N^2} \ldots y_N^{a_N^N}
\end{aligned}
$$

Embedded resolution of singularities of toric varieties (2)

If we set $\langle a^i, m\rangle = \sum_{k=1}^N a^i_k m_k$, the transform of the monomial u^m is

$$u^m \mapsto y_1^{\langle a^1, m\rangle} \dots y_N^{\langle a^N, m\rangle}.$$

And so

$$u^m - \lambda_{mn} u^n \mapsto y_1^{\langle a^1, m\rangle} \dots y_N^{\langle a^N, m\rangle} - \lambda_{mn} y_1^{\langle a^1, n\rangle} \dots y_N^{\langle a^N, n\rangle}.$$

Now comes the important

REMARK

If the cone σ is compatible with the hyperplane $H_{m-n} \subset \check{\mathbf{R}}^N$ which is the linear dual of the vector $m - n \in \mathbf{R}^N$, then by definition it is entirely on one side of this hyperplane so that all the $\langle a^i, m - n\rangle$ which are not zero are of the same sign.
Up to exchanging m and n and reordering the a^i we may assume that $\langle a^i, m - n\rangle = 0$ for $1 \leq i \leq t$ and that $\langle a^{t+1}, m - n\rangle, \ldots, \langle a^N, m - n\rangle$ are > 0. But then we can rewrite:

$$u^m - \lambda_{mn} u^n \mapsto y_1^{\langle a^1, n\rangle} \ldots y_N^{\langle a^N, n\rangle} (y_{t+1}^{\langle a^{t+1}, m-n\rangle} \ldots y_N^{\langle a^N, m-n\rangle} - \lambda_{mn}).$$

And if $u^m - \lambda_{mn} u^n$ is one of the generators of a prime binomial ideal, the vector $m - n \in \mathbf{Z}^N$ has to be primitive (its non zero components are coprime) and because the a^i are a basis of the integral lattice, the vector with coordinates $\langle a^i, m - n\rangle$ is also primitive. But since $\lambda_{mn} \neq 0$ this implies that the hypersurface

$$y_{t+1}^{\langle a^{t+1}, m-n\rangle} \dots y_N^{\langle a^N, m-n\rangle} - \lambda_{mn} = 0,$$

which is the strict transform of our binomial, is non singular, *whatever the characteristic of k is*.

Embedded resolution of singularities of toric varieties (End)

Now you will find it easy to believe that things work in the same way for a prime binomial ideal

$$(u^{m^\ell} - \lambda_\ell u^{n^\ell})_{\ell \in L} \subset k[u_1, \ldots, u_N]$$

by taking a regular fan in $\check{\mathbf{R}}^N_{\geq 0}$ compatible with all the hyperplanes $H_{m^\ell - n^\ell}$.

This is always possible because of the theorem of resolution of normal toric varieties.

The fact that the vector $m - n$ is primitive has to be replaced by the fact that the lattice $\mathcal{L} \subset \mathbf{Z}^N$ generated by the vectors $m^\ell - n^\ell$ is *saturated*, which means that it is a direct factor in $\mathbf{Z}^N$.

The fact that $\mathcal{L}$ is saturated if the ideal is prime uses that k is algebraically closed.

Finally, the jacobian minors of a binomial ideal are related in a very simple manner with the minors of the matrix of the vectors $m^\ell - n^\ell$.

Then, the overweight condition implies, by a purely combinatorial argument, that one can find regular fans in $\check{\mathbf{R}}^N_{\geq 0}$ defining birational toric maps which not only resolve the singularities of the toric variety associated to the graded ring $\mathrm{gr}_\nu R$ but also uniformize the valuation ν on R.

The idea is very simple to explain in the case of a single equation

$$F = u^m - \lambda u^n + \sum_{w(p)>w(m)} c_p u^p.$$

Let

$$E' = \langle \{p - n / c_p \neq 0\}, m - n \rangle \subset \mathbf{R}^N,$$

where $\langle a, b, ... \rangle$ denotes the cone generated by $a, b, ...$. Since there may be infinitely many exponents p, the smallest closed convex cone containing E' may not be rational. However, the power series ring being noetherian, there exist finitely many exponents $(p_f - n)_{f \in F}$ as above, with F finite, such that E' is contained in the rational cone E generated by $m - n$, the vectors $(p_f - n)_{f \in F}$ and the basis vectors of $\mathbf{R}^N$. This cone is strictly convex because all its elements have a strictly positive weight except the positive multiples of $m - n$.
We can define the *weight vector* $\mathbf{w} = (w(u_1), \ldots, w(u_N)) \in \check{\mathbf{R}}^N$. The weight of a monomial u^m is then the evaluation, or scalar product, $\langle \mathbf{w}, m \rangle$. We note that, by construction, we have $\mathbf{w} \in \check{E}$.

So if Σ is a regular fan subdividing $\check{\mathbf{R}}^N_{\geq 0}$ which is compatible with H and $\check{E}$, it will contain a regular cone σ of dimension N whose intersection with H is of dimension $N-1$, which contains $\mathbf{w}$ and is contained in $\check{E}$. By the resolution theorem for normal toric varieties, since $\check{E}$ is a rational convex cone and H a rational hyperplane, we know that there exist such regular fans.

As a first step, let us examine the transforms in the charts $Z(\sigma)$ corresponding to cones $\sigma = \langle a^1, \ldots, a^N \rangle$ which contain **w** and are compatible with $\check{E}$ and H.

• *Because our fan is compatible with H, the convex cone σ has to be entirely on one side of H and its intersection with H is a face*. We may assume that $a^1, \ldots, a^t$ are those among the a^j which lie in the hyperplane H and all the other $\langle a^j, m - n \rangle$ are of the same sign, say $\langle a^j, m - n \rangle > 0$. We have then

$$u^m - \lambda u^n \longmapsto y_1^{\langle a^1, n \rangle} \cdots y_N^{\langle a^N, n \rangle} (y_{t+1}^{\langle a^{t+1}, m-n \rangle} \cdots y_N^{\langle a^N, m-n \rangle} - \lambda).$$

• *By compatibility with $\check{E}$ and since it contains* $\mathbf{w}$ *which is in* $\check{E}$, the cone σ is contained in $\check{E} \subseteq \check{E}'$ so that *all* $\langle a^i, p-n\rangle$ *are* ≥ 0. After perhaps re-subdividing σ and choosing a smaller regular cone containing $\mathbf{w}$ and whose intersection with H does not meet the boundary of $\check{E}$, we have that the $\langle a^i, p-n\rangle$ are > 0 at least for those i such that $a^i \in H$.

In the corresponding chart $Z(\sigma)$ the transform of our equation F by the monomial map can then be written:

$$y_1^{\langle a^1,n\rangle}\dots y_N^{\langle a^N,n\rangle}\big(y_{t+1}^{\langle a^{t+1},m-n\rangle}\dots y_N^{\langle a^N,m-n\rangle}-\lambda+\sum_p c_p y_1^{\langle a^1,p-n\rangle}\dots y_N^{\langle a^N,p-n\rangle}\big).$$

Since $\sum_{i=1}^N \langle a^i, p-n\rangle w(y_i) = \langle \mathbf{w}, p-n\rangle$, this shows that the strict transform F' of F by the monomial map $Z(\sigma)\to \mathbf{A}^N(k)$, which is the quantity between parenthesis, is an overweight deformation of the strict transform, of weight zero, of the initial part of F.
This implies the result we seek since the hypersurface defined by the initial part of F' is non singular: it implies that $F'=0$ is non singular and transversal to the toric boundary *at the point picked by the valuation*.

The charts where the strict transform intersects the maximal number of components of the toric boundary are obtained by choosing the regular cone $\sigma \in \Sigma$ in such a way that its intersection with the hyperplane H is of maximal dimension $N-1$, which means that $N-1$ of the vectors a^i are in H. The $N-1$ corresponding coordinates y_i will be of positive value and provide a system of local coordinates for the strict transform of our hypersurface at the point picked by the valuation. In fact, if a^N is the vector which is not in H, according to what we saw above we must have $\langle a^N, m-n\rangle = 1$ and our local equation becomes

$$F' = y_N - \lambda + \sum_{w(p)>w(n)} c_p y_1^{\langle a^1, p-n\rangle} \dots y_N^{\langle a^N, p-n\rangle}.$$

Since the weight of y_N is zero and since this is an overweight deformation, we see immediately that F' is a power series in $y_1, \dots, y_{N-1}$ and $w_N = y_N - \lambda$ and the hypersurface $F' = 0$ is non singular and transversal to the toric boundary at the point $y_1 = \dots = y_{N-1} = 0, y_N = \lambda$, with local coordinates $y_1, \dots, y_{N-1}$. This point is the point picked by the valuation because on $F' = 0$ the valuation of $y_N - \lambda$ has to be positive.

Now you will find it easy to believe that things work essentially in the same way whene there is a finite number of equations.
Again the fact that an affine toric variety corresponds to a saturated lattice and the special form of the jacobian minors play an essential role.

This shows local uniformization in the case where $\mathrm{gr}_\nu R$ is finitely generated.
Let us say that the semigroup of our rational valuation on R is *quasi-finitely generated* if there is a local ring R' essentially of finite type over R and dominated by R_ν such that the semigroup of the valuation ν on R' is finitely generated.

Theorem: The semigroup of an Abhyankar valuation of a complete noetherian domain with an algebraically closed residue field is quasi finitely generated.

The proof uses the valuative Cohen theorem and flattening, as well as local uniformization in the case where the semigroup is finitely generated.

Conclusion

Combining the two results gives a proof of local uniformization for Abhyankar valuations of complete equicharacteristic noetherian domains with an algebraically closed residue field. If the conjecture about extending valuations to a complete local ring without changing the value group is true, it should also provide a proof for excellent equicharacteristic local domains with an algebraically closed residue field.

All this leads to the following question: Given a singular algebraic subvariety X of an affine space $\mathbf{A}^N(k)$ over an algebraically closed field k, can one re-embed X in another affine space $\mathbf{A}^M(k)$ is such a way that:

• There exists a system of coordinates $z_1, \ldots, z_M$ for $\mathbf{A}^M(k)$ such that there are regular fans Σ subdividing $\check{\mathbf{R}}^M_{\geq 0}$ and such that the corresponding birational toric map $Z(\Sigma) \longrightarrow \mathbf{A}^M(k)$ gives an embedded resolution (or pseudo-resolution) of X.

• The singular locus of X is a union of intersections of X with strata of the canonical stratification of the union of coordinate hyperplanes (the toric boundary, in the toric jargon).

Very optimistic, some will say. Well...

Tevelev's Theorem, Collectanea Math., 2014

Let k be an algebraically closed field of characteristic zero.
Let $X \subset \mathbf{P}^n$ be an irreducible algebraic variety. For a sufficiently high order Veronese re-embedding $X \subset \mathbf{P}^N$ one can choose homogeneous coordinates $z_0, \ldots, z_N$, a smooth toric variety Z' of the algebraic torus $T = \mathbf{P}^N \setminus \bigcup\{z_i = 0\}$ and a toric birational morphism $Z' \to \mathbf{P}^N$ such that the following conditions are satisfied: $X \cap T$ is non-empty, the strict transform of X in Z' is smooth and intersects the toric boundary transversally, and $Z' \to \mathbf{P}^N$ is a composition of blowing-ups with smooth torus-invariant centers.

Very roughly speaking, a suitable embedding of a singular space in some affine space must have the property that there are enough coordinate hyperplanes for the strict transforms of the corresponding hyperplane sections to ”read” all the important information of the exceptional divisor of an embedded resolution in that embedding.

There is both local and global information along the exceptional divisor, and it is really complicated.
But things are easier if we use valuations, one at a time, as probes.

How to prove the existence of suitable embeddings without having an embedded resolution?

Prove the existence of a suitable embedding for each valuation, for which there is a toric modification of the new ambient space which uniformizes the valuation, then use the compactness of the space of valuations to prove that there exist a finite collection of such suitable embeddings having the property that for every valuation, at least one of these embeddings is suitable. Then glue up those finitely many embeddings into a single one.

Well, we are not there yet!

After what we have seen, we only know how to prove the existence of suitable embeddings for the simplest valuations, called ”Abhyankar valuations”, but they are very important since they can be used to approximate any valuation.

Thank you for your attention

(Bibliography on the next page)

R. Goldin and B. Teissier, Resolving singularities of plane analytic branches with one toric morphism, in *Resolution of Singularities, a research textbook in tribute to Oscar Zariski*, Progress in Math. Vol. 181, Birkhäuser, Basel, 2000, 315-340.

P. González Pérez and B. Teissier, Embedded resolutions of non necessarily normal affine toric varieties, *Comptes-rendus Acad. Sci. Paris, Ser.1*., **(334)**, (2002), 379-382.

J. Herrera, M. A. Olalla, M. Spivakovsky, B. Teissier, Extending a valuation centered in a local domain to its formal completion. *Proceedings of the London Mathematical Society* (3) 105 (2012) 571-621,

B. Teissier, Valuations, deformations, and toric geometry. In *Valuation Theory and its applications, Vol. II*, Fields Inst. Commun. 33, AMS., Providence, RI., 2003, 361-459. Available at http://people.math.jussieu.fr/~teissier/

B. Teissier, Overweight deformations of affine toric varieties and local uniformization, in *Valuation theory in interaction*, Proceedings of the second international conference on valuation theory, Segovia–El Escorial, 2011. Edited by A. Campillo, F-V. Kuhlmann and B. Teissier. European Math. Soc. Publishing House, Congress Reports Series, Sept. 2014, 474-565.

J. Tevelev, On a question of B. Teissier, *Collectanea Math.*, Vol. 65, Issue 1, January 2014, 61-66. (Published on line February 2013. DOI 10.1007/s13348-013-0080-9)