The reduced Bautin index of planar vector fields

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Introduction

The motivation of this paper comes from the so-called “Local version of the 16-th Hilbert problem”. Consider a polynomial vector field of degree q

$$W_{\underline{a},\underline{b}} = x\partial_y - y\partial_x + \sum_{2\le i+j\le q} a_{ij}x^iy^j\partial_x + b_{ij}x^iy^j\partial_y,$$

the a_{ij} and b_{ij} being real or complex. This vector field is a deformation of the vector field $x\partial_y - y\partial_x$ whose trajectories are concentric circles around 0. We will prove in this paper a precise version of the following assertion: for any compact K in the space of the (a_{ij}, b_{ij}), there exist a number $p(q)$ and a neighborhood $U(q, K)$ of 0 such that for $(\underline{a}, \underline{b}) \in K$:

- either 0 is again a center of $W_{\underline{a},\underline{b}}$ (i.e., 0 is an elliptic non-degenerate singular point of W, and W is integrable near 0),
- or $W_{\underline{a},\underline{b}}$ has at most $p(q)$ limit cycles in $U(q, K)$.

The local 16-th Hilbert problem consists in finding explicit expressions for $U(q, K)$ and $p(q)$. This problem is solved only for $q = 2$ by the so-called “Bautin Theorem”,(cf. [B], [Ya]). Bautin considered the Poincaré first return map around the origin restricted to a line with coordinate X as a series $F_z(X)$ in X with coefficients depending on the parameters $z = (a_{ij}, b_{ij})$. The limit cycles correspond to the zeroes of $F_z(X) - X$. Given a series

$$S_z(X) = \sum_{k=0}^{\infty} a_k(z)X^k$$

in one variable X with polynomial coefficients $a_k(z) \in \mathbf{K}[z_1, \ldots, z_n]$, $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, Bautin then considered in [B] the ideal I of $\mathbf{K}[z]$ generated by all $a_k(z)$; since the polynomial ring is nœtherian there is a smallest integer d such that $a_0, \ldots, a_d$ generate I. This number is the Bautin index of the series $S_z(X)$. In special cases Bautin was able to bound the number of zeroes of $S_z(X)$, hence the number of limit cycles,in function of d, and then to bound d itself when $q = 2$. More generally, when the series is an A_0-series in the sense of Briskin-Yomdin (see section 2 and [B-Y]), for each z one can bound by d the number of zeroes in X of the series $S_z(X)$ which lie inside a disk of radius $\mu_1(1 + |z|)^{-\mu_2}$ centered at 0, where μ_1, μ_2 are positive constants depending on $S_z(X)$, see [F-Y].

In this paper, following [B-Y], we retain the fact that the Poincaré first return map (we call it simply the “Poincaré return map” in this paper) is an A_0-series, and bound the number of zeroes of an A_0-series in a controlled neighbourhood of 0; as we already

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mentionned, this number is equal to the number of limit cycles in the case of the Poincaré return map. The bound is given by the reduced Bautin index $\overline{d}$, which is the smallest integer $\overline{d}$ such that $(a_0, \ldots, a_{\overline{d}})$ generate an ideal with the same integral closure as I. Similar results are proved in [F-Y], with $\bar{d}$ replaced by the Bautin index d. Since $\overline{d} \leq d$, our bound is better. Also our proof is more direct.

In fact, we prove the following result (see Theorem 3.1):

Theorem *if the A_0-series $S_z(X)$ is not identically zero, for each $z \in \mathbf{K}$ it is convergent and the number of its zeroes is bounded by the reduced Bautin index $\overline{d}$, in a disk of radius $R(z)$ with*

$$R(z) = \mu_1(1 + |z|)^{-\mu_2},$$

where $|z|$ denote the usual norm of a vector $z \in \mathbf{C}^n$ (or $\mathbf{R}^n$) and μ_1, μ_2 are positive constants depending only on the series.

Precise estimations of μ_1 and μ_2 are given in terms of certain parameters of the A_0-series (see (17) and Remark 4.4). When $S_z(X)$ is the Poincaré return map of a vector field $W_{\underline{a},\underline{b}}$ we are able to estimate μ_2 in terms of q, d, and $\overline{d}$, but there remains work to do for the constant μ_1.

The main open question in this context is to estimate $\overline{d}$ in terms of q. This should *a priori* be less difficult for $\overline{d}$ than for d, since it is much easier to determine whether ideals have the same integral closure than to determine whether they are equal.

The content of the paper is as follows: in Section 1, we study the integral closure of ideals in a polynomial ring, which is the tool which permits the replacement of d by $\overline{d}$ in the bound for the number of limit cycles. Section 2 introduces A_0-series, and proves that the Poincaré return map of a polynomial vector field is an A_0-series. Its parameters are computed in terms of q. In Section 3, properties of A_0-series (in relation to the Bautin index) are studied, and the main result (Theorem 3.1) is stated, with a sketch of proof. In Section 4, a proposition due to A. Douady is used to find a lower bound for the absolute value of a complex polynomial on a circle of controlled radius, in order to apply Rouché's principle. The proof of the main result follows and an Appendix gives some precisions about the Division Theorem needed in the proof.

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1. Integral closure of ideals in a polynomial ring

In this section we translate into inequalities the condition of integral dependence over an ideal in a ring of polynomials with real or complex coefficients. A similar result is already well known in local analytic geometry (see [L-T], [Li-T]).

Let I be an ideal of a ring A; an element $P \in A$ is said to be *integral* over I if it satisfies an integral dependence relation:

$$(*) \qquad P^k + b_1 P^{k-1} + \cdots + b_k = 0 \text{ with } \; b_i \in I^i.$$

Recall (see [L-T], [Li-T]) that the set of elements integral over I is again an ideal, denoted here by $\overline{I}$ and called the integral closure of I. It is contained in the radical of I. If $A = \mathbf{C}\{z_1, \ldots, z_n\}$ is the ring of convergent power series, $P \in A$ is integral over $I = (a_0, \ldots, a_d)$ if and only if there exist a constant C and a neighborhood U of 0 in $\mathbf{C}^n$ such that

$$|P(z)| \leq C \cdot \sup_i |a_i(z)|, \quad \text{for all } z \in U, \tag{1}$$

identifying a germ with a suitable representative.

Let us assume now that A is the polynomial algebra $\mathbf{C}[z_1, \ldots, z_n]$, and let $(c_0, \ldots, c_r)$ be a system of generators for the ideal I. We shall say that $(c_0, \ldots, c_r)$ is a *Macaulay basis* of I if the homogenizations $\tilde{c}_i(Z, T)$ of the polynomials $c_i(z)$ generate the homogenization $\tilde{I} \subset \mathbf{C}[Z_1, \ldots, Z_n, T]$ of the ideal I, i.e the homogeneous ideal generated by the homogeneizations of elements of I. Let us denote by p the degree of P, β_i the degree of b_i and γ_i the degree of c_i. The equation $(*)$ implies the inequality $kp \leq \sup_i((k-i)p + \beta_i)$ or $\sup_i(\beta_i - ip) \geq 0$.

Let us denote by ℓ the smallest integer satisfying the inequality

$$\ell \geq \sup_i \frac{\beta_i}{i} - p.$$

1.1 Proposition *Set $A = \mathbf{C}[z_1, \ldots, z_n]$, and let $P \in A$. The following conditions are equivalent :*

(a) *P is integral over I.*

(b) *Given a Macaulay basis $(c_0, \ldots, c_r)$ of the ideal I, there exists a constant $C > 0$ such that*

$$|P(z)| \leq C \cdot \sup_j \big((1 + |z|)^{\ell + p - \gamma_j} \cdot |c_j(z)|\big) \quad \textit{for all } z \in \mathbf{C}^n.$$

(c) *For any system of generators $(a_0, \ldots, a_d)$ of I, there exist constants $C_1 > 0$ and $\mu \in \mathbf{N}$ such that*

$$|P(z)| \leq C_1 \cdot (1 + |z|)^{\mu} \cdot \sup_j |a_j(z)| \quad \textit{for all } z \in \mathbf{C}^n.$$

Proof **(a) $\Rightarrow$ (b)**. We have $\ell \geq 0$, and if we replace z_j by Z_j/T in $(*)$ and multiply the result by $T^{k(\ell+p)}$, we get a homogeneous integral dependence relation, where for each polynomial $G(z)$ of degree γ, we denote by $\tilde{G}(Z, T)$ the homogeneous polynomial $T^{\gamma} G(Z_1/T, \ldots, Z_n/T)$:

$$(T^{\ell} \tilde{P})^k + \cdots + T^{i(\delta+\ell) - \beta_i} \tilde{b}_i (T^{\ell} \tilde{P})^{k-i} + \cdots + T^{k(\delta+\ell) - \beta_k} \tilde{b}_k = 0. \tag{$\tilde{*}$}$$

For each i, the homogeneous polynomial $\tilde{b}_i$ belongs to $(\tilde{I})^i$; if $(c_0, \ldots, c_r)$ is a Macaulay basis of I, any element $G \in I$ can be written

$$G = \sum_j d_j c_j \quad \text{with } \deg(d_j c_j) \leq \deg G.$$

Then the homogeneizations of the "monomials" $c^m = c_1^{m_1} \cdots c_r^{m_r}$ of total degree i in the c_j's generate the ideal $(\tilde{I})^i$ and therefore if $b_i \in I^i$, we have $\tilde{b}_i \in (\tilde{I})^i$. Therefore $(\tilde{*})$ is an integral dependence relation for the homogeneous polynomial $T^\ell \tilde{P}$ over the homogeneous ideal $\tilde{I}$. Viewing this as an integral dependence relation in $\mathbf{C}^{n+1}$ and using (1) and homogeneity, we deduce that, for a Macaulay basis $(c_0, \ldots, c_r)$ of I, for each relatively compact neighborhood U of 0 there is a constant $C(U) > 0$ such that for $(Z,T) \in U$ one has

$$|T^\ell \tilde{P}(Z,T)| \leq C(U) \cdot \sup_j |\tilde{c}_j(Z,T)|.$$

Now we may restrict this inequality to the open set $T \neq 0$, set $z_i = Z_i/T$ and restrict again to the hypersurface $T^{-1} = 1 + |z|$; we obtain the existence of a constant $C > 0$ such that for all $z \in \mathbf{C}^n$ we have

$$|P(z)| \leq C \cdot \sup_j \big((1+|z|)^{\ell+p-\gamma_j} |c_j(z)|\big). \tag{2}$$

(b) ⇒ (c). This follows upon expressing the generators c_j of a Macaulay basis in terms of the a_k, say $c_j = \sum m_{jk} a_k$, where the m_{jk} are polynomials, and noticing that for each m_{jk}, if its degree is d_{jk}, there is a positive constant C_{jk} such that

$$|m_{jk}(z)| \leq C_{jk}(1+|z|)^{d_{jk}}.$$

(c) ⇒ (a). Let us set $z_i = T^{-1} Z_i$. Denote by α_j the degree of a_j, choose an integer $r \geq \sup(p, \sup_j(\mu + \gamma_j))$, and multiply the inequality of (c) by $|T|^r$; we obtain the following inequality for $T \neq 0$:

$$|T|^{r-p} |\tilde{P}(Z,T)| \leq C_1(|T| + |Z|)^\mu \sup_j |T^{r-\mu-\alpha_j} \tilde{a}_j(Z,T)|.$$

Since both sides are continuous, this inequality is also valid for $T = 0$, and from [L-T] or [Li-T] we deduce that $T^{r-p}\tilde{P}(Z,T)$ is integral in the ring $\mathbf{C}\{Z,T\}$ over the product of the ideal $(Z,T)^\mu$ and the homogeneous ideal J generated by the $(T^{r-\mu-\gamma_j} \cdot \tilde{a}_j(Z,T))_{0 \leq j \leq r}$. We can write in $\mathbf{C}\{Z,T\}$ an integral dependence relation

$$(T^{r-p}\tilde{P})^k + A_1(Z,T)(T^{r-p}\tilde{P})^{k-1} + \cdots + A_k(Z,T) = 0$$

with $A_i(Z,T) \in (Z,T)^{\mu i} J^i$. Taking the homogeneous component of degree kr in this equation, we obtain a homogeneous integral dependence relation for $T^{r-p}\tilde{P}$ over $(Z,T)^\mu J$ in $\mathbf{C}[T, Z_1, \ldots, Z_n]$, with the same expression.

Setting now $T = 1$ in this last relation establishes the asserted integral dependence relation for P over I in $\mathbf{C}[z_1, \ldots, z_n]$.

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1.2 Remark Observe that (b) also holds in the more general case where $(c_0, \ldots, c_r)$ are a Gröbner basis of I with respect to a monomial order on $\mathbf{N}^n$ defined by positive integer weights $\tau_1, \ldots, \tau_n$. The degrees have to be replaced accordingly by the weighted degrees w.r.t. $\tau_1, \ldots, \tau_n$. Macaulay bases correspond to $\tau_i = 1$ for all i.

In the real case, it is then natural to define the real integral closure $\overline{I}$ of an ideal $I \subset \mathbf{R}[x]$, $x = (x_1, \ldots, x_n)$, as the set of polynomials P for which there exist constants C_1 and μ such that

$$|P(x)| \leq C_1 \cdot (1 + |x|)^{\mu} \cdot \sup_j |a_j(x)|$$

for all $x \in \mathbf{R}^n$, see [Fe].

Notice that in the real and complex case, it is possible to give an explicit bound for the constant μ: if q is an upper bound for the degrees p and α_i, one can take $\mu = q^{\beta n}$, where β is a universal constant. This is proved in [So], Lemma 5, for a continuous semi-algebraic function $f(x)$, but the same proof works for a locally bounded semi-algebraic function, which is the case for $|P(x)|/\sup_j |a_j(x)|$. If the coefficients of P and of the a_i's are integers, it is also possible to estimate the constant C_1 by the same kind of bound, where now q depends also on $\|P\|$ and the $\|a_j\|$'s (see the next section for the definition of the norm $\|P\|$ of a polynomial P).

2. The Poincaré return map and A_0-series

For a polynomial $a \in \mathbf{C}[z]$, $a = \sum a_{\underline{i}} z^{\underline{i}}$, $\underline{i} := (i_1, \ldots, i_n)$, we set $\|a\| = \sum_{\underline{i}} |a_{\underline{i}}|$, and denote by $\deg(a)$ the degree of a. Let us recall, after Briskin-Yomdin [B-Y], the following definition:

2.1 Definition Let

$$S_z(X) = \sum_{k \geq 0} a_k(z) X^k$$

be a power series with coefficients $a_k(z) \in \mathbf{C}[z]$. Then $S_z(X)$ is an A_0-series if there exist constants $\lambda_i \geq 0$, $1 \leq i \leq 4$, such that

$$(3) \qquad \begin{cases} \deg(a_k) \leq \lambda_1 k + \lambda_2 \\ \|a_k\| \leq \lambda_3 \lambda_4^k. \end{cases}$$

The λ_i are called the (growth) parameters of the A_0-series. Note that if $S_z(X)$ is an A_0-series, its radius of convergence $R(z)$ satisfies the inequality

$$(4) \qquad R(z) \geq \frac{1}{\lambda_4 (1 + |z|)^{\lambda_1}}.$$

The growth conditions on the a_k's are rather natural. They appear also in other circumstances, e.g. in Monsky-Washnitzer's construction of a formal cohomology theory [M-W].

We shall prove, following the classical method (cf. [F-Y]), that the Poincaré return map associated to a vector field of type :

$$(5) \qquad x\partial_y - y\partial_x + \sum_{2 \leq i+j \leq q} a_{ij} x^i y^j \partial_x + b_{ij} x^i y^j \partial_y,$$

is an A_0-series $S_z(X) = \sum a_k(z)X^k$, setting $z = (\underline{a}, \underline{b}) = ((a_{ij}), (b_{ij}))$. Moreover, we will bound the constants λ_i in terms of q.

Let us recall how the Poincaré return map is defined. Take a line through 0, e.g. the x-axis. Then, given a compact set K in the z-space, there exists a positive real number x_0 such that for any $z \in K$ and any real $X \leq x_0$, the trajectory $(r(t), \theta(t))$ of the vector field starting at $(X, 0)$ has strictly increasing angle θ between 0 and 2π. Therefore θ can be taken as a parameter along this trajectory, which makes r a function $r(\theta, X)$ of θ and the initial value X. The return map is then defined for $X \in [0, x_0]$ by $S_z(X) = r(2\pi, X)$.

2.2 Proposition *In the situation just described, the power series $S_z(X)$ is an A_0-series with the following parameters : $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 1, \lambda_4 = 33\pi q^4$.*

Proof In polar coordinates $x = r\cos\theta,\ y = r\sin\theta$, we have

$$dr = \cos\theta dx + \sin\theta dy, \quad rd\theta = -\sin\theta dx + \cos\theta dy,$$

and the trajectories of the vector field (5) satisfy

$$\begin{aligned}\frac{dr}{d\theta} &= r\frac{\cos\theta(-y + \sum a_{ij}x^iy^j) + \sin\theta(x + \sum b_{ij}x^iy^j)}{-\sin\theta(-y + \sum a_{ij}x^iy^j) + \cos\theta(x + \sum b_{ij}x^iy^j)} \qquad (6)\\ &= r\frac{\cos\theta(-r\sin\theta + \sum a_{ij}r^{i+j}(\cos\theta)^i(\sin\theta)^j) + \sin\theta(r\cos\theta + \sum b_{ij}r^{i+j}(\cos\theta)^i(\sin\theta)^j)}{-\sin\theta(-r\sin\theta + \sum a_{ij}r^{i+j}(\cos\theta)^i(\sin\theta)^j) + \cos\theta(r\cos\theta + \sum b_{ij}r^{i+j}(\cos\theta)^i(\sin\theta)^j)}\\ &= \frac{\sum_{i=2}^{q} r^iP_i}{1 - \sum_{i=1}^{q-1} r^iQ_i}.\end{aligned}$$

where P_i, Q_i are linear forms in $z = (\underline{a}, \underline{b})$, and where for any $\theta \in [0, 2\pi]$,

$$\|P_i\| \leq 2(i+1), \quad \|Q_i\| \leq 2(i+2). \qquad (7)$$

Moreover, P_i and Q_i are homogeneous polynomials in $(\sin\theta, \cos\theta)$, with

$$\deg(P_i) = i+1,\ \deg(Q_i) = i+2.$$

There exists $\rho > 0$ such that $|\sum r^iQ_i(z,\theta)| < 1$ for all $0 \leq r \leq \rho,\ z \in K,\ \theta \in [0, 2\pi]$. Write

$$\frac{dr}{d\theta} = \sum_{k=2}^{+\infty} r^kR_k. \qquad (8)$$

as a series in r.

2.3 Lemma *R_k is a polynomial in z of degree $\leq k-1$, with $\|R_k\| \leq (2q^2)^k$. It is a polynomial in $(\sin\theta, \cos\theta)$ of degree $\leq 3(k-1)$. Moreover, as a polynomial in $(\sin\theta, \cos\theta)$, R_k is homogeneous of degree $k+1$ mod 2 .*

Here, $\|R_k\|$ is computed for θ fixed, considering R_k as a polynomial in z only.

Proof Let us first prove the assertion on the degree of R_k in $\sin\theta$ and $\cos\theta$. By definition, we have

$$\sum_{k=2}^{\infty} r^k R_k = (\sum_{i=2}^{q} r^i P_i)(1 + \Big(\sum_1^{q-1} r^i Q_i\Big) + \cdots + \Big(\sum_1^{q-1} r^i Q_i\Big)^p + \cdots).$$

A monomial of R_k is of the form $P_i Q_1^{\alpha_1} \ldots Q_{q-1}^{\alpha_{q-1}}$, with $2 \le i \le q$, $\alpha_1 + \cdots + \alpha_{q-1} = p$ for some p, $1 \le p \le k-i$, and $i + \alpha_1 + 2\alpha_2 + \cdots + (q-1)\alpha_{q-1} = k$. Its degree in $\cos\theta, \sin\theta$ is

$$(i+1) + 3\alpha_1 + 4\alpha_2 + \cdots + (q+1)\alpha_{q-1} = k + 1 + 2p.$$

Then the maximum of the degrees of such a monomial is obtained for p maximum, i.e., $p = k-2$ which gives the bound $3 + 3(k-2) = 3(k-1)$ for the degree in $\cos\theta, \sin\theta$ (this bound is reached for the monomial $P_2 Q_1^{k-2}$). The assertion about the homogeneity (mod 2) of R_k is easily proved by induction on k.

Let us now prove the following claim: *Let $\alpha_{j,p}$ be the norm of the coefficient of r^j in $(\sum_1^{q-1} r^i Q_i)^p$. Then $\alpha_{j,p} \le (2q^2)^p$.*
This is shown by induction on p, using (7) : For $p = 1$ we have $\alpha_{j,1} \le 2(i+2) \le 2(q+1) \le 2q^2$ $(q \ge 2)$. For the induction step write

$$\Big(\sum r^i Q_i\Big)^{p+1} = \Big(\sum_1^{q-1} r^i Q_i\Big)^p \Big(\sum_1^{q-1} r^i Q_i\Big)$$

and get $\alpha_{j,p+1} \le \sum_{j-q+1}^{j-1} \alpha_{i,p} 2(j-i+2) \le (2q^2)^p 2(q+1)(q-2) \le (2q^2)^{p+1}$, which proves the claim.

The norm of the coefficient of r^k in

$$\frac{1}{1 - \sum_1^{q-1} r^i Q_i} = 1 + (\sum r^i Q_i) + \cdots + (\sum r^i Q_i)^p + \cdots$$

is $\le \sum_1^k (2q^2)^i = 2q^2 \frac{(2q^2)^k - 1}{2q^2 - 1} \le (2q^2)^{k+1}$. We now multiply $\frac{1}{1 - \sum_1^{q-1} r^i Q_i}$ by $(r^2 P_2 + \cdots + r^q P_q)$. The norm of the coefficient of r^k is then $\le \sum_{k-q}^{k-2} (2q^2)^{i+1} 2(k-i+1) \le (q-2)(2q^2)^{k-1} 2(q+1) \le (2q^2)^k$.

■

We want to find the solution of (8) with $r(0) = X$, expressed as a power series in X:

(9) $$r = r(\theta, z, X) = X + \sum_2^{+\infty} a_k(z,\theta) X^k, \quad a_k(z,0) = 0.$$

The Poincaré return map $S_z(X)$ will then be obtained by setting $\theta = 2\pi$ in (9), say $S_z(X) = r(2\pi, z, X)$. For $k \geq 2$, we get from (8):

$$\begin{cases} a_k' = \sum_{i=2}^{k} R_i \cdot G_{ik}(a_2, \ldots, a_{k-i+1}), \\ a_k(z, 0) = 0, \end{cases} \tag{10}$$

where $G_{ik}(a_2, \ldots, a_{k-i+1})$ is the coefficient of X^k in $(X + a_2 X^2 + \cdots + a_p X^p + \cdots)^i$ and where a_k' denotes $\partial a_k / \partial \theta$. Integration of (10) with initial conditions $a_k(z, 0) = 0$ gives the series (9). Note that $a_2(z, 2\pi) = 0$ (see Claim 5.4). Proposition 2.2 is then immediate from the following:

2.4 Lemma *With the notations introduced above, $a_k(z, \theta)$ is a polynomial in z of degree $\leq k - 1$, such that for any θ, $0 \leq \theta \leq 2\pi$, the inequality $\|a_k\| \leq (33\pi q^4)^k$ holds.*

Proof The fact that $\deg(a_k) \leq k - 1$ is easy to prove by induction: use the equations (17) which appear below, after claim 5.4. Set $M_k(z) = (2q^2 \sup(1, |z|))^k$. We then have $|R_k(z, \theta)| \leq M_k(z)$, and $\|R_k\| \leq M_k(0)$ (Lemma 2.3). To estimate $\|a_k\|$ we use the following lemma.

2.5 Lemma *Let $T_z(X) = X + \sum_{i \geq 2} b_i(z) X^i$ be the power series defined by the functional equation*

$$T = X + 2\pi \sum_{k \geq 2} M_k(z) T^k. \tag{11}$$

Then :

a) *The series $T_z(X)$ is a majorizing series for $S_z(X)$, i.e., $|a_k(z, \theta)| \leq |b_k(z)|$ for all z and $0 \leq \theta \leq 2\pi$.*

b) *$\|a_k\| \leq b_k(\underline{1})$ for $k \geq 2$, where $\underline{1} = (1, \ldots, 1)$.*

Proof of 2.5 Formula (11) gives

$$X + \sum_{i \geq 2} b_i(z) X^i = X + 2\pi \sum_{k \geq 2} M_k(z) (X + \sum_{i \geq 2} b_i(z) X^i)^k,$$

which implies that

$$b_k(z) = 2\pi \sum_{i=2}^{k} M_i(z) H_{ik}(b_2, \ldots, b_{k-i+1}), \tag{12}$$

where $H_{ik}(s_2, \ldots, s_{k-i+1})$ is the coefficient of X^k in $(X + \sum_{j \geq 2} s_j X^j)^i$. Then the two assertions of Lemma 2.5 follow by induction, comparing (12) with (10), and using the inequality $\|PQ\| \leq \|P\| \cdot \|Q\|$.

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Proof of 2.4 We have from (11): $X = T - 2\pi \sum_{k\geq 2} M_k(z)T^k$. Set $c = 2\pi$, $M_k = \alpha^k$, with $\alpha = 2q^2\sup(1, |z|)$. We get $X = T - (c\alpha^2 T^2)(\frac{1}{1-\alpha T})$ for $|T| < 1/\alpha$, which gives $\alpha T^2(1 + c\alpha) - T(1 + \alpha X) + X = 0$, and

$$T = \frac{1 + \alpha X \pm \sqrt{(1 + \alpha X)^2 - 4X\alpha(1 + c\alpha)}}{2\alpha(1 + c\alpha)},$$

where we must take the minus sign in view of (11). Then T is a power series in X, $T = X + \sum_{k\geq 2} b_k X^k$. Let us prove that $|b_k| \leq (33\pi q^4)^k$. Set $T = \frac{1+\alpha X-\sqrt{1+u}}{2\alpha(1+c\alpha)}$ with $u = \alpha X(a + \alpha X)$ and $a = -2(1+2c\alpha)$. We have $(1+u)^{1/2} = \sum n_k u^k = \sum n_k \alpha^k X^k (a+\alpha X)^k$, with binomial coefficients $|n_k| \leq 1/2$. Now the modulus of the coefficient of X^k in $(1+u)^{1/2}$ is smaller than $\alpha^k(1 + a)^k \leq \alpha^k(1 + 4c\alpha)^k \leq (33\pi q^4 \sup(1, |z|))^k$. This proves Lemma 2.4 and Proposition 2.2, after setting $z = \underline{1}$.

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3. A_0-series and the Bautin index

Definitions For any series $\sum a_k(z)X^k$, the ideal $I = (a_k(z),\ k \geq 0)$ of $\mathbf{K}[z]$ is called the *Bautin ideal* of the series. This ideal is finitely generated since $\mathbf{K}[z]$ is nœtherian. The least integer d such that $I = (a_0, \dots, a_d)$ is called the *Bautin index* of the series (see [B]).

Let us denote by $\overline{d}$ the least integer such that I and $(a_0, \dots, a_{\overline{d}})$ have the same integral closure (resp. the same real integral closure; see Section 1). One clearly has $\overline{d} \leq d$. We call $\overline{d}$ the *reduced Bautin index* of the series. Note that in the real case, $\overline{d}$ can be smaller than the reduced Bautin index of the complexification.

The following theorem is the main result of this paper. It generalizes Theorem 2.3.7 of [F-Y].

3.1 Theorem *Let $S_z(X) = \sum_{k\geq 0} a_k(z)X^k$ be a non-zero A_0-series, with polynomial coefficients $a_k(z)$ in $\mathbf{C}[z]$ or $\mathbf{R}[z]$. Let $\overline{d}$ be its reduced Bautin index.*
1) There exist positive constants μ_1 and μ_2 depending on the series such that for z in $\mathbf{C}^n$, respectively $\mathbf{R}^n$, and setting $R(z) = \mu_1(1 + |z|)^{-\mu_2}$, the series $S_z(X)$ converges for X in the disk $D(0, R(z))$ and has at most $\overline{d}$ distinct zeroes there.
2) The constants μ_1 and μ_2 may be taken to have the following form:

$$\mu_1 = \left(4 \cdot 5^{2(\overline{d}+1)} C_3 \lambda_4^{\overline{d}+1}\right)^{-1}, \quad \mu_2 = \lambda_1(\overline{d} + 1) + \lambda_2 + \alpha.$$

Here the λ_i are the parameters of the A_0-series and the new constants C_3 and α essentially describe the growth as $|z| \to \infty$ of the $|a_k(z)|$ in comparison to that of

$|a_i(z)|$, $0 \le i \le \overline{d}$; see Corollary 3.3. More precise estimates for the constants μ_1 and μ_2 are given at the end of the paper in Section 5.

The main idea of the proof is to apply Rouché Theorem to bound the number of zeroes of $S_z(X)$ in some disk. Write $S_z(X) = P_z(X) + Q_z(X)$, with $P_z(X) = \sum_{k \le \overline{d}} a_k(z) X^k$, $Q_z(X) = \sum_{k > \overline{d}} a_k(z) X^k$.

First, we prove in Section 4, Corollary 4.3, that for given R, $0 < R \le 1$, there exists a constant $\eta = \eta(\overline{d}) = 5^{-2(\overline{d}+1)}$ and a radius R_1 with $\eta R < R_1 < R$, such that for $|X| = R_1$, we have:

$$|P_z(X)| \ge \frac{1}{2} \cdot R_1^{\overline{d}} \cdot \sup_{0 \le j \le \overline{d}} |a_j(z)|$$

for any z. In order to apply Rouché's Theorem on $|X| = R_1$, we need that

$$|Q_z(X)| < \frac{1}{2} \cdot R_1^{\overline{d}} \cdot \sup_{0 \le j \le \overline{d}} |a_j(z)|$$

for $|X| = R_1$. This is fulfilled if

$$|a_k(z)| R_1^{k-\overline{d}} < \frac{1}{2^{k-\overline{d}+1}} \cdot \sup_{0 \le j \le \overline{d}} |a_j(z)| \quad \text{for } k > \overline{d}. \tag{13}$$

Corollary 3.3 proves, using the Division Theorem of the Appendix, that

(14). $$|a_k(z)| \le C_3 \cdot (1 + |z|)^{\lambda_1 k + \lambda_2 + \alpha} \cdot \lambda_4^k \cdot \sup_{0 \le j \le \overline{d}} |a_j(z)|.$$

We may increase the value of C_3 so that $2C_3 \lambda_4^{\overline{d}} \ge 1$. A direct computation using (14) shows that (13) is satisfied for any $R_1 < R_0(z)$, with

$$R_0(z) = \frac{1}{4 \cdot C_3 \cdot (1 + |z|)^{\lambda_1(\overline{d}+1) + \lambda_2 + \alpha} \cdot \lambda_4^{\overline{d}+1}}.$$

Now we choose $R_0(z)$ as our R, and we can apply Rouché's Theorem on a circle of radius

$$R_1 := R(z) \ge \eta R = \frac{1}{5^{2(\overline{d}+1)} \cdot 4 \cdot C_3 \cdot \lambda_4^{\overline{d}+1} \cdot (1 + |z|)^{\lambda_1(\overline{d}+1) + \lambda_2 + \alpha}}. \tag{15}$$

This is the value asserted in Theorem 3.1. Note that inequality (13) implies that the series in X converges in the disk of radius R_1.

The key point is the proof of inequality (14); it consists of the following steps:

a) Let $c_j(z)$ be a Gröbner basis of the ideal $I = (a_0, \ldots, a_d)$. In Proposition 3.2 we bound $|a_k(z)|$ for $k \ge d+1$ in terms of $(c_0, \ldots, c_r)$ by the Division Theorem.
b) We bound $|c_j(z)|$ in terms of $(a_0, \ldots, a_d)$, using the norm of the transformation matrix M.
c) We bound $|a_{\overline{d}+1}|, \ldots, |a_d|$ in terms of $(a_0, \ldots, a_{\overline{d}})$ using Section 1 about the integral closure.

Steps b) and c) are carried out simultaneously using Proposition 1.1, c).

Let us now begin the proof of 3.1. We first relate A_0-series to the Bautin index. Let $\tau = (\tau_1, \ldots, \tau_n)$ be real numbers ≥ 1 which are linearly independent over $\mathbf{Q}$. Then, for $P = \sum_{\alpha \in \mathbf{N}^n} P_\alpha z^\alpha$ in $\mathbf{C}[z]$ and $t > 0$, we define the norm

$$|P|_t = \sum_\alpha |P_\alpha| \cdot t^{<\tau, \alpha>}.$$

Let $S_z(X)$ be an A_0-series, and let c_i denote a generator system or a Gröbner basis of the Bautin ideal I of $S_z(X)$ w.r.t. the monomial order on $\mathbf{N}^n$ given by τ.

3.2 Proposition *Let $S_z(X) = \sum_k a_k(z) X^k$ be an A_0-series with Bautin ideal $I = (a_0, \ldots, a_d)$. Let $c_0, \ldots, c_r$ be a Gröbner basis of I with respect to τ. There exist constants $C_2 > 0$ and $t_0 \geq 1$ such that for $|z| \geq t_0$ and $k \in \mathbf{N}$ one has*

$$|a_k(z)| \leq C_2 \cdot |z|^{\lambda_1 k + \lambda_2} \cdot \lambda_4^k \cdot \sup_{1 \leq i \leq r} |c_i(z)|.$$

Proof We apply the Division Theorem for polynomials to $a_k(z) \in I$. There exist constants $t_0 \geq 1$ and $C > 0$ and polynomials $b_{ki}(z) \in \mathbf{C}[z]$ such that $a_k(z) = \sum_{i=0}^r b_{ki}(z) c_i(z)$ and such that for $t \geq t_0$

$$\sum |b_{ki}|_t \cdot |c_i|_t \leq C \cdot |a_k|_t.$$

We may assume $c_i \neq 0$ for all i and get

$$\sum |b_{ki}|_t \leq C \cdot |a_k|_t \cdot (\inf_i |c_i|_t)^{-1} \leq C \cdot |a_k|_t \cdot (\inf_i |c_i|_{t_0})^{-1}.$$

Set $C_{t_0} = C \cdot (\inf_i |c_i|_{t_0})^{-1}$ and get $\sum |b_{ki}|_t \leq C_{t_0} \cdot |a_k|_t$. Let $z \in \mathbf{C}^n$ with $|z| \geq t_0^{\tau_0}$ be fixed, where τ_0 denotes the minimum of the components of τ. Then we can choose $t \geq t_0$ such that $|z_i|^{1/\tau_i} \leq t \leq |z|$ for all i (since $\tau_i \geq 1$, set e.g. $t = |z|$). Now,

$$|a_k(z)| \leq \sum |b_{ik}(z)| \cdot |c_i(z)| \leq \sum |b_{ik}(z)| \cdot \sup_i |c_i(z)|.$$

One has $|b_{ik}(z)| \leq |b_{ik}|_t$, by the choice of t, and $\sum |b_{ik}|_t \leq C_{t_0} \cdot |a_k|_t$, which gives

$$|a_k(z)| \leq C_{t_0} \cdot |a_k|_t \cdot \sup_i |c_i(z)|.$$

But $|a_k|_t \leq \lambda_3 \cdot \lambda_4^k \cdot t^{\tau_o(\lambda_1 k + \lambda_2)}$ by (3), and $t \leq |z|$ by the choice of t. Therefore,

$$|a_k(z)| \leq C_{t_0} \cdot \lambda_3 \cdot \lambda_4^k \cdot t^{\tau_o(\lambda_1 k + \lambda_2)} \cdot \sup_i |c_i(z)|$$

for $|z| \geq t_0$. Now, τ_0 can be chosen arbitrarily close to 1, which proves the proposition, setting $C_2 = \lambda_3 \cdot C_{t_0}$.

∎

3.3 Corollary *Let $S_z(X) = \sum a_k(z)X^k$ be an A_0-series with reduced Bautin index $\overline{d}$. There exist constants $C_3 > 0$ and $\alpha > 0$ such that for $z \in \mathbf{C}^n$ and $k \in \mathbf{N}$,*

$$|a_k(z)| \leq C_3 \cdot (1+|z|)^{\lambda_1 k + \lambda_2 + \alpha} \cdot \lambda_4^k \cdot \sup_{0 \leq i \leq \overline{d}} |a_i(z)|.$$

Proof By Proposition 1.1 (c), there exist constants $D_j > 0$ such that

$$|c_j(z)| \leq D_j \cdot (1 + |z|^{\mu_j}) \cdot \sup_{0 \leq i \leq \overline{d}} |a_i| \, .$$

Then set $C_4 = \sup_j D_j$, $\alpha = \sup(\mu_j)$, $C_3 = C_2 C_4$.

∎

In the case of a principal ideal I, we can give an explicit expression for all the constants involved in Proposition 3.2, in terms of $\|a_d\|$ and the constants λ_i of the series $S_z(X)$.

3.4 Proposition *Assume that the Bautin ideal $I = (a_0, \ldots, a_d)$ is principal, generated by a_d. Then $\overline{d} = d$, and for all k,*

$$|a_k(z)| \leq \frac{1}{\|a_d\|} \cdot \lambda_3 \cdot 2^{dn} \cdot \lambda_4^k \cdot (1+|z|)^{\lambda_1 k + \lambda_2 - \delta}$$

with $\delta = \deg\, a_d$.

Proof By hypothesis, we have $a_k(z) = a_d(z) m_k(z)$, with $m_k(z)$ a polynomial of degree

$$\deg\, a_k - \deg\, a_d \leq \lambda_1 k + \lambda_2 - \delta,$$

and of norm

$$\|m_k\| \leq \frac{1}{\|a_d\|} \cdot 2^{dn} \cdot \|a_k\|,$$

see [M], Théorème 4 bis, p. 172 (recall that by definition, $\|a_k\| \leq \lambda_3 \lambda_4^k$).

∎

4. Zeroes of A_0-series and proof of Theorem 3.1

Recall that

$$S_z(X) = \sum_{k=0}^{\overline{d}} a_k(z) X^k + \sum_{k \geq \overline{d}+1} a_k(z) X^k := P_z(X) + Q_z(X).$$

For a fixed $z \in \mathbf{C}^n$, we will apply Rouché's Theorem in a disk of radius $R_1 \leq R$, where $R \geq \frac{1}{\lambda_4 (1+|z|)^{\lambda_1}}$ is the radius of convergence of $S_z(X)$. We want to find a circle Γ_1 of radius R_1 such that on Γ_1 we have $|P_z(X)| > |Q_z(X)|$. Then Rouché's Theorem will imply that the number of zeroes of $S_z(X)$ in the interior of Γ_1 is less than the number of zeroes of $P_z(X)$, therefore less than $\overline{d}$.

We thank A. Douady for providing the arguments below.

4.1 Proposition (A. Douady) *Let $P(X) = \sum_{i=0}^{\overline{d}} a_i X^i$ be a polynomial with complex coefficients, let $\gamma \in \mathbf{R}^+$, and set $\eta = (2\gamma+1)^{-2(\overline{d}+1)}$. Then given $R > 0$, there exist R_1, $\eta R < R_1 < R$, and i, $0 \leq i \leq \overline{d}$, such that*

$$|a_i|R_1^i > \gamma \cdot \sum_{j \neq i} |a_j| R_1^j.$$

For the proof we need the following:

4.2 Lemma *In the first quadrant $\mathbf{R}_+^2$ with coordinates (x, y), consider the lines*

$$D_i : \, y = \log|a_i| + i\, x.$$

For any positive $\lambda \in \mathbf{R}^+$, and any interval I of length λ, there exists an index i such that, setting $y_j(x) = \log|a_j| + jx$, we have for all $0 \leq j \leq \overline{d}$ and some $x_1 \in I$:

$$y_i(x_1) - y_j(x_1) \geq \frac{\lambda}{2(\overline{d}+1)} |j - i|.$$

Proof of 4.2 Let E be the convex subset of $\mathbf{R}_+^2$ defined by the inequality

$$y \geq \sup_j y_j(x).$$

The set E has at most $\overline{d}+1$ extreme points. Therefore there exists at least one interval $I' \subset I$ of length $\frac{\lambda}{\overline{d}+1}$ which does not contain the abscissa of any of these extreme points.

Let x_1 be the abscissa of the middle point of such an interval I'. Then there exists i such that $y_i(x_1) > y_j(x_1), \quad j \neq i$, since this is true for any x in the interior of I'. Since the slope of the line D_j is j, we have the inequality

$$y_i(x_1) - y_j(x_1) \geq \frac{\lambda}{2(\overline{d}+1)} |j - i|.$$

■

Proof of 4.1 Given $R > 0$ and a number $\eta < 1$, let us consider the interval

$$I = [\log \eta R, \log R]$$

of length $\lambda = \log(\eta^{-1})$. Applying Lemma 4.2 we obtain an index i, $0 \leq i \leq \overline{d}$, and a number R_1 with $\log \eta R < \log R_1 < \log R$ such that

$$|a_i|R_1^i > |a_j|R_1^j \ell^{\frac{|j-i|}{2}}$$

with

$$\ell = \exp \frac{\lambda}{\overline{d}+1}.$$

This gives

$$\sum_{j\neq i} |a_j|R_1^j < |a_i|R_1^i 2\frac{\ell^{-1/2}}{1-\ell^{-1/2}}$$

provided that $\ell > 1$. Now, in order to have the inequality of Proposition 4.1, it suffices to have $2\frac{\ell^{-1/2}}{1-\ell^{-1/2}} \leq \gamma^{-1}$, that is,

$$\ell \geq (2\gamma+1)^2,$$

which is indeed > 1. From $\ell = \exp \frac{\lambda}{\overline{d}+1} = \exp(\frac{\log(\eta^{-1})}{\overline{d}+1}) = \eta^{-\frac{1}{\overline{d}+1}}$, it follows that this is achieved if

$$\eta \leq \frac{1}{(2\gamma+1)^{2(\overline{d}+1)}} \quad ,$$

which is the result.

■

Corollary 4.3 *We have* $|P(X)| \geq \frac{1}{2} \cdot \sup_{0\leq j\leq \overline{d}}|a_j| \cdot R_1^j$ *for* $|X| = R_1$*, and* $\eta R < R_1 < R$*, with* $\eta \geq \frac{1}{25^{\overline{d}+1}}$.

Proof We apply Proposition 4.1 to the polynomial $P(X)$: there exists i, $0 \leq i \leq d$, such that

$$|a_i| \cdot R_1^i > \gamma \sum_{j\neq i} |a_j| \cdot R_1^j.$$

This implies

$$|P(X)| \geq (\gamma - 1) \sum_{j\neq i} |a_j| \cdot R_1^j$$

and

$$|P(X)| \geq (1 - \frac{1}{\gamma}) \cdot R_1^i \cdot |a_i|$$

for $|X| = R_1$, since we have $|P(X)| \geq |a_i| \cdot R_1^i - \sum_{j\neq i} |a_j| \cdot R_1^j$ for $|X| = R_1$. Taking $\gamma = 2$, we find $|P(X)| \geq \frac{1}{2}\, |a_j| \cdot R_1^j$ for any j, $0 \leq j \leq \overline{d}$. Therefore $|P(X)| \geq \frac{1}{2} \sup_j |a_j| \cdot R_1^j$, and η can be chosen such that $\eta \geq 25^{-\overline{d}-1}$.

■

Let us now end the proof of Theorem 3.1. We assume $R_1 \leq 1$ for simplicity. We have by the corollary above that

$$|P_z(X)| \geq \frac{1}{2} \sup_{0\leq j\leq \overline{d}}|a_j| \cdot R_1^j \geq \frac{1}{2} \sup_{0\leq j\leq \overline{d}}|a_j| \cdot \inf(R_1^{\overline{d}}, 1) = \frac{1}{2} R_1^{\overline{d}} \cdot \sup_j |a_j|.$$

To apply Rouché's Theorem on the circle of radius R_1 we need that:

$$Q_z(X) = \sum_{k=\overline{d}+1}^{\infty} R_1^k \cdot |a_k(z)| < \frac{1}{2} R_1^{\overline{d}} \cdot \sup_{0\leq j\leq \overline{d}} |a_j(z)|,$$

for which it suffices that $R_1^{k-\overline{d}} \cdot |a_k(z)| < 2^{-k+\overline{d}-1} \cdot \sup_{0\leq j\leq \overline{d}} |a_j(z)|$ for $|X| = R_1$.

We now apply the computations which give us the inequalities (13)-(15). We see that the series $S_z(X)$ has at most $\overline{d}$ zeroes in the disk of radius $R(z) := R_1$, $R_1 > \eta R$, which gives

$$R(z) \geq \left(4 \cdot 5^{2(\overline{d}+1)} \cdot C_3 \cdot \lambda_4^{\overline{d}+1}\right)^{-1} \cdot (1 + |z|)^{-(\lambda_1(\overline{d}+1)+\lambda_2+\alpha)}. \tag{16}$$

This proves Theorem 3.1, with $\mu_1 = \left(4 \cdot 5^{2(\overline{d}+1)} \cdot C_3 \cdot \lambda_4^{\overline{d}+1}\right)^{-1}$ and $\mu_2 = \lambda_1(\overline{d}+1) + \lambda_2 + \alpha$. ■

5. Remarks on estimates and questions

We briefly discuss how to control the constants involved. Unfortunately the estimates depend on the Bautin index and not just on the reduced Bautin index. Let (c_i) be a Gröbner basis of the ideal I for the order described in the Appendix, M the transformation matrix from the a_i's to the c_j's, and let g be a bound for the degrees (in z) of the c_i's.

Also, there are other parameters of an A_0-series than the $\lambda_i's$ which will enter in the evaluation of the constants μ_1 and μ_2 of Theorem 3.1 :

One should try to estimate μ_1 and μ_2 in terms of the Bautin index d, the reduced Bautin index $\overline{d}$, and the norm of the transformation matrix M with entries in $\mathbf{K}[z]$ between the basis $(a_0, \ldots, a_d)$ of the ideal I they span, and a Gröbner basis $(c_0, \ldots, c_r)$ of I.
The main remaining open question in the local version of Hilbert's 16-th problem is to relate the degree q of the original plane vector field (and the size of its coefficients) to these parameters of the Poincaré return map.
At the end of this section, we compute a lower bound for the absolute value of the non-zero coefficients of the a_k's (considered as polynomials in z). This should be useful for the estimation of the constant μ_2 in Theorem 3.1.
1) **Estimating μ_1 and μ_2 in terms of the parameters** $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, d, e, \|M\|)$ **of the A_0-series**, where e is an integer such that $a_j(z) \in \frac{1}{e}\mathbf{Z}[z]$ for $0 \leq j \leq d$.
a) Exponents : To estimate the constant α, notice first that we can take the degrees of the elements of a Gröbner basis $g = (\sup_{1\leq i\leq d} \deg a_i + 1)^{3n2^{n-1}}$ by ([G-M], Theorem 11). We then have $\alpha \leq g^{\beta n}$, where β is a universal constant (see Remark 1.2), and n is the dimension of the z-space. Finally we get:

5.1 Proposition *There exists a universal constant β such that for an A_0-series with coefficients in $K[z_1, \ldots, z_n]$ and with parameters λ_i the constant α is bounded in terms of the Bautin index d and the number n of variables by:*

$$\alpha \leq (\lambda_1 d + \lambda_2)^{3\beta n^2 2^{n-1}}.$$

It would be preferable to have an estimate in terms of $\overline{d}$ instead of d. The behaviour of Gröbner basis with respect to integral closure is still mysterious.

b) The constant C_3 : We have $C_3 = C_2C_4$. First, we have $C_2 = \lambda_3 C_{t_0} = \lambda_3 C\cdot(\inf |c_i|_{t_0})^{-1}$. It is easy to see that we may choose $t_0 \geq 1$ such that $C \leq 2$. The bound e is also valid for the c_i's, which gives $C_{t_0} \leq \frac{2}{e}$.

c) The constant C_4 : we have $C_4 = \sup_j D_j$ (Corollary 3.3). Assuming that for $0 \leq j \leq d$ we have $a_j(z) \in \frac{1}{e}\mathbf{Z}[z]$, we have by [So] that $D_j \leq h^{\gamma n}$, where γ is a universal constant and h is a function of g, e, and the $\|c_i\|$'s.

5.2 Proposition *For an A_0-series such that $a_j(z) \in \frac{1}{e}\mathbf{Z}[z]$ for $0 \leq j \leq d$, we have:*

$$C_4 \leq 2e^{-1}h^{\gamma n},$$

where γ is a universal constant and h is a function of g, e and the $\|c_i\|$'s. This follows from [So] after multiplication of all coefficients by e.

2) **Bounding μ_1, μ_2 in terms of the degree q of the original vector field (5): open problems**. Since the λ_i's are bounded in terms of q (Proposition 2.2), we see that the exponent of $1+|z|$ in (15) is bounded in terms of q, d and $\overline{d}$; as noticed above, it is an open problem to estimate d and $\overline{d}$ in terms of q only. The constant C_2 is also bounded in terms of q and d (we have $C_2 = \lambda_3 C_{t_0}$ with $\lambda_3 = 1$, $C_{t_0} \leq \frac{2}{\zeta}$; a lower bound for ζ is given by Proposition 5.3); the constant C_4 is more delicate to estimate : it depends on the norm of the transformation matrix M (which we do not know how to control in function of q), and on the constant of Proposition 1.1, c), estimated by Solernó when the coefficients of the polynomials are in $\mathbf{Z}$. Here the $a_j(z)$ have their coefficients in $\frac{1}{\tilde{e}}\cdot\mathbf{Z}[\pi]$, where $\tilde{e} \leq \rho_d$ according to Claim 5.4 below.

5.3 Proposition *Let $S_z(X) = \sum a_k(z)X^k$ be the Poincaré return map for the vector field*

$$x\partial_y - y\partial_x + \sum_{2\leq i+j\leq q} a_{ij}x^iy^j\partial_x + b_{ij}x^iy^j\partial_y,$$

of degree q. Set $a_k(z) = \sum a_{k,\underline{i}}z^{\underline{i}}$. If $a_{k,\underline{i}} \neq 0$, then

$$|a_{k,\underline{i}}| \geq \frac{\beta_k(q)}{\rho_k},$$

with

$$\beta_k(q) = \exp\{-2\,.\,10^6\ \left[\tfrac{k-1}{2}\right]\left(\log[(2^k-1)\rho_k(33\pi q^4)^k] + \left[\tfrac{k-1}{2}\right]\log\left[\tfrac{k-1}{2}\right]\right)\left(1+\log\left[\tfrac{k-1}{2}\right]\right)\},$$

and ρ_k is an effectively computable function of k defined below.

Proof By linearization of a polynomial in $(\sin s_i\theta, \cos t_j\theta)$, we mean the replacement of each monomial $(\sin s_1\theta)^{d_1}\cdots(\sin s_p\theta)^{d_p}(\cos t_1\theta)^{f_1}\cdots(\cos t_q\theta)^{f_q}$ by a linear term

$$\sum \lambda_i \sin\alpha_i\theta + \sum \mu_j \cos\beta_j\theta$$

with α_i, β_j bounded by $\sum s_i d_i + \sum t_j f_j$. Let us first look at the term $a_2(z,\theta)$. We have $a'_2 = R_2 = P_2$; it is therefore a homogeneous polynomial in $(\sin\theta, \cos\theta)$ of degree 3, which gives by linearization a polynomial in $z, \cos j\theta, \sin j\theta$, $1 \leq j \leq 3$ with coefficients in $\frac{1}{4}\mathbf{Z}$, with no term in θ. By integration in θ, we get a polynomial with coefficients in $\frac{1}{12}\mathbf{Z}$, which gives $\rho_2 = 12$ (and $a_2(z, 2\pi) = 0$).
For the induction step, we consider the function $a_k(z,\theta)$ given by (10). It is a polynomial in $z, \theta, \cos\theta, \sin\theta$. We have seen that its degree in z is $\leq k-1$. We first prove the following:

5.4 Claim *The degree in θ of $a_k(z,\theta)$ is $\leq \left[\frac{k-1}{2}\right]$. After linearization, $a_k(z,\theta)$ becomes a polynomial of degree ≤ 1 in $\sin j\theta$, $\cos j\theta$, with $j \leq 3(k-1)$. As a polynomial in $(z, \theta, \sin j\theta, \cos j\theta)$, its coefficients belong to $\frac{1}{\rho_k}\mathbf{Z}$, where the integers ρ_k satisfy the inequalities*:

$$\rho_k \leq \big((3k-3)!\big)^{\left[\frac{k-1}{2}\right]} \cdot 2^{3(k-2)} \cdot (\rho_2 \cdots \rho_{k-1})^k, \quad \rho_2 = 12. \tag{16}$$

Proof Let us look at equation (10) :

$$a'_k = G_{2k}R_2 + \cdots + G_{ik}(a_2, \ldots, a_{k-1})R_i + \cdots + R_k.$$

For a term $1^{d_1} a_2^{d_2} \cdots a_{k-1}^{d_{k-1}} R_i$ of $G_{ik}R_i$, we have

$$\begin{cases} d_1 + 2d_2 + \cdots + (k-1)d_{k-1} = k \\ d_1 + d_2 + \cdots + d_{k-1} = i. \end{cases} \tag{17}$$

Its degree in θ is, by the induction hypothesis, bounded by $\sum_{j=1}^{k} \left[\frac{j-1}{2}\right] d_j \leq \frac{k-i}{2}$. After integration, the degree in θ of each term increases at most by one, and it follows from the homogeneity result of Lemma 2.3 that if i is even, the degree in θ of $G_{ik}(a_2, \ldots, a_{k-1})R_i$ does not increase; therefore, the degree in θ of a_k is bounded by

$$\left[\sup\left(\tfrac{k-2}{2}, \tfrac{k-3}{2} + 1\right)\right] = \left[\tfrac{k-1}{2}\right].$$

By the induction hypothesis, each a_j is linear in $\cos s\theta, \sin s\theta$, $s \leq 3(j-1)$, with coefficients in $\frac{1}{\rho_j}\mathbf{Z}$ and R_i has $\mathbf{Z}$-coefficients, and degree $\leq 3(i-1)$ (Lemma 2.3). Linearization of the terms in $(\sin\theta, \cos\theta)$ gives linear terms in $(\sin\alpha\theta, \cos\alpha\theta)$, with

$$\alpha \leq 3(i-1) + \sum_{j=1}^{k-1} 3(j-1)d_j \leq 3(i-1) + 3k - 3i = 3(k-1).$$

Let us now estimate ρ_k: by induction hypothesis, each term $a_2^{d_2} \ldots a_{k-1}^{d_{k-1}} R_i$ has coefficients in $\frac{1}{(\rho_1 \cdots \rho_{k-1})^k}\mathbf{Z}$ (each a_j being linear in $(\sin s\theta, \cos s\theta)$, $s \leq 3(j-1)$). Linearization multiplies the coefficients at most by 2^{-r}, $0 \leq r \leq 3(k-1)$, and integration with respect to θ multiplies at most by a factor of $\big((3k-3)!\big)^{\left[\frac{k-1}{2}\right]}$.

■

To get the Poincaré return map, we have to set $\theta = 2\pi$ in $a_k(z, \theta)$. The only terms which give a nonzero contribution are of the form $(\cos j\theta)\theta^l$, $l \geq 1$, due to the initial condition $a_k(z, 0) = 0$. We may consider the polynomials $\rho_k \tilde{a}_k(z, \theta) = \sum_i \tilde{a}_{k,i}(\theta) z^i$ obtained from $a_k(z, \theta)$ by setting $\cos j\theta = 1$, $\sin j\theta = 0$ and multiplying by ρ_k; the $\tilde{a}_{k,i}(\theta)$ are polynomials with integral coefficients, of degree $\leq k - 1$ and size $\|\tilde{a}_{k,i}\| \leq \rho_k (33\pi q^4)^k$ by Lemma 2.4. The value $\tilde{a}_{k,i}(2\pi)$ is the value for $\theta' = \pi$ of the polynomial $\tilde{a}_{k,i}(2\theta')$ which is integral, of degree $\leq k - 1$ and size $\leq (2^k - 1)\rho_k (33\pi q^4)^k$.

Now, we can apply:

Theorem (Nesterenko-Waldschmidt, Theorem 2 of [N-W])
Given a nonzero polynomial $P \in \mathbf{Z}[X]$ with $\|P\| \leq L, \deg P \leq d$, and $L \geq 3$, then:

$$|P(\pi)| \geq \exp\{-2 \,.\, 10^6 d(\log L + d \log d)(1 + \log d)\}.$$

■

In our case the size is clearly ≥ 3, and we get

$$|\tilde{a}_{k,i}(2\pi)| \geq \exp\{-2 \,.\, 10^6 \left[\tfrac{k-1}{2}\right] \big(\log[(2^k - 1)\rho_k (33\pi q^4)^k] + \left[\tfrac{k-1}{2}\right] \log \left[\tfrac{k-1}{2}\right]\big)\big(1 + \log \left[\tfrac{k-1}{2}\right]\big)\}.$$

This ends the proof of the proposition.

■

Let us remark that when the A_0-series stems from a vector field, there is an explicit lower bound ζ on the absolute values of the non zero coefficients of the a_i's, $0 \leq i \leq d$. We may, by Proposition 5.3, take

$$\zeta = \frac{1}{\rho_d} \exp\{-2 \,.\, 10^6 \left[\tfrac{d-1}{2}\right] \big(\log[(2^d - 1)\rho_d (33\pi q^4)^d] + \left[\tfrac{d-1}{2}\right] \log \left[\tfrac{d-1}{2}\right]\big)\big(1 + \log \left[\tfrac{d-1}{2}\right]\big)\}.$$

Solernó's proof has yet to be adapted to this case, using the evaluation of ζ given above.

Appendix

We give a version of the Division Theorem for polynomials with norm estimates analogous to the case of convergent power series [Ga, H-M]. Consider a $\mathbf{C}[z]$-linear map $l : \mathbf{C}[z]^r \longrightarrow \mathbf{C}[z]$, say $l(b) = b \cdot c = \sum_i b_i c_i$. The objective is to describe explicitly direct complements L and J of its kernel K and image I and to give norm estimates for the induced projections. On the way one constructs a continuous scission σ of l, i.e. a map $\sigma : \mathbf{C}[z] \longrightarrow \mathbf{C}[z]^r$ with $l\sigma l = l$, giving an upper bound on its norm.

Let $\tau = (\tau_1, \ldots, \tau_n)$ be real numbers ≥ 1 which are linearly independent over $\mathbf{Q}$ and equip $\mathbf{C}[z]$ with the norms

$$|P|_t = \sum_\alpha |P_\alpha| \cdot t^{<\tau,\alpha>}.$$

The vector τ induces a total ordering on the monomials in $\mathbf{C}[z]$ by comparing their weighted degrees $< \tau, \alpha >$. Let c_i^o be the initial monomials of c_i, i.e. the largest monomial of the expansion of c_i. The map l is then approximated by the monomial $\mathbf{C}[z]$-linear map $l^o : \mathbf{C}[z]^r \longrightarrow \mathbf{C}[z]$ given by $l(b) = b \cdot c^o = \sum_i b_i c_i^o$.

Now, the kernel and the image of l^o have natural direct complements L and J in $\mathbf{C}[z]^r$ and $\mathbf{C}[z]$ given by support conditions. Let α_i be the exponent of c_i and set $E = \bigcup_i \alpha_i + \mathbf{N}^n$. Let $E = \bigcup_i E_i$ be a partition of E with $\alpha_i \in E_i$. Then

$$J = \{a \in \mathbf{C}[z],\ \text{supp}\ a \subset \mathbf{N}^n \setminus E\} \quad \text{and} \quad L = \{b \in \mathbf{C}[z]^r,\ \text{supp}\ b_i \subset E_i - \alpha_i\}$$

are direct complements of I^o in $\mathbf{C}[z]$ and of K^o in $\mathbf{C}[z]^r$.

Division Theorem *Let* $l : \mathbf{C}[z]^r \longrightarrow \mathbf{C}[z]$ *be a* $\mathbf{C}[z]$*-linear map, say* $l(b) = b \cdot c = \sum_i b_i c_i$*. Set* $K = \text{Ker}\ l$*,* $I = \text{Im}\ l$ *and let* $L \subset \mathbf{C}[z]^r$ *and* $J \subset \mathbf{C}[z]$ *be direct complements of* K^0 *and* I^0 *as defined above.*

(a) *Assume that the* c_i*'s form a Gröbner basis of* $I = \text{Im}\ l$*. Then* $I \oplus J = \mathbf{C}[z]$*,* $K \oplus L = \mathbf{C}[z]^r$*.*

(b) *Let the* c_i*'s be arbitrary. There are constants* $C > 0$ *and* $t_0 \geq 1$ *such that for all* $t \geq t_0$ *the following holds: For any* $a \in \mathbf{C}[z]$ *the unique elements* $b \in L$ *and* $b' \in J$ *with* $a = \sum b_i c_i + b'$ *satisfy*

$$\sum |b_i|_t \cdot |c_i|_t + |b'|_t \ \leq \ C \cdot |a|_t.$$

(c) *The map* l *admits a scission* $\sigma : \mathbf{C}[z] \longrightarrow \mathbf{C}[z]^r$ *with norm estimate* $|\sigma|_t \leq C \cdot t^d$ *for all* $t \geq t_0$*, where* d *is the highest degree occurring in the minimal monomial generator system of* I^0*.*

Proof We adapt the proof of the Division Theorem for convergent power series, Thm. 5.1 of [H-M], p. 107, to the polynomial context.

The initial monomial of a polynomial is the largest of its monomials w.r.t. the norm $|\ |_t$. Assume that the c_i are monic and decompose them into $c_i = x^{\alpha_i} + c_i'$ with $c_i^o = x^{\alpha_i}$. Let $\varepsilon > 0$ be such that $< \tau, \alpha_i > \geq < \tau, \alpha > + \varepsilon$ for all i and all α in the support of c_i'. There exists a $t_0 \geq 1$ such that for all $t \geq t_0$ and all i

$$t^{<\tau,\alpha_i>} \geq t^\varepsilon \cdot |c_i'|_t.$$

The constant t_0 depends on an upper bound for the norms of the coefficients of the c_i'. Fix $t \geq t_0$. Equip $\mathbf{C}[z]^r$ with the norm $|b|_t = \sum_i |b_i|_t$. We may assume that all $c_i \neq 0$. The continuous linear map

$$u : L \oplus J \longrightarrow \mathbf{C}[z] : (b, b') \longrightarrow b \cdot c + b'$$

will be shown to be bijective. Supply the vector space $L \oplus J$ with the norm:

$$|(b, b')|_t = \sum_i |b_i|_t \cdot |c_i^o|_t + |b'|_t.$$

By definition of J and L the map

$$v : L \oplus J \longrightarrow \mathbf{C}[z] : (b, b') \longrightarrow \sum b_i \cdot c_i^o + b'$$

is bijective, bicontinuous of norm 1, and its inverse v^{-1} has norm 1 as well. Decompose u into $u = v + w$ where $w(b, b') = b \cdot c' = \sum b_i \cdot c_i'$. This yields

$$|w|_t \leq t^{-\varepsilon} \quad \text{and} \quad |wv^{-1}|_t \leq t^{-\varepsilon} < 1$$

for $t \geq t_0$. The geometric series defining the inverse of $uv^{-1} = \mathrm{id} + wv^{-1}$ is locally finite (i.e., finite when evaluated on a polynomial) since the monomial order given by τ is a well-ordering and wv^{-1} decreases the degree of a polynomial w.r.t. τ. It therefore defines a map from $\mathbf{C}[z]$ to $\mathbf{C}[z]$. Moreover

$$|(uv^{-1})^{-1}| \leq \frac{1}{1 - t_0^{-\varepsilon}} =: C.$$

Consequently u is invertible and

$$|u^{-1}| \leq C \quad \text{for} \quad t \geq t_0.$$

This proves the assertion.

■

Bibliography

[B] N.N. Bautin, On the number of limit cycles which appear with the variation of coefficients from an equilibrium state of the type focus or center, *Amer. Math. Soc. Transl.*, **100** (1954).

[B-Y] M. Briskin, Y. Yomdin, *Algebraic Families of Analytic Functions, I*, J. Diff. Equations, **136, 2**,(1997), 248-267 .

[Fe] A. Fekak : *Interprétation algébrique de l'exposant de Lojasiewicz*, CRAS Paris, **310**, 193-196 (1990).

[F-Y] J-P. Françoise, Y. Yomdin : *Bernstein Inequalities and Applications to Analytic Geometry and Differential Equations*, J. of Funct. Analysis, **145**, 185-205 (1997).
[Ga] A. Galligo : *Théorème de Division et stabilité en géométrie analytique locale*, Ann. Inst. Fourier (Grenoble)**29** (1979), 107-184.
[G-M] M. Giusti, T. Mora : *The complexity of Gröbner basis*, preprint (1994).
[H-M] H. Hauser, G. Müller : *A Rank Theorem for Analytic Maps Between Power Series Spaces*, Publ. Math. IHES, **80**, 95-115, (1994).
[L-T] M. Lejeune-Jalabert, B. Teissier : *Clôture intégrale des idéaux et équisingularité*, Séminaire Ecole Polytechnique 1974, Publ. Institut Fourier, Grenoble.
[Li-T] J. Lipman, B. Teissier : *Pseudo-rational singularities and a theorem of Briançon-Skoda about integral closures of ideals*, Michigan Math. J., **28**, 97-116 (1981).
[M] M. Mignotte : *Mathématiques pour le calcul formel*, Presses Universitaires de France, 1989.
[M-W] P. Monsky, G. Washnitzer: *Formal cohomology: I*, A. Math., **88**, 181-217 (1968).
[N-W] Y. Nesterenko, M. Waldschmidt : *On the approximation of the values of exponential function and logarithm by algebraic numbers*, (in russian), Mat. Zapiski, **2** *Diophantine approximations, Proceedings of papers dedicated to the memory of Prof. N.I. Feldman*, ed. Yu.V.Nesterenko, Centre for applied research under Mech.-Math. Faculty of MSU, Moscow , 23-42, (1996).
[So] P. Solernó : *Effective Łojasiewicz Inequalities in Semi-Algebraic Geometry*, Appl. Algebra Engnrg. Conn. Comput., **2** (1991), 2-14.
[Ya] S. Yakovenko : *A geometric proof of the Bautin theorem*, **23**, Amer. Math. Soc. Transl. Ser. 2, **165** (1995), 203-219,.

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