Extending a valuation centred in a local domain to the formal completion

F. J. Herrera Govantes, M. A. Olalla Acosta, M. Spivakovsky and B. Teissier

Abstract

Let (R, m, k) be a local noetherian domain with field of fractions K and R_ν a valuation ring, dominating R (not necessarily birationally). Let $\nu|_K : K^* \twoheadrightarrow \Gamma$ be the restriction of ν to K; by definition, $\nu|_K$ is centred at R. Let $\hat{R}$ denote the m-adic completion of R. In the applications of valuation theory to commutative algebra and the study of singularities, one is often induced to replace R by its m-adic completion $\hat{R}$ and ν by a suitable extension $\hat{\nu}_-$ to $\hat{R}/P$ for a suitably chosen prime ideal P, such that $P \cap R = (0)$. The purpose of this paper is to give, assuming that R is excellent, a systematic description of all such extensions $\hat{\nu}_-$ and to identify certain classes of extensions which are of particular interest for applications.

1. Introduction

All the rings in this paper will be commutative with 1.

Let (R, m, k) be a local noetherian domain with field of fractions K and R_ν a valuation ring, dominating R (not necessarily birationally). Let $\nu|_K : K^* \twoheadrightarrow \Gamma$ be the restriction of ν to K; by definition, $\nu|_K$ is centred at R. Let $\hat{R}$ denote the m-adic completion of R. In the applications of valuation theory to commutative algebra and the study of singularities, one is often induced to replace R by its m-adic completion $\hat{R}$ and ν by a suitable extension $\hat{\nu}_-$ to $\hat{R}/P$ for a suitably chosen prime ideal P, such that $P \cap R = (0)$ (one specific application we have in mind has to do with the approaches to proving the Local Uniformization Theorem in arbitrary characteristic such as that found in a forthcoming paper of Spivakovsky (Resolution of singularities I: Local Uniformization of an equicharacteristic quasi-excellent local domain whose residue field k satisfies $[k : k^p] < \infty$) and in [**13**]). The first reason is that the ring $\hat{R}$ is not, in general, an integral domain, so that we can only hope to extend ν to a *pseudo-valuation* on $\hat{R}$, which means precisely a valuation $\hat{\nu}_-$ on a quotient $\hat{R}/P$ as above. The prime ideal P is called the *support* of the pseudo-valuation. It is well known and easy to prove that such extensions $\hat{\nu}_-$ exist for some minimal prime ideals P of $\hat{R}$. Although, as we shall see, the datum of a valuation ν determines a unique minimal prime of $\hat{R}$ when R is excellent, in general, there are many possible primes P as above and for a fixed P many possible extensions $\hat{\nu}_-$. This is the second reason to study extensions $\hat{\nu}_-$.

The purpose of this paper is to give, assuming that R is excellent, a systematic description of all such extensions $\hat{\nu}_-$ and to identify certain classes of extensions which are of particular interest for applications. In fact, the only assumption about R we ever use in this paper is a weaker and a more natural condition than excellence, called the G condition, but we chose to talk about excellent rings since this terminology seems to be more familiar to most of the people. For the reader's convenience, the definitions of excellent and G-rings are recalled in the appendix. Under this assumption, we study extensions to (an integral quotient of) the completion $\hat{R}$ of a valuation ν and give descriptions of the valuations with which such extensions

are composed. In particular, we give criteria for the uniqueness of the extension if certain simple data on these composed valuations are fixed.

We conjecture (see [**13**, statement 5.19] and Conjecture 1.1 for a stronger and more precise statement) that given an excellent local ring R and a valuation ν of R which is positive on its maximal ideal m, there exists a prime ideal H of the m-adic completion $\hat{R}$ such that $H \cap R = (0)$ and an extension of ν to $\hat{R}/H$ which has the same value group as ν.

When studying extensions of ν to the completion of R, one is led to the study of its extensions to the henselization $\tilde{R}$ of R as a natural first step. This, in turn, leads to the study of extensions of ν to finitely generated local strictly étale extensions R^e of R. We therefore start out by letting $\sigma : R \to R^\dagger$ denote one of the three operations of completion, (strict) henselization, or a finitely generated local strictly étale extension:

$$R^\dagger = \hat{R} \tag{1.1}$$

or

$$R^\dagger = \tilde{R} \tag{1.2}$$

or

$$R^\dagger = R^e. \tag{1.3}$$

The ring $R^\dagger$ is local; let $m^\dagger$ denote its maximal ideal. The homomorphisms

$$R \longrightarrow \tilde{R} \quad \text{and} \quad R \longrightarrow R^e$$

are regular for any ring R; by definition, if R is an excellent ring, then the completion homomorphism is regular (in fact, regularity of the completion homomorphism is precisely the defining property of G-rings; see the appendix for the definition of regular homomorphism).

Let r denote the (real) rank of ν. Let $(0) = \Delta_r \subsetneq \Delta_{r-1} \subsetneq \ldots \subsetneq \Delta_0 = \Gamma$ be the isolated subgroups of Γ and $P_0 = (0) \subsetneq P_1 \subseteq \ldots \subseteq P_r = m$ the prime valuation ideals of R, which need not, in general, be distinct. In this paper, we will assume that R is excellent. Under this assumption, we will canonically associate to ν a chain $H_1 \subset H_3 \subset \ldots \subset H_{2r+1} = mR^\dagger$ of ideals of $R^\dagger$, numbered by odd integers from 1 to $2r + 1$, such that $H_{2\ell+1} \cap R = P_\ell$ for $0 \leqslant \ell \leqslant r$. We will show that all the ideals $H_{2\ell+1}$ are prime. We will define $H_{2\ell}$ to be the unique minimal prime ideal of $P_\ell R^\dagger$, contained in $H_{2\ell+1}$ (that such a minimal prime is unique follows from the regularity of the homomorphism σ).

We will thus obtain, in the cases (1.1)–(1.3), a chain of $2r + 1$ prime ideals

$$H_0 \subset H_1 \subset \ldots \subset H_{2r} = H_{2r+1} = mR^\dagger,$$

satisfying $H_{2\ell} \cap R = H_{2\ell+1} \cap R = P_\ell$ and such that $H_{2\ell}$ is a minimal prime of $P_\ell R^\dagger$ for $0 \leqslant \ell \leqslant r$. Moreover, if $R^\dagger = \tilde{R}$ or $R^\dagger = R^e$, then $H_{2\ell} = H_{2\ell+1}$. We call H_i the *ith implicit prime ideal* of $R^\dagger$, associated to R and ν. The ideals H_i behave well under local blowing ups along ν (that is, birational local homomorphisms $R \to R'$ such that ν is centred in R'), and more generally under ν-*extensions* of R defined below in Subsection 1.1. This means that given any local blowing up along ν or ν-extension $R \to R'$, the ith implicit prime ideal H_i' of $R'^\dagger$ has the property that $H_i' \cap R^\dagger = H_i$. This intersection has a meaning in view of Lemma 1.1.

For a prime ideal P in a ring R, $\kappa(P)$ will denote the residue field R_P/PR_P.

Let $(0) \subsetneq \mathbf{m}_1 \subsetneq \ldots \subsetneq \mathbf{m}_{r-1} \subsetneq \mathbf{m}_r = \mathbf{m}_\nu$ be the prime ideals of the valuation ring R_ν. By definitions, our valuation ν is a composition of r rank 1 valuations $\nu = \nu_1 \circ \nu_2 \circ \ldots \circ \nu_r$, where ν_ℓ is a valuation of the field $\kappa(\mathbf{m}_{\ell-1})$, centred at $(R_\nu)_{\mathbf{m}_\ell}/\mathbf{m}_{\ell-1}$ (see [**17**, Chapter VI, § 10, p. 43], for the definition of composition of valuations; more information and a simple example of composition is given in Subsection 1.1, where we interpret each $\mathbf{m}_\ell$ as the limit of a tree of ideals).

If $R^\dagger = \tilde{R}$, we will prove that there is a unique extension $\tilde{\nu}_-$ of ν to $\tilde{R}/H_0$. If $R^\dagger = \hat{R}$, the situation is more complicated. First, we need to discuss the behaviour of our constructions under ν-extensions.

1.1. *Local blowings up and trees*

We consider *extensions* $R \to R'$ of local rings, that is, injective morphisms such that R' is an R-algebra essentially of finite type and $m' \cap R = m$. In this paper, we consider only extensions with respect to ν; that is, both R and R' are contained in a fixed valuation ring R_ν. Such extensions form a direct system $\{R'\}$. We will consider many direct systems of rings and of ideals indexed by $\{R'\}$; direct limits will always be taken with respect to the direct system $\{R'\}$. Unless otherwise specified, we will assume that

$$\varinjlim_{R'} R' = R_\nu. \tag{1.4}$$

Note that by the fundamental properties of valuation rings [**17**, § VI], assuming the equality (1.4) is equivalent to assuming that $\varinjlim_{R'} K' = K_\nu$, where K' stands for the field of fractions of R' and K_ν for that of R_ν, and that $\varinjlim_{R'} R'$ is a valuation ring.

DEFINITION 1.1. A *tree* of R'-algebras is a direct system $\{S'\}$ of rings, indexed by the directed set $\{R'\}$, where S' is an R'-algebra. Note that the maps are not necessarily injective. A morphism $\{S'\} \to \{T'\}$ of trees is the datum of a map of R'-algebras $S' \to T'$ for each R' commuting with the tree morphisms for each map $R' \to R''$.

LEMMA 1.1. *Let $R \to R'$ be an extension of local rings. We have the following.*

(1) *The ideal $N := m^\dagger \otimes_R 1 + 1 \otimes_R m'$ is maximal in the R-algebra $R^\dagger \otimes_R R'$.*
(2) *The natural map of completions (resp. henselizations) $R^\dagger \to R'^\dagger$ is injective.*

Proof. (1) follows from that fact that $R^\dagger/m^\dagger = R/m$. The proof of (2) relies on a construction which we shall use often: the map $R^\dagger \to R'^\dagger$ can be factored as

$$R^\dagger \to (R^\dagger \otimes_R R')_N \to R'^\dagger, \tag{1.5}$$

where the first map sends x to $x \otimes 1$ and the second is determined by $x \otimes x' \mapsto \hat{b}(x).c(x')$ where $\hat{b}$ is the natural map $R^\dagger \to R'^\dagger$ and c is the canonical map $R' \to R'^\dagger$. The first map is injective because $R^\dagger$ is a flat R-algebra and it is obtained by tensoring the injection $R \to R'$ by the R-algebra $R^\dagger$; furthermore, elements of $R^\dagger$ whose image in $R^\dagger \otimes_R R'$ lie outside of N are precisely units of $R^\dagger$, hence they are not zero divisors in $R^\dagger \otimes_R R'$ and $R^\dagger$ injects in every localization of $R^\dagger \otimes_R R'$.

Since $m' \cap R = m$, we see that the inverse image by the natural map of R'-algebras

$$\iota \colon R' \to (R^\dagger \otimes_R R')_N,$$

defined by $x' \mapsto 1 \otimes_R x'$, of the maximal ideal $M = (m^\dagger \otimes_R 1 + 1 \otimes_R m')(R^\dagger \otimes_R R')_N$ of $(R^\dagger \otimes_R R')_N$ is the ideal m' and that ι induces a natural isomorphism $R'/m'^i \xrightarrow{\sim} (R^\dagger \otimes_R R')_N/M^i$ for each i. From this it follows by the universal properties of completion and henselization that the second map in the sequence (1.5) is the completion (resp. the henselization inside the completion) of $R^\dagger \otimes_R R'$ with respect to the ideal M. It is therefore also injective. $\square$

DEFINITION 1.2.   Let $\{S'\}$ be a tree of R'-algebras. For each S', let I' be an ideal of S'. We say that $\{I'\}$ is a tree of ideals if for any arrow $b_{S'S''}\colon S' \to S''$ in our direct system, we have $b_{S'S''}^{-1} I'' = I'$. We have the obvious notion of inclusion of trees of ideals. In particular, we may speak about chains of trees of ideals.

EXAMPLES.   The maximal ideals of the local rings of our system $\{R'\}$ form a tree of ideals.

For any non-negative element $\beta \in \Gamma$, the valuation ideals $\mathcal{P}'_\beta \subset R'$ of value β form a tree of ideals of $\{R'\}$. Similarly, the ith prime valuation ideals $P'_i \subset R'$ form a tree. If $\operatorname{rk} \nu = r$, the prime valuation ideals P'_i give rise to a chain

$$P'_0 = (0) \subsetneq P'_1 \subseteq \ldots \subseteq P'_r = m' \tag{1.6}$$

of trees of prime ideals of $\{R'\}$.

We discuss this last example in a little more detail and generality in order to emphasize our point of view, crucial throughout this paper: the datum of a composite valuation is equivalent to the datum of its components. Namely, suppose we are given a chain of trees of ideals as in (1.6), where we relax our assumptions of the P'_i as follows. We no longer assume that the chain (1.6) is maximal, nor that $P'_i \subsetneq P'_{i+1}$, even for R' sufficiently large; in particular, for the purposes of this example we momentarily drop the assumption that $\operatorname{rk} \nu = r$. We will still assume, however, that $P'_0 = (0)$ and that $P'_r = m'$.

Taking the limit in (1.6), we obtain a chain

$$(0) = \mathbf{m}_0 \subsetneq \mathbf{m}_1 \subseteq \ldots \subseteq \mathbf{m}_r = \mathbf{m}_\nu \tag{1.7}$$

of prime ideals of the valuation ring R_ν.

Similarly, for each $1 \leqslant \ell \leqslant r$ one has the equality

$$\varinjlim_{R'} \frac{R'}{P'_\ell} = \frac{R_\nu}{\mathbf{m}_\ell}.$$

Then specifying the valuation ν is equivalent to specifying valuations $\nu_0, \nu_1, \ldots, \nu_r$, where ν_0 is the trivial valuation of K and, for $1 \leqslant \ell \leqslant r$, ν_ℓ is a valuation of the residue field $k_{\nu_{\ell-1}} = \kappa(\mathbf{m}_{\ell-1})$, centred at the local ring $\varinjlim R'_{P'_\ell}/P'_{\ell-1}R'_{P'_\ell} = (R_\nu)_{\mathbf{m}_\ell}/\mathbf{m}_{\ell-1}$ and taking its values in the totally ordered group $\Delta_{\ell-1}/\Delta_\ell$.

The relationship between ν and the ν_ℓ is that ν is the composition

$$\nu = \nu_1 \circ \nu_2 \circ \ldots \circ \nu_r. \tag{1.8}$$

For example, the datum of the valuation ν, or of its valuation ring R_ν, is equivalent to the datum of the valuation ring $R_\nu/\mathbf{m}_{r-1} \subset (R_\nu)_{\mathbf{m}_{r-1}}/\mathbf{m}_{r-1}(R_\nu)_{\mathbf{m}_{r-1}} = \kappa(\mathbf{m}_{r-1})$ of the valuation ν_r of the field $\kappa(\mathbf{m}_{r-1})$ and the valuation ring $(R_\nu)_{\mathbf{m}_{r-1}}$. If we assume, in addition, that for R sufficiently large the chain (1.6) (equivalently, (1.7)) is a maximal chain of distinct prime ideals, then $\operatorname{rk} \nu = r$ and $\operatorname{rk} \nu_\ell = 1$ for each ℓ.

REMARK 1.   Another way to describe the same property of valuations is that, given a prime ideal H of the local integral domain R one builds all valuations centred in R having H as one of the P_ℓ by choosing a valuation ν_1 of R centred at H, so that $\mathbf{m}_{\nu_1} \cap R = H$ and choosing a valuation subring $\bar{R}_{\bar{\nu}}$ of the field $R_{\nu_1}/\mathbf{m}_{\nu_1}$ centred at R/H. Then $\nu = \nu_1 \circ \bar{\nu}$.

Note that choosing a valuation of R/H determines a valuation of its field of fractions $\kappa(H)$, which is, in general, much smaller than $R_{\nu_1}/\mathbf{m}_{\nu_1}$. Given a valuation of R with centre H, in order to determine a valuation of R with centre m inducing on R/H a given valuation μ we must choose an extension $\bar{\nu}$ of μ to $R_{\nu_1}/\mathbf{m}_{\nu_1}$, and there are, in general, many possibilities.

This will be used in the sequel. In particular, it will be applied to the case where a valuation ν of R extends uniquely to a valuation $\hat{\nu}_-$ of $\hat{R}/H$ for some prime H of $\hat{R}$. Assuming that $\hat{R}$ is an integral domain, this determines a unique valuation of $\hat{R}$ only if the height ht H of H in $\hat{R}$ is at most one. In all other cases the dimension of $\hat{R}_H$ is at least 2 and we have infinitely many valuations with which to compose $\hat{\nu}_-$. This is the source of the height conditions we shall see in § 6.

EXAMPLE 1. Let k_0 be a field and $K = k_0((u,v))$ the field of fractions of the complete local ring $R = k_0[\![u,v]\!]$. Let $\Gamma = \mathbf{Z}^2$ with lexicographical ordering. The isolated subgroups of Γ are $(0) \subsetneq (0) \oplus \mathbf{Z} \subsetneq \mathbf{Z}^2$. Consider the valuation $\nu : K^* \to \mathbf{Z}^2$, centred at R, given by

$$\nu(v) = (0,1), \tag{1.9}$$
$$\nu(u) = (1,0), \tag{1.10}$$
$$\nu(c) = 0 \quad \text{for any } c \in k_0^*. \tag{1.11}$$

This information determines ν completely; namely, for any power series

$$f = \sum_{\alpha,\beta} c_{\alpha\beta} u^\alpha v^\beta \in k_0[\![u,v]\!],$$

we have

$$\nu(f) = \min\{(\alpha,\beta) \,|\, c_{\alpha\beta} \neq 0\}.$$

We have rk ν = rat.rk $\nu = 2$. Let $\Delta = (0) \oplus \mathbf{Z}$. Let Γ_+ denote the semigroup of all the non-negative elements of Γ. Let $k_0[\![\Gamma_+]\!]$ denote the R-algebra of power series $\sum c_{\alpha,\beta} u^\alpha v^\beta$ where $c_{\alpha,\beta} \in k_0$ and the exponents (α,β) form a well-ordered subset of Γ_+. By classical results (see [**7**, **8**]), it is a valuation ring with maximal ideal generated by all the monomials $u^\alpha v^\beta$, where $(\alpha,\beta) > (0,0)$ (in other words, either $\alpha > 0, \beta \in \mathbf{Z}$ or $\alpha = 0, \beta > 0$). Then

$$R_\nu = k_0[\![\Gamma_+]\!] \cap k_0((u,v))$$

is a valuation ring of K, and contains $k[\![u,v]\!]$; it is the valuation ring of the valuation ν. The prime ideal $\mathbf{m}_1$ is the ideal of R_ν generated by all the uv^β, $\beta \in \mathbf{Z}$. The valuation ν_1 is the discrete rank 1 valuation of K with valuation ring

$$(R_\nu)_{\mathbf{m}_1} = k_0[\![u,v]\!]_{(u)}$$

and ν_2 is the discrete rank 1 valuation of $k_0((v))$ with valuation ring $R_\nu/\mathbf{m}_1 \cong k_0[\![v]\!]$.

EXAMPLE 2. To give a more interesting example, let k_0 be a field of characteristic zero and

$$K = k_0(x,y,z)$$

a purely transcendental extension of k_0 of degree 3. Let w be an independent variable and put $k = \bigcup_{j=1}^\infty k_0(w^{1/j})$. Let $\Gamma = \mathbf{Z} \oplus \mathbf{Q}$ with the lexicographical ordering and $\Delta = (0) \oplus \mathbf{Q}$ the non-trivial isolated subgroup of Γ. Let u and v be new variables and let $\mu_1 : k((u,v)) \to \mathbf{Z}^2_{\text{lex}}$ be the valuation of the previous example. Let μ_2 denote the x-adic valuation of k and put $\mu = \mu_1 \circ \mu_2$. Consider the map $\iota : k_0[x,y,z] \to k[\![u,v]\!]$ which sends x to w, y to v and z to $u - \sum_{j=1}^\infty w^{1/j} v^j$. Let $\nu_1 = \mu_1|_K$ and $\nu = \mu|_K$.

The valuation $\nu : K^* \to \Gamma$ is centred at the local ring $R = k_0[x,y,z]_{(x,y,z)}$; we have

$$\nu(x) = (0,1), \tag{1.12}$$
$$\nu(y) = (1,0), \tag{1.13}$$
$$\nu(z) = (1,1). \tag{1.14}$$

Write as a composition of two rank 1 valuations: $\nu = \nu_1 \circ \nu_2$. We have natural inclusions $R_{\nu_1} \subset R_{\mu_1}$ and $k_{\nu_1} \subset k_{\mu_1} = k$. We claim that k_{ν_1} is not finitely generated over k_0. Indeed, if this were not the case, then there would exist a prime number p such that $w^{1/p} \notin k_{\nu_1}$. Let $k' = k_0(x^{1/(p-1)!})$. Let $L = k'(y, z)$. Consider the tower of field extensions $K \subset L \subset k[\![u, v]\!]$ and let ν' denote the restriction of μ to L. Let Γ' be the value group of ν' and $k_{\nu'}$ the residue field of its valuation ring. Now, L contains the element $z_p := z - \sum_{j=1}^{p-1} x^{1/j} y^j$ as well as z_p/y^p. We have

$$\nu'(z_p) = \left(p, \frac{1}{p} \right), \tag{1.15}$$

$\nu'(z_p/y^p) = 0$ and the natural image of z_p/y^p in $k_{\mu_1} = k$ is $w^{1/p}$. Now, $p \nmid [L : K]$, $[\Gamma' : \Gamma] \mid [L : K]$ and $[k_{\nu'} : k_\nu] \mid [L : K]$. This implies that $z_p \in L$ and $w^{1/p} \in k_{\nu_1}$, which gives the desired contradiction.

It is not hard to show that for each j, there exists a local blowing up $R \to R'$ of R such that, in the notation of (1.6), we have $\kappa(P_1') = k_0(w^{1/j!})$ and that $\kappa(\mathbf{m}_1) = \lim_{j \to \infty} \kappa(P_1') = k$. The first one is the blowing up of the ideal $(y, z)R$, localized at the point $y = 0, z/y = x$. Then one blows up the ideal $(z/y - x, y)$, and so on.

Another way to see the valuation $\nu = \nu_1 \circ \nu_2$ is to note that ν_1 is the restriction to K of the v-adic valuation under the inclusion of fields deduced from the inclusion of rings

$$k_0[\![x, y, z]\!]_{(y,z)} \hookrightarrow k[\![v^{\mathbf{Z}_+}]\!]$$

which sends x to w, y to v and z to $\sum_{j=1}^\infty w^{1/j} v^j$. Recall that the ring on the right is made of power series with non-negative rational exponents whose set of exponents is well ordered. We have $k_{\nu_1} = k$.

REMARK 2. The point of the last example is to show that, given a composed valuation as in (1.8), ν_ℓ is a valuation of the field $k_{\nu_{\ell-1}}$, which may *properly* contain $\kappa(P_{\ell-1}')$, for *every* $R' \in \mathcal{T}$. This fact will be a source of complication later on and we prefer to draw attention to it from the beginning.

Coming back to the implicit prime ideals, we will see that the implicit prime ideals H_i' form a tree of ideals of $R^\dagger$.

We will show that if ν extends to a valuation of $\hat{\nu}_-$ centred at $\hat{R}/P$ with $P \cap R = (0)$, then the prime P must contain the minimal prime H_0 of $\hat{R}$. We will then show that specifying an extension $\hat{\nu}_-$ of ν as above is equivalent to specifying a chain of prime valuation ideals

$$\tilde{H}_0' \subset \tilde{H}_1' \subset \ldots \subset \tilde{H}_{2r}' = m'\hat{R}' \tag{1.16}$$

of $\hat{R}'$ such that $H_\ell' \subset \tilde{H}_\ell'$ for all $\ell \in \{0, \ldots, 2r\}$, and valuations $\hat{\nu}_1, \hat{\nu}_2, \ldots, \hat{\nu}_{2r}$, where $\hat{\nu}_i$ is a valuation of the field $k_{\hat{\nu}_{i-1}}$ (the residue field of the valuation ring $R_{\hat{\nu}_{i-1}}$), arbitrary when i is odd and satisfying certain conditions, coming from the valuation $\nu_{i/2}$, when i is even.

The prime ideals H_i are defined as follows.

Recall that given a valued ring (R, ν), that is a subring $R \subseteq R_\nu$ of the valuation ring R_ν of a valuation with value group Γ, one defines for each $\beta \in \Gamma$ the valuation ideals of R associated to β:

$$\mathcal{P}_\beta(R) = \{x \in R / \nu(x) \geqslant \beta\},$$
$$\mathcal{P}_\beta^+(R) = \{x \in R / \nu(x) > \beta\}$$

and the associated graded ring

$$\mathrm{gr}_\nu R = \bigoplus_{\beta \in \Gamma} \frac{\mathcal{P}_\beta(R)}{\mathcal{P}_\beta^+(R)} = \bigoplus_{\beta \in \Gamma_+} \frac{\mathcal{P}_\beta(R)}{\mathcal{P}_\beta^+(R)}.$$

The second equality comes from the fact that if $\beta \in \Gamma_- \setminus \{0\}$, we have $\mathcal{P}_\beta^+(R) = \mathcal{P}_\beta(R) = R$. If $R \to R'$ is an extension of local rings such that $R \subset R' \subset R_\nu$ and $m_\nu \cap R' = m'$, we will write $\mathcal{P}'_\beta$ for $\mathcal{P}_\beta(R')$.

Fix a valuation ring R_ν dominating R, and a tree $\mathcal{T} = \{R'\}$ of noetherian local R-subalgebras of R_ν, having the following properties: for each ring $R' \in \mathcal{T}$, all the birational ν-extensions of R' belong to $\mathcal{T}$. Moreover, we assume that the field of fractions of R_ν equals $\varinjlim_{R'} K'$, where K' is the field of fractions of R'. The tree $\mathcal{T}$ will stay constant throughout this paper. In the special case when R happens to have the same field of fractions as R_ν, we may take $\mathcal{T}$ to be the tree of all the birational ν-extensions of R.

NOTATION.   For a ring $R' \in \mathcal{T}$, we shall denote by $\mathcal{T}(R')$ the subtree of $\mathcal{T}$ consisting of all the ν-extensions R'' of R'.

We now define

$$H_{2\ell+1} = \bigcap_{\beta \in \Delta_\ell} \left(\left(\varinjlim_{R'} \mathcal{P}'_\beta R'^\dagger \right) \bigcap R^\dagger \right), \quad 0 \leqslant \ell \leqslant r-1 \tag{1.17}$$

(in the beginning of §3 we provide some motivation for this definition and give several elementary examples of H'_i and $\tilde{H}'_i$).

The questions answered in this paper originally arose from our work on the Local Uniformization Theorem, where passage to completion is required in both the approaches of Spivakovsky (forthcoming paper, Resolution of singularities I: Local Uniformization of an equicharacteristic quasi-excellent local domain whose residue field k satisfies $[k : k^p] < \infty$) and Teissier [13]. In [13], one really needs to pass to completion for valuations of arbitrary rank. One of the main intended applications of the theory of implicit prime ideals is the following conjecture. Let

$$\Gamma \hookrightarrow \hat{\Gamma} \tag{1.18}$$

be an extension of ordered groups of the same rank. Let

$$(0) = \Delta_r \subsetneqq \Delta_{r-1} \subsetneqq \ldots \subsetneqq \Delta_0 = \Gamma \tag{1.19}$$

be the isolated subgroups of Γ and

$$(0) = \hat{\Delta}_r \subsetneqq \hat{\Delta}_{r-1} \subsetneqq \ldots \subsetneqq \hat{\Delta}_0 = \hat{\Gamma}$$

the isolated subgroups of $\hat{\Gamma}$, so that the inclusion (1.18) induces inclusions

$$\Delta_\ell \hookrightarrow \hat{\Delta}_\ell \tag{1.20}$$

and

$$\frac{\Delta_\ell}{\Delta_{\ell+1}} \hookrightarrow \frac{\hat{\Delta}_\ell}{\hat{\Delta}_{\ell+1}}. \tag{1.21}$$

Let $G \hookrightarrow \hat{G}$ be an extension of graded algebras without zero divisors, such that G is graded by Γ_+ and $\hat{G}$ by $\hat{\Gamma}_+$. The graded algebra G is endowed with a natural valuation with value group Γ and similarly for $\hat{G}$ and $\hat{\Gamma}$. These natural valuations will both be denoted by ord.

DEFINITION 1.3.   We say that the extension $G \hookrightarrow \hat{G}$ is *scalewise birational* if for any $x \in \hat{G}$ and $\ell \in \{1, \ldots, r\}$ such that $\operatorname{ord} x \in \hat{\Delta}_\ell$ there exists $y \in G$ such that $\operatorname{ord} y \in \Delta_\ell$ and $xy \in G$.

Of course, scalewise birational implies birational and also that $\hat{\Gamma} = \Gamma$.

While the main result of this paper is the primality of the implicit ideals associated to a valuation, and the subsequent description of the extensions of the valuation to the completion, the main conjecture stated here is the following:

CONJECTURE 1.1.   Assume that $\dim R' = \dim R$ for all $R' \in \mathcal{T}$. Then there exists a tree of prime ideals H' of $\hat{R}'$ with $H' \cap R' = (0)$ and a valuation $\hat{\nu}_-$, centred at $\varinjlim(\hat{R}'/H')$ and having the following property:

For any $R' \in \mathcal{T}$ the graded algebra $\mathrm{gr}_{\hat{\nu}_-}(\hat{R}'/H')$ is a scalewise birational extension of $\mathrm{gr}_\nu R'$.

The example given in [**13**, Remark 5.20(4)] shows that the morphism of associated graded rings is not an isomorphism in general.

The approach to the Local Uniformization Theorem taken in Spivakovsky (forthcoming paper) is to reduce the problem to the case of rank 1 valuations. The theory of implicit prime ideals is much simpler for valuations of rank 1 and takes only a few pages in § 2.

The paper is organized as follows. In § 3, we define the odd-numbered implicit ideals $H_{2\ell+1}$ and prove that $H_{2\ell+1} \cap R = P_\ell$. We observe that by their very definition, the ideals $H_{2\ell+1}$ behave well under ν-extensions; they form a tree. Proving that $H_{2\ell+1}$ is indeed prime is postponed until later sections; it will be proved gradually in Sections 4–8. In the beginning of § 3 we will explain in more detail the respective roles played by the odd-numbered and the even-numbered implicit ideals, give several examples (among other things, to motivate the need for taking the limit with respect to R' in (1.17)) and say one or two words about the techniques used to prove our results.

In § 4, we prove the primality of the implicit prime ideals assuming a certain technical condition, called *stability*, about the tree $\mathcal{T}$ and the operation †. It follows from the noetherianity of $R^\dagger$ that there exists a specific R' for which the limit in (1.17) is attained. One of the main points of § 4 is to prove properties of stable rings which guarantee that this limit is attained whenever R' is stable. We then use the excellence of R to define the even-numbered implicit prime ideals: for $i = 2\ell$ the ideal $H_{2\ell}$ is defined to be the unique minimal prime of $P_\ell R^\dagger$, contained in $H_{2\ell+1}$ (in the case $R^\dagger = \hat{R}$ it is the excellence of R which implies the uniqueness of such a minimal prime). We have

$$H_{2\ell} \cap R = P_\ell$$

for $\ell \in \{0, \ldots, r\}$. The results of § 4 apply equally well to completions, henselizations and other local étale extensions; to complete the proof of the primality of the implicit ideals in various contexts such as henselization or completion, it remains to show the existence of stable ν-extensions in the corresponding context.

In § 5, we describe the set of extensions $\nu_-^\dagger$ of ν to $\varinjlim_{R'}(R'^\dagger/P'R'^\dagger)$, where P' is a tree of prime ideals of $R'^\dagger$ such that $P' \cap R' = (0)$. We show (Theorem 5.4) that specifying such a valuation $\nu_-^\dagger$ is equivalent to specifying the following data.

(1) A chain (1.16) of trees of prime ideals $\tilde{H}'_i$ of $R'^\dagger$ (where $\tilde{H}'_0 = P'$), such that $H'_i \subset \tilde{H}'_i$ for each i and each $R' \in \mathcal{T}$, satisfying one additional condition (we will refer to the chain (1.16) as the chain of trees of ideals, *determined by* the extension $\nu_-^\dagger$)

(2) A valuation $\nu_i^\dagger$ of the residue field $k_{\nu_{i-1}^\dagger}$ of $\nu_{i-1}^\dagger$, whose restriction to the field $\varinjlim_{R'} \kappa(\tilde{H}'_{i-1})$ is centred at the local ring $\varinjlim_{R'}(R'^\dagger_{\tilde{H}'_i}/\tilde{H}'_{i-1}R'^\dagger_{\tilde{H}'_i})$.

If $i = 2\ell$ is even, the valuation $\nu_i^\dagger$ must be of rank 1 and its restriction to $\kappa(\mathbf{m}_{\ell-1})$ must coincide with ν_ℓ.

Note the recursive nature of this description of $\nu_-^\dagger$: in order to describe $\nu_i^\dagger$, we must know $\nu_{i-1}^\dagger$ in order to talk about its residue field $k_{\nu_{i-1}^\dagger}$.

In §6, we address the question of uniqueness of $\nu_-^\dagger$. We describe several classes of extensions $\nu_-^\dagger$ which are particularly useful for the applications: *minimal* and *evenly minimal* extensions, and also those $\nu_-^\dagger$ for which, denoting by $\mathrm{ht}\, I$ the height of an ideal, we have

$$\mathrm{ht}\, \tilde{H}'_{2\ell+1} - \mathrm{ht}\, \tilde{H}'_{2\ell} \leqslant 1 \quad \text{for } 0 \leqslant \ell \leqslant r; \tag{1.22}$$

in fact, the special case of (1.22) which is of most interest for the applications is

$$\tilde{H}'_{2\ell} = \tilde{H}'_{2\ell+1} \quad \text{for } 1 \leqslant \ell \leqslant r. \tag{1.23}$$

We prove some necessary and some sufficient conditions under which an extension $\nu_-^\dagger$ whose corresponding ideals $\tilde{H}'_i$ satisfy (1.23) is uniquely determined by the ideals $\tilde{H}'_i$. We also give sufficient conditions for the graded algebra $\mathrm{gr}_\nu R'$ to be scalewise birational to $\mathrm{gr}_{\hat{\nu}_-} \hat{R}'$ for each $R' \in \mathcal{T}$. These sufficient conditions are used in §9 to prove some partial results towards Conjecture 1.1.

In §7, we show the existence of ν-extensions in $\mathcal{T}$, stable for henselization, thus reducing the proof of the primality of $H_{2\ell+1}$ to the results of §4. We study the extension of ν to $\tilde{R}$ modulo its first prime ideal and prove that such an extension is unique.

In §8, we use the results of §7 to prove the existence of ν-extensions in $\mathcal{T}$, stable for completion. Combined with the results of §4 this proves that the ideals $H_{2\ell+1}$ are prime.

In §9, we describe a possible approach and prove some partial results towards constructing a chain of trees (1.16) of prime ideals of $\hat{R}'$ satisfying (1.23) and a corresponding valuation $\hat{\nu}_-$ which satisfies the conclusion of Conjecture 1.1. We also prove a necessary and a sufficient condition for the uniqueness of $\hat{\nu}_-$, assuming Conjecture 1.1.

2. *Extending a valuation of rank 1 centred in a local domain to its formal completion*

Let (R, M, k) be a local noetherian domain, K its field of fractions, and $\nu : K \to \Gamma_+ \cup \{\infty\}$ a rank 1 valuation, centred at R (that is, non-negative on R and positive on M).

Let $\hat{R}$ denote the formal completion of R. It is convenient to extend ν to a valuation centred at $\hat{R}/H$, where H is a prime ideal of $\hat{R}$ such that $H \cap R = (0)$. In this section, we will assume that ν is of rank 1, so that the value group Γ is archimedian. We will explicitly describe a prime ideal H of $\hat{R}$, canonically associated to ν, such that $H \cap R = (0)$ and such that ν has a unique extension $\hat{\nu}_-$ to $\hat{R}/H$.

Let $\Phi = \nu(R \setminus (0))$, let $\mathcal{P}_\beta$ denote the ν-ideal of R of value β and $\mathcal{P}_\beta^+$ the greatest ν-ideal, properly contained in $\mathcal{P}_\beta$. We now define the main object of study of this section. Let

$$H := \bigcap_{\beta \in \Phi} (\mathcal{P}_\beta \hat{R}). \tag{2.1}$$

REMARK 3. Since R is noetherian, we have $\nu(M) > 0$ and since the ordered group Γ is archimedian, for every $\beta \in \Phi$, there exists $n \in \mathbf{N}$ such that $M^n \subset \mathcal{P}_\beta$. In other words, the M-adic topology on R is finer than (or equal to) the ν-adic topology. Therefore an element $x \in \hat{R}$ lies in $\mathcal{P}_\beta \hat{R} \iff$ there exists a Cauchy sequence $\{x_n\} \subset R$ in the M-adic topology, converging to x, such that $\nu(x_n) \geqslant \beta$ for all $n \iff$ for *every* Cauchy sequence $\{x_n\} \subset R$, converging to x, $\nu(x_n) \geqslant \beta$ for all $n \gg 0$. By the same token, $x \in H \iff$ there exists a Cauchy sequence $\{x_n\} \subset R$, converging to x, such that $\lim_{n \to \infty} \nu(x_n) = \infty \iff$ for *every* Cauchy sequence $\{x_n\} \subset R$, converging to x, $\lim_{n \to \infty} \nu(x_n) = \infty$.

EXAMPLE 3. Let $R = k[u,v]_{(u,v)}$. Then $\hat{R} = k[\![u,v]\!]$. Consider an element $w = u - \sum_{i=1}^{\infty} c_i v^i \in \hat{R}$, where $c_i \in k^*$ for all $i \in \mathbf{N}$, such that w is transcendental over $k(u,v)$. Consider the injective map $\iota : k[u,v]_{(u,v)} \to k[\![t]\!]$ which sends v to t and u to $\sum_{i=1}^{\infty} c_i t^i$. Let ν be the valuation induced from the t-adic valuation of $k[\![t]\!]$ via ι. The value group of ν is $\mathbf{Z}$ and $\Phi = \mathbf{N}_0$. For each $\beta \in \mathbf{N}$, $\mathcal{P}_\beta = (v^\beta, u - \sum_{i=1}^{\beta-1} c_i v^i)$. Thus, $H = (w)$.

We come back to the general theory. Since the formal completion homomorphism $R \to \hat{R}$ is faithfully flat,

$$\mathcal{P}_\beta \hat{R} \cap R = \mathcal{P}_\beta \quad \text{for all } \beta \in \Phi. \tag{2.2}$$

Taking the intersection over all $\beta \in \Phi$, we obtain

$$H \cap R = \left(\bigcap_{\beta \in \Phi} (\mathcal{P}_\beta \hat{R}) \right) \cap R = \bigcap_{\beta \in \Phi} \mathcal{P}_\beta = (0). \tag{2.3}$$

In other words, we have a natural inclusion $R \hookrightarrow \hat{R}/H$.

THEOREM 2.1. (i) H *is a prime ideal of* $\hat{R}$.
(ii) ν *extends uniquely to a valuation* $\hat{\nu}_-$*, centred at* $\hat{R}/H$.

Proof. Let $\bar{x} \in \hat{R}/H \setminus \{0\}$. Pick a representative x of $\bar{x}$ in $\hat{R}$, so that $\bar{x} = x \bmod H$. Since $x \notin H$, we have $x \notin \mathcal{P}_\alpha \hat{R}$ for some $\alpha \in \Phi$.

LEMMA 2.2 (See [**17**, Appendix 5, Lemma 3]). *Let* ν *be a valuation of rank 1 centred in a local noetherian domain* (R, M, k). *Let*

$$\Phi = \nu(R \setminus (0)) \subset \Gamma.$$

Then Φ *contains no infinite bounded sequences.*

Proof. An infinite ascending sequence $\alpha_1 < \alpha_2 < \ldots$ in Φ, bounded above by an element $\beta \in \Phi$, would give rise to an infinite descending chain of ideals in $R/\mathcal{P}_\beta$. Thus it is sufficient to prove that $R/\mathcal{P}_\beta$ has finite length.

Let $\delta := \nu(M) \equiv \min(\Phi \setminus \{0\})$. Since Φ is archimedian, there exists $n \in \mathbf{N}$ such that $\beta \leqslant n\delta$. Then $M^n \subset \mathcal{P}_\beta$, so that there is a surjective map $R/M^n \twoheadrightarrow R/\mathcal{P}_\beta$. Thus $R/\mathcal{P}_\beta$ has finite length, as desired. $\square$

By Lemma 2.2, the set $\{\beta \in \Phi \,|\, \beta < \alpha\}$ is finite. Hence there exists a unique $\beta \in \Phi$ such that

$$x \in \mathcal{P}_\beta \hat{R} \setminus \mathcal{P}_\beta^+ \hat{R}. \tag{2.4}$$

Note that β depends only on $\bar{x}$, but not on the choice of the representative x. Define the function $\hat{\nu}_- : \hat{R}/H \setminus \{0\} \to \Phi$ by

$$\hat{\nu}_-(\bar{x}) = \beta. \tag{2.5}$$

By (2.2), if $x \in R \setminus \{0\}$, then

$$\hat{\nu}_-(x) = \nu(x). \tag{2.6}$$

It is obvious that

$$\hat{\nu}_-(x+y) \geqslant \min\{\hat{\nu}_-(x), \hat{\nu}_-(y)\}, \tag{2.7}$$
$$\hat{\nu}_-(xy) \geqslant \hat{\nu}_-(x) + \hat{\nu}_-(y) \tag{2.8}$$

for all $x, y \in \hat{R}/H$. The point of the next lemma is to show that $\hat{R}/H$ is a domain and that $\hat{\nu}_-$ is, in fact, a valuation (that is, that the inequality (2.8) is, in fact, an equality).

LEMMA 2.3.   *For any non-zero $\bar{x}, \bar{y} \in \hat{R}/H$, we have $\bar{x}\bar{y} \neq 0$ and $\hat{\nu}_-(\bar{x}\bar{y}) = \hat{\nu}_-(\bar{x}) + \hat{\nu}_-(\bar{y})$.*

Proof.   Let $\alpha = \hat{\nu}_-(\bar{x})$, $\beta = \hat{\nu}_-(\bar{y})$. Let x and y be representatives in $\hat{R}$ of $\bar{x}$ and $\bar{y}$, respectively. We have $M\mathcal{P}_\alpha \subset \mathcal{P}_\alpha^+$, so that

$$\frac{\mathcal{P}_\alpha}{\mathcal{P}_\alpha^+} \cong \frac{\mathcal{P}_\alpha}{\mathcal{P}_\alpha^+ + M\mathcal{P}_\alpha} \cong \frac{\mathcal{P}_\alpha}{\mathcal{P}_\alpha^+} \otimes_R k \cong \frac{\mathcal{P}_\alpha}{\mathcal{P}_\alpha^+} \otimes_R \frac{\hat{R}}{M\hat{R}} \cong \frac{\mathcal{P}_\alpha \hat{R}}{(\mathcal{P}_\alpha^+ + M\mathcal{P}_\alpha)\hat{R}} \cong \frac{\mathcal{P}_\alpha \hat{R}}{\mathcal{P}_\alpha^+ \hat{R}}, \tag{2.9}$$

and similarly for β. By (2.9) there exist $z \in \mathcal{P}_\alpha$, $w \in \mathcal{P}_\beta$, such that $z \equiv x \bmod \mathcal{P}_\alpha^+ \hat{R}$ and $w \equiv y \bmod \mathcal{P}_\beta^+ \hat{R}$. Then

$$xy \equiv zw \bmod \mathcal{P}_{\alpha+\beta}^+ \hat{R}. \tag{2.10}$$

Since ν is a valuation, $\nu(zw) = \alpha + \beta$, so that $zw \in \mathcal{P}_{\alpha+\beta} \setminus \mathcal{P}_{\alpha+\beta}^+$. By (2.2) and (2.10), this proves that $xy \in \mathcal{P}_{\alpha+\beta}\hat{R} \setminus \mathcal{P}_{\alpha+\beta}^+ \hat{R}$. Thus, $xy \notin H$ (hence $\bar{x}\bar{y} \neq 0$ in $\hat{R}/H$) and $\hat{\nu}_-(\bar{x}\bar{y}) = \alpha + \beta$, as desired. $\square$

By Lemma 2.3, H is a prime ideal of $\hat{R}$. By (2.7) and Lemma 2.3, $\hat{\nu}_-$ is a valuation, centred at $\hat{R}/H$. To complete the proof of Theorem 2.1, it remains to prove the uniqueness of $\hat{\nu}_-$. Let $x, \bar{x}$, the element $\alpha \in \Phi$ and also

$$z \in \mathcal{P}_\alpha \setminus \mathcal{P}_\alpha^+ \tag{2.11}$$

be as in the proof of Lemma 2.3. Then there exist

$$u_1, \ldots, u_n \in \mathcal{P}_\alpha^+$$

and

$$v_1, \ldots, v_n \in \hat{R} \tag{2.12}$$

such that $x = z + \sum_{i=1}^n u_i v_i$. Letting $\bar{v}_i := v_i \bmod H$, we obtain $\bar{x} = \bar{z} + \sum_{i=1}^n \bar{u}_i \bar{v}_i$ in $\hat{R}/H$. Therefore, by (2.11)–(2.12), for any extension of ν to a valuation $\hat{\nu}'_-$, centred at $\hat{R}/H$, we have

$$\hat{\nu}'_-(\bar{x}) = \alpha = \hat{\nu}_-(\bar{x}), \tag{2.13}$$

as desired. This completes the proof of Theorem 2.1. $\square$

DEFINITION 2.1.   The ideal H is called the *implicit prime ideal* of $\hat{R}$, associated to ν. When dealing with more than one ring at a time, we will sometimes write $H(R, \nu)$ for H.

More generally, let ν be a valuation centred at R, not necessarily of rank 1. In any case, we may write ν as a composition $\nu = \mu_2 \circ \mu_1$, where μ_2 is centred at a non-maximal prime ideal P of R and $\mu_1|_{R/P}$ is of rank 1. The valuation $\mu_1|_{R/P}$ is centred at R/P. We define the *implicit prime ideal* of R with respect to ν, denoted $H(R, \nu)$, to be the inverse image in $\hat{R}$ of the implicit prime ideal of $\hat{R}/P$ with respect to $\mu_1|_{R/P}$. For the rest of this section, we will continue to assume that ν is of rank 1.

REMARK 4.   By (2.9), we have the following natural isomorphisms of graded algebras:

$$\operatorname{gr}_\nu R \cong \operatorname{gr}_{\hat{\nu}_-} \frac{\hat{R}}{H},$$

$$G_\nu \cong G_{\hat{\nu}_-}.$$

We will now study the behaviour of H under local blowings up of R with respect to ν and, more generally, under local homomorphisms. Let $\pi : (R, M) \to (R', M')$ be a local homomorphism of local noetherian domains. Assume that ν extends to a rank 1 valuation $\nu' : R' \setminus \{0\} \to \Gamma'$, where $\Gamma' \supset \Gamma$. The homomorphism π induces a local homomorphism $\hat{\pi} : \hat{R} \to \hat{R}'$ of formal completions. Let $\Phi' = \nu'(R' \setminus \{0\})$. For $\beta \in \Phi'$, let $\mathcal{P}'_\beta$ denote the ν'-ideal of $R_{\nu'}$ of value β, as above. Let $H' = H(R', \nu')$.

LEMMA 2.4.   *Let $\beta \in \Phi$. Then*

$$(\mathcal{P}'_\beta \hat{R}') \cap \hat{R} = \mathcal{P}_\beta \hat{R}. \tag{2.14}$$

Proof.   Since by assumption ν' extends ν we have $\mathcal{P}'_\beta \cap R = \mathcal{P}_\beta$ and the inclusion

$$(\mathcal{P}'_\beta \hat{R}') \cap \hat{R} \supseteq \mathcal{P}_\beta \hat{R}. \tag{2.15}$$

We will now prove the opposite inclusion. Take an element $x \in (\mathcal{P}'_\beta \hat{R}') \cap \hat{R}$. Let $\{x_n\} \subset R$ be a Cauchy sequence in the M-adic topology, converging to x. Then $\{\pi(x_n)\}$ converge to $\hat{\pi}(x)$ in the M'-adic topology of $\hat{R}'$. Applying Remark 3 to R', we obtain

$$\nu(x_n) \equiv \nu'(\pi(x_n)) \geqslant \beta \quad \text{for } n \gg 0. \tag{2.16}$$

By (2.16) and Remark 4, applied to R, we have $x \in \mathcal{P}_\beta \hat{R}$. This proves the opposite inclusion in (2.15), as desired. $\square$

COROLLARY 2.5.   *We have*

$$H' \cap \hat{R} = H.$$

Proof.   Since ν' is of rank 1, Φ is cofinal in Φ'. Now the corollary follows by taking the intersection over all $\beta \in \Phi$ in (2.14). $\square$

Let J be a non-zero ideal of R and let $R \to R'$ be the local blowing up along J with respect to ν. Take an element $f \in J$, such that $\nu(f) = \nu(J)$. By the *strict transform* of J in $\hat{R}'$ we will mean the ideal

$$J^{\mathrm{str}} := \bigcup_{i=1}^\infty ((J\hat{R}') : f^i) \equiv (J\hat{R}'_f) \cap \hat{R}'.$$

If g is another element of J such that $\nu(g) = \nu(J)$, then $\nu(f/g) = 0$, so that f/g is a unit in R'. Thus the definition of strict transform is independent of the choice of f.

COROLLARY 2.6. $H^{\mathrm{str}} \subset H'$.

Proof.   Since $H\hat{R}' \subset H'$, we have $H^{\mathrm{str}} = (H\hat{R}'_f) \cap \hat{R}' \subset (H'\hat{R}'_f) \cap \hat{R}' = H'$, where the last equality holds because H' is a prime ideal of $\hat{R}'$, not containing f. $\square$

Using Zariski's Main Theorem, it can be proved that H^{str} is prime. Since this fact is not used in the sequel, we omit the proof.

COROLLARY 2.7.   *Let the notation and assumptions be as in Corollary 2.6. Then*

$$\mathrm{ht}\, H' \geqslant \mathrm{ht}\, H. \tag{2.17}$$

In particular,

$$\dim \frac{\hat{R}'}{H'} \leqslant \dim \frac{\hat{R}}{H}. \tag{2.18}$$

Proof.   Let $\bar{R} := (\hat{R} \otimes_R R')_{M'\hat{R}' \cap (\hat{R} \otimes_R R')}$. Let ϕ denote the natural local homomorphism

$$\bar{R} \longrightarrow \hat{R}'.$$

Let $\bar{H} := H' \cap \bar{R}$. Now, take $f \in J$ such that $\nu(f) = \nu(J)$. Then $f \notin H'$ and, in particular, $f \notin \bar{H}$. Since $R'_f \cong R_f$, we have $\hat{R}_f = \bar{R}_f$. In view of Corollary 2.5, we obtain $H\hat{R}_f \cong \bar{H}\bar{R}_f$, so

$$\mathrm{ht}\, H = \mathrm{ht}\, \bar{H}. \tag{2.19}$$

Now, $\bar{R}$ is a local noetherian ring, whose formal completion is $\hat{R}'$. Hence ϕ is faithfully flat and therefore satisfies the going down theorem. Thus we have $\mathrm{ht}\, H' \geqslant \mathrm{ht}\, \bar{H}$. Combined with (2.19), this proves (2.17). As for the last statement of the corollary, it follows from the well-known fact that dimension does not increase under blowing up [**12**, Lemma 2.2]: we have $\dim R' \leqslant \dim R$, hence

$$\dim \hat{R}' = \dim R' \leqslant \dim R = \dim \hat{R},$$

and (2.18) follows from (2.17) and from the fact that complete local rings are catenarian. $\square$

It may well happen that the containment of Corollary 2.6 and the inequality in (2.17) are strict. The possibility of strict containment in Corollary 2.6 is related to the existence of subanalytic functions, which are not analytic. We illustrate this statement by an example in which $H^{\mathrm{str}} \subsetneq H'$ and $\mathrm{ht}\, H < \mathrm{ht}\, H'$.

EXAMPLE 4.   Let k be a field and let

$$R = k[x, y, z]_{(x,y,z)},$$
$$R' = k[x', y', z']_{(x',y',z')},$$

where $x' = x$, $y' = y/x$ and $z' = z$. We have $K = k(x, y, z)$, $\hat{R} = k[\![x, y, z]\!]$, $\hat{R}' = k[\![x', y', z']\!]$. Let t_1 and t_2 be auxiliary variables and let $\sum_{i=1}^{\infty} c_i t_1^i$ (with $c_i \in k$) be an element of $k[\![t_1]\!]$, transcendental over $k(t_1)$. Let θ denote the valuation, centred at $k[\![t_1, t_2]\!]$, defined by $\theta(t_1) = 1$, $\theta(t_2) = \sqrt{2}$ (the value group of θ is the additive subgroup of $\mathbf{R}$, generated by 1 and $\sqrt{2}$). Let $\iota : R' \hookrightarrow k[\![t_1, t_2]\!]$ denote the injective map defined by $\iota(x') = t_2$, $\iota(y') = t_1$, $\iota(z') = \sum_{i=1}^{\infty} c_i t_1^i$. Let ν denote the restriction of θ to K, where we view K as a subfield of $k((t_1, t_2))$ via ι. Let $\Phi = \nu(R \setminus \{0\})$; $\Phi' = \nu(R' \setminus \{0\})$. For $\beta \in \Phi'$, P'_β is generated by all the monomials of the form $x'^\alpha y'^\gamma$ such that $\sqrt{2}\alpha + \gamma \geqslant \beta$, together with $z' - \sum_{j=1}^{i} c_j y'^j$, where i is the greatest non-negative integer such that $i < \beta$.

Let $w' := z' - \sum_{i=1}^{\infty} c_i y'^i$. Then $H' = (w')$, but $H = H' \cap \hat{R} = (0)$, so that $H^{\mathrm{str}} = (0) \subsetneq H'$ and $\mathrm{ht}\, H = 0 < 1 = \mathrm{ht}\, H'$.

Recall the following basic result of the theory of G-rings:

PROPOSITION 2.8.   *Assume that R is a reduced G-ring. Then $\hat{R}_H$ is a regular local ring.*

Proof.   Let $K = R_{\mathcal{P}_\infty} = \kappa(\mathcal{P}_\infty)$ (here we are using that R is reduced and that $\mathcal{P}_\infty$ is a minimal prime of R). By definition of G-ring, the map $R \to \hat{R}$ is a regular homomorphism. Then by (2.3) $\hat{R}_H$ is geometrically regular over K, hence regular. $\square$

REMARK 5.   Having extended in a unique manner the valuation ν to a valuation $\hat{\nu}_-$ of $\hat{R}/H$, we see that if R is a G-ring, by Proposition 2.8 there is a unique minimal prime $\hat{\mathcal{P}}_\infty$ of $\hat{R}$ contained in H, corresponding to the ideal (0) in $\hat{R}_H$. Since $H \cap R = (0)$, we have the equality $\hat{\mathcal{P}}_\infty \cap R = (0)$. Choosing a valuation μ of the fraction field of $\hat{R}_H/\hat{\mathcal{P}}_\infty \hat{R}_H$ centred at $\hat{R}_H/\hat{\mathcal{P}}_\infty \hat{R}_H$ whose value group Ψ is a free abelian group produces a composed valuation $\hat{\nu}_- \circ \mu$ on $\hat{R}/\hat{\mathcal{P}}_\infty$ with value group $\Psi \oplus \Gamma$ ordered lexicographically, as follows:

Given $x \in \hat{R}/\hat{\mathcal{P}}_\infty$, let $\psi = \mu(x)$ and blow up in R the ideal $\mathcal{P}_\psi$ along our original valuation, obtaining a local ring R'. According to what we have seen so far in this section, in its completion $\hat{R}'$, we can write $x = ye$ with $\mu(e) = \psi$ and $y \in \hat{R}' \setminus H'$. The valuation ν on R' extends uniquely to a valuation of $\hat{R}'/H'$, which we may still denote by $\hat{\nu}_-$ because it induces $\hat{\nu}_-$ on $\hat{R}/H$. Let us consider the image $\bar{y}$ of y in $\hat{R}'/H'$. Setting $(\hat{\nu}_- \circ \mu)(x) = \psi \oplus \hat{\nu}_-(\bar{y}) \in \Psi \oplus \Gamma$ determines a valuation of $\hat{R}/\hat{\mathcal{P}}_\infty$ as required.

If we drop the assumption that Ψ is a free abelian group, the above construction still works, but the value group $\bar{\Gamma}$ of $\hat{\nu}_- \circ \mu$ need not be isomorphic to the direct sum $\Psi \oplus \Gamma$. Rather, we have an exact sequence $0 \to \Gamma \to \bar{\Gamma} \to \Psi \to 0$, which need not, in general, be split; see [**14**, Proposition 4.3].

In the sequel, we shall reduce to the case where $\hat{R}$ is an integral domain, so that $\hat{\mathcal{P}}_\infty = (0)$ and we will have constructed a valuation of $\hat{R}$.

3. *Definition and first properties of implicit ideals*

Let the notation be as above. Before plunging into technical details, we would like to give a brief and informal overview of our constructions and the motivation for them. Above, we recalled the well-known fact that if $\operatorname{rk} \nu = r$, then for every ν-extension $R \to R'$ the valuation ν canonically determines a flag (1.6) of r subschemes of $\operatorname{Spec} R'$. This paper shows the existence of subschemes of $\operatorname{Spec} \hat{R}$, determined by ν, which are equally canonical and which become explicit only after completion. To see what they are, first of all note that the ideal $P_l' \hat{R}'$, for $R' \in \mathcal{T}$ and $0 \leqslant \ell \leqslant r-1$, need not in general be prime (although it is prime whenever R' is henselian). Another way of saying the same thing is that the ring R'/P_ℓ' need not be analytically irreducible in general. However, we will see in §8 (resp. §7) that the valuation ν picks out in a canonical way one of the minimal primes of $P_l' \hat{R}'$ (resp. $P_l' \tilde{R}'$). We call this minimal prime $H_{2\ell}'$ for reasons which will become apparent later. By the flatness of completion (resp. henselization), we have $H_{2\ell}' \cap R' = P_\ell'$. We will show that the ideals $H_{2\ell}'$ form a tree.

Let

$$(0) = \Delta_r \subsetneqq \Delta_{r-1} \subsetneqq \ldots \subsetneqq \Delta_0 = \Gamma \tag{3.1}$$

be the isolated subgroups of Γ. There are other ideals of $\hat{R}$, apart from the $H_{2\ell}$, canonically associated to ν, whose intersection with R equals P_ℓ, for example, the ideal $\bigcap_{\beta \in \Delta_\ell} \mathcal{P}_\beta \hat{R}$. The same is true of the even larger ideal

$$H_{2\ell+1} = \bigcap_{\beta \in \Delta_\ell} \left(\left(\varinjlim_{R'} \mathcal{P}_\beta' \hat{R}' \right) \bigcap \hat{R} \right), \tag{3.2}$$

(that $H_{2\ell+1} \cap R = P_\ell$ is easy to see and will be shown later in this section, in Proposition 3.1). While the examples below show that the ideal $\bigcap_{\beta \in \Delta_\ell} \mathcal{P}_\beta \hat{R}$ need not, in general, be prime, the ideal $H_{2\ell+1}$ always is (this is the main theorem of this paper; it will be proved in §8). The ideal $H_{2\ell+1}$ contains $H_{2\ell}$ but is not, in general equal to it. To summarize, we will show that the valuation ν picks out in a canonical way a generic point $H_{2\ell}$ of the formal fibre over P_ℓ and also another point $H_{2\ell+1}$ in the formal fibre, which is a specialization of $H_{2\ell}$.

The main technique used to prove these results is to analyse the set of zero divisors of $R'^\dagger / P'_\ell R'^\dagger$ (where $R^\dagger$ stands for either $\hat{R}$, $\tilde{R}$, or a finite type étale extension R^e of R), as follows. We show that the reducibility of $R'^\dagger / P'_\ell R'^\dagger$ is related to the existence of non-trivial algebraic extensions of $\kappa(P_\ell)$ inside $\kappa(P_\ell) \otimes_R R^\dagger$. More precisely, in the next section, we define R to be *stable* if $R^\dagger / P_{\ell+1} R^\dagger$ is a domain and there does not exist a non-trivial algebraic extension of $\kappa(P_{\ell+1})$ which embeds both into $\kappa(P_{\ell+1}) \otimes_R R^\dagger$ and into $\kappa(P'_{\ell+1})$ for some $R' \in \mathcal{T}$. We show that if R is stable then $R'^\dagger / P'_{\ell+1} R'^\dagger$ is a domain for all $R' \in \mathcal{T}$. For $\bar{\beta} \in \Gamma/\Delta_{\ell+1}$, let

$$\mathcal{P}_{\bar{\beta}} = \{x \in R \,|\, \nu(x) \bmod \Delta_{\ell+1} \geqslant \bar{\beta}\}. \tag{3.3}$$

If Φ denotes the semigroup $\nu(R \setminus \{0\}) \subset \Gamma$, which is well ordered since R is noetherian (see [**17**, Appendix 4, Proposition 2]), and

$$\beta(\ell) = \min\{\gamma \in \Phi \,|\, \beta - \gamma \in \Delta_{\ell+1}\}$$

then $\mathcal{P}_{\bar{\beta}} = \mathcal{P}_{\beta(l)}$.

We have the inclusions

$$P_\ell \subset \mathcal{P}_{\bar{\beta}} \subset P_{\ell+1},$$

and $\mathcal{P}_{\bar{\beta}}$ is the inverse image in R by the canonical map $R \to R/P_\ell$ of a valuation ideal $\bar{\mathcal{P}}_{\bar{\beta}} \subset R/P_\ell$ for the rank 1 valuation $R/P_\ell \setminus \{0\} \to \Delta_\ell/\Delta_{\ell+1}$ induced by $\nu_{\ell+1}$.

We will deduce from the above that if R is stable, then for each $\bar{\beta} \in \Delta_\ell/\Delta_{\ell+1}$ and each ν-extension $R \to R'$, we have $\mathcal{P}'_{\bar{\beta}} R'^\dagger \cap R^\dagger = \mathcal{P}_{\bar{\beta}} R^\dagger$, which gives us a very good control of the limit in the definition of $H_{2\ell+1}$ and of the ν-extensions R' for which the limit is attained.

We then show, separately in the cases when $R^\dagger = \tilde{R}$ (§7) and $R^\dagger = \hat{R}$ (§8), that there always exists a stable ν-extension $R' \in \mathcal{T}$.

We are now ready to go into details, after giving several examples of implicit ideals and the phenomena discussed above.

Let $0 \leqslant \ell \leqslant r$. We define our main object of study, the $(2\ell + 1)$st implicit prime ideal $H_{2\ell+1} \subset R^\dagger$, by

$$H_{2\ell+1} = \bigcap_{\beta \in \Delta_\ell} \left(\left(\varinjlim_{R'} \mathcal{P}'_\beta R'^\dagger \right) \bigcap R^\dagger \right), \tag{3.4}$$

where R' ranges over $\mathcal{T}$. As usual, we think of (3.4) as a tree equation: if we replace R by any other $R'' \in \mathcal{T}$ in (3.4), it defines the corresponding ideal $H''_{2\ell+1} \subset \hat{R}''^\dagger$. Note that for $\ell = r$ (3.4) reduces to

$$H_{2r+1} = mR^\dagger.$$

We start by giving several examples of the ideals H'_i (and also of $\tilde{H}'_i$, which will appear a little later in the paper).

EXAMPLE 5.   Let $R = k[x, y, z]_{(x,y,z)}$. Let ν be the valuation with value group $\Gamma = \mathbf{Z}^2_{\text{lex}}$, defined as follows. Take a transcendental power series $\sum_{j=1}^\infty c_j u^j$ in a variable u over k. Consider the homomorphism $R \hookrightarrow k[\![u, v]\!]$ which sends x to v, y to u and z to $\sum_{j=1}^\infty c_j u^j$. Consider the valuation ν, centred at $k[\![u, v]\!]$, defined by $\nu(v) = (0, 1)$ and $\nu(u) = (1, 0)$; its restriction to R will also be denoted by ν, by abuse of notation. Let R_ν denote the valuation ring of ν in

$k(x, y, z)$ and let $\mathcal{T}$ be the tree consisting of all the local rings R' essentially of finite type over R, birationally dominated by R_ν. Let $^\dagger = \hat{\ }$ denote the operation of formal completion. Given $\beta = (a, b) \in \mathbf{Z}^2_{\text{lex}}$, we have $\mathcal{P}_\beta = x^b(y^a, z - c_1 y - \ldots - c_{a-1} y^{a-1})$. The first isolated subgroup $\Delta_1 = (0) \oplus \mathbf{Z}$. Then $\bigcap_{\beta \in (0) \oplus \mathbf{Z}}(\mathcal{P}_\beta \hat{R}) = (y, z)$ and $\bigcap_{\beta \in \Gamma = \Delta_0}(\mathcal{P}_\beta \hat{R}) = (z - \sum_{j=1}^\infty c_j y^j)$. It is not hard to show that, for any $R' \in \mathcal{T}$, we have $H'_1 = (z - \sum_{j=1}^\infty c_j y^j)R'$ and that $H_3 = (y, z)\hat{R}$. It will follow from the general theory developed in §6 that ν admits a unique extension $\hat{\nu}$ to $\varinjlim_{R'} \hat{R'}$. This extension has value group $\hat{\Gamma} = \mathbf{Z}^3_{\text{lex}}$ and is defined by $\hat{\nu}(x) = (0, 0, 1)$, $\hat{\nu}(y) = (0, 1, 0)$ and $\hat{\nu}(z - \sum_{j=1}^\infty c_j y^j) = (1, 0, 0)$. For each $R' \in \mathcal{T}$ the ideal H'_1 is the prime valuation ideal corresponding to the isolated subgroup $(0) \oplus \mathbf{Z}^2_{\text{lex}}$ of $\hat{\Gamma}$ (that is, the ideal whose elements have values outside of $(0) \oplus \mathbf{Z}^2_{\text{lex}}$) while H'_3 is the prime valuation ideal corresponding to the isolated subgroup $(0) \oplus (0) \oplus \mathbf{Z}$.

EXAMPLE 6.   Let $R = k[x, y, z]_{(x,y,z)}$, $\Gamma = \mathbf{Z}^2_{\text{lex}}$, the power series $\sum_{j=1}^\infty c_j u^j$ and the operation $^\dagger = \hat{\ }$ be as in the previous example. This time, let ν be defined as follows. Consider the homomorphism $R \hookrightarrow k[\![u, v]\!]$ which sends x to u, y to $\sum_{j=1}^\infty c_j u^j$ and z to v. Consider the valuation ν, centred at $k[\![u, v]\!]$, defined by $\nu(v) = (1, 0)$ and $\nu(u) = (0, 1)$; its restriction to R will also be denoted by ν. Let R_ν denote the valuation ring of ν in $k(x, y, z)$ and let $\mathcal{T}$ be the tree consisting of all the local rings R' essentially of finite type over R, birationally dominated by R_ν. Given $\beta = (a, b) \in \mathbf{Z}^2_{\text{lex}}$, we have $\mathcal{P}_\beta = z^a(x^b, y - c_1 x - \ldots - c_{b-1} x^{b-1})$. The first isolated subgroup $\Delta_1 = (0) \oplus \mathbf{Z}$. Then $\bigcap_{\beta \in (0) \oplus \mathbf{Z}}(\mathcal{P}_\beta \hat{R}) = (y - \sum_{j=1}^\infty c_j x^j, z)$ and $\bigcap_{\beta \in \Gamma = \Delta_0}(\mathcal{P}_\beta \hat{R}) = (0)$. It is not hard to show that, for any $R' \in \mathcal{T}$, we have $H'_1 = (0)$ and that $H_3 = (y - \sum_{j=1}^\infty c_j x^j, z)\hat{R'}$. In this case, the extension $\hat{\nu}$ to $\varinjlim_{R'} \hat{R'}$ is not unique. Indeed, one possible extension $\hat{\nu}^{(1)}$ has value group $\hat{\Gamma} = \mathbf{Z}^3_{\text{lex}}$ and is defined by $\hat{\nu}^{(1)}(x) = (0, 0, 1)$, $\hat{\nu}^{(1)}(y - \sum_{j=1}^\infty c_j x^j) = (0, 1, 0)$ and $\hat{\nu}^{(1)}(z) = (1, 0, 0)$. In this case, for any $R' \in \mathcal{T}$, the ideal H'_3 is the prime valuation ideal corresponding to the isolated subgroup $(0) \oplus (0) \oplus \mathbf{Z}$ of $\hat{\Gamma}$.

Another extension $\hat{\nu}^{(2)}$ of ν is defined by $\hat{\nu}^{(2)}(x) = (0, 0, 1)$, $\hat{\nu}^{(2)}(y - \sum_{j=1}^\infty c_j x^j) = (1, 0, 0)$ and $\hat{\nu}^{(2)}(z) = (0, 1, 0)$. In this case, the tree of ideals corresponding to the isolated subgroup $(0) \oplus (0) \oplus \mathbf{Z}$ is H'_3 (exactly the same as for $\hat{\nu}^{(1)}$) while that corresponding to $(0) \oplus \mathbf{Z}^2_{\text{lex}}$ is $\tilde{H}'_1 = (y - \sum_{j=1}^\infty c_j x^j)$. The tree $\tilde{H}'_1$ of prime $\hat{\nu}^{(2)}$-ideals determines the extension $\hat{\nu}^{(2)}$ completely.

The following two examples illustrate the need for taking the limit over the tree $\mathcal{T}$.

EXAMPLE 7.   Let us consider the local domain $S = k[x, y]_{(x,y)}/(y^2 - x^2 - x^3)$. There are two distinct valuations centred in (x, y). Let $a_i \in k$, $i \geqslant 2$ be such that

$$\left(y + x + \sum_{i \geqslant 2} a_i x^i\right)\left(y - x - \sum_{i \geqslant 2} a_i x^i\right) = y^2 - x^2 - x^3.$$

We shall denote by ν_+ the rank 1 discrete valuation defined by

$$\nu_+(x) = \nu_+(y) = 1,$$
$$\nu_+(y + x) = 2,$$
$$\nu_+\left(y + x + \sum_{i=2}^{b-1} a_i x^i\right) = b.$$

Now let $R = k[x, y, z]_{(x,y,z)}/(y^2 - x^2 - x^3)$. Let $\Gamma = \mathbf{Z}^2$ with the lexicographical ordering. Let ν be the composite valuation of the (z)-adic one with ν_+, centred in $R/(z)$. The point of this example is to show that

$$H^*_{2\ell+1} = \bigcap_{\beta \in \Delta_\ell} \mathcal{P}_\beta \hat{R}$$

does not work as the definition of the $(2\ell + 1)$st implicit prime ideal because the resulting ideal $H^*_{2\ell+1}$ is not prime. Indeed, as $\mathcal{P}_{(a,0)} = (z^a)$, we have

$$H^*_1 = \bigcap_{(a,b) \in \mathbf{Z}^2} \mathcal{P}_{(a,b)} \hat{R} = (0).$$

Let $f = y + x + \sum_{i \geqslant 2} a_i x^i, g = y - x - \sum_{i \geqslant 2} a_i x^i \in \hat{R}$. Clearly, $f, g \notin H^*_1 = (0)$, but $f \cdot g = 0$, so the ideal H^*_1 is not prime.

One might be tempted (as we were) to correct this problem by localizing at $H^*_{2\ell+3}$. Indeed, if we take the new definition of $H^*_{2\ell+1}$ to be, recursively in the descending order of ℓ,

$$H^*_{2\ell+1} = \left(\bigcap_{\beta \in \Delta_\ell} \mathcal{P}_\beta \hat{R}_{H^*_{2\ell+3}} \right) \cap \hat{R}, \tag{3.5}$$

then in the present example the resulting ideals $H^*_3 = (z, f)$ and $H^*_1 = (f)$ are prime. However, the next example shows that the definition (3.5) also does not, in general, give rise to prime ideals.

EXAMPLE 8.   Let $R = k[x, y, z]_{(x,y,z)}/(z^2 - y^2(1 + x))$. Let $\Gamma = \mathbf{Z}^2$ with the lexicographical ordering. Let t be an independent variable and let ν be the valuation, centred in R, induced by the t-adic valuation of $k[\![t^\Gamma]\!]$ under the injective homomorphism $\iota : R \hookrightarrow k[\![t^\Gamma]\!]$, defined by $\iota(x) = t^{(0,1)}, \iota(y) = t^{(1,0)}$ and $\iota(z) = t^{(1,0)}\sqrt{1 + t^{(0,1)}}$. The prime ν-ideals of R are $(0) \subsetneq P_1 \subsetneq m$, with $P_1 = (y, z)$. We have $\bigcap_{\beta \in \Delta_1} \mathcal{P}_\beta \hat{R} = (y, z)\hat{R} = P_1 \hat{R}$ and $\bigcap_{\beta \in \Gamma} \mathcal{P}_\beta \hat{R}_{(y,z)} = \bigcap_{\beta \in \Gamma} \mathcal{P}_\beta \hat{R} = (0)$. Note that the ideal (0) is not prime in $\hat{R}$. Now, let $R' = R[z/y]_{m'}$, where $m' = (x, y, z/y - 1)$ is the centre of ν in $R[z/y]$. We have $z - y\sqrt{1 + x} \in \hat{R} \setminus \mathcal{P}_{(2,0)}\hat{R}$. On the other hand, $z - y\sqrt{1 + x} = y(z/y - \sqrt{1 + x}) = 0$ in $\hat{R}'$; in particular, $z - y\sqrt{1 + x} \in \bigcap_{\beta \in \Gamma} \mathcal{P}'_\beta \hat{R}'$. Thus this example also shows that the ideals $\mathcal{P}_\beta \hat{R}, \bigcap_{\beta \in \Delta_\ell} \mathcal{P}_\beta \hat{R}$ and $\bigcap_{\beta \in \Delta_\ell} \mathcal{P}_\beta \hat{R}_{H_{2\ell+3}}$ do not behave well under blowing up.

Note that both Examples 7 and 8 occur not only for the completion $\hat{R}$ but also for the henselization $\tilde{R}$.

We come back to the general theory of implicit ideals.

PROPOSITION 3.1.   We have $H_{2\ell+1} \cap R = P_\ell$.

Proof.   Recall that $P_\ell = \{x \in R \mid \nu(x) \notin \Delta_\ell\}$. If $x \in P_\ell$, then, since Δ_ℓ is an isolated subgroup, we have $x \in \mathcal{P}_\beta$ for all $\beta \in \Delta_\ell$. The same inclusion holds for the same reason in all extensions $R' \subset R_\nu$ of R, and this implies the inclusion $P_\ell \subseteq H_{2\ell+1} \cap R$. Now let x be in $H_{2\ell+1} \cap R$ and assume $x \notin P_\ell$. Then there is a $\beta \in \Delta_\ell$ such that $x \notin \mathcal{P}_\beta$. By faithful flatness of $R^\dagger$ over R we have $\mathcal{P}_\beta R^\dagger \cap R = \mathcal{P}_\beta$. This implies that $x \notin \mathcal{P}_\beta R^\dagger$, and the same argument holds in all the extensions $R' \in \mathcal{T}$, so x cannot be in $H_{2\ell+1} \cap R$. This contradiction shows the desired equality.   □

PROPOSITION 3.2.   *The ideals $H'_{2\ell+1}$ behave well under ν-extensions $R \to R'$ in $\mathcal{T}$. In other words, let $R \to R'$ be a ν-extension in $\mathcal{T}$ and let $H'_{2\ell+1}$ denote the $(2\ell+1)$st implicit prime ideal of $\hat{R}'$. Then $H_{2\ell+1} = H'_{2\ell+1} \cap R^{\dagger}$.*

Proof.   Immediate from the definitions. $\qquad\square$

To study the ideals $H_{2\ell+1}$, we need to understand more explicitly the nature of the limit appearing in (3.4). To study the relationship between the ideals $P_\beta R^{\dagger}$ and $P'_\beta {R'}^{\dagger} \bigcap R^{\dagger}$, it is useful to factor the natural map $R^{\dagger} \to {R'}^{\dagger}$ as $R^{\dagger} \to (R^{\dagger} \otimes_R R')_{M'} \xrightarrow{\phi} {R'}^{\dagger}$ as we did in the proof of Lemma 1.1. In general, the ring $R^{\dagger} \otimes_R R'$ is not local (see the above examples), but it has one distinguished maximal ideal M', namely, the ideal generated by $mR^{\dagger} \otimes 1$ and $1 \otimes m'$, where m' denotes the maximal ideal of R'. The map ϕ factors through the local ring $(R^{\dagger} \otimes_R R')_{M'}$ and the resulting map $(R^{\dagger} \otimes_R R')_{M'} \to {R'}^{\dagger}$ is either the formal completion or the henselization; in either case, it is faithfully flat. Thus, $P'_\beta {R'}^{\dagger} \cap (R^{\dagger} \otimes_R R')_{M'} = P'_\beta (R^{\dagger} \otimes_R R')_{M'}$. This shows that we may replace ${R'}^{\dagger}$ by $(R^{\dagger} \otimes_R R')_{M'}$ in (3.4) without affecting the result, that is,

$$H_{2\ell+1} = \bigcap_{\beta \in \Delta_\ell} \left(\left(\varinjlim_{R'} \mathcal{P}'_\beta (R^{\dagger} \otimes_R R')_{M'} \right) \bigcap R^{\dagger} \right). \tag{3.6}$$

From now on, we will use (3.6) as our working definition of the implicit prime ideals. One advantage of the expression (3.6) is that it makes sense in a situation more general than the completion and the henselization. Namely, to study the case of the henselization $\tilde{R}$, we will need to consider local étale extensions R^e of R, which are contained in $\tilde{R}$ (particularly, those which are essentially of finite type). The definition (3.6) of the implicit prime ideals makes sense also in that case.

4.   *Stable rings and primality of their implicit ideals.*

Let the notation be as in the preceding sections. As usual, $R^{\dagger}$ will denote one of $\hat{R}$, $\tilde{R}$ or R^e (a local étale ν-extension essentially of finite type). Take an $R' \in \mathcal{T}$ and $\bar{\beta} \in \Delta_\ell / \Delta_{\ell+1}$. We have the obvious inclusion of ideals

$$\mathcal{P}_{\bar{\beta}} R^{\dagger} \subset \mathcal{P}_{\bar{\beta}} {R'}^{\dagger} \cap R^{\dagger} \tag{4.1}$$

(where $\mathcal{P}_{\bar{\beta}}$ is defined in (3.3)). A useful subtree of $\mathcal{T}$ is formed by the ℓ-stable rings, which we now define. An important property of stable rings, proved below, is that the inclusion (4.1) is an equality whenever R' is stable.

DEFINITION 4.1.   A ring $R' \in \mathcal{T}(R)$ is said to be *ℓ-stable* if the following two conditions hold.

(1) The ring

$$\kappa(P'_\ell) \otimes_R (R' \otimes_R R^{\dagger})_{M'} \tag{4.2}$$

is an integral domain.

(2) There do not exist an $R'' \in \mathcal{T}(R')$ and a non-trivial algebraic extension L of $\kappa(P'_\ell)$ which embeds both into $\kappa(P'_\ell) \otimes_R (R' \otimes_R R^{\dagger})_{M'}$ and $\kappa(P''_\ell)$.

We say that R is *stable* if it is ℓ-stable for each $\ell \in \{0, \ldots, r\}$.

REMARK 6.   (1) Rings of the form (4.2) will be a basic object of study in this paper. Another way of looking at the same ring, which we will often use, comes from interchanging the order of tensor product and localization. Namely, let T' denote the image of the multiplicative system $(R' \otimes_R R^\dagger) \setminus M'$ under the natural map $R' \otimes_R R^\dagger \to \kappa(P'_\ell) \otimes_R R^\dagger$. Then the ring (4.2) equals the localization $(T')^{-1}(\kappa(P'_\ell) \otimes_R R^\dagger)$.

(2) In the special case $R' = R$ in Definition 4.1, we have

$$\kappa(P'_\ell) \otimes_R (R' \otimes_R R^\dagger)_{M'} = \kappa(P_\ell) \otimes_R R^\dagger.$$

If, moreover, R/P_ℓ is analytically irreducible, then the hypothesis that $\kappa(P_\ell) \otimes_R R^\dagger$ is a domain holds automatically; in fact, this hypothesis is equivalent to analytic irreducibility of R/P_ℓ if $R^\dagger = \hat{R}$ or $R^\dagger = \tilde{R}$.

(3) Consider the special case when R' is Henselian and $\dagger = \hat{}$. Excellent Henselian rings are algebraically closed inside their formal completions, so both (1) and (2) of Definition 4.1 hold automatically for this R'. Thus excellent Henselian local rings are always stable.

In this section, we study ℓ-stable rings. We prove that if R is ℓ-stable, then so is any $R' \in \mathcal{T}(R)$ (justifying the name 'stable'). The main result of this section, Theorem 4.6, says that if R is stable, then the implicit ideal $H'_{2\ell+1}$ is prime for each $\ell \in \{0, \dots, r\}$ and each $R' \in \mathcal{T}(R)$.

REMARK 7.   In the next two sections, we will show that there exist stable rings $R' \in \mathcal{T}$ for both $R^\dagger = \hat{R}$ and $R^\dagger = R^e$. However, the proof of this is different depending on whether we are dealing with completion or with an étale extension, and will be carried out separately in two separate sections: one devoted to henselization, the other to completion.

PROPOSITION 4.1.   Fix an integer ℓ, $0 \leqslant \ell \leqslant r$. Assume that R' is ℓ-stable and let $R'' \in \mathcal{T}(R')$. Then R'' is ℓ-stable.

Proof.   We have to show that (1) and (2) of Definition 4.1 for R' imply (1) and (2) of Definition 4.1 for R''. The ring

$$\kappa(P''_\ell) \otimes_R (R'' \otimes_R R^\dagger)_{M''} \tag{4.3}$$

is a localization of $\kappa(P''_\ell) \otimes_R (\kappa(P'_\ell) \otimes_R (R' \otimes_R R^\dagger)_{M'})$. Hence (1) and (2) of Definition 4.1, applied to R', imply that $\kappa(P''_\ell) \otimes_R (R'' \otimes_R R^\dagger)_{M''}$ is an integral domain, so (1) of Definition 4.1 holds for R''. Replacing R' by R'' clearly does not affect the hypotheses about the non-existence of the extension L, so (2) of Definition 4.1 also holds for R''. $\square$

Next, we prove a technical result on which much of the rest of the paper is based. For $\bar{\beta} \in \Gamma/\Delta_{\ell+1}$, let

$$\mathcal{P}_{\bar{\beta}+} = \{x \in R \,|\, \nu(x) \bmod \Delta_{\ell+1} > \bar{\beta}\}. \tag{4.4}$$

As usual, $\mathcal{P}'_{\bar{\beta}+}$ will stand for the analogous notion, but with R replaced by R', etc.

PROPOSITION 4.2.   Assume that R itself is $(\ell+1)$-stable and let $R' \in \mathcal{T}(R)$.

(i) For any $\bar{\beta} \in \Delta_\ell/\Delta_{\ell+1}$

$$\mathcal{P}'_{\bar{\beta}} R'^\dagger \cap R^\dagger = \mathcal{P}_{\bar{\beta}} R^\dagger. \tag{4.5}$$

(ii) *For any $\bar{\beta} \in \Gamma/\Delta_{\ell+1}$ the natural map*

$$\frac{\mathcal{P}_{\bar{\beta}} R^{\dagger}}{\mathcal{P}_{\bar{\beta}+} R^{\dagger}} \longrightarrow \frac{\mathcal{P}'_{\bar{\beta}} R'^{\dagger}}{\mathcal{P}'_{\bar{\beta}+} R'^{\dagger}} \tag{4.6}$$

   is injective.

As explained at the end of the previous section, since $R'^{\dagger}$ is faithfully flat over the ring $(R^{\dagger} \otimes_R R')_{M'}$, we may replace $R'^{\dagger}$ by $(R^{\dagger} \otimes_R R')_{M'}$ in both (i) and (ii) of Proposition 4.2.

Proof of (i) *of Proposition* 4.2. It is sufficient to prove that

$$\mathcal{P}'_{\bar{\beta}}(R^{\dagger} \otimes_R R')_{M'} \cap R^{\dagger} = \mathcal{P}_{\bar{\beta}} R^{\dagger}. \tag{4.7}$$

One inclusion in (4.7) is trivial; we must show that

$$\mathcal{P}'_{\bar{\beta}}(R^{\dagger} \otimes_R R')_{M'} \cap R^{\dagger} \subset \mathcal{P}_{\bar{\beta}} R^{\dagger}. \tag{4.8}$$

LEMMA 4.3. *Let T' denote the image of the multiplicative set $(R' \otimes_R R^{\dagger}) \setminus M'$ under the natural map of R-algebras $R' \otimes_R R^{\dagger} \to R'_{P'_{\ell+1}}/\mathcal{P}'_{\bar{\beta}} R'_{P'_{\ell+1}} \otimes_R R^{\dagger}$. Then the map of R-algebras*

$$\bar{\pi} : \frac{R_{P_{\ell+1}}}{\mathcal{P}_{\bar{\beta}} R_{P_{\ell+1}}} \otimes_R R^{\dagger} \longrightarrow (T')^{-1} \left(\frac{R'_{P'_{\ell+1}}}{\mathcal{P}'_{\bar{\beta}} R'_{P'_{\ell+1}}} \otimes_R R^{\dagger} \right) \tag{4.9}$$

induced by $\pi : R \to R'$ is injective.

Proof of Lemma 4.3. We start with the field extension

$$\kappa(P_{\ell+1}) \hookrightarrow \kappa(P'_{\ell+1})$$

induced by π. Since $R^{\dagger}$ is flat over R, the induced map $\pi_1 : \kappa(P_{\ell+1}) \otimes_R R^{\dagger} \to \kappa(P'_{\ell+1}) \otimes_R R^{\dagger}$ is also injective. By (1) of Definition 4.1, $\kappa(P'_{\ell+1}) \otimes_R R^{\dagger}$ is a domain. In particular,

$$\kappa(P'_{\ell+1}) \otimes_R R^{\dagger} = \left(\frac{R'_{P'_{\ell+1}}}{\mathcal{P}'_{\bar{\beta}} R'_{P'_{\ell+1}}} \otimes_R R^{\dagger} \right)_{red}. \tag{4.10}$$

The local ring $R'_{P'_{\ell+1}}/\mathcal{P}'_{\bar{\beta}} R'_{P'_{\ell+1}}$ is artinian because it can be seen as the quotient of $R'_{P'_{\ell+1}}/P'_{\ell} R'_{P'_{\ell+1}}$ by a valuation ideal corresponding to a rank 1 valuation. Since the ring is noetherian the valuation of the maximal ideal is positive, and since the group is archimedian, a power of the maximal ideal is contained in the valuation ideal.

Therefore, its only associated prime is its nilradical, the ideal $P'_{\ell+1} R'_{P'_{\ell+1}}/\mathcal{P}'_{\bar{\beta}} R'_{P'_{\ell+1}}$; in particular, the (0) ideal in this ring has no embedded components. Since $R^{\dagger}$ is flat over R, $R'_{P'_{\ell+1}}/\mathcal{P}'_{\bar{\beta}} R'_{P'_{\ell+1}} \otimes_R R^{\dagger}$ is flat over $R'_{P'_{\ell+1}}/\mathcal{P}'_{\bar{\beta}} R'_{P'_{\ell+1}}$ by base change. Hence the (0) ideal of $R'_{P'_{\ell+1}}/\mathcal{P}'_{\bar{\beta}} R'_{P'_{\ell+1}} \otimes_R R^{\dagger}$ has no embedded components. In particular, since the multiplicative system T' is disjoint from the nilradical of $R'_{P'_{\ell+1}}/\mathcal{P}'_{\bar{\beta}} R'_{P'_{\ell+1}} \otimes_R R^{\dagger}$, the set T' contains no zero divisors, so localization by T' is injective.

By the definition of $\mathcal{P}_{\bar{\beta}}$, the map $R_{P_{\ell+1}}/\mathcal{P}_{\bar{\beta}} R_{P_{\ell+1}} \to R'_{P'_{\ell+1}}/\mathcal{P}'_{\bar{\beta}} R'_{P'_{\ell+1}}$ is injective, hence so is

$$\frac{R_{P_{\ell+1}}}{\mathcal{P}_{\bar{\beta}} R_{P_{\ell+1}}} \otimes_R R^{\dagger} \longrightarrow \frac{R'_{P'_{\ell+1}}}{\mathcal{P}'_{\bar{\beta}} R'_{P'_{\ell+1}}} \otimes_R R^{\dagger}$$

by the flatness of $R^\dagger$ over R. Combining this with the injectivity of the localization by T', we obtain that $\bar{\pi}$ is injective, as desired. Lemma 4.3 is proved. $\qquad\square$

Again by the definition of $\mathcal{P}_{\bar{\beta}}$, the localization map $R/\mathcal{P}_{\bar{\beta}} \to R_{P_{\ell+1}}/\mathcal{P}_{\bar{\beta}} R_{P_{\ell+1}}$ is injective, hence so is the map

$$\frac{R}{\mathcal{P}_{\bar{\beta}}} \otimes_R R^\dagger \longrightarrow \frac{R_{P_{\ell+1}}}{\mathcal{P}_{\bar{\beta}} R_{P_{\ell+1}}} \otimes_R R^\dagger \tag{4.11}$$

by the flatness of $R^\dagger$ over R. Combining this with Lemma 4.3, we see that the composition

$$\frac{R}{\mathcal{P}_{\bar{\beta}}} \otimes_R R^\dagger \longrightarrow (T')^{-1}\left(\frac{R'_{P'_\ell}}{\mathcal{P}'_{\bar{\beta}} R'_{P'_\ell}} \otimes_R R^\dagger \right) \tag{4.12}$$

of (4.11) with $\bar{\pi}$ is also injective. Now, (4.12) factors through $(R'/\mathcal{P}'_{\bar{\beta}} \otimes_R R^\dagger)_{M'}$ (here we are guilty of a slight abuse of notation: we denote the natural image of M' in $R'/\mathcal{P}'_{\bar{\beta}} \otimes_R R^\dagger$ also by M'). Hence the map

$$\frac{R}{\mathcal{P}_{\bar{\beta}}} \otimes_R R^\dagger \longrightarrow \left(\frac{R'}{\mathcal{P}'_{\bar{\beta}}} \otimes_R R^\dagger \right)_{M'} \tag{4.13}$$

is injective. Since $R/\mathcal{P}_{\bar{\beta}} \otimes_R R^\dagger \cong R^\dagger/\mathcal{P}_{\bar{\beta}} R^\dagger$ and $(R'/\mathcal{P}'_{\bar{\beta}} \otimes_R R^\dagger)_{M'} \cong (R' \otimes_R R^\dagger)_{M'}/\mathcal{P}'_{\bar{\beta}}(R' \otimes_R R^\dagger)_{M'}$, the injectivity of (4.13) is the same as (4.8). This completes the proof of (i) of Proposition 4.2. $\qquad\square$

Proof of (ii) *of Proposition* 4.2. We start with the injective homomorphism

$$\frac{\mathcal{P}_{\bar{\beta}}}{\mathcal{P}_{\bar{\beta}+}} \otimes_R \kappa(P_{\ell+1}) \longrightarrow \frac{\mathcal{P}'_{\bar{\beta}}}{\mathcal{P}'_{\bar{\beta}+}} \otimes_R \kappa(P'_{\ell+1}) \tag{4.14}$$

of $\kappa(P_{\ell+1})$-vector spaces. Since $R^\dagger$ is flat over R, tensoring (4.14) produces an injective homomorphism

$$\frac{\mathcal{P}_{\bar{\beta}} R^\dagger}{\mathcal{P}_{\bar{\beta}+} R^\dagger} \otimes_R \kappa(P_{\ell+1}) \longrightarrow \frac{\mathcal{P}'_{\bar{\beta}}}{\mathcal{P}'_{\bar{\beta}+}} \otimes_R \kappa(P'_{\ell+1}) \otimes_R R^\dagger \tag{4.15}$$

of $R^\dagger$-modules. Now, the $\kappa(P_{\ell+1})$-vector space $\mathcal{P}'_{\bar{\beta}}/\mathcal{P}'_{\bar{\beta}+} \otimes_R \kappa(P'_{\ell+1})$ is, in particular, a torsion-free $\kappa(P_{\ell+1})$-module. Since $\kappa(P_{\ell+1}) \otimes_R R^\dagger$ is a domain by definition of $(\ell+1)$-stable and by the flatness of $R^\dagger \otimes_R \kappa(P_{\ell+1})$ over $\kappa(P_{\ell+1})$, the $R^\dagger \otimes_R \kappa(P_{\ell+1})$-module $\mathcal{P}'_{\bar{\beta}}/\mathcal{P}'_{\bar{\beta}+} \otimes_R \kappa(P'_{\ell+1}) \otimes_R R^\dagger$ is also torsion-free; in particular, its localization map by any multiplicative system is injective. Let S' denote the image of the multiplicative set $(R' \otimes_R R^\dagger) \setminus M'$ under the natural map of R-algebras $R' \otimes_R R^\dagger \to \kappa(P'_{\ell+1}) \otimes_R R^\dagger$. By the above, the composition

$$\frac{\mathcal{P}_{\bar{\beta}} R^\dagger}{\mathcal{P}_{\bar{\beta}+} R^\dagger} \otimes_R \kappa(P_{\ell+1}) \longrightarrow (S')^{-1}\left(\frac{\mathcal{P}'_{\bar{\beta}}}{\mathcal{P}'_{\bar{\beta}+}} \otimes_R \kappa(P'_{\ell+1}) \otimes_R R^\dagger \right) \tag{4.16}$$

of (4.15) with the localization by S' is injective.

By the definition of $\mathcal{P}_{\bar{\beta}}$, the localization map $\mathcal{P}_{\bar{\beta}}/\mathcal{P}_{\bar{\beta}+} \to \mathcal{P}_{\bar{\beta}}/\mathcal{P}_{\bar{\beta}+} \otimes_R \kappa(P_{\ell+1})$ is injective, hence so is the map

$$\frac{\mathcal{P}_{\bar{\beta}} R^\dagger}{\mathcal{P}_{\bar{\beta}+} R^\dagger} = \frac{\mathcal{P}_{\bar{\beta}}}{\mathcal{P}_{\bar{\beta}+}} \otimes_R R^\dagger \longrightarrow \frac{\mathcal{P}_{\bar{\beta}} R^\dagger}{\mathcal{P}_{\bar{\beta}+} R^\dagger} \otimes_R \kappa(P_{\ell+1}) \tag{4.17}$$

by the flatness of $R^\dagger$ over R. Combining this with the injectivity of (4.16), we see that the composition

$$\frac{\mathcal{P}_{\bar\beta} R^\dagger}{\mathcal{P}_{\bar\beta+} R^\dagger} \longrightarrow (S')^{-1}\left(\frac{\mathcal{P}'_{\bar\beta}}{\mathcal{P}'_{\bar\beta+}} \otimes_R \kappa(P'_{\ell+1}) \otimes_R R^\dagger\right) \tag{4.18}$$

of (4.17) with (4.16) is also injective. Now, (4.18) factors through $\mathcal{P}'_{\bar\beta}/\mathcal{P}'_{\bar\beta+} \otimes_{R'} (R' \otimes_R R^\dagger)_{M'}$. Hence the map

$$\frac{\mathcal{P}_{\bar\beta} R^\dagger}{\mathcal{P}_{\bar\beta+} R^\dagger} \longrightarrow \frac{\mathcal{P}'_{\bar\beta}}{\mathcal{P}'_{\bar\beta+}} \otimes_{R'} (R' \otimes_R R^\dagger)_{M'} \tag{4.19}$$

is injective. Since $\mathcal{P}'_{\bar\beta}/\mathcal{P}'_{\bar\beta+} \otimes_{R'} R'^\dagger \cong \mathcal{P}'_{\bar\beta} R'^\dagger / \mathcal{P}'_{\bar\beta+} R'^\dagger$ and by faithful flatness of $R'^\dagger$ over $(R' \otimes_R R^\dagger)_{M'}$, the injectivity of (4.19) implies the injectivity of the map (4.6) required in (ii) of Proposition 4.2.

This completes the proof of Proposition 4.2. $\qquad\square$

COROLLARY 4.4. *Take an integer* $\ell \in \{0, \ldots, r-1\}$ *and assume that* R *is* $(\ell+1)$-*stable. Then*

$$H_{2\ell+1} = \bigcap_{\beta \in \Delta_\ell} \mathcal{P}_\beta R^\dagger. \tag{4.20}$$

Proof. By Lemma 4 of Appendix 4 of [**17**], the ideals $\mathcal{P}_{\bar\beta}$ are cofinal among the ideals $\mathcal{P}_\beta$ for $\beta \in \Delta_\ell$. $\qquad\square$

COROLLARY 4.5. *Assume that* R *is stable. Take an element* $\beta \in \Gamma$. *Then* $\mathcal{P}'_\beta R'^\dagger \cap R^\dagger = \mathcal{P}_\beta$.

Proof. It is sufficient to prove that for each $\ell \in \{0, \ldots, r-1\}$ and $\bar\beta \in \Gamma/\Delta_{\ell+1}$, we have

$$\mathcal{P}'_{\bar\beta} R'^\dagger \cap R^\dagger = \mathcal{P}_{\bar\beta}; \tag{4.21}$$

Corollary 4.5 is just the special case of (4.21) when $\ell = r-1$. We prove (4.21) by contradiction. Assume the contrary and take the smallest ℓ for which (4.21) fails to be true. Let $\Phi' = \nu(R' \setminus \{0\})$. We will denote by $\Phi/\Delta_{\ell+1}$ the image of Φ under the composition of natural maps $\Phi \hookrightarrow \Gamma \to \Gamma/\Delta_{\ell+1}$ and similarly for $\Phi'/\Delta_{\ell+1}$. Clearly, if (4.21) fails for a certain $\bar\beta$, it also fails for some $\bar\beta \in \Phi'/\Delta_{\ell+1}$; take the smallest $\bar\beta \in \Phi'/\Delta_{\ell+1}$ with this property. If we had $\bar\beta = \min\{\tilde\beta \in \Phi'/\Delta_{\ell+1} \mid \tilde\beta - \bar\beta \in \Delta_\ell\}$, then (4.21) would also fail with $\bar\beta$ replaced by $\bar\beta \bmod \Delta_\ell \in \Gamma/\Delta_\ell$, contradicting the minimality of ℓ. Thus

$$\bar\beta > \min\left\{\tilde\beta \in \frac{\Phi'}{\Delta_{\ell+1}} \,\middle|\, \tilde\beta - \bar\beta \in \Delta_\ell\right\}. \tag{4.22}$$

Let $\bar\beta-$ denote the immediate predecessor of $\bar\beta$ in $\Phi'/\Delta_{\ell+1}$. By (4.22), we have $\bar\beta - \bar\beta- \in \Delta_\ell$. By the choice of $\bar\beta$, we have $\mathcal{P}_{\bar\beta-} = \mathcal{P}'_{\bar\beta-} R'^\dagger \cap R^\dagger$ but $\mathcal{P}_{\bar\beta} \subsetneq \mathcal{P}'_{\bar\beta} R'^\dagger \cap R^\dagger$. This contradicts Proposition 4.2, applied to $\bar\beta-$. Corollary 4.5 is proved. $\qquad\square$

Now we are ready to state and prove the main theorem of this section.

THEOREM 4.6. (1) *Fix an integer* $\ell \in \{1, \ldots, r+1\}$. *Assume that there exists* $R' \in \mathcal{T}(R)$ *which is* $(\ell+1)$-*stable. Then the ideal* $H_{2\ell+1}$ *is prime.*

(2) *Let $i = 2\ell + 2$. There exists an extension $\nu_{i0}^{\dagger}$ of $\nu_{\ell+1}$ to $\varinjlim_{R'} \kappa(H'_{i-1})$, with value group*

$$\Delta_{i-1,0} = \frac{\Delta_{\ell}}{\Delta_{\ell+1}}, \tag{4.23}$$

whose valuation ideals are described as follows. For an element $\bar{\beta} \in \Delta_{\ell}/\Delta_{\ell+1}$, the $\nu_{i0}^{\dagger}$-ideal of $R^{\dagger}/H_{i-1}$ of value $\bar{\beta}$, denoted by $\mathcal{P}_{\bar{\beta}\ell}^{\dagger}$, is given by the formula

$$\mathcal{P}_{\bar{\beta},\ell+1}^{\dagger} = \left(\varinjlim_{R'} \frac{\mathcal{P}'_{\bar{\beta}} R'^{\dagger}}{H'_{i-1}} \right) \cap \frac{R^{\dagger}}{H_{i-1}}. \tag{4.24}$$

REMARK 8.   Once the even-numbered implicit prime ideals $H'_{2\ell}$ are defined below, we will show that $\nu_{i0}^{\dagger}$ is the unique extension of $\nu_{\ell+1}$ to $\varinjlim_{R'} \kappa(H'_{i-1})$, centred in the local ring $\varinjlim_{R'} (R'^{\dagger}_{H'_{2\ell+2}} / H'_{2\ell+1} R'^{\dagger}_{H'_{2\ell+2}})$.

Proof of Theorem 4.6.   Let R' be a stable ring in $\mathcal{T}(R)$. Once Theorem 4.6 is proved for R', the same results for R will follow easily by intersecting all the ideals of $R'^{\dagger}$ in sight with $R^{\dagger}$. Therefore from now on we will replace R by R', that is, we will assume that R itself is stable.

Let Φ_{ℓ} denote the image of the semigroup $\nu(R \setminus \{0\})$ in $\Gamma/\Delta_{\ell+1}$. As we saw above, Φ_{ℓ} is well ordered. For an element $\bar{\beta} \in \Phi_{\ell}$, let $\bar{\beta}+$ denote the immediate successor of $\bar{\beta}$ in Φ_{ℓ}.

Take any element $x \in R^{\dagger} \setminus H_{i-1}$. By Corollary 4.4, there exists (a unique) $\bar{\beta} \in \Phi_{\ell} \cap \Delta_{\ell}/\Delta_{\ell+1}$ such that

$$x \in \mathcal{P}_{\bar{\beta}} R^{\dagger} \setminus \mathcal{P}_{\bar{\beta}+} R^{\dagger} \tag{4.25}$$

(where, of course, we allow $\bar{\beta} = 0$). Let $\bar{x}$ denote the image of x in $R^{\dagger}/H_{i-1}$. We define

$$\nu_{i0}^{\dagger}(\bar{x}) = \bar{\beta}.$$

Next, take another element $y \in R^{\dagger} \setminus H_{2\ell+1}$ and let $\gamma \in \Phi_{\ell} \cap \Delta_{\ell}/\Delta_{\ell+1}$ be such that

$$y \in \mathcal{P}_{\bar{\gamma}} R^{\dagger} \setminus \mathcal{P}_{\bar{\gamma}+} R^{\dagger}. \tag{4.26}$$

Let $(a_1, \ldots, a_n)$ be a set of generators of $\mathcal{P}_{\bar{\beta}}$ and $(b_1, \ldots, b_s)$ a set of generators of $\mathcal{P}_{\bar{\gamma}}$, with $\nu_{\ell+1}(a_1) = \bar{\beta}$ and $\nu_{\ell+1}(b_1) = \bar{\gamma}$. Let R' be a local blowing up along ν such that R' contains all the fractions a_i/a_1 and b_j/b_1. By Proposition 4.1 and Definition 4.1(1), the ideal $P'_{\ell+1} R'^{\dagger}$ is prime. By construction, we have $a_1 \mid x$ and $b_1 \mid y$ in $R'^{\dagger}$. Write $x = za_1$ and $y = wb_1$ in $R'^{\dagger}$. The equality (4.5), combined with (4.25) and (4.26), implies that $z, w \notin P'_{\ell+1} R'^{\dagger}$, hence

$$zw \notin P'_{\ell+1} R'^{\dagger} \tag{4.27}$$

by the primality of $P'_{\ell+1} R'^{\dagger}$. We obtain

$$xy = a_1 b_1 zw. \tag{4.28}$$

Since ν is a valuation on R', we have $(\mathcal{P}'_{\bar{\beta}+\bar{\gamma}+} : (a_1 b_1) R') \subset P'_{\ell+1}$. By faithful flatness of $R'^{\dagger}$ over R' we obtain

$$(\mathcal{P}'_{\bar{\beta}+\bar{\gamma}+} R'^{\dagger} : (a_1 b_1) R'^{\dagger}) \subset P'_{\ell+1} R'^{\dagger}. \tag{4.29}$$

Combining this with (4.27) and (4.28), we obtain

$$xy \notin \mathcal{P}_{\bar{\beta}+\bar{\gamma}+} R^{\dagger}, \tag{4.30}$$

in particular, $xy \notin H_{2\ell+1}$. We started with two arbitrary elements $x, y \in R^{\dagger} \setminus H_{2\ell+1}$ and showed that $xy \notin H_{2\ell+1}$. This proves (1) of Theorem 4.6.

Furthermore, (4.30) shows that $\nu_{i0}^\dagger(\bar{x}\bar{y}) = \bar{\beta} + \bar{\gamma}$, so $\nu_{i0}^\dagger$ induces a valuation of $\kappa(H_{i-1})$ and hence also of $\varinjlim_{R'} \kappa(H'_{i-1})$. Equality (4.23) holds by definition and (4.24) by the assumed stability of R. $\qquad\square$

Next, we define the even-numbered implicit prime ideals $H'_{2\ell}$. The only information we need to use to define the prime ideals $H'_{2\ell} \subset H'_{2\ell+1}$ and to prove that $H'_{2\ell-1} \subset H'_{2\ell}$ are the facts that $H_{2\ell+1}$ is a prime lying over P_ℓ and that the ring homomorphism $R' \to R'^\dagger$ is regular.

PROPOSITION 4.7. *There exists a unique minimal prime ideal $H_{2\ell}$ of $P_\ell R^\dagger$, contained in $H_{2\ell+1}$.*

Proof. Since $H_{2\ell+1} \cap R = P_\ell$, $H_{2\ell+1}$ belongs to the fibre of the map $\operatorname{Spec} R^\dagger \to \operatorname{Spec} R$ over P_ℓ. Since R was assumed to be excellent, $S := R^\dagger \otimes_R \kappa(P_\ell)$ is a regular ring (note that the excellence assumption is needed only in the case $R^\dagger = \hat{R}$; the ring homomorphism $R \to R^\dagger$ is automatically regular if $R^\dagger = \tilde{R}$ or $R^\dagger = R^e$). Hence its localization $\bar{S} := S_{H_{2\ell+1}S} \cong R^\dagger_{H_{2\ell+1}}/P_\ell R^\dagger_{H_{2\ell+1}}$ is a regular *local* ring. In particular, $\bar{S}$ is an integral domain, so (0) is its unique minimal prime ideal. The set of minimal prime ideals of $\bar{S}$ is in one-to-one correspondence with the set of minimal primes of P_ℓ, contained in $H_{2\ell+1}$, which shows that such a minimal prime $H_{2\ell}$ is unique, as desired. $\qquad\square$

We have $P_\ell \subset H_{2\ell} \cap R \subset H_{2\ell+1} \cap R = P_\ell$, so $H_{2\ell} \cap R \subset P_\ell$.

PROPOSITION 4.8. *We have $H_{2\ell-1} \subset H_{2\ell}$.*

Proof. Take an element $\beta \in \Delta_{\ell-1}/\Delta_\ell$ and a stable ring $R' \in \mathcal{T}$. Then $\mathcal{P}'_\beta \subset P'_\ell$, so

$$H'_{2\ell-1} \subset \mathcal{P}'_\beta R'^\dagger \subset P'_\ell R'^\dagger \subset H'_{2\ell}. \tag{4.31}$$

Intersecting (4.31) back with $R^\dagger$ we obtain the result. $\qquad\square$

In §7, we will see that if $R^\dagger = \tilde{R}$ or $R^\dagger = R^e$, then $H_{2\ell} = H_{2\ell+1}$ for all ℓ.
Let the notation be the same as in Theorem 4.6.

PROPOSITION 4.9. *The valuation $\nu_{i0}^\dagger$ is the unique extension of ν_ℓ to a valuation of $\varinjlim_{R'} \kappa(H'_{i-1})$, centred in the local ring $\varinjlim_{R'} (R'^\dagger_{H'_{2\ell}}/H'_{2\ell-1}R'^\dagger_{H'_{2\ell}})$.*

Proof. As usual, without loss of generality, we may assume that R is stable. Take an element $x \in R^\dagger \setminus H_{2\ell-1}$. Let $\beta = \nu_{i0}^\dagger(\bar{x})$ and let R' be the blowing up of the ideal $\mathcal{P}_\beta = (a_1, \ldots, a_n)$, as in the proof of Theorem 4.6. Write

$$x = za_1 \tag{4.32}$$

in R'. We have $z \in R'^\dagger \setminus P'_\ell R'^\dagger$, hence

$$\bar{z} \in \frac{R'^\dagger_{H'_{2\ell}}}{H'_{2\ell-1}R'^\dagger_{H'_{2\ell}}} \setminus \frac{P'_\ell R'^\dagger_{H'_{2\ell}}}{H'_{2\ell-1}R'^\dagger_{H'_{2\ell}}} = \frac{R'^\dagger_{H'_{2\ell}}}{H'_{2\ell-1}R'^\dagger_{H'_{2\ell}}} \setminus \frac{H'_{2\ell}R'^\dagger_{H'_{2\ell}}}{H'_{2\ell-1}R'^\dagger_{H'_{2\ell}}}. \tag{4.33}$$

If ν^* is any other extension of ν_ℓ to $\lim_{\overrightarrow{R'}} \kappa(H'_{i-1})$, centred in $\lim_{\overrightarrow{R'}}(R'^\dagger_{H'_{2\ell}}/H'_{2\ell-1}R'^\dagger_{H'_{2\ell}})$, then $\nu^*(\bar{a}_1) = \beta$, $\nu^*(z) = 0$ by (4.33), so $\nu^*(\bar{x}) = \beta = \nu^\dagger_{i0}(\bar{x})$. This completes the proof of the uniqueness of $\nu^\dagger_{i0}$. $\qquad\square$

REMARK 9. If R' is stable, we have a natural isomorphism of graded algebras

$$\mathrm{gr}_{\nu^\dagger_{i0}} \frac{R'^\dagger_{H'_{2\ell}}}{H'_{2\ell-1}R'^\dagger_{H'_{2\ell}}} \cong \mathrm{gr}_{\nu_\ell} \frac{R'_{P'_\ell}}{P'_{\ell-1}R'_{P'_\ell}} \otimes_{R'} \kappa(H'_{2\ell}).$$

In particular, the residue field of $\nu^\dagger_{i0}$ is $k_{\nu^\dagger_{i0}} = \lim_{\overrightarrow{R'}} \kappa(H'_{2\ell})$.

5. A classification of extensions of ν to $\hat{R}$

The purpose of this section is to give a systematic description of all the possible extensions $\nu^\dagger_-$ of ν to a quotient of $R^\dagger$ by a minimal prime as compositions of $2r$ valuations,

$$\nu^\dagger_- = \nu^\dagger_1 \circ \ldots \circ \nu^\dagger_{2r}, \tag{5.1}$$

satisfying certain conditions. One is naturally led to consider the more general problem of extending ν not only to rings of the form $R^\dagger/P$ but also to the ring $\lim_{\to} R'^\dagger/P'$, where P' is a tree of prime ideals of $R'^\dagger$, such that $P' \cap R' = (0)$. We deal in a uniform way with all the three cases $R^\dagger = \hat{R}$, $R^\dagger = \tilde{R}$ and $R^\dagger = R^e$, in order to be able to apply the results proved here to all three later in the paper. However, the reader should think of the case $R^\dagger = \hat{R}$ as the main case of interest and the cases $R^\dagger = \tilde{R}$ and $R^\dagger = R^e$ as auxiliary and slightly degenerate, since, as we shall see, in these cases the equality $H_{2\ell} = H_{2\ell+1}$ is satisfied for all ℓ and the extension $\nu^\dagger_-$ will later be shown to be unique.

We will associate to each extension $\nu^\dagger_-$ of ν to $R^\dagger$ a chain

$$\tilde{H}'_0 \subset \tilde{H}'_1 \subset \ldots \subset \tilde{H}'_{2r} = m'R'^\dagger \tag{5.2}$$

of prime $\nu^\dagger_-$-ideals, corresponding to the decomposition (5.1) and prove some basic properties of this chain of ideals.

Now for the details. We wish to classify all the pairs $(\{\tilde{H}'_0\}, \nu^\dagger_+)$, where $\{\tilde{H}'_0\}$ is a tree of prime ideals of $R'^\dagger$, such that $\tilde{H}'_0 \cap R' = (0)$, and $\nu^\dagger_+$ is an extension of ν to the ring $\lim_{\to} R'^\dagger/\tilde{H}'_0$.

Pick and fix one such pair $(\{\tilde{H}'_0\}, \nu^\dagger_+)$. We associate to it the following collection of data, which, as we will see, will in turn determine the pair $(\{\tilde{H}'_0\}, \nu^\dagger_+)$.

First, we associate to $(\{\tilde{H}'_0\}, \nu^\dagger_-)$ a chain (5.2) of $2r$ trees of prime $\nu^\dagger_-$-ideals. Let $\Gamma^\dagger$ denote the value group of $\nu^\dagger_-$. Defining (5.2) is equivalent to defining a chain

$$\Gamma^\dagger = \Delta^\dagger_0 \supset \Delta^\dagger_1 \supset \ldots \supset \Delta^\dagger_{2r} = \Delta^\dagger_{2r+1} = (0) \tag{5.3}$$

of $2r$ isolated subgroups of $\Gamma^\dagger$ (the chain (5.3) will not, in general, be maximal, and $\Delta^\dagger_{2\ell+1}$ need not be distinct from $\Delta^\dagger_{2\ell}$).

We define the $\Delta^\dagger_i$ as follows. For $0 \leqslant \ell \leqslant r$, let $\Delta^\dagger_{2\ell}$ and $\Delta^\dagger_{2\ell+1}$ denote, respectively, the greatest and the smallest isolated subgroups of $\Gamma^\dagger$ such that

$$\Delta^\dagger_{2\ell} \cap \Gamma = \Delta^\dagger_{2\ell+1} \cap \Gamma = \Delta_\ell. \tag{5.4}$$

LEMMA 5.1. We have

$$\mathrm{rk}\, \frac{\Delta^\dagger_{2\ell-1}}{\Delta^\dagger_{2\ell}} = 1 \tag{5.5}$$

for $1 \leqslant \ell \leqslant r$.

Proof. Since by construction $\Delta_{2\ell}^\dagger \neq \Delta_{2\ell-1}^\dagger$, equality (5.5) is equivalent to saying that there is no isolated subgroup $\Delta^\dagger$ of $\Gamma^\dagger$ which is properly contained in $\Delta_{2\ell-1}^\dagger$ and properly contains $\Delta_{2\ell}^\dagger$. Suppose such an isolated subgroup $\Delta^\dagger$ existed. Then

$$\Delta_\ell = \Delta_{2\ell}^\dagger \cap \Gamma \subsetneq \Delta^\dagger \cap \Gamma \subsetneq \Delta_{2\ell-1}^\dagger \cap \Gamma = \Delta_{\ell-1}, \tag{5.6}$$

where the first inclusion is strict by the maximality of $\Delta_{2\ell}^\dagger$ and the second by the minimality of $\Delta_{2\ell-1}^\dagger$. Thus $\Delta^\dagger \cap \Gamma$ is an isolated subgroup of Γ, properly containing Δ_ℓ and properly contained in Δ_{l-1}, which is impossible since $\mathrm{rk}\,(\Delta_{\ell-1}/\Delta_\ell) = 1$. This is a contradiction, hence $\mathrm{rk}\,(\Delta_{2\ell-1}^\dagger/\Delta_{2\ell}^\dagger) = 1$, as desired. $\qquad\square$

DEFINITION 5.1.   Let $0 \leqslant i \leqslant 2r$. The ith prime ideal *determined* by $\nu_-^\dagger$ is the prime $\nu_-^\dagger$-ideal $\tilde{H}_i'$ of $R'^\dagger$, corresponding to the isolated subgroup $\Delta_i^\dagger$ (that is, the ideal $\tilde{H}_i'$ consisting of all the elements of $R'^\dagger$ whose values lie outside of $\Delta_i^\dagger$). The chain of trees (5.2) of prime ideals of $R'^\dagger$ formed by the $\tilde{H}_i'$ is referred to as the chain of trees *determined* by $\nu_-^\dagger$.

The equality (5.4) says that

$$\tilde{H}_{2\ell}' \cap R' = \tilde{H}_{2\ell+1}' \cap R' = P_\ell' \tag{5.7}$$

By definitions, for $1 \leqslant i \leqslant 2r$, $\nu_i^\dagger$ is a valuation of the field $k_{\nu_{i-1}^\dagger}$. In the sequel, we will find it useful to talk about the restriction of $\nu_i^\dagger$ to a smaller field, namely, the field of fractions of the ring $\varinjlim_{R'}(R'^\dagger_{\tilde{H}_i'}/\tilde{H}_{i-1}'R'^\dagger_{\tilde{H}_i'})$; we will denote this restriction by $\nu_{i0}^\dagger$. The field of fractions of $R'^\dagger_{\tilde{H}_i'}/\tilde{H}_{i-1}'R'^\dagger_{\tilde{H}_i'}$ is $\kappa(\tilde{H}_{i-1}')$, hence that of $\varinjlim_{R'}(R'^\dagger_{\tilde{H}_i'}/\tilde{H}_{i-1}'R'^\dagger_{\tilde{H}_i'})$ is $\varinjlim_{R'}\kappa(\tilde{H}_{i-1}')$, which is a subfield of $k_{\nu_{i-1}^\dagger}$. The value group of $\nu_{i0}^\dagger$ will be denoted by $\Delta_{i-1,0}$; we have $\Delta_{i-1,0} \subset \Delta_{i-1}^\dagger/\Delta_i^\dagger$. If $i = 2\ell$ is even, then $R'_{P_l'}/P_{l-1}'R'_{P_l'} < R'^\dagger_{\tilde{H}_i'}/\tilde{H}_{i-1}'R'^\dagger_{\tilde{H}_i'}$, so $\varinjlim_{R'}(R'_{P_l'}/P_{l-1}'R'_{P_l'}) < \varinjlim_{R'}(R'^\dagger_{\tilde{H}_i'}/\tilde{H}_{i-1}'R'^\dagger_{\tilde{H}_i'})$. In this case, $\mathrm{rk}\,\nu_i^\dagger = 1$ and $\nu_i^\dagger$ an extension of the rank 1 valuation ν_ℓ from $\kappa(P_{\ell-1})$ to $k_{\nu_{i-1}^\dagger}$; we have

$$\frac{\Delta_{\ell-1}}{\Delta_\ell} \subset \Delta_{i-1,0} \subset \frac{\Delta_{i-1}^\dagger}{\Delta_i^\dagger}. \tag{5.8}$$

PROPOSITION 5.2.   Let $i = 2\ell$. As usual, for an element $\bar{\beta} \in (\Delta_\ell/\Delta_{\ell+1})_+$, let $\mathcal{P}_{\bar{\beta}}$ (resp. $\mathcal{P}_{\bar{\beta}}'$) denote the preimage in R (resp. in R') of the $\nu_{\ell+1}$-ideal of R/P_ℓ (resp. R'/P_ℓ') of value greater than or equal to $\bar{\beta}$. Then

$$\bigcap_{\bar{\beta} \in (\Delta_\ell/\Delta_{\ell+1})_+} \varinjlim_{R'}(\mathcal{P}_{\bar{\beta}}'R'^\dagger + \tilde{H}_{i+1}')R'^\dagger_{\tilde{H}_{i+2}'} \cap R^\dagger \subset \tilde{H}_{i+1}'. \tag{5.9}$$

The inclusion (5.9) should be understood as a condition on the tree of ideals. In other words, it is equally valid if we replace R' by any other ring $R'' \in \mathcal{T}$.

Proof of Proposition 5.2.   Since $\mathrm{rk}\,(\Delta_{i+1}^\dagger/\Delta_{i+2}^\dagger) = 1$ by Lemma 5.1, $\Delta_\ell/\Delta_{\ell+1}$ is cofinal in $\Delta_{i+1}^\dagger/\Delta_{i+2}^\dagger$. Then for any $x \in \bigcap_{\bar{\beta} \in (\Delta_\ell/\Delta_{\ell+1})_+} \varinjlim_{R'}(\mathcal{P}_{\bar{\beta}}'R'^\dagger + \tilde{H}_{i+1}')R'^\dagger_{\tilde{H}_{i+2}'} \cap R^\dagger$ we have $\nu_-^\dagger(x) \notin \Delta_i^\dagger$, hence $x \in \tilde{H}_{i+1}'$, as desired. $\qquad\square$

From now to the end of §6, we will assume that $\mathcal{T}$ contains a stable ring R', so that we can apply the results of the previous section, in particular, the primality of the ideals H_i'.

PROPOSITION 5.3.   *We have*

$$H_i' \subset \tilde{H}_i' \quad \text{for all } i \in \{0, \ldots, 2r\}. \tag{5.10}$$

Proof.   For $\beta \in \Gamma^\dagger$ and $R' \in \mathcal{T}$, let $\mathcal{P}_\beta'^\dagger$ denote the $\nu_-^\dagger$-ideal of $R'^\dagger$ of value β. Fix an integer $\ell \in \{0, \ldots, r\}$. For each $R' \in \mathcal{T}$, each $\beta \in \Delta_\ell$ and $x \in \mathcal{P}_\beta'$ we have $\nu_-^\dagger(x) = \nu(x) \geqslant \beta$, hence

$$\mathcal{P}_\beta' R'^\dagger \subset \mathcal{P}_\beta'^\dagger. \tag{5.11}$$

Taking the inductive limit over all $R' \in \mathcal{T}$ and the intersection over all $\beta \in \Delta_\ell$ in (5.11), and using the cofinality of Δ_ℓ in $\Delta_{2\ell+1}^\dagger$ and the fact that $\bigcap_{\beta \in \Delta_{2\ell}^\dagger} (\varinjlim_{R'} \mathcal{P}_\beta'^\dagger) = \varinjlim_{R'} \tilde{H}_{2\ell+1}'$, we obtain the inclusion (5.10) for $i = 2\ell + 1$. To prove (5.10) for $i = 2\ell$, note that $\tilde{H}_{2\ell}' \cap R' = \tilde{H}_{2\ell+1}' \cap R' = P_\ell$. By the same argument as in Proposition 4.7, excellence of R' implies that there is a unique minimal prime $H_{2\ell}^*$ of $P_\ell' R'^\dagger$, contained in $\tilde{H}_{2\ell+1}'$ and a unique minimal prime $H_{2\ell}^{**}$ of $P_\ell' R'^\dagger$, contained in $\tilde{H}_{2\ell}'$. Now, Proposition 4.7 and the facts that $H_{2\ell+1}' \subset \tilde{H}_{2\ell+1}'$ and $\tilde{H}_{2\ell}' \subset \tilde{H}_{2\ell+1}'$ imply that $H_{2\ell}' = H_{2\ell}^* = H_{2\ell}^{**}$, hence $H_{2\ell}' = H_{2\ell}^{**} \subset \tilde{H}_{2\ell}'$, as desired. $\square$

DEFINITION 5.2.   A chain of trees (5.2) of prime ideals of $R'^\dagger$ is said to be *admissible* if $H_i' \subset \tilde{H}_i'$ and (5.7) and (5.9) hold.

Equalities (5.7), Proposition 5.2 and Proposition 5.3 say that a chain of trees (5.2) of prime ideals of $R'^\dagger$, determined by $\nu_-^\dagger$, is admissible.

Summarizing all of the above results, and keeping in mind the fact that specifying a composition of $2r$ valuation is equivalent to specifying all of its $2r$ components, we arrive at one of the main theorems of this paper:

THEOREM 5.4.   *Specifying the valuation $\nu_-^\dagger$ is equivalent to specifying the following data. The data will be described recursively in i, that is, the description of $\nu_i^\dagger$ assumes that $\nu_{i-1}^\dagger$ is already defined.*

(1) *An admissible chain of trees (5.2) of prime ideals of $R'^\dagger$.*

(2) *For each i, $1 \leqslant i \leqslant 2r$, a valuation $\nu_i^\dagger$ of $k_{\nu_{i-1}^\dagger}$ (where $\nu_0^\dagger$ is taken to be the trivial valuation by convention), whose restriction to $\varinjlim_{R'} \kappa(\tilde{H}_{i-1}')$ is centred at the local ring $\varinjlim_{R'} (R'^\dagger_{\tilde{H}_i'} / \tilde{H}_{i-1}' R'^\dagger_{\tilde{H}_i'})$.*

The data $\{\nu_i^\dagger\}_{1 \leqslant i \leqslant 2r}$ are subject to the following additional condition: if $i = 2\ell$ is even then $\operatorname{rk} \nu_i^\dagger = 1$ and $\nu_i^\dagger$ is an extension of ν_ℓ to $k_{\nu_{i-1}^\dagger}$ (which is naturally an extension of $k_{\nu_{\ell-1}}$).

In particular, note that such extensions $\nu_-^\dagger$ always exist, and usually there are plenty of them. The question of uniqueness of $\nu_-^\dagger$ and the related question of uniqueness of $\nu_i^\dagger$, especially in the case when i is even, will be addressed in the next section.

6. Uniqueness properties of $\nu_-^\dagger$

In this section, we address the question of uniqueness of the extension $\nu_-^\dagger$. One result in this direction, which will be very useful here, was already proved in § 4: Proposition 4.9. We give some necessary and some sufficient conditions both for the uniqueness of $\nu_-^\dagger$ once the chain (5.2) of prime ideals determined by $\nu_-^\dagger$ has been fixed, and also for the unconditional uniqueness of $\nu_-^\dagger$. In § 7, we will use one of these uniqueness criteria to prove uniqueness of $\nu_-^\dagger$ in the cases $R^\dagger = \tilde{R}$ and $R^\dagger = R^e$. At the end of this section, we generalize and give a new point of view of an old result of Heinzer and Sally (Proposition 6.9), which provides a sufficient condition for the uniqueness of $\nu_-^\dagger$; see also [**13**, Remark 5.22].

For a ring $R' \in \mathcal{T}$, let K' denote the field of fractions of R'. For some results in this section we will need to impose an additional condition on the tree $\mathcal{T}$: we will assume that there exists $R_0 \in \mathcal{T}$ such that, for all $R' \in \mathcal{T}(R_0)$, the field K' is algebraic over K_0. This assumption is needed in order to be able to control the height of all the ideals in sight. Without loss of generality, we may take $R_0 = R$.

PROPOSITION 6.1.   *Assume that for all $R' \in \mathcal{T}$ the field K' is algebraic over K. Consider a ring homomorphism $R' \to R''$ in $\mathcal{T}$. Take an $\ell \in \{0, \ldots, r\}$. We have*

$$\operatorname{ht} H_{2\ell}'' \leqslant \operatorname{ht} H_{2\ell}'. \tag{6.1}$$

If equality holds in (6.1), *then*

$$\operatorname{ht} H_{2\ell+1}'' \geqslant \operatorname{ht} H_{2\ell+1}'. \tag{6.2}$$

Proof.   We start by recalling a well-known lemma (for a proof see [**17**, Appendix 1, Propositions 2 and 3, p. 326]):

LEMMA 6.2.   *Let $R \hookrightarrow R'$ be an extension of integral domains, essentially of finite type. Let K and K' be the respective fields of fractions of R and R'. Consider prime ideals $P \subset R$ and $P' \subset R'$ such that $P = P' \cap R$. Then*

$$\operatorname{ht} P' + \operatorname{tr.deg.}(\kappa(P')/\kappa(P)) \leqslant \operatorname{ht} P + \operatorname{tr.deg.}(K'/K). \tag{6.3}$$

Moreover, equality holds in (6.3) *whenever R is universally catenarian.*

Apply Lemma 6.2 to the rings R' and R'' and the prime ideals $P_\ell' \subset R'$ and $P_\ell'' \subset R''$. In the case at hand, we have $\operatorname{tr.deg.}(K''/K') = 0$ by assumption. Hence

$$\operatorname{ht} P_\ell'' \leqslant \operatorname{ht} P_\ell'. \tag{6.4}$$

Since $H_{2\ell}'$ is a minimal prime of $P_\ell' R'^\dagger$ and $R'^\dagger$ is faithfully flat over R', we have $\operatorname{ht} P_\ell' = \operatorname{ht} H_{2\ell}'$. Similarly, $\operatorname{ht} P_\ell'' = \operatorname{ht} H_{2\ell}''$, and (6.1) follows. Furthermore, equality in (6.1) is equivalent to equality in (6.4).

To prove (6.2), let $\bar{R} = (R'' \otimes_{R'} R'^\dagger)_{M''}$, where $M'' = (m'' \otimes 1 + 1 \otimes m'R'^\dagger)$ and let $\bar{m}$ denote the maximal ideal of $\bar{R}$. We have the natural maps $R'^\dagger \xrightarrow{\iota} \bar{R} \xrightarrow{\sigma} R''^\dagger$. The homomorphism σ is nothing but the formal completion of the local ring $\bar{R}$; in particular, it is faithfully flat. Let

$$\bar{H} = H_{2\ell+1}'' \cap \bar{R}, \tag{6.5}$$

$\bar{H}_0 = H_0'' \cap \bar{R}$. Since H_0'' is a minimal prime of $R''^\dagger$ and σ is faithfully flat, $\bar{H}_0$ is a minimal prime of $\bar{R}$.

Assume that equality holds in (6.1) (and hence also in (6.4)). Since equality holds in (6.4), by Lemma 6.2 (applied to the ring extension $R' \to R''$) the field $\kappa(P'')$ is algebraic over $\kappa(P')$.

Apply Lemma 6.2 to the ring extension $R'^\dagger/H_0' \hookrightarrow \bar{R}/\bar{H}_0$ and the prime ideals $H'_{2\ell+1}/H_0'$ and $\bar{H}/\bar{H}_0$. Since K'' is algebraic over K', $\kappa(\bar{H}_0)$ is algebraic over $\kappa(H_0')$. Since $\kappa(P'')$ is algebraic over $\kappa(P')$, $\kappa(\bar{H})$ is algebraic over $\kappa(H'_{2\ell+1})$. Finally, $\hat{R}'$ is universally catenarian because it is a complete local ring. Now in the case $^\dagger = \hat{}$ Lemma 6.2 says that $\mathrm{ht}(H'_{2\ell+1}/H_0') = \mathrm{ht}(\bar{H}/\bar{H}_0)$. Since both $\hat{R}'$ and $\bar{R}$ are catenarian, this implies that

$$\mathrm{ht}\, H'_{2\ell+1} = \mathrm{ht}\, \bar{H}. \tag{6.6}$$

In the case where $^\dagger$ stands for henselization or a finite étale extension, (6.6) is an immediate consequence of (6.5). Thus (6.6) is true in all the cases. Since σ is faithfully flat and in view of (6.5), $\mathrm{ht}\, \bar{H} \leqslant \mathrm{ht}\, H''_{2\ell+1}$. Combined with (6.6), this completes the proof. $\qquad\square$

COROLLARY 6.3.   *For each i, $0 \leqslant i \leqslant 2r$, the quantity $\mathrm{ht}\, H_i'$ stabilizes for R' sufficiently far out in $\mathcal{T}$.*

The following proposition is an immediate consequence of Theorem 5.4.

PROPOSITION 6.4.   *Suppose given an admissible chain of trees (5.2) of prime ideals of $R'^\dagger$. For each $\ell \in \{0, \ldots, r-1\}$, consider the set of all $R' \in \mathcal{T}$ such that*

$$\mathrm{ht}\, \tilde{H}'_{2\ell+1} - \mathrm{ht}\, \tilde{H}'_{2\ell} \leqslant 1 \quad \text{for all even } i \tag{6.7}$$

and, in case of equality, the 1-dimensional local ring $R'^\dagger_{\tilde{H}'_{2\ell+1}}/\tilde{H}'_{2\ell}R'^\dagger_{\tilde{H}'_{2\ell+1}}$ is unibranch (that is, analytically irreducible). Assume that, for each ℓ, the set of such R' is cofinal in $\mathcal{T}$. Let $\nu^\dagger_{2\ell+1,0}$ denote the unique valuation centred at $\varinjlim_{R'}(R'^\dagger_{\tilde{H}'_{2\ell+1}}/\tilde{H}'_{2\ell}R'^\dagger_{\tilde{H}'_{2\ell+1}})$.

Assume that for each even $i = 2\ell$, ν_ℓ admits a unique extension $\nu^\dagger_{i0}$ to a valuation of $\varinjlim_{R'} \kappa(\tilde{H}'_{i-1})$, centred in $\varinjlim_{R'}(R'^\dagger_{\tilde{H}'_i}/\tilde{H}'_{i-1}R'^\dagger_{\tilde{H}'_i})$. Then specifying the valuation $\nu^\dagger_-$ is equivalent to specifying for each odd i, $2 < i < 2r$, an extension $\nu^\dagger_i$ of the valuation $\nu^\dagger_{i0}$ of $\varinjlim_{R'} \kappa(\tilde{H}'_{i-1})$ to its field extension $k_{\nu^\dagger_{i-1}}$ (in particular, such extensions $\nu^\dagger_-$ always exist). If for each odd i, $2 < i < 2r$, the field extension $\varinjlim_{R'} \kappa(\tilde{H}'_{i-1}) \to k_{\nu^\dagger_{i-1}}$ is algebraic and the extension $\nu^\dagger_i$ of $\nu^\dagger_{i0}$ to $k_{\nu^\dagger_{i-1}}$ is unique, then there is a unique extension $\nu^\dagger_-$ of ν such that the $\tilde{H}'_i$ are the prime ideals, determined by $\nu^\dagger_-$.

Conversely, assume that K' is algebraic over K and that there exists a unique extension $\nu^\dagger_-$ of ν such that the $\tilde{H}'_i$ are the prime $\nu^\dagger_-$-ideals, determined by $\nu^\dagger_-$. Then for each $\ell \in \{0, \ldots, r-1\}$ and for all R' sufficiently far out in $\mathcal{T}$ the inequality (6.7) holds. For each even $i = 2\ell$, ν_ℓ admits a unique extension $\nu^\dagger_{i0}$ to a valuation of $\varinjlim_{R'} \kappa(\tilde{H}'_{i-1})$, centred in $\varinjlim_{R'}(R'^\dagger_{\tilde{H}'_i}/\tilde{H}'_{i-1}R'^\dagger_{\tilde{H}'_i})$; we have $\mathrm{rk}\, \nu^\dagger_{i0} = 1$. For each odd i, the ring $\varinjlim_{R'}(R'^\dagger_{\tilde{H}'_i}/\tilde{H}'_{i-1}R'^\dagger_{\tilde{H}'_i})$ is a valuation ring of a (not necessarily discrete) rank 1 valuation. For each odd i, $1 \leqslant i < 2r$, the field extension $\varinjlim_{R'} \kappa(\tilde{H}'_{i-1}) \to k_{\nu^\dagger_{i-1}}$ is algebraic and the extension $\nu^\dagger_i$ of $\nu^\dagger_{i0}$ to $k_{\nu^\dagger_{i-1}}$ is unique.

REMARK 10.   We do not know of a simple criterion to decide when, given an algebraic field extension $K \hookrightarrow L$ and a valuation ν of K, is the extension of ν to L unique. See [6, 15] for more information about this question and an algorithm for arriving at the answer using MacLane's key polynomials.

Next we describe three classes of extensions of ν to $\varinjlim_{R'} R'^\dagger$, which are of particular interest for applications, and which we call *minimal, evenly minimal* and *tight* extensions.

DEFINITION 6.1.　Let $\nu_-^\dagger$ be an extension of ν to $\varinjlim_{R'} R'^\dagger$ and let the notation be as above. We say that $\nu_-^\dagger$ is *evenly minimal* if whenever $i = 2\ell$ is even, the following two conditions hold:

(1)
$$\Delta_{i-1,0} = \frac{\Delta_{\ell-1}}{\Delta_\ell}. \tag{6.8}$$

(2) For an element $\bar\beta \in \Delta_{\ell-1}/\Delta_\ell$, the $\nu_{i0}^\dagger$-ideal of $R_{\tilde H_i}^\dagger / \tilde H_{i-1} R_{\tilde H_i}^\dagger$ of value $\bar\beta$, denoted by $\mathcal{P}_{\bar\beta,\ell}^\dagger$, is given by the formula

$$\mathcal{P}_{\bar\beta,\ell}^\dagger = \left(\varinjlim_{R'} \frac{\mathcal{P}_{\bar\beta}' R_{\tilde H_i'}'^\dagger}{\tilde H_{i-1}' R_{\tilde H_i'}'^\dagger} \right) \cap \frac{R_{\tilde H_i}^\dagger}{\tilde H_{i-1} R_{\tilde H_i}^\dagger}. \tag{6.9}$$

We say that $\nu_-^\dagger$ is *minimal* if $\tilde H_i' = H_i'$ for each R' and each $i \in \{0, \dots, 2r+1\}$. We say that $\nu_-^\dagger$ is *tight* if it is evenly minimal and

$$\tilde H_i' = \tilde H_{i+1}' \quad \text{for all even } i. \tag{6.10}$$

REMARK 11.　(1) The valuation $\nu_{i0}^\dagger$ is uniquely determined by conditions (6.8) and (6.9). Recall also that if $i = 2\ell$ is even and we have:

$$\tilde H_i' = H_i' \tag{6.11}$$

and

$$\tilde H_{i-1}' = H_{i-1}' \tag{6.12}$$

then $\nu_{i0}^\dagger$ is uniquely determined by ν_ℓ by Proposition 4.9. In particular, if $\nu_-^\dagger$ is minimal (that is, if (6.11)–(6.12) hold for all i), then $\nu_-^\dagger$ is evenly minimal.

(2) The definition of evenly minimal extensions can be rephrased as follows in terms of the associated graded algebras of ν_ℓ and $\nu_{i0}^\dagger$. First, consider a homomorphism in $\mathcal{T}$ and the following diagram, composed of natural homomorphisms:

$$\frac{\mathcal{P}_{\bar\beta}'}{\mathcal{P}_{\bar\beta+}'} \otimes_{R'} R_{\tilde H_i'}'^\dagger \xrightarrow{\ \lambda'\ } \frac{\mathcal{P}_{\bar\beta}' R_{\tilde H_i'}'^\dagger}{\mathcal{P}_{\bar\beta+}' R_{\tilde H_i'}'^\dagger} \xrightarrow{\ \phi'\ } \frac{\mathcal{P}_{\bar\beta}'^\dagger}{\mathcal{P}_{\bar\beta+}'^\dagger} \tag{6.13}$$
$$\Big\uparrow {\scriptstyle \psi'}$$
$$\frac{\mathcal{P}_{\bar\beta}^\dagger}{\mathcal{P}_{\bar\beta+}^\dagger}$$

It follows from Nakayama's Lemma that equality (6.9) is equivalent to saying that

$$\frac{\mathcal{P}_{\bar\beta}^\dagger}{\mathcal{P}_{\bar\beta+}^\dagger} = \varinjlim_{R'} \psi'^{-1} \left((\phi' \circ \lambda') \left(\frac{\mathcal{P}_{\bar\beta}'}{\mathcal{P}_{\bar\beta+}'} \otimes_{R'} \kappa(\tilde H_i') \right) \right). \tag{6.14}$$

Taking the direct sum in (6.14) over all $\bar\beta \in \Delta_{\ell-1}/\Delta_\ell$ and passing to the limit on the both sides, we see that the extension $\nu_-^\dagger$ is evenly minimal if and only if we have the following equality of graded algebras:

$$\mathrm{gr}_{\nu_\ell} \left(\varinjlim_{R'} \frac{R'}{P_\ell'} \right) \otimes_{R'} \kappa(\tilde H_i') = \mathrm{gr}_{\nu_{i0}^\dagger} \left(\varinjlim_{R'} \frac{R_{\tilde H_i'}'^\dagger}{\tilde H_{i-1}' R_{\tilde H_i'}'^\dagger} \right).$$

EXAMPLES.   The extension $\hat{\nu}$ of Example 5 (p. 15) is minimal, but not tight. The valuation ν admits a unique tight extension $\hat{\nu}_2 \circ \hat{\nu}_3$ to $\varinjlim_{R'}(\hat{R}'/H'_1)$; the valuation $\hat{\nu}$ is the composition of the discrete rank 1 valuation $\hat{\nu}_1$, centred in $\varinjlim_{R'} \hat{R}'_{H'_1}$ with $\hat{\nu}_2 \circ \hat{\nu}_3$.

The extension $\hat{\nu}^{(1)}$ of Example 6 (p. 16) is minimal. The extension $\hat{\nu}^{(2)}$ is evenly minimal but not minimal. Neither $\hat{\nu}^{(1)}$ nor $\hat{\nu}^{(2)}$ is tight. The valuation ν admits a unique tight extension $\hat{\nu}_2 \circ \hat{\nu}_3$ to $\varinjlim_{R'}(R'/\tilde{H}'_1)$, where $\tilde{H}'_1 = (y - \sum_{j=1}^{\infty} c_j x^j)$; the valuation $\hat{\nu}^{(2)}$ is the composition of the discrete rank 1 valuation $\hat{\nu}_1$, centred in $\varinjlim_{R'} \hat{R}'_{\tilde{H}'_1}$ with $\hat{\nu}_2 \circ \hat{\nu}_3$.

REMARK 12.   As of this moment, we do not know of any examples of extensions $\hat{\nu}_-$ which are not evenly minimal. Thus, formally, the question of whether every extension $\hat{\nu}_-$ is evenly minimal is open, though we strongly suspect that counterexamples do exist.

PROPOSITION 6.5.   *Let $i = 2\ell$ be even and let $\nu_{i0}^{\dagger}$ be the extension of ν_{ℓ} to $\varinjlim_{R'} \kappa(\tilde{H}'_{i-1})$, centred at the local ring $\varinjlim_{R'}(R'^{\dagger}_{\tilde{H}'_i}/\tilde{H}'_{i-1}R'^{\dagger}_{\tilde{H}'_i})$, defined by (6.9). Then*

$$k_{\nu_{i0}^{\dagger}} = \varinjlim_{R'} \kappa(\tilde{H}'_i). \tag{6.15}$$

Proof.   Take two elements $x, y \in \varinjlim_{R'}(R'^{\dagger}_{\tilde{H}'_i}/\tilde{H}'_{i-1}R'^{\dagger}_{\tilde{H}'_i})$, such that $\nu_{i0}^{\dagger}(x) = \nu_{i0}^{\dagger}(y)$. We must show that the image of x/y in $k_{\nu_{i0}^{\dagger}}$ belongs to $\varinjlim_{R'} \kappa(\tilde{H}'_i)$. Without loss of generality, we may assume that $x, y \in R^{\dagger}_{\tilde{H}_i}/\tilde{H}_{i-1}R^{\dagger}_{\tilde{H}_i}$. Let $\beta = \nu_{i0}^{\dagger}(x) = \nu_{i0}^{\dagger}(y)$. Choose $R' \in \mathcal{T}$ sufficiently far out in the direct system so that $x, y \in \mathcal{P}'_{\beta} R'^{\dagger}_{\tilde{H}'_i}/\tilde{H}'_{i-1}R'^{\dagger}_{\tilde{H}'_i}$. Let $R' \to R''$ be the blowing up of the ideal $\mathcal{P}'_{\beta} R'$. Then in $\mathcal{P}''_{\beta} R''^{\dagger}_{\tilde{H}''_i}/\tilde{H}''_{i-1}R''^{\dagger}_{\tilde{H}''_i}$ we can write

$$x = az \tag{6.16}$$

and

$$y = aw, \tag{6.17}$$

where $\nu_{i0}^{\dagger}(a) = \beta$ and $\nu_{i0}^{\dagger}(z) = \nu_{i0}^{\dagger}(w) = 0$. Let $\bar{z}$ be the image of z in $\kappa(\tilde{H}''_i)$ and similarly for $\bar{w}$. Then the image of x/y in $k_{\nu_{i0}^{\dagger}}$ equals $\bar{z}/\bar{w} \in \kappa(\tilde{H}''_i)$, and the result is proved.   □

REMARK 13.   Theorem 5.4 and the existence of the extension $\nu_{2\ell,0}^{\dagger}$ of ν_{ℓ} in the case when $\tilde{H}'_{2\ell} = H'_{2\ell}$ and $\tilde{H}'_{2\ell-1} = H'_{2\ell-1}$ guaranteed by Theorem 4.6(2) allow us to give a fairly explicit description of the totality of minimal extensions as compositions of $2r$ valuations and, in particular, to show that they always exist. Indeed, minimal extensions $\nu_-^{\dagger}$ can be constructed at will, recursively in i, as follows. Assume that the valuations $\nu_1^{\dagger}, \ldots, \nu_{i-1}^{\dagger}$ are already constructed. If i is odd, let $\nu_i^{\dagger}$ be an arbitrary valuation of the residue field $k_{\nu_{i-1}^{\dagger}}$ of the valuation ring $R_{\nu_{i-1}^{\dagger}}$. If $i = 2\ell$ is even, let $\nu_{i0}^{\dagger}$ be the extension of ν_{ℓ} to $\varinjlim_{R'} \kappa(H'_{i-1})$, centred at the local ring $\varinjlim_{R'}(R'^{\dagger}_{H'_i}/H'_{i-1}R'^{\dagger}_{H'_i})$, whose existence and uniqueness are guaranteed by Theorem 4.6(2) and Proposition 4.9, respectively. Let $\nu_i^{\dagger}$ be an arbitrary extension of $\nu_{i0}^{\dagger}$ to the field $k_{\nu_{i-1}^{\dagger}}$. It is clear that all the minimal extensions $\nu_-^{\dagger}$ of ν are obtained in this way.

In the next section, we will use this remark to show that if $R^{\dagger} = \tilde{R}$ or $R^{\dagger} = R^e$, then ν admits a unique extension to $R^{\dagger}/H_0$, which is necessarily minimal.

We end this section by giving some sufficient conditions for the uniqueness of $\nu_-^\dagger$.

PROPOSITION 6.6.   *Suppose given an admissible chain of trees* (5.2) *of prime ideals of* $R'^\dagger$. *For each* $\ell \in \{0, \ldots, r-1\}$, *consider the set of all* $R' \in \mathcal{T}$ *such that*

$$\operatorname{ht} \tilde{H}'_{2\ell+1} - \operatorname{ht} \tilde{H}'_{2\ell} \leqslant 1 \quad \text{for all even } i \tag{6.18}$$

and, in case of equality, the 1-dimensional local ring $R'^\dagger_{\tilde{H}'_{2\ell+1}} / \tilde{H}'_{2\ell} R'^\dagger_{\tilde{H}'_{2\ell+1}}$ *is unibranch* (*that is, analytically irreducible*). *Assume that for each* ℓ *the set of such* R' *is cofinal in* $\mathcal{T}$.

Let $\nu_-^\dagger$ *be an extension of* ν *such that the* $\tilde{H}'_i$ *are prime* $\nu_-^\dagger$-*ideals. Assume that* $\nu_-^\dagger$ *is evenly minimal. Then there is at most one such extension* $\nu_-^\dagger$ *and exactly one such* $\nu_-^\dagger$ *if*

$$\tilde{H}'_i = H'_i \quad \text{for all } i. \tag{6.19}$$

(*in the latter case* $\nu_-^\dagger$ *is minimal by definition*).

Proof.   By Theorem 4.6(2) and Proposition 4.9, if (6.19) holds, then $\nu_-^\dagger$ is minimal and for each even i the extension $\nu_{i0}^\dagger$ exists and is unique. Therefore, we may assume that in all the cases $\nu_-^\dagger$ is evenly minimal and that $\nu_{i0}^\dagger$ exists whenever (6.19) holds.

The valuation $\nu_-^\dagger$, if it exists, is a composition of $2r$ valuations: $\nu_-^\dagger = \nu_1^\dagger \circ \nu_2^\dagger \circ \ldots \circ \nu_{2r}^\dagger$, subject to the conditions of Theorem 5.4. We prove the uniqueness of $\nu_-^\dagger$ by induction on r. Assume the result is true for $r-1$. This means that there is at most one evenly minimal extension $\nu_3^\dagger \circ \nu_4^\dagger \circ \ldots \circ \nu_{2r}^\dagger$ of $\nu_2 \circ \nu_3 \circ \ldots \circ \nu_r$ to $\varinjlim_{R'} \kappa(\tilde{H}'_2)$, and exactly one in the case when (6.19) holds. To complete the proof of uniqueness of $\nu_-^\dagger$, it is sufficient to show that both $\nu_1^\dagger$ and $\nu_2^\dagger$ are unique and that the residue field of $\nu_2^\dagger$ equals $\varinjlim_{R'} \kappa(\tilde{H}'_2)$.

We start with the uniqueness of $\nu_1^\dagger$. If (6.10) holds, then $\nu_1^\dagger$ is the trivial valuation. Suppose, on the other hand, that equality holds in (6.18). Then the restriction of $\nu_1^\dagger$ to each $R' \in \mathcal{T}$ such that the local ring $\varinjlim_{R'} (\hat{R}'_{\tilde{H}'_1} / \tilde{H}'_0 \hat{R}'_{\tilde{H}'_1})$ is 1-dimensional and unibranch is the unique divisorial valuation centred in that ring (in particular, its residue field is $\kappa(\tilde{H}'_1)$). By the assumed cofinality of such R', the valuation $\nu_1^\dagger$ is unique and its residue field equals $\varinjlim_{R'} \kappa(\tilde{H}'_1)$. Thus, regardless of whether or not the inequality in (6.18) is strict, $\nu_1^\dagger$ is unique and we have the equality of residue fields

$$k_{\nu_1^\dagger} = \varinjlim_{R'} \kappa(\tilde{H}'_1). \tag{6.20}$$

This equality implies that $\nu_2^\dagger = \nu_{20}^\dagger$. Now, the valuation $\nu_2^\dagger = \nu_{20}^\dagger$ is uniquely determined by the conditions (6.8) and (6.9), and its residue field is

$$k_{\nu_2^\dagger} = k_{\nu_{20}^\dagger} = \varinjlim_{R'} \kappa(\tilde{H}'_2) \tag{6.21}$$

by Proposition 6.5. Furthermore, by Theorem 4.6 exactly one such $\nu_2^\dagger$ exists whenever (6.19) holds. This proves that there is at most one possibility for $\nu_-^\dagger$: the composition of $\nu_1^\dagger \circ \nu_2^\dagger$ with $\nu_3^\dagger \circ \nu_4^\dagger \circ \ldots \circ \nu_{2r}^\dagger$, and exactly one if (6.19) holds. $\qquad \square$

PROPOSITION 6.7.   *The extension* $\nu_-^\dagger$ *is tight if and only if for each* R' *in our direct system the natural graded algebra extension* $\operatorname{gr}_\nu R' \to \operatorname{gr}_{\nu_-^\dagger} R'^\dagger$ *is scalewise birational.*

REMARK 14.   Proposition 6.7 allows us to rephrase Conjecture 1.1 as follows: the valuation ν admits at least one tight extension $\nu_-^\dagger$.

Proof of Proposition 6.7. 'If' Assume that for each R' in our direct system the natural graded algebra extension $\mathrm{gr}_\nu R' \to \mathrm{gr}_{\nu_-^\dagger} R'^\dagger$ is scalewise birational. Then

$$\Gamma^\dagger = \Gamma. \tag{6.22}$$

Together with (5.4) this implies that, for each $l \in \{1, \ldots, r+1\}$, we have $\Delta_{2\ell-2}^\dagger = \Delta_{2\ell-1}^\dagger = \Delta_{\ell-1}$ under the identification (6.22). Then $\Delta_{2\ell-1}^\dagger/\Delta_{2\ell}^\dagger = (0)$, so for all odd i the valuation $\nu_i^\dagger$ is trivial. This proves the equality (6.10) in the definition of tight. It remains to show that $\nu_-^\dagger$ is evenly minimal. We will prove the even minimality in the form of equality (6.14) for each $\bar\beta \in \Delta_{\ell-1}/\Delta_\ell$. The right-hand side of (6.14) is trivially contained on the left-hand side; we must prove the opposite inclusion. To do that, take a non-zero element $x \in \mathcal{P}_{\bar\beta}^\dagger/\mathcal{P}_{\bar\beta+}^\dagger$. By scalewise birationaliy, there exist non-zero elements $\bar y, \bar z \in \mathrm{gr}_\nu R$, with $\mathrm{ord}\,\bar y, \mathrm{ord}\,\bar z \in \Delta_\ell$, such that $x\bar y = \bar z$. Let y be a representative of $\bar y$ in R, and similarly for z. Let $R \to R'$ be the local blowing up with respect to ν along the ideal (y, z). Then, in the notation of (6.14), we have

$$x = \psi'^{-1}\left((\phi' \circ \lambda')\left(\frac{\bar z}{\bar y} \otimes_{R'} 1\right)\right) \in \psi'^{-1}\left((\phi' \circ \lambda')\left(\frac{\mathcal{P}_{\bar\beta}'}{\mathcal{P}_{\bar\beta+}'} \otimes_{R'} \kappa(\tilde H_i')\right)\right). \tag{6.23}$$

This proves (6.14). 'If' is proved.

'Only if'. Assume that $\nu_-^\dagger$ is tight (that is, it is evenly minimal and (6.10) holds) and take $R' \in \mathcal{T}$. Then the valuation $\nu_{2\ell+1}$ is trivial for all ℓ, so $\nu_-^\dagger = \nu_2^\dagger \circ \nu_4^\dagger \circ \ldots \circ \nu_{2r}^\dagger$. We must show that the graded algebra extension $\mathrm{gr}_\nu R' \to \mathrm{gr}_{\nu_-^\dagger} R'^\dagger$ is scalewise birational. Again, we use induction on r. Take an element $x \in R'^\dagger$. If $\nu_-^\dagger(x) \in \Delta_1$, then $\mathrm{in}_{\nu_-^\dagger} x \in \mathrm{gr}_{\nu_4^\dagger \circ \ldots \circ \nu_{2r}^\dagger}(R'^\dagger/\tilde H_2')$, hence by the induction assumption there exists $y \in R'$ with $\nu_-^\dagger(x) \in \Delta_1$ and $\mathrm{in}_{\nu_-^\dagger}(xy) \in \mathrm{gr}_\nu(R'/P_1)$. In this case, there is nothing more to prove. Thus we may assume that $\nu_-^\dagger(x) \notin \Delta_1$. It remains to show that there exists $y \in R'$ such that $\mathrm{in}_{\nu_-^\dagger}(xy) \in \mathrm{gr}_\nu R'$. Since the natural map sending each element of the ring to its image in the graded algebra behaves well with respect to multiplication and division, local blowings up induce birational transformations of graded algebras, and it is enough to find a local blowing up $R'' \in \mathcal{T}(R')$ and $y \in R''$ such that $\mathrm{in}_{\nu_-^\dagger}(xy) \in \mathrm{gr}_\nu R''$.

Now, Proposition 6.5 shows that there exists a local blowing up $R' \to R''$ such that $x = az$ (6.16), with $z \in R''$ and $\nu_2^\dagger(a) = \nu_{2,0}^\dagger(a) = 0$. The last equality means that $\nu_-^\dagger(a) \in \Delta_1$, and the result follows from the induction assumption, applied to a. □

The argument above also shows the following. Let $\Phi'^\dagger = \nu_-^\dagger(R'^\dagger \setminus \{0\})$, take an element $\beta \in \Phi'^\dagger$ and let $\mathcal{P}'_\beta^\dagger$ denote the $\nu_-^\dagger$-ideal of $R'^\dagger$ of value β.

COROLLARY 6.8.   *Take an element* $x \in \mathcal{P}'_\beta^\dagger$. *There exists a local blowing up* $R' \to R''$ *such that* $\beta \in \nu(R'') \setminus \{0\}$ *and* $x \in \mathcal{P}''_\beta R''_i^\dagger$.

The following proposition gives a sufficient condition for the uniqueness of $\nu_-^\dagger$ (this result is due to Heinzer and Sally [**5**]).

PROPOSITION 6.9.   *Assume that* K' *is algebraic over* K *for all* $R' \in \mathcal{T}$ *and that the following conditions hold*:

(1) $\mathrm{ht}\, H_1' \leqslant 1$;
(2) $\mathrm{ht}\, H_1' + \mathrm{rat.rk}\,\nu = \dim R'$, *where* R' *is taken to be sufficiently far out in the direct system.*

Let $\nu_-^\dagger$ be an extension of ν to a ring of the form $\varinjlim_{R'}(R'^\dagger/\tilde{H}_0')$. Then either

$$\tilde{H}_0' = H_0' \tag{6.24}$$

or

$$\tilde{H}_0' = H_1'. \tag{6.25}$$

The valuation ν admits a unique extension to $\varinjlim_{R'}(R'^\dagger/H_0')$ and a unique extension to $\varinjlim_{R'}(R'^\dagger/H_1')$. The first extension is minimal and the second is tight.

Proof. For $1 \leqslant \ell \leqslant r$, let r_ℓ denote the rational rank of ν_ℓ. Let $\nu_-^\dagger$ be an extension of ν to a ring of the form $\varinjlim_{R'}(R'^\dagger/\tilde{H}_0')$, where $\tilde{H}_0'$ is a tree of prime ideals of $R'^\dagger$ such that $\tilde{H}_0' \cap R' = (0)$.

By Corollary 6.3 $\operatorname{ht} H_i'$ stabilizes for $1 \leqslant i \leqslant 2r$ and R' sufficiently far out in the direct system. From now on, we will assume that R' is chosen sufficiently far so that the stable value of $\operatorname{ht} H_i'$ is attained. Now, let $i = 2\ell$. The valuation $\nu_{i0}^\dagger$ is centred in the local noetherian ring $R'^\dagger_{\tilde{H}_i'}/\tilde{H}_{i-1}' R'^\dagger_{\tilde{H}_i'}$, hence by Abhyankar's inequality

$$\operatorname{rat.rk} \nu_{i0}^\dagger \leqslant \dim \frac{R'^\dagger_{\tilde{H}_i'}}{\tilde{H}_{i-1}' R'^\dagger_{\tilde{H}_i'}} \leqslant \operatorname{ht} \tilde{H}_i' - \operatorname{ht} \tilde{H}_{i-1}'. \tag{6.26}$$

Since this inequality is true for all even i, summing over all i we obtain:

$$\dim R' = \dim R'^\dagger = \sum_{i=1}^{2r}(\operatorname{ht} \tilde{H}_i' - \operatorname{ht} \tilde{H}_{i-1}') \geqslant \operatorname{ht} \tilde{H}_1' + \sum_{\ell=1}^{r}(\operatorname{ht} \tilde{H}_{2\ell}' - \operatorname{ht} \tilde{H}_{2\ell-1}')$$

$$\geqslant \operatorname{ht} H_1' + \sum_{\ell=1}^{r} \operatorname{rat.rk} \nu_{2\ell,0}^\dagger \geqslant \operatorname{ht} H_1' + \sum_{\ell=1}^{r} r_\ell = \operatorname{ht} H_1' + \operatorname{rat.rk} \nu = \dim R'. \tag{6.27}$$

Hence all the inequalities in (6.26) and (6.27) are equalities. In particular, we have

$$\operatorname{ht} \tilde{H}_1' = \operatorname{ht} H_1';$$

combined with Proposition 5.3 this shows that

$$\tilde{H}_1' = H_1'. \tag{6.28}$$

Together with the hypothesis (1) of Proposition 6.9, this already proves that at least one of (6.24)–(6.25) holds. Furthermore, equalities in (6.26) and (6.27) prove that

$$\operatorname{ht} \tilde{H}_i' = \operatorname{ht} \tilde{H}_{i-1}'$$

for all odd $i > 1$, so that

$$\tilde{H}_i' = \tilde{H}_{i-1}' \quad \text{whenever } i > 1 \text{ is odd} \tag{6.29}$$

and that

$$r_i = \operatorname{ht} \tilde{H}_i' - \operatorname{ht} \tilde{H}_{i-1}' \tag{6.30}$$

whenever i is even.

Now, consider the special case when $\tilde{H}_i' = H_i'$ for $i \geqslant 1$ and $\tilde{H}_0'$ is as in (6.24)–(6.25). According to Proposition 4.9 for each even $i = 2\ell$ there exists a unique extension $\nu_{i0}^\dagger$ of ν_l to a valuation of $\varinjlim_{R'} \kappa(H_{i-1}')$, centred in the local ring $\varinjlim_{R'}(R'^\dagger_{H_{2\ell}'}/H_{2\ell-1}')$. Moreover, we have

$$k_{\nu_{2\ell,0}^\dagger} = \varinjlim_{R'} \kappa(H_{2\ell}') \tag{6.31}$$

by Remark 9. By Theorem 5.4, there exists an extension $\nu_-^\dagger$ of ν to $\varinjlim_{R'}(R'^\dagger/\tilde{H}_0')$ such that the $\{\tilde{H}_i'\}$ as above is the chain of trees of prime ideals, determined by $\nu_-^\dagger$. In particular, (6.29) and (6.30) hold with $\tilde{H}_i'$ replaced by H_i'.

Now (6.30) and Proposition 5.3 imply that, for any extension $\nu_-^\dagger$, we have $\tilde{H}_i' = H_i'$ for $i > 0$, so that the special case above is, in fact, the only case possible. Furthermore, by (6.29) we have $H_{2\ell+1}' = H_{2\ell}'$ for all $\ell \in \{1,\ldots,r\}$. This implies that for all such ℓ the valuation $\nu_{2\ell+1,0}^\dagger$ is the trivial valuation of $\varinjlim_{R'} \kappa(H_{2\ell}')$; in particular,

$$k_{\nu_{2\ell+1,0}^\dagger} = \varinjlim_{R'} \kappa(H_{2\ell}'), \tag{6.32}$$

for all $\ell \in \{1,\ldots,r-1\}$.

If $\tilde{H}_0' = H_1' = \tilde{H}_1'$, then the only possibility for $\nu_{10}^\dagger = \nu_1^\dagger$ is the trivial valuation of $\varinjlim_{R'} \kappa(H_1')$; we have

$$k_{\nu_1^\dagger} = k_{\nu_{10}^\dagger} = \varinjlim_{R'} \kappa(H_1'). \tag{6.33}$$

If $\tilde{H}_0' = H_0'$, then by the hypothesis (1) of Proposition 6.9 and the excellence of R the ring $R'^\dagger_{H_1'}/H_0'R'^\dagger_{H_1'}$ is a regular 1-dimensional local ring (in particular, unibranch), hence the valuation $\nu_1^\dagger = \nu_{10}^\dagger$ centred at $\varinjlim_{R'}(R'^\dagger_{H_1'}/H_0'R'^\dagger_{H_1'})$ is unique and (6.33) holds also in this case.

By induction on i, it follows from (6.31) and (6.32), the uniqueness of $\nu_{2\ell,0}^\dagger$ and the triviality of $\nu_{2\ell+1,0}^\dagger$ for $\ell \geq 1$ that $\nu_i^\dagger$ is uniquely determined for all i and $k_{\nu_i^\dagger} = \varinjlim_{R'} \kappa(H_i')$. This proves that in both cases (6.24) and (6.25) the valuation $\nu_-^\dagger = \nu_1^\dagger \circ \ldots \circ \nu_{2r}^\dagger$ is unique. The last statement of Proposition 6.9 is immediate from definitions. $\qquad\square$

A related necessary condition for the uniqueness of $\nu_-^\dagger$ will be proved in §9.

7. *Extending a valuation centred in an excellent local domain to its henselization*

Let $\tilde{R}$ denote the henselization of R, as above. The completion homomorphism $R \to \hat{R}$ factors through the henselization: $R \to \tilde{R} \to \hat{R}$. In this section, we will show that H_1 is a minimal prime of $\tilde{R}$, that ν extends uniquely to a valuation $\tilde{\nu}_-$ of rank r centred at $\tilde{R}/H_1$, and that H_1 is the unique prime ideal P of $\tilde{R}$ such that ν extends to a valuation of $\tilde{R}/P$. Furthermore, we will prove that $H_{2\ell+1}$ is a minimal prime of $P_\ell\tilde{R}$ for all ℓ and that these are precisely the prime $\tilde{\nu}$-ideals of $\tilde{R}$.

Studying the implicit prime ideals of $\tilde{R}$ and the extension of ν to $\tilde{R}$ is a logical intermediate step before attacking the formal completion, for the following reason. As we will show in the next section, if R is already henselian in (3.4), then $\mathcal{P}_\beta'\hat{R}_{H_{2\ell+1}'}' \cap \hat{R} = \mathcal{P}_\beta\hat{R}$ for all β and R' and thus we have $H_{2\ell+1} = \bigcap_{\beta \in \Delta_\ell}(\mathcal{P}_\beta\hat{R})$.

We state the main result of this section. In the case when R^e is an étale extension of R, contained in $\tilde{R}$, we use (3.6) with $R^\dagger = R^e$ as our definition of the implicit prime ideals.

THEOREM 7.1. *Let R^e be a local étale extension of R, contained in $\tilde{R}$. Then we have the following.*

(1) *The ideal $H_{2\ell+1}$ is prime for $0 \leq l \leq r$; it is a minimal prime of $P_\ell R^e$. In particular, H_1 is a minimal prime of R^e. We have $H_{2\ell} = H_{2\ell+1}$ for $0 \leq l \leq r$.*

(2) *The ideal H_1 is the unique prime P of R^e such that there exists an extension ν_-^e of ν to R^e/P; the extension ν_-^e is unique. The graded algebra $\mathrm{gr}_{\nu_-^e}(R^e/H_1)$ is scalewise birational to $\mathrm{gr}_\nu R$; in particular, $\mathrm{rk}\,\nu_-^e = r$.*

(3) *The ideals $H_{2\ell+1}$ are precisely the prime ν_-^e-ideals of R^e.*

Proof. By assumption, the ring R^e is a direct limit of local, strict étale extensions of R which are essentially of finite type. All the assertions (1)–(3) behave well under taking direct limits, so it is sufficient to prove Theorem 7.1 in the case when R^e is essentially of finite type over R. From now on, we will restrict attention to this case.

The next step is to describe explicitly those local blowings up $R \to R'$ for which R' is ℓ-stable. Their interest to us is that, according to Proposition 4.2, if R' is ℓ-stable, then for all $R'' \in \mathcal{T}(R')$ and all $\beta \in \Delta_\ell / \Delta_{\ell+1}$, we have the equality

$$\mathcal{P}_\beta''(R'' \otimes_R R^e) \cap R^e = \mathcal{P}_\beta R^e; \tag{7.1}$$

in particular, the limit in (3.6) is attained, that is, we have the equality

$$H_{2\ell+1} = \bigcap_{\beta \in \Delta_\ell} ((\mathcal{P}_\beta'(R^e \otimes_R R'))_{M'}) \cap R^e). \tag{7.2}$$

LEMMA 7.2.   *Let $R/P_\ell \to T$ be a finitely generated extension of R/P_ℓ, contained in $R_\nu/\mathbf{m}_\ell$. Let*

$$\mathbf{q} = \frac{\mathbf{m}_\nu}{\mathbf{m}_\ell} \cap T.$$

There exists a ν-extension $R \to R'$ of R such that $R'/P_\ell' = T_\mathbf{q}$.

Proof. Write $T = (R/P_\ell)[\bar{a}_1, \ldots, \bar{a}_k]$, with $\bar{a}_i \in R_\nu/\mathbf{m}_\ell$, that is, $\nu_{\ell+1}(\bar{a}_i) \geqslant 0$, $1 \leqslant i \leqslant k$. We can lift the $\bar{a}_i$ to elements a_i in R_ν such that $\nu(a_i) \geqslant 0$. Let us consider the ring $R'' = R[a_1, \ldots, a_k] \subset R_\nu$ and its localization $R' = R''_{\mathbf{m}_\nu \cap R''}$. The ideal P_ℓ' is the kernel of the natural map $R' \to R_\nu/\mathbf{m}_\ell$. Thus, both R'/P_ℓ' and $T_\mathbf{q}$ are equal to the R/P_l-subalgebra of $R_\nu/\mathbf{m}_\ell$, obtained by adjoining $\bar{a}_1, \ldots, \bar{a}_k$ to R/P_l inside $R_\nu/\mathbf{m}_\ell$ and then localizing at the preimage of the ideal $\mathbf{m}_\nu/\mathbf{m}_\ell$. This proves Lemma 7.2. $\square$

Let us now go back to our étale extension $R \to R^e$.

LEMMA 7.3.   *Fix an integer $l \in \{0, \ldots, r\}$. There exists a local blowing up $R \to R'$ along ν having the following property: let P_ℓ' denote the ℓth prime ν-ideal of R'. Then the ring R'/P_ℓ' is analytically irreducible; in particular, $R'/P_\ell' \otimes_R R^e$ is an integral domain.*

REMARK 15.   We are not claiming that there exists $R' \in \mathcal{T}$ such that R'/P_ℓ' is analytically irreducible for all ℓ (and we do not know how to prove such a claim), only that, for each ℓ, there exists an R', which may depend on ℓ, such that R'/P_ℓ' is analytically irreducible. On the other hand, below we will prove that there exists an ℓ-stable $R' \in \mathcal{T}$. According to Definition 4.1(2) and Proposition 4.1, such a stable R' has the property that $\kappa(P_\ell'') \otimes_R (R'' \otimes_R R^e)_{M''}$ is a domain for all $R'' \in \mathcal{T}(R')$. For a given R'', this property is weaker than the analytic irreducibility of R''/P_ℓ''. The latter is equivalent to saying that $\kappa(P_\ell'') \otimes_R (R'' \otimes_R R^\sharp)_{M''}$ is a domain for every local étale extension $R^\sharp$ of R''.

Proof of Lemma 7.3. Since R is an excellent local ring, every homomorphic image of R is Nagata [**10**, (Theorems 72 (31.H), 76 (33.D) and 78 (33.H))]. Let $\pi : R/P_\ell \to S$ be the normalization of R/P_ℓ. Then S is a finitely generated R/P_ℓ-algebra contained in $R_\nu/\mathbf{m}_\ell$, to which we can apply Lemma 7.2. We obtain a ν-extension $R \to R'$ such that the ring $R'/P_\ell' \cong$

$R'/P_\ell R'$ is a localization of S at a prime ideal, hence it is an excellent normal local ring. In particular, it is analytically irreducible [**11**, Theorem (43.20), p. 187 and Corollary (44.3), p. 189], as desired. $\qquad\square$

Next, we fix $\ell \in \{0, \ldots, r\}$ and study the ring $(T')^{-1}(\kappa(P'_\ell) \otimes_R R^e)$, in particular, the structure of the set of its zero divisors, as R' runs over $\mathcal{T}(R)$ (here T' is as in Remark 6). Since R^e is separable algebraic, essentially of finite type over R, the ring $(T')^{-1}(\kappa(P'_\ell) \otimes_R R^e)$ is finite over $\kappa(P'_\ell)$; this ring is reduced, but it may contain zero divisors. In fact, it is a direct product of fields which are finite separable extensions of $\kappa(P'_\ell)$ because R^e is separable and essentially of finite type over R.

Consider a chain $R \to R' \to R''$ of ν-extensions in $\mathcal{T}$. Let

$$\kappa(P_\ell) \otimes_R R^e = \prod_{j=1}^{n} K_j, \tag{7.3}$$

$$(T')^{-1}(\kappa(P'_\ell) \otimes_R R^e) = \prod_{j=1}^{n'} K'_j, \tag{7.4}$$

$$(T'')^{-1}(\kappa(P''_\ell) \otimes_R R^e) = \prod_{j=1}^{n''} K''_j \tag{7.5}$$

be the corresponding decompositions as products of finite field extensions of $\kappa(P_\ell)$ (resp. $\kappa(P'_\ell)$, resp. $\kappa(P''_\ell)$). We want to compare $(T')^{-1}(\kappa(P'_\ell) \otimes_R R^e)$ with $(T'')^{-1}(\kappa(P''_\ell) \otimes_R R^e)$.

REMARK 16.   The ring $\kappa(P'_\ell) \otimes_R R^e$ is itself a direct product of finite extensions of $\kappa(P'_\ell)$; say $\kappa(P'_\ell) = \prod_{j \in S'} K'_j$ for a certain set S'. In this situation, localization is the same thing as the natural projection to the product of the K'_j over a certain subset $\{1, \ldots, n'\}$ of S'. Thus the passage from $(T')^{-1}(\kappa(P'_\ell) \otimes_R R^e)$ to $(T'')^{-1}(\kappa(P''_\ell) \otimes_R R^e)$ can be viewed as follows: first, tensor each K'_j with $\kappa(P''_\ell)$ over $\kappa(P'_\ell)$; then, in the resulting direct product of fields, remove a certain number of factors.

Let $\bar{K}'_1, \ldots, \bar{K}'_{\bar{n}'}$ be the distinct isomorphism classes of finite extensions of $\kappa(P'_\ell)$ appearing among $K'_1, \ldots, K'_{n'}$, arranged in such a way that $[\bar{K}'_j : \kappa(P'_\ell)]$ is non-increasing with j, and similarly for $\bar{K}''_1, \ldots, \bar{K}''_{\bar{n}''}$.

LEMMA 7.4.   We have the inequality

$$([\bar{K}''_1 : \kappa(P''_\ell)], \ldots, [\bar{K}''_{\bar{n}''} : \kappa(P''_\ell)], n'') \leqslant ([\bar{K}'_1 : \kappa(P'_\ell)], \ldots, [\bar{K}'_{\bar{n}'} : \kappa(P'_\ell)], n') \tag{7.6}$$

in the lexicographical ordering. Furthermore, either R' is ℓ-stable or there exists $R'' \in \mathcal{T}$ such that strict inequality holds in (7.6).

Proof.   Fix a $q \in \{1, \ldots, \bar{n}'\}$ and consider the tensor product $\bar{K}'_q \otimes_R \kappa(P''_\ell)$. Since $\bar{K}'_q$ is separable over $\kappa(P'_\ell)$, the ring $\bar{K}'_q \otimes_R \kappa(P''_\ell) = \prod_{j \in S''_q} K''_j$ is a product of fields. Moreover, the following two cases are possible.

(a) There exists a non-trivial extension L of $\kappa(P'_\ell)$ which embeds both into $\kappa(P''_\ell)$ and $\bar{K}'_q$. In this case,

$$[K''_j : \kappa(P''_\ell)] < [\bar{K}'_q : \kappa(P'_\ell)] \quad \text{ for all } j \in S''_q. \tag{7.7}$$

(b) There is no field extension L as in (a). In this case $\bar{K}'_q \otimes_R \kappa(P''_\ell)$ is a field, so

$$\#S''_q = 1 \tag{7.8}$$

and

$$[K''_j : \kappa(P''_\ell)] = [\bar{K}'_q : \kappa(P'_\ell)] \quad \text{for } j \in S''_q. \tag{7.9}$$

Now, if there exists $q \in \{1, \ldots, \bar{n}'\}$ for which (a) holds, take the smallest such q. Then (7.7)–(7.9) imply that strict inequality holds in (7.6). On the other hand, if (b) holds for all $q \in \{1, \ldots, \bar{n}'\}$, then (7.8) and (7.9) imply that

$$([\bar{K}''_1 : \kappa(P''_\ell)], \ldots, [\bar{K}''_{\bar{n}''} : \kappa(P''_\ell)]) = ([\bar{K}'_1 : \kappa(P'_\ell)], \ldots, [\bar{K}'_{\bar{n}'} : \kappa(P'_\ell)]) \tag{7.10}$$

and $n'' \leqslant n'$, so again (7.6) holds.

Finally, assume that R' is not ℓ-stable. If there exists $R'' \in \mathcal{T}$ and $q \in \{1, \ldots, \bar{n}'\}$ for which (a) holds, then by the above we have strict inequality in (7.6) and there is nothing more to prove. Assume that there are no such R'' and q. Then $(T')^{-1}(\kappa(P'_\ell) \otimes_R R^e)$ is not a domain, so $n' > 1$.

Take $R'' \in \mathcal{T}(R')$ such that $(R''/P''_l \otimes_R R^e)_{M''}$ is an integral domain; such an R'' exists by Lemma 7.3. Then $n'' = 1 < n'$, as desired.   □

COROLLARY 7.5.   *There exists a stable $R' \in \mathcal{T}$. The limit in (3.6) is attained for this R'.*

Proof.   In view of Proposition 4.1, it is sufficient to prove that there exists $R' \in \mathcal{T}$ which is ℓ-stable for all $\ell \in \{0, 1, \ldots, r\}$. First, we fix $\ell \in \{0, 1, \ldots, r\}$. Lemma 7.4 implies that there exists $R' \in \mathcal{T}(R)$ which is ℓ-stable.

By Proposition 4.1, repeating the procedure above for each ℓ we can successively enlarge R' in such a way that it becomes stable.

The last statement follows from Proposition 4.2.   □

We are now in a position to prove Theorem 7.1.

By Theorem 4.6(1), $H_{2\ell-1}$ is prime. By Proposition 3.1, $H_{2\ell+1}$ maps to P_ℓ under the map $\pi^e : \operatorname{Spec} R^e \to \operatorname{Spec} R$. Since this map is étale, its fibres are 0-dimensional, which shows that $H_{2\ell+1}$ is a minimal prime of P_ℓ. This proves (1) of Theorem 7.1.

By Proposition 5.3, for $0 \leqslant i \leqslant 2r$, $\tilde{H}_i$ is a prime ideal of R^e, containing H_i. Since the fibres of π^e are 0-dimensional, we must have $\tilde{H}_i = H_i$, so $\tilde{H}_{2\ell} = \tilde{H}_{2\ell+1} = H_{2\ell} = H_{2\ell+1}$ for $0 \leqslant \ell \leqslant r$. In particular, $\tilde{H}_0 = H_1$. This shows that the unique prime $\tilde{H}_0$ of R^e such that there exists an extension ν^e_- of ν to $R^e/\tilde{H}_0$ is $\tilde{H}_0 = H_1$. Now (2) of Theorem 7.1 is given by Proposition 6.6.

(3) of Theorem 7.1 is now immediate. This completes the proof of Theorem 7.1.   □

We note the following corollary of the proof of (2) of Theorem 7.1 and Corollary 6.8. Let $\Phi^e = \nu^e_-(R^e \setminus \{0\})$, take an element $\beta \in \Phi^e$ and let $\mathcal{P}^e_\beta$ denote the ν^e_--ideal of R^e of value β.

COROLLARY 7.6.   *Take an element $x \in \mathcal{P}^e_\beta$. There exists a local blowing up $R \to R'$ such that $\beta \in \nu(R') \setminus \{0\}$ and $x \in \mathcal{P}'_\beta R'^e$.*

8.   *The Main Theorem: the primality of implicit ideals*

In this section, we study the ideals H_j for $\hat{R}$ instead of $\tilde{R}$. The main result of this section is as follows:

THEOREM 8.1.   *The ideal $H_{2\ell-1}$ is prime.*

Proof.   For the purposes of this proof, let $H_{2\ell-1}$ denote the implicit ideals of $\hat{R}$ and $\tilde{H}_{2\ell-1}$ the implicit prime ideals of the henselization $\tilde{R}$ of R.

Let S be a local domain. By [**11**, (Theorem (43.20), p. 187)], there exists bijective maps between the set of minimal prime ideals of the henselization $\tilde{S}$ and the maximal ideals of the normalization S^n. If, in addition, S is excellent, these two sets also admit a natural bijection to the set of minimal primes of $\hat{S}$ (see [**11**, (Corollary (44.3), p. 189)]). If S is a henselian local domain, its only minimal prime is the (0) ideal, hence by the above the same is true of $\hat{S}$. Thus $\hat{S}$ is also a domain.

This shows that any excellent henselian local domain is analytically irreducible, hence $\tilde{H}_{2\ell-1}\hat{R}$ is prime for all $\ell \in \{1, \ldots, r+1\}$. Let $\tilde{\nu}_-$ denote the unique extension of ν to $\tilde{R}/\tilde{H}_1$, constructed in the previous section. Let $H^*_{2\ell-1} \subset \tilde{R}/\tilde{H}_1$ denote the implicit ideals associated to the henselian ring $\tilde{R}/\tilde{H}_1$ and the valuation $\tilde{\nu}_-$.

CLAIM.   We have $H^*_{2\ell-1} = H_{2\ell-1}/\tilde{H}_1$.

Proof of the claim.   For $\beta \in \Gamma$, let $\tilde{P}_\beta$ denote the $\tilde{\nu}_-$-ideal of $\tilde{R}/\tilde{H}_1$ of value β. For all β, we have $P_\beta/\tilde{H}_1 \subset \tilde{P}_\beta$, and the same inclusion holds for all the local blowings up of R, hence $H_{2\ell-1}/\tilde{H}_1 \subset H^*_{2\ell-1}$. To prove the opposite inclusion, we may replace $\tilde{R}$ by a finitely generated strict étale extension R^e of R. Now let $\Phi^e = \nu^e_-(R^e \setminus \{0\})$ and take an element $\beta \in \Phi^e \cap \Delta_{\ell-1}$. By Corollary 7.6, there exists a local blowing up $R \to R'$ such that $x \in P'_\beta R'^e$. Letting β vary over $\Phi^e \cap \Delta_{\ell-1}$, we obtain that if $x \in H^*_{2\ell-1}$, then $x \in H_{2\ell-1}/\tilde{H}_1$, as desired. This completes the proof of the claim. $\square$

The claim shows that replacing R by $\tilde{R}/\tilde{H}_1$ in Theorem 8.1 does not change the problem. In other words, we may assume that R is a henselian domain and, in particular, that $\hat{R}$ is also a domain. Similarly, the ring $R/P_\ell \otimes_R \hat{R} \cong \hat{R}/P_\ell$ is a domain, hence so is its localization $\kappa(P_\ell) \otimes_R \hat{R}$.

Since R is a henselian excellent ring, it is algebraically closed in $\hat{R}$ (see [**11**, Corollary (44.3), p. 189] and Corollary A.4 of the appendix); of course, the same holds for R/P_ℓ for all ℓ. Then $\kappa(P_\ell)$ is algebraically closed in $\kappa(P_\ell) \otimes_R \hat{R}$. This shows that the ring R is stable. Now Theorem 8.1 follows from Theorem 4.6. This completes the proof of Theorem 8.1. $\square$

9.   *Towards a proof of Conjecture* 1.1, *assuming local uniformization in lower dimension*

Let the notation be as in the previous sections. In this section, we assume that the Local Uniformization Theorem holds and propose an approach to proving Conjecture 1.1. We prove a Corollary of Conjecture 1.1 which gives a sufficient condition for $\hat{\nu}_-$ to be unique, which also turns out to be necessary under the additional assumption that $\hat{\nu}_-$ is minimal. We will assume that all the $R' \in \mathcal{T}$ are birational to each other, so that all the fraction fields $K' = K$ and the homomorphisms $R' \to R''$ are local blowings up with respect to ν. Finally, we assume that R contains a field k_0 and a local k_0-subalgebra S essentially of finite type, over which R is strictly étale. In particular, all the rings in sight are equicharacteristic.

First, we state the Local Uniformization Theorem in the precise form in which we are going to use it.

DEFINITION 9.1.   We say that *the embedded Local Uniformization theorem holds in $\mathcal{T}$* if the following conditions are satisfied.

Take an integer $\ell \in \{1, \ldots, r-1\}$. Let $\mu_{\ell+1} := \nu_{\ell+1} \circ \nu_{\ell+2} \circ \ldots \circ \nu_r$. Consider a tree $\{H'\}$ of prime ideals of $\hat{R}'/P'_\ell$ such that $H' \cap R'/P'_\ell = (0)$ and a tight extension $\hat{\mu}_{2\ell+2}$ of $\mu_{\ell+1}$ to $\varinjlim_{R'}(\hat{R}'/H')$.

(1) There exists a local blowing up $\pi : R \to R'$ in $\mathcal{T}$, which induces an isomorphism at the centre of ν_ℓ, such that R'/P'_ℓ is a regular local ring.

(2) Assume that R'/P'_ℓ is a regular local ring. Then there exists in $\mathcal{T}$ a sequence $\pi : R \to R'$ of local blowings up along non-singular centres not containing the centre of ν_l such that $\hat{R}'/H'$ is a regular local ring.

It is well known [**1**, **9**, **16**] that the embedded Local Uniformization Theorem holds if R is an excellent local domain such that either char $k = 0$ or dim $R \leqslant 3$ (to be precise, (1) of Definition 9.1 is well known and (2) is an easy consequence of known results While the Local Uniformization Theorem in full generality is still an open problem, it is widely believed to hold for arbitrary quasi-excellent local domains. Proving this is an active field of current research in algebraic geometry. Proving local uniformization for rings of arbitrary characteristic is one of the intended applications of Conjecture 1.1. Note that in Definition 9.1 we require only local uniformization of rings of dimension strictly less than dim R; the idea is to use induction on dim R to prove local uniformization of rings of dimension dim R.

We begin by stating a strengthening of Conjecture 1.1 (using Remark 14):

CONJECTURE 9.1. The valuation ν admits at least one tight extension $\hat{\nu}_-$. This tight extension $\hat{\nu}_-$ can be chosen to have the following additional property: for rings R' sufficiently far in the tree $\mathcal{T}$ we have the equality of semigroups $\hat{\nu}_-(\hat{R}'/\tilde{H}'_0 \setminus \{0\}) = \nu(R' \setminus \{0\})$ and for $\beta \in \nu(R' \setminus \{0\})$ the $\hat{\nu}_-$-ideal of value β is $\mathcal{P}_\beta \hat{R}'/\tilde{H}'_0$. In particular, we have the equality of graded algebras $\mathrm{gr}_\nu R' = \mathrm{gr}_{\hat{\nu}_-}(\hat{R}'/\tilde{H}'_0)$.

Below, we give an explicit construction of a valuation $\hat{\nu}_-$ whose existence is asserted in the Conjecture by describing the trees of ideals $\tilde{H}'_i$, $0 \leqslant i \leqslant 2r$ and, for each i, a valuations $\hat{\nu}_i$ of the residue field $k_{\nu_{i-1}}$, such that $\hat{\nu}_- = \hat{\nu}_1 \circ \ldots \hat{\nu}_{2r}$. More precisely, for $\ell \in \{0, \ldots, r-1\}$, we will construct, recursively in the descending order of ℓ, a tree $J'_{2\ell+1}$ of prime ideals of $\hat{R}'/H'_{2\ell}$, $R' \in \mathcal{T}$, such that $J'_{2\ell+1} \cap R'/P_\ell = (0)$, and an extension $\hat{\mu}_{2\ell+2}$ of $\mu_{\ell+1}$ to $\varinjlim_{R' \in \mathcal{T}}(\hat{R}'/J'_{2\ell+1}\hat{R}')$; the valuation $\hat{\mu}_2$ will be our candidate for the desired tight extension $\hat{\nu}_-$ of $\mu_1 = \nu$. Unfortunately, two steps in this construction still remain conjectural, namely, proving that $\hat{\mu}_{2\ell+2}$ is, indeed, a valuation, and that it is tight (this is essentially the content of Conjectures 9.2 and 9.3). Once these conjectures are proved, our recursive construction will be complete and Conjecture 9.1 will follow by setting $\hat{\nu}_- = \hat{\mu}_2$.

Let us now describe the construction in detail. According to Corollary 6.3, we may assume that ht H'_i is constant for each i after replacing R by some other ring sufficiently far in $\mathcal{T}$. From now on, we will make this assumption without always stating it explicitly.

By (1) of Definition 9.1, applied successively to the trees of ideals

$$P'_\ell \subset R', \quad \ell \in \{1, \ldots, r-1\},$$

there exists $R'' \in \mathcal{T}$ such that R''/P''_ℓ is regular for all $\ell \in \{1, \ldots, r-1\}$. Without loss of generality, we may also assume that R'' is stable.

For $\ell \in \{1, \ldots, r-1\}$ and $R'' \in \mathcal{T}$, let $\mathcal{T}_\ell(R'')$ denote the subtree of $\mathcal{T}$, consisting of all the local blowings up of R'' along ideals not contained in P''_ℓ (such local blowings up induce an isomorphism at the point $P''_\ell \in \mathrm{Spec}\, R''$). Below, we will sometimes work with trees of rings

and ideals indexed by $\mathcal{T}_\ell(R'')$ for suitable ℓ and R'' (instead of trees indexed by all of $\mathcal{T}$); the precise tree with which we are working will be specified in each case.

For $\ell = r - 1$, we define $J'_{2r-1} := H'_{2r-1}$ and $\hat{\mu}_{2r} := \hat{\nu}_{2r,0}$; according to Proposition 4.9, $\hat{\nu}_{2r,0} = \hat{\nu}_{2r}$ is the unique extension of ν_r to $\varinjlim_{R'}(\hat{R}'/H'_{2r-1}\hat{R}')$.

Next, assume that $\ell \in \{1, \ldots, r - 1\}$, that the tree $J'_{2\ell+1}$ of prime ideals of $\hat{R}'/H'_{2\ell}\hat{R}'$ and a tight extension $\hat{\mu}_{2\ell+2}$ of $\mu_{\ell+1}$ to $\varinjlim_{R'}(\hat{R}'/J'_{2\ell+1})$ are already constructed for $R' \in \mathcal{T}$ and that $J'_{2\ell+1} \cap R'/P_\ell = (0)$. It remains to construct the ideals $J'_{2\ell-1} \subset \hat{R}'/H'_{2\ell-2}\hat{R}'$ and a tight extension $\hat{\mu}_{2\ell}$ of μ_ℓ to $\varinjlim_{R'}(\hat{R}'/J'_{2\ell-1})$ for $R' \in \mathcal{T}$.

We will assume, inductively, that for all $R' \in \mathcal{T}$ the quantity $\operatorname{ht} J'_{2\ell+1}$ is constant and the following conditions hold.

(1) We have the equality of semigroups $\hat{\mu}_{2\ell+2}(\hat{R}'/J'_{2\ell+1} \setminus \{0\}) \cong \mu_{\ell+1}(R'/P'_\ell \setminus \{0\})$.

(2) For all $\beta \in \mu_{\ell+1}(R'/P'_\ell \setminus \{0\})$ the $\hat{\mu}_{2\ell+2}$-ideal of $\hat{R}'/J'_{2\ell+1}$ of value β is the extension to $\hat{R}'/J'_{2\ell+1}$ of the $\mu_{\ell+1}$-ideal of R'/P'_ℓ of value β.

(3) In particular, we have a canonical isomorphism $\operatorname{gr}_{\hat{\mu}_{2\ell+2}}(\hat{R}'/J'_{2\ell+1}) \cong \operatorname{gr}_{\mu_{\ell+1}}(R'/P'_\ell)$ of graded algebras.

By (2) of Definition 9.1 applied to the prime ideals $J'_{2\ell+1} \subset \hat{R}'/H'_{2\ell}\hat{R}'$, there exists $R' \in \mathcal{T}$ such that both R'/P'_ℓ and $\hat{R}'/J'_{2\ell+1}$ are regular. The fact that R'/P'_ℓ is regular implies that so is $\hat{R}'/P'_\ell\hat{R}'$. In particular, $\hat{R}'/P'_\ell\hat{R}'$ is a domain, so $H'_{2\ell} = P'_\ell\hat{R}'$. Take a regular system of parameters

$$\bar{u}' = (\bar{u}'_1, \ldots, \bar{u}'_{n_\ell})$$

of R'/P'_ℓ. Let k' denote the common residue field of R', $R'/P'_{\ell-1}$ and R'/P'_ℓ. Fix an isomorphism $R'/P'_\ell \cong k'[\![\bar{u}']\!]$. Renumbering the variables, if necessary, we may assume that there exists $s_\ell \in \{1, \ldots, n_\ell\}$ such that $\bar{u}'_1, \ldots, \bar{u}'_{s_\ell}$ are k'-linearly independent modulo $(m'^2 + J'_{2\ell+1})\hat{R}'/H'_{2\ell}$. Since $\hat{R}'/J'_{2\ell+1}$ is regular, the ideal $J'_{2\ell+1}$ is generated by a set of the form $\bar{v}' = (\bar{v}'_{s_\ell+1}, \ldots, \bar{v}'_{n_\ell})$, where

$$\bar{v}'_j = \bar{u}'_j - \bar{\phi}_j(\bar{u}'_1, \ldots, \bar{u}'_{s_\ell}), \ \bar{\phi}_j(\bar{u}'_1, \ldots, \bar{u}'_{s_\ell}) \in k'[\![\bar{u}'_1, \ldots, \bar{u}'_{s_\ell}]\!].$$

Let $\bar{w}' = (\bar{w}'_1, \ldots, \bar{w}'_{s_\ell}) = (\bar{u}'_1, \ldots, \bar{u}'_{s_\ell})$. Let z' be a minimal set of generators of $P'_\ell/P'_{\ell-1}$. Let k'_0 be a quasi-coefficient field of R' (that is, a subfield of R' over which k' is formally étale; such a quasi-coefficient field exists by [**10**], moreover, since R' is algebraic over a finite type algebra over a field by hypotheses, k' is finite over k'_0). By the hypotheses on R and since R'/P'_ℓ is a regular local ring and $\bar{u}'$ is a minimal set of generators of its maximal ideal m'/P'_ℓ, there exists an ideal $I \subset k'_0[z']$ such that $R'/P'_{\ell-1}$ is an étale extension of $k'_0[z', \bar{u}']_{(z', \bar{u}')}/I$. By assumptions, we have $\operatorname{ht} P'_{\ell-1} < \operatorname{ht} P'_\ell$, so $0 < \operatorname{ht} P'_\ell - \operatorname{ht} P'_{\ell-1} = \operatorname{ht}(z') - \operatorname{ht} I$, in other words,

$$\operatorname{ht} I < \operatorname{ht}(z'). \tag{9.1}$$

Next, we prove two general lemmas about ring extensions.

NOTATION. Let k_0 be a field and (S, m, k) a local noetherian k_0-algebra. For a field extension

$$k_0 \hookrightarrow L \tag{9.2}$$

such that $k \otimes_{k_0} L$ is a domain, let $S(L)$ denote the localization of the ring $S \otimes_{k_0} L$ at the prime ideal $m(S \otimes_{k_0} L)$.

LEMMA 9.1. Let k_0, (S, m, k) and L be as above. The ring $S(L)$ is noetherian.

Proof.   If the field extension (9.2) is finitely generated, Lemma 9.1 is obvious. In the general case, write $L = \lim_{\overrightarrow{i}} L_i$ as a direct limit of its finitely generated subextensions. For each L_i, let k_i denote the residue field of $S(L_i)$; k_i is nothing but the field of fractions of $k \otimes_{k_0} L_i$. Write $\hat{S} = k[\![x]\!]/H$, where x is a set of generators of m and H a certain ideal of $k[\![x]\!]$. Then $\widehat{S(L_i)} \cong k_i[\![x]\!]/Hk_i[\![x]\!]$. Given two finitely generated extensions $L_i \subset L_j$ of k_0, contained in L, we have a commutative diagram:

$$
\begin{array}{ccccc}
S(L_j) & \overset{\pi_j}{\to} & \widehat{S(L_j)} & \cong & \dfrac{k_j[\![x]\!]}{Hk_j[\![x]\!]} \\[2mm]
\psi_{ij} \uparrow & & \uparrow & & \uparrow \phi_{ij} \\[2mm]
S(L_i) & \overset{\pi_i}{\to} & \widehat{S(L_i)} & \cong & \dfrac{k_i[\![x]\!]}{Hk_i[\![x]\!]}
\end{array}
$$

where ϕ_{ij} is the map induced by the natural inclusion $k_i \hookrightarrow k_j$ and the identity map of x to itself. Let $k_\infty = \lim_{\overrightarrow{i}} k_i$. Then, for each i, we have the obvious faithfully flat map $\rho_i : \widehat{S(L_i)} \to k_\infty[\![x]\!]/Hk_\infty[\![x]\!]$, defined by the natural inclusion $k_i \hookrightarrow k_\infty$ and the identity map of x to itself; the maps ρ_i commute with the ϕ_{ij}. Thus, we have constructed a faithfully flat map $\rho_i \circ \pi_i$ from each element of the direct system $S(L_i)$ to the fixed noetherian ring $k_\infty[\![x]\!]/Hk_\infty[\![x]\!]$; moreover, the maps $\rho_i \circ \pi_i$ are compatible with the homomorphisms ψ_{ij} of the direct system. This implies that the ring $S(L) = \lim_{\overrightarrow{i}} S(L_i)$ is noetherian. $\square$

LEMMA 9.2.   *Let (S, m, k) be a local noetherian ring. Let t be an arbitrary collection of independent variables. Consider the rings $S[t]$ and $S(t) := S[t]_{mS[t]}$. Let I be an ideal of S. Then*

$$
IS(t) \cap S[t] = IS[t]. \tag{9.3}
$$

Proof.   First, assume the collection t consists of a single variable. Consider elements $f, g \in S[t]$ such that

$$
f \notin mS[t] \tag{9.4}
$$

and

$$
fg \in IS[t]. \tag{9.5}
$$

Proving Equation (9.3) amounts to proving that

$$
g \in IS[t]. \tag{9.6}
$$

We prove (9.6) by contradiction. Assume that $g \notin IS[t]$. Then there exists $n \in \mathbb{N}$ such that $g \notin (I + m^n)S[t]$. Take the smallest such n, so that

$$
g \in (I + m^{n-1})S[t] \setminus (I + m^n)S[t]. \tag{9.7}
$$

Write $f = \sum_{j=0}^{q} a_j t^j$ and $g = \sum_{j=0}^{l} b_j t^j$. Let

$$
l_0 := \max\{j \in \{0, \ldots, l\} \mid b_j \notin I + m^n\} \tag{9.8}
$$

and

$$
q_0 := \max\{j \in \{0, \ldots, q\} \mid a_j \notin m\}. \tag{9.9}
$$

Let $c_{l_0+q_0}$ denote the $(l_0 + q_0)$th coefficient of fg. We have

$$
c_{l_0+q_0} = \sum_{i+j=l_0+q_0} a_i b_j = a_{q_0} b_{l_0} + \sum_{\substack{i+j=l_0+q_0 \\ i>q_0}} a_i b_j + \sum_{\substack{i+j=l_0+q_0 \\ j>l_0}} a_i b_j.
$$

By definition of l_0 and q_0 and (9.7) we have:

$$a_{q_0} b_{l_0} \notin I + m^n \tag{9.10}$$

and

$$\sum_{\substack{i+j=l_0+q_0 \\ i>q_0}} a_i b_j + \sum_{\substack{i+j=l_0+q_0 \\ j>l_0}} a_i b_j \in I + m^n. \tag{9.11}$$

Hence $c_{l_0+q_0} \notin I + m^n$, which contradicts (9.5). This completes the proof of Lemma 9.2 in the case when t is a single variable. The case of a general t now follows by transfinite induction on the collection t. $\square$

LEMMA 9.3. *There exist sets of representatives*

$$u' = (u'_1, \ldots, u'_{n_\ell})$$

of $\bar{u}'$ and ϕ_j of $\bar{\phi}_j$, $s_\ell < j \leqslant n_\ell$, in $\hat{R}'/H'_{2\ell-2}\hat{R}'$, having the following properties. Let

$$w' = (w'_1, \ldots, w'_{s_\ell}) = (u'_1, \ldots, u'_{s_\ell}), \tag{9.12}$$
$$v' = (v'_{s_\ell+1}, \ldots, v'_{n_\ell}) = (u'_{s_\ell+1} - \phi_{s_\ell+1}, \ldots, u'_{n_\ell} - \phi_{n_\ell}). \tag{9.13}$$

Let $J'_{2\ell-1} = H'_{2\ell-1}/H'_{2\ell-2} + (v') \subset \hat{R}'/H'_{2\ell-2}$. Then

$$w' \subset \frac{R'}{P'_{\ell-1}} \tag{9.14}$$

and

$$J'_{2\ell-1} \cap \frac{R'}{P'_{\ell-1}} = (0). \tag{9.15}$$

Proof of Lemma 9.3. There is no problem choosing w' to satisfy (9.14).

As for (9.15), we first prove Lemma 9.3 under the assumption that k is countable. We choose the representatives u' arbitrarily and let $\bar{\phi}_j(u') \in k[\![u']\!]$ denote the formal power series obtained by substituting u' for $\bar{u}'$ in $\bar{\phi}_j$. Any representative ϕ_j of $\bar{\phi}_j$, $s_\ell < j \leqslant n_\ell$ has the form $\phi_j = \bar{\phi}_j(u') + h_j$ with $h_j \in (z')(\hat{R}'/H'_{2\ell-2})$. We define the h_j required in Lemma 9.3 recursively in j. Take $j \in \{s_\ell + 1, \ldots, n_\ell\}$. Assume that $h_{s_\ell+1}, \ldots, h_{j-1}$ are already defined and that

$$(v'_{s_\ell+1}, \ldots, v'_{j-1}) \cap \frac{R'}{P'_{\ell-1}} = (0), \tag{9.16}$$

where we view $R'/P'_{\ell-1}$ as a subring of $\hat{R}'/H'_{2\ell-1}$. Since the ring $R'/P'_{\ell-1}$ is countable, there are countably many ideals in $\hat{R}'/(H'_{2\ell-1} + (v'_{s_\ell+1}, \ldots, v'_{j-1}))$, not contained in $(z')\hat{R}'/(H'_{2\ell-1} + (v'_{s_\ell+1}, \ldots, v'_{j-1}))$, which are minimal primes of ideals of the form $(f)(\hat{R}'/(H'_{2\ell-1} + (v'_{s_\ell+1}, \ldots, v'_{j-1})))$, where f is a non-zero element of $m'/P'_{\ell-1}$. Let us denote these ideals by $\{I_q\}_{q \in \mathbb{N}}$; we have

$$\operatorname{ht} I_q = 1 \quad \text{for all } q \in \mathbb{N}. \tag{9.17}$$

We note that

$$\frac{(z')\hat{R}'}{H'_{2\ell-1} + (v'_{s_\ell+1}, \ldots, v'_{j-1})} \not\subset I_q \quad \text{for all } q \in \mathbb{N}. \tag{9.18}$$

Indeed, by (9.1) and (9.16) we have $\operatorname{ht}((z')\hat{R}'/(H'_{2\ell-1} + (v'_{s_\ell+1}, \ldots, v'_{j-1}))) \geqslant 1$. In view of (9.17), containment in (9.18) would imply equality, which contradicts the definition of I_q.

Since $H'_{2\ell-1} \subsetneq H'_{2\ell}$ and $J'_{2\ell+1} \subsetneq m'\hat{R}'/H'_{2\ell}$, we have

$$\dim \frac{\hat{R}'}{H'_{2\ell-1} + (v'_{s_\ell+1}, \ldots, v'_{j-1})} \geqslant (\operatorname{ht} H'_{2\ell} - \operatorname{ht} H'_{2\ell-1}) + \operatorname{ht} \frac{m'\hat{R}'}{H'_{2\ell}} - (j - s_\ell - 1)$$

$$\geqslant (\operatorname{ht} H'_{2\ell} - \operatorname{ht} H'_{2\ell-1}) + \operatorname{ht} \frac{m'\hat{R}'}{H'_{2\ell}} - \operatorname{ht} J'_{2\ell+1} + 1 \geqslant 3. \quad (9.19)$$

Let $\tilde{u}_j$ denote the image of $u'_j - \bar{\phi}_j(u')$ in $\hat{R}'/(H'_{2\ell-1} + (v'_{s_\ell+1}, \ldots, v'_{j-1}))$. Next, we construct an element $\tilde{h}_j \in (z')\hat{R}'/(H'_{2\ell-1} + (v'_{s_\ell+1}, \ldots, v'_{j-1}))$ such that

$$\tilde{u}_j - \tilde{h}_j \notin \bigcup_{q=1}^{\infty} I_q. \quad (9.20)$$

The element $\tilde{h}_j$ will be given as the sum of an infinite series $\sum_{t=0}^{\infty} h_{jt}^t$ in $(z')(\hat{R}'/(H'_{2\ell-1} + (v'_{s_\ell+1}, \ldots, v'_{j-1})))$, convergent in the m'-adic topology, which we will now construct recursively in t. Put $h_{j0} = 0$. Assume that $t > 0$, that $h_{j0}, \ldots, h_{j,t-1}$ are already defined and that for $q \in \{1, \ldots, t-1\}$ we have $u'_j - \bar{\phi}_j(u') - \sum_{l=0}^{q} h_{jl} \notin \bigcup_{l=1}^{q} I_l$ and $h_{jq} \in (z') \cap (\bigcap_{l=1}^{q-1} I_l)$. If $u'_j - \bar{\phi}_j(u') - \sum_{l=0}^{t-1} h_{jl} \notin I_t$, put $h_{jt} = 0$. If $u'_j - \bar{\phi}_j(u') - \sum_{l=0}^{t-1} h_{jl} \in I_t$, let h_{jt} be any element of $(z') \cap (\bigcap_{l=1}^{t-1} I_l) \setminus I_t$ (such an element exists because I_t is prime, in view of (9.18)). This completes the definition of $\tilde{h}_j$. Let h_j be an arbitrary representative of $\tilde{h}_j$ in $\hat{R}'/H'_{2\ell-2}$.

We claim that

$$(H'_{2\ell-1} + (v'_{s_\ell+1}, \ldots, v'_j)) \cap \frac{R'}{P'_{\ell-1}} = (0). \quad (9.21)$$

Indeed, suppose that the above intersection contained a non-zero element f. Then any minimal prime $\tilde{I}$ of the ideal $(v'_j)(\hat{R}'/(H'_{2\ell-1} + (v'_{s_\ell+1}, \ldots, v'_{j-1})))$ is also a minimal prime of $(f)(\hat{R}'/(H'_{2\ell-1} + (v'_{s_\ell+1}, \ldots, v'_{j-1})))$. Since $v_j \notin ((z')\hat{R}'/H'_{2\ell-1} + (v'_{s_\ell+1}, \ldots, v'_{j-1}))$, we have $\tilde{I} \not\subset (z')\hat{R}'/(H'_{2\ell-1} + (v'_{s_\ell+1}, \ldots, v'_{j-1}))$. Hence $\tilde{I} = I_q$ for some $q \in \mathbb{N}$. Then $v_j \in I_q$, which contradicts (9.20).

Carrying out the above construction for all $j \in \{s_\ell + 1, \ldots, n_\ell\}$ produces the elements ϕ_j required in Lemma 9.3. This completes the proof of Lemma 9.3 in the case when k is countable.

Next, assume that k is uncountable. Let u' be chosen as above.

By assumption, $R'/P'_{\ell-1}$ contains a k_0-subalgebra S essentially of finite type, over which $R'/P'_{\ell-1}$ is strictly étale. Take a countable subfield $L_1 \subset k_0$ such that the algebra S is defined already over L_1 (this means that S has the form

$$S = (S'_1 \otimes_{L_1} k_0)_{m'_1(S'_1 \otimes_{L_1} k_0)}, \quad (9.22)$$

where (S'_1, m'_1, k'_1) is a local L_1-algebra essentially of finite type). Next, let $L_1 \subset L_2 \subset \ldots$ be an increasing chain of finitely generated field extensions of L_1, contained in k_0, having the following property. Let (S'_q, m'_q, k'_q) denote the localization of $S'_1 \otimes_{k'_1} L_q$ at the maximal ideal $m'_1(S'_1 \otimes_{k'_1} L_q)$. We require that

$$k'_\infty := \bigcup_{q=1}^{\infty} k'_q$$

contain all the coefficients of all the formal power series $\bar{\phi}_{s_\ell+1}, \ldots, \bar{\phi}_{n_\ell}$ and such that the ideal $H'_{2\ell-1}/H'_{2\ell-2}$ is generated by elements of $(k'_\infty[\![z']\!]/I_\infty)[\![u']\!]$, where I_∞ is the kernel of the natural homomorphism $k'_\infty[\![z']\!] \to \hat{R}'/H'_{2\ell-2}$. Let $H'_{2\ell-1,\infty} = H'_{2\ell-1}/H'_{2\ell-2} \cap (k'_\infty[\![z']\!]/I_\infty)[\![u']\!]$. We have constructed an increasing chain $S'_1 \subset S'_2 \subset \ldots$ of local L_1-algebras essentially of finite type such that k'_q is the residue field of S'_q. Then $S'_\infty := \bigcup_{q=1}^{\infty} S'_q$ is a local noetherian ring whose completion is $k'_\infty[\![z', u']\!]/(P'_{\ell-1} \cap S'_\infty)k'_\infty[\![z', u']\!]$. Let m'_∞ denote the maximal ideal of S'_∞. The

above argument in the countable case shows that there exist representatives $\phi_{s_\ell+1}, \ldots, \phi_{n_\ell}$ of $\bar{\phi}_{s_\ell+1}, \ldots, \bar{\phi}_{n_\ell}$ in $\hat{S}'_\infty/H'_{2\ell-2} \cap \hat{S}'_\infty$ such that, defining $v' = (v'_{s_\ell+1}, \ldots, v'_{n_\ell})$ as in (9.13), we have

$$((v') + H'_{2\ell-1,\infty}) \cap S'_\infty = (0). \tag{9.23}$$

Let $L_\infty = \bigcup_{q=1}^\infty L_q$ and let t denote a transcendence base of k_0 over L_∞. Let the notation be as in Lemma 9.1 with k_0 replaced by L_∞. For example, $S'_\infty(L_\infty(t))$ will denote the localization of the ring $S'_\infty \otimes_{L_\infty} L_\infty(t)$ at the prime ideal $m'_\infty(S'_\infty \otimes_{L_\infty} L_\infty(t))$.

By (9.23),

$$((v') + H'_{2\ell-1,\infty})\hat{S}'_\infty[t] \cap S'_\infty[t] = (0). \tag{9.24}$$

Now Lemma 9.2 and the fact that $S'_\infty[t]$ is a domain imply that

$$((v') + H'_{2\ell-1,\infty})\hat{S}'_\infty(L_\infty(t)) \cap S'_\infty(L_\infty(t)) = (0). \tag{9.25}$$

Next, let $\tilde{L}$ be a finite extension of $L_\infty(t)$, contained in k_0; then $S'_\infty(\tilde{L})$ is finite over $S'_\infty(L_\infty(t))$. Since $\hat{S}'_\infty(\tilde{L})$ is faithfully flat over $\hat{S}'_\infty(L_\infty(t))$ and in view of (9.25), we have

$$(((v') + H'_{2\ell-1,\infty})\hat{S}'_\infty(\tilde{L}) \cap S'_\infty(\tilde{L})) \cap S'_\infty(L_\infty(t)) = (0).$$

Hence $ht((v') + H'_{2\ell-1,\infty})\hat{S}'_\infty(\tilde{L}) \cap S'_\infty(\tilde{L}) = 0$. Since $S'_\infty(\tilde{L})$ is a domain, this implies that

$$((v') + H'_{2\ell-1,\infty})\hat{S}'_\infty(\tilde{L}) \cap S'_\infty(\tilde{L}) = (0). \tag{9.26}$$

Since k_0 is algebraic over $L_\infty(t)$, it is the limit of the direct system of all the finite extensions of $L_\infty(t)$ contained in it. We pass to the limit in (9.26). By (9.22), we have $S = S'_\infty(k_0)$; we also note that $\hat{S} = \hat{R}'/H'_{2\ell-2}$.

Since the natural maps $\hat{S}'_\infty(\tilde{L}) \to \hat{S}'_\infty(k_0)$ are all faithfully flat, we obtain

$$((v') + H'_{2\ell-1,\infty})\hat{S}'_\infty(k_0) \cap S = (0). \tag{9.27}$$

Since $\hat{S} = \hat{R}'/H'_{2\ell-2}$ is also the formal completion of $\hat{S}'_\infty(k_0)$, it is faithfully flat over $\hat{S}'_\infty(k_0)$. Hence

$$J'_{2\ell-1} \cap \hat{S}'_\infty(k_0) = ((v') + H'_{2\ell-1,\infty})\frac{\hat{R}'}{H'_{2\ell-2}} \cap \hat{S}'_\infty(k_0) = ((v') + H'_{2\ell-1,\infty})\hat{S}'_\infty(k_0). \tag{9.28}$$

Combining this with (9.27), we obtain

$$J'_{2\ell-1} \cap S = (0). \tag{9.29}$$

Thus the ideal $J'_{2\ell-1} \cap R'/P'_{\ell-1}$ contracts to (0) in S. Since $R'/P'_{\ell-1}$ is étale over S, this implies the desired equality (9.15). This completes the proof of Lemma 9.3. □

Since $\hat{R}'/H'_{2\ell}\hat{R}'$ is a complete regular local ring and (w', v') is a set of representatives of a minimal set of generators of its maximal ideal $m'\hat{R}'/H'_{2\ell}$, there exists a complete local domain R'_ℓ (not necessarily regular) such that $\hat{R}'/H'_{2\ell-1} \cong R'_\ell[\![w', v']\!]$. Consider the ring homomorphism

$$R'_\ell[\![w', v']\!] \longrightarrow R'_\ell[\![w']\!], \tag{9.30}$$

obtained by taking the quotient modulo (v'). By (9.30), the quotient of $\hat{R}'/H'_{2\ell-2}$ by $J'_{2\ell-1}$ is the integral domain $R'_\ell[\![w']\!]$, hence $J'_{2\ell-1}$ is prime.

Consider a local blowing up $R' \to R''$ in $\mathcal{T}$. Because of the stability assumption on R, the ring $\hat{R}''/H''_{2\ell-2} \otimes_R \kappa(P''_{l-1})$ is finite over $\hat{R}'/H'_{2\ell-2} \otimes_R \kappa(P'_{l-1})$; hence the ring $\lim_{\overrightarrow{R'' \in \mathcal{T}}}(\hat{R}''/H''_{2\ell-2} \otimes_R \kappa(P''_{l-1}))$ is integral over $\hat{R}'/H'_{2\ell-2} \otimes_R \kappa(P'_{l-1})$. In particular, there exists a prime ideal in

$$\lim_{\overrightarrow{R'' \in \mathcal{T}}} \left(\frac{\hat{R}''}{H''_{2\ell-2}} \otimes_R \kappa(P''_{l-1}) \right),$$

lying over $J'_{2\ell-1}(\hat{R}'/H'_{2\ell-2}) \otimes_R \kappa(P'_{l-1})$. Pick and fix one such prime ideal. Intersecting this ideal with $\hat{R}''/H''_{2\ell-2}$ for each $R'' \in \mathcal{T}$, we obtain a tree $J''_{2\ell-1}$ of prime ideals of $\hat{R}''/H''_{2\ell-2}$, $R'' \in \mathcal{T}$.

Our next task is to define the restriction of the valuation $\hat{\mu}_{2\ell}$ to the ring $\hat{R}'/J'_{2\ell-1}$. By the induction assumption, $\hat{\mu}_{2\ell+2}$ is already defined on $\varinjlim_{R' \in \mathcal{T}} (\hat{R}'/J'_{2\ell+1}\hat{R}')$. For all *stable* $R'' \in \mathcal{T}$, we have the isomorphism $\mathrm{gr}_{\hat{\mu}_{2\ell+2}}(\hat{R}''/J''_{2\ell+1}) \cong \mathrm{gr}_{\mu_{\ell+1}}(R''/P''_\ell)$ of graded algebras (in particular, $\mathrm{gr}_{\hat{\mu}_{2\ell+2}}(\hat{R}''/J''_{2\ell+1})$ is scalewise birational to $\mathrm{gr}_{\mu_{\ell+1}}(R''/P''_\ell)$ for any $R'' \in \mathcal{T}$ and $\hat{\mu}_{2\ell+2}$ has the same value group Δ_ℓ as $\mu_{\ell+1}$).

Define the prime ideals $\tilde{H}''_{2\ell-2} = \tilde{H}''_{2\ell-1}$ to be equal to the preimage of $J''_{2\ell-1}$ in $\hat{R}''$ and $\tilde{H}''_{2\ell} = \tilde{H}''_{2\ell+1}$ the preimage of $J''_{2\ell+1}$ in $\hat{R}''$. By definition of tight extensions, the valuation $\hat{\nu}_{2\ell+1}$ must be trivial. It remains to describe the valuation $\hat{\mu}_{2\ell}$ on $\hat{R}''/J''_{2\ell-1}$, $R'' \in \mathcal{T}$. We will first define $\hat{\nu}_{2\ell}$ and then put $\hat{\mu}_{2\ell} = \hat{\nu}_{2\ell} \circ \hat{\mu}_{2\ell+2}$.

By definition of tight extensions, the value group of $\hat{\mu}_{2\ell}$ must be equal to $\Delta_{\ell-1}$ and that of $\hat{\nu}_{2\ell}$ to $\Delta_{\ell-1}/\Delta_\ell$. For a positive element $\bar{\beta} \in \Delta_{\ell-1}/\Delta_\ell$, define the candidate for $\hat{\nu}_{2\ell}$-ideal of $\hat{R}''_{\tilde{H}''_{2\ell}}/\tilde{H}''_{2\ell-1}\hat{R}''_{\tilde{H}''_{2\ell}}$ of value $\bar{\beta}$, denoted by $\hat{\mathcal{P}}''_{\bar{\beta}\ell}$, by the formula

$$\hat{\mathcal{P}}''_{\bar{\beta}\ell} = \frac{\mathcal{P}''_{\bar{\beta}}\hat{R}''_{\tilde{H}''_{2\ell}}}{\tilde{H}''_{2\ell-1}\hat{R}''_{\tilde{H}''_{2\ell}}}. \tag{9.31}$$

CONJECTURE 9.2.   The elements ϕ_j of Lemma 9.3 can be chosen in such a way that the following condition holds. For each positive element $\beta \in \Delta_{\ell-1}/\Delta_\ell$ and each tree morphism $R' \to R''$ in $\mathcal{T}$, we have

$$\hat{\mathcal{P}}''_{\beta\ell} \cap \hat{R}'_{\tilde{H}'_{2\ell}} = \hat{\mathcal{P}}'_{\beta\ell}.$$

CONJECTURE 9.3.   The elements ϕ_j of Lemma 9.3 can be chosen in such a way that

$$\bigcap_{\bar{\beta} \in (\Delta_{\ell-1}/\Delta_\ell)_+} (\mathcal{P}'_{\bar{\beta}} + \tilde{H}'_{2\ell-1})\hat{R}'_{\tilde{H}'_{2\ell}} \subset \tilde{H}'_{2\ell-1}. \tag{9.32}$$

For the rest of this section assume that Conjectures 9.2 and 9.3 are true.
For all $\bar{\beta} \in (\Delta_{\ell-1}/\Delta_\ell)_+$, we have the natural isomorphism

$$\lambda_{\bar{\beta}} : \frac{\mathcal{P}'_{\bar{\beta}}}{\mathcal{P}'_{\bar{\beta}+}} \otimes_{\kappa(P'_{\ell-1})} \kappa(\tilde{H}'_{2\ell}) \longrightarrow \frac{\hat{\mathcal{P}}'_{\bar{\beta}}}{\hat{\mathcal{P}}'_{\bar{\beta}+}}$$

of $\kappa(\tilde{H}'_{2\ell})$-vector spaces. The following fact is an easy consequences of Conjecture 9.2:

COROLLARY 9.4 (conditional on Conjecture 9.2).   *If the elements ϕ_j of Lemma 9.3 can be chosen as in Conjecture 9.2 then the graded algebra*

$$\mathrm{gr}_{\nu_\ell} \frac{R'_{P'_\ell}}{P'_{\ell-1}R'_{P'_\ell}} \otimes_{\kappa(P'_{\ell-1})} \kappa(\tilde{H}'_{2\ell}) \cong \bigoplus_{\bar{\beta} \in (\Delta_{\ell-1}/\Delta_\ell)_+} \frac{\hat{\mathcal{P}}'_{\bar{\beta}}}{\hat{\mathcal{P}}'_{\bar{\beta}+}}$$

is an integral domain.

For a non-zero element $x \in \hat{R}'_{\tilde{H}'_{2\ell}}/\tilde{H}'_{2\ell-1}$, let $\mathrm{Val}_\ell(x) = \{\beta \in \nu_\ell(R'/P'_{\ell-1} \setminus \{0\}) \mid x \in \hat{\mathcal{P}}'_{\beta\ell}\}$. We define $\hat{\nu}_{2\ell}$ by the formula

$$\hat{\nu}_{2\ell}(x) = \max \mathrm{Val}_\ell(x). \tag{9.33}$$

Since ν_ℓ is a rank 1 valuation, centred in a local noetherian domain $R'/P'_{\ell-1}$, the semigroup $\nu_\ell(R'/P'_{\ell-1} \setminus \{0\})$ has order type $\mathbb{N}$, so by (9.32) the set $Val_\ell(x)$ contains a maximal element. This proves that the valuation $\hat{\nu}_{2\ell}$ is well defined by the formula (9.33), and that we have a natural isomorphism of graded algebras

$$\mathrm{gr}_{\nu_\ell} \frac{R'_{P'_\ell}}{P'_{\ell-1} R'_{P'_\ell}} \otimes_{\kappa(P'_{\ell-1})} \kappa(\tilde{H}'_{2\ell}) \cong \mathrm{gr}_{\hat{\nu}_{2\ell}} \frac{\hat{R}_{\tilde{H}'_{2\ell}}}{\tilde{H}'_{2\ell-1}}.$$

Since the above construction is valid for all $R \in \mathcal{T}$, $\hat{\nu}_{2\ell}$ extends naturally to a valuation centred in the ring $\varinjlim_{R'}(\hat{R}''/\tilde{H}''_{2\ell-1}\hat{R}'')$ (by abuse of notation, this extension will also be denoted by $\hat{\nu}_{2\ell}$).

The extension $\hat{\mu}_{2\ell}$ of μ_ℓ to $\varinjlim_{R'' \in \mathcal{T}(R')}(\hat{R}''/\tilde{H}''_{2\ell-1}\hat{R}')$ is defined by $\hat{\mu}_{2\ell} = \hat{\nu}_{2\ell} \circ \hat{\mu}_{2\ell+2}$.

This completes the proof of Conjecture 9.1 (assuming Conjectures 9.2 and 9.3) by descending induction on ℓ.

The following corollary of Conjecture 9.1 gives necessary conditions for $\hat{\nu}_-$ to be uniquely determined by ν; it also shows that the same conditions are sufficient for $\hat{\nu}_-$ to be the unique minimal extension of ν, that is, to satisfy

$$\tilde{H}'_i = H'_i, \quad 0 \leqslant i \leqslant 2r. \tag{9.34}$$

Suppose given a tree $\{\tilde{H}'_0\}$ of minimal prime ideals of $\hat{R}'$ (in particular, $R' \cap \tilde{H}'_0 = (0)$). If the valuation ν admits an extension to a valuation $\hat{\nu}_-$ of $\varinjlim_{R'}(\hat{R}'/\tilde{H}'_0)$, then $\tilde{H}'_0$ is the 0th prime ideal of $\hat{R}'$, determined by $\hat{\nu}_-$. Since $\tilde{H}'_0$ is assumed to be a *minimal* prime, we have $\tilde{H}'_0 = H'_0$ by Proposition 5.3.

REMARK 17. Let the notation be as in Conjecture 9.1. Denote the tree of prime ideals $\{\tilde{H}'_0\}$ by $\{H'\}$ for short. Consider a homomorphism

$$R' \longrightarrow R'' \tag{9.35}$$

in $\mathcal{T}$. Assume that the local rings R' and $\hat{R}'/H'$ are regular, and let $V = (V_1, \ldots, V_s)$ be a minimal set of generators of H'. Then V can be extended to a regular system of parameters for $\hat{R}'$. We have an isomorphism $\hat{R}' \cong (\hat{R}'/H')\llbracket V \rrbracket$. The morphism (9.35) induces an isomorphism $\hat{R}'_{H'} \cong \hat{R}''_{H''}$, so that V induces a regular system of parameters of $\hat{R}''_{H''}$. In particular, the H''-adic valuation of $\hat{R}''_{H''}$ coincides with the H'-adic valuation of $\hat{R}'_{H'}$. On the other hand, we do not know, assuming that R'' and $\hat{R}''/H''$ are regular and $\mathrm{ht}\, H'' = \mathrm{ht}\, H'$, whether V induces a minimal set of generators of H''; we suspect that the answer is 'no'.

COROLLARY 9.5 (conditional on Conjecture 9.1). *If the valuation ν admits a unique extension to a valuation $\hat{\nu}_-$ of $\varinjlim_{R'}(\hat{R}'/H'_0)$, then the following conditions hold:*

(1) $\mathrm{ht}\, H'_1 \leqslant 1$;
(2) $H'_i = H'_{i-1}$ *for all odd $i > 1$.*

Moreover, this unique extension $\hat{\nu}_-$ is minimal.

Conversely, assume that (1)–(2) hold. Then there exists a unique minimal extension $\hat{\nu}_-$ of ν to $\varinjlim_{R'}(\hat{R}'/H'_0)$.

Proof. The fact that conditions (1), (2) and equation (9.34) determine $\hat{\nu}_-$ uniquely is nothing but Proposition 6.6. Conversely, assume that there exists a unique extension $\hat{\nu}_-$ of ν to $\varinjlim_{R'}(\hat{R}'/H_0')$. By Remark 13, there exist minimal extensions of ν to $\varinjlim_{R'}(\hat{R}'/H_0')$, hence $\hat{\nu}_-$ must be minimal.

Next, by Conjecture 9.1, there exists a tree of prime ideals $\tilde{H}'$ with $H' \cap R' = (0)$ and a tight extension $\hat{\mu}_-$ of ν to $\varinjlim_{R'}(\hat{R}'/H')$. The ideals H' are both the zeroth and the first ideals determined by $\hat{\mu}_-$; in particular, we have

$$H_0' \subset H_1' \subset H' \tag{9.36}$$

by Proposition 5.3. Now, take any valuation θ, centred in the regular local ring $R'_{H'}/H_0'$, such that the residue field $k_\theta = \kappa(H')$. Then the composition $\hat{\mu}_- \circ \theta$ is an extension of ν to $\varinjlim_{R'}(\hat{R}'/H_0')$, hence

$$\hat{\mu}_- \circ \theta = \hat{\nu}_- \tag{9.37}$$

by uniqueness. For $i \geqslant 1$, the ith prime ideal, determined by $\hat{\mu}_- \circ \theta = \hat{\nu}_-$ coincides with that determined by $\hat{\mu}_-$. Since ν is minimal and $\hat{\mu}_-$ is tight, we obtain condition (2) of Corollary 9.5. Finally, if we had $\operatorname{ht} H' > 1$, there would be infinitely many choices for θ, contradicting (9.37) and the uniqueness of $\hat{\nu}_-$. Thus $\operatorname{ht} H' \leqslant 1$. Combined with (9.36), this proves (1) of Corollary 9.5. This completes the proof of Corollary 9.5, assuming Conjecture 9.1. $\qquad\square$

Appendix. *Regular morphisms and G-rings*

In this appendix, we recall the definitions of regular homomorphism, G-rings and excellent and quasi-excellent rings. We also summarize some of their basic properties used in the rest of the paper.

DEFINITION A.1 [**10**, Chapter 13, (33.A), p. 249]. Let $\sigma : A \to B$ be a homomorphism of noetherian rings. We say that σ is *regular* if it is flat, and for every prime ideal $P \subset A$, the ring $B \otimes_A \kappa(P)$ is geometrically regular over $\kappa(P)$ (this means that for any finite field extension $\kappa(P) \to k'$, the ring $B \otimes_A k'$ is regular).

REMARK A.1. If $\kappa(P)$ is perfect, the ring $B \otimes_A \kappa(P)$ is geometrically regular over $\kappa(P)$ if and only if it is regular.

REMARK A.2. It is known that a morphism of finite type is regular in the above sense if and only if it is smooth (that is, formally smooth in the sense of Grothendieck with respect to the discrete topology), though we do not use this fact in this paper.

Regular morphisms come up in a natural way when one wishes to pass to the formal completion of a local ring:

DEFINITION A.2 [**10**, (33.A) and (34.A)]. Let R be a noetherian ring. For a maximal ideal m of R, let $\hat{R}_m$ denote the m-adic completion of R. We say that R is a *G-ring* if for every maximal ideal m of R, the natural map $R \to \hat{R}_m$ is a regular homomorphism.

The property of being a G-ring is preserved by localization and passing to rings essentially of finite type over R.

DEFINITION A.3 [**10**, Definition 2.5, (34.A), p. 259].   Let R be a noetherian ring. We say that R is *quasi-excellent* if the following two conditions hold.

(1) R is J-2, that is, for any scheme X, which is reduced and of finite type over $\operatorname{Spec} R$, $\operatorname{Reg}(X)$ is open in the Zariski topology.
(2) For every maximal ideal $m \subset R$, R_m is a G-ring.

It is known [**10**] that a *local* G-ring is automatically J-2, hence automatically quasi-excellent. Thus for local rings 'G-ring' and 'quasi-excellent' are one and the same thing. A ring is said to be *excellent* if it is quasi-excellent and universally catenary, but we do not need the catenary condition in this paper.

Both excellence and quasi-excellence are preserved by localization and passing to rings of finite type over R (see [**10**, Chapter 13, (33.G), Theorem 77, p. 254]). In particular, any ring essentially of finite type over a field, $\mathbf{Z}$, $\mathbf{Z}_{(p)}$ and $\mathbf{Z}_p$, the Witt vectors or any other excellent Dedekind domain is excellent. See [**11**, (Appendix A.1, p. 203)] for some examples of non-excellent rings.

Rings which arise from natural constructions in algebra and geometry are excellent. Complete and complex-analytic local rings are excellent (see [**10**, Theorem 30.D]) for a proof that any complete local ring is excellent).

Finally, we remark that the category of quasi-excellent rings is a natural one for doing algebraic geometry, since it is the largest reasonable class of rings for which resolution of singularities can hold. Namely, let R be a noetherian ring. Grothendieck [**4**, IV.7.9] proves that if all of the irreducible closed subschemes of $\operatorname{Spec} R$ and all of their finite purely inseparable covers admit resolution of singularities, then R must be quasi-excellent. Grothendieck's result means that the largest *class* of noetherian rings, closed under homomorphic images and finite purely inseparable extensions, for which resolution of singularities could possibly exist, is *quasi-excellent* rings.

We now summarize the specific uses we make of quasi-excellence in the present paper. We begin by recalling three results from [**10**, **11**]. As a point of terminology, we note that Nagata's 'pseudo-geometric' rings are now commonly known as Nagata rings. Quasi-excellent rings are Nagata [**10**, (33.H), Theorem 78].

THEOREM A.1 [**10**, (34.C), Theorem 79].   *Let R be an excellent normal local ring. Then R is analytically normal* (*this means that its formal completion $\hat{R}$ is normal*).

THEOREM A.2 [**11**, (43.20), p. 187].   *Let R be a local integral domain, $\tilde{R}$ its Henselization and R' its normalization. There is a natural one-to-one correspondence between the minimal primes of $\tilde{R}$ and the maximal ideals of R'.*

PROPOSITION A.3 [**11**, Corollary (44.3), p. 189].   *Let R be a quasi-excellent analytically normal local ring. Then its Henselization $\tilde{R}$ is analytically irreducible and is algebraically closed in its formal completion.*

From the above results we deduce the following.

COROLLARY A.4.   *Let $(R, \mathbf{m})$ be a Henselian excellent local domain. Then R is analytically irreducible and is algebraically closed in $\hat{R}$.*

Proof. If, in addition, we assume R to be normal, the result follows from Theorems A.1 and A.3. In the general case, let R' denote the normalization of R. Then R' is a Henselian normal quasi-excellent local ring, so it satisfies the conclusions of Corollary A.4. Consider the commutative diagram:

$$
\begin{array}{ccc}
R & \xrightarrow{\phi} & \hat{R} \\
{\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\hat{\psi}} \\
R' & \xrightarrow{\phi'} & \hat{R}'
\end{array}
\tag{A.1}
$$

where $\hat{R}'$ stands for the formal completion of R'. Since R is Nagata, R' is a finite R-module. Thus ϕ' coincides with the **m**-adic completion of R', viewed as an R-module. Hence $\hat{R}' \cong R' \otimes_R \hat{R}$. Since ψ is injective and $\hat{R}$ is flat over r, the map $\hat{\psi}$ is also injective. Since R' is analytically irreducible, $\hat{R}'$ is a domain, and therefore so is its subring $\hat{R}$. This proves that R is analytically irreducible.

To prove that R is algebraically closed in $\hat{R}$, take an element $x \in \hat{R}$, algebraic over R. Since all the maps in (A.1) are injective, let us view all the rings involved as subrings of $\hat{R}'$. Since R' is algebraically closed in $\hat{R}'$, we have $x \in R'$, in particular, we may write $x = a/b$ with $a, b \in R$. Now, since $(a)\hat{R} \subset (b)\hat{R}$ and $\hat{R}$ is faithfully flat over R, we have $(a) \subset (b)$ in R, so $x = a/b \in R$. This proves that R is algebraically closed in $\hat{R}$. Corollary A.4 is proved. $\square$

Next we summarize, in a more specific manner, the way in which these results are applied in the present paper. The main applications are as follows.

(1) Let R be an excellent local domain, P a prime ideal of R and $H_i \subset H_{i+1}$ two prime ideals of $\hat{R}$ such that

$$
H_i \cap R = H_{i+1} \cap R = P.
\tag{A.2}
$$

Then R/P is also excellent. Definitions A.1–A.3 imply that the ring $\hat{R} \otimes_R \kappa(P)$ is geometrically regular over P, in particular, regular. Moreover, (A.2) implies that the ideal $H_{i+1}/P\hat{R}$ is a prime ideal of $\hat{R}/P\hat{R}$, disjoint from the natural image of $R \setminus P$ in $\hat{R}/P\hat{R}$. Thus the local ring $\hat{R}_{H_{i+1}}/P\hat{R}_{H_{i+1}}$ is a localization of $\hat{R} \otimes_R \kappa(P)$ at the prime ideal $H_{i+1}(\hat{R} \otimes_R \kappa(P))$ and so is a local ring, geometrically regular over $\kappa(P)$, in particular, a regular local ring and, in particular, a domain.

(2) Assume, in addition, that H_i is a minimal prime of $P\hat{R}$. Since $\hat{R}_{H_{i+1}}/P\hat{R}_{H_{i+1}}$ is a domain, H_i is the only minimal prime of $P\hat{R}$, contained in H_{i+1}. We have $P\hat{R}_{H_{i+1}} = H_i\hat{R}_{H_{i+1}}$.

References

1. S. ABHYANKAR, 'Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0$', *Ann. of Math.* 63 (1956) 491–526.

2. S. D. CUTKOSKY and S. EL HITTI, 'Formal prime ideals of infinite value and their algebraic resolution', *Ann. Fac. Sci. Toulouse Math.* (3–4) 19 (2010) 635–649.

3. S. CUTKOSKY and L. GHEZZI, 'Completions of valuation rings', *Contemp. Math.* 386 (2005) 13–34.

4. A. GROTHENDIECK and J. DIEUDONNÉ, *Eléments de Géométrie algébrique*, Publications Mathématiques. Institut de Hautes Études Scientifiques 24 (Springer, Berlin, 1965) Chapter IV.

5. W. Heinzer and J. Sally, 'Extensions of valuations to the completion of a local domain', *J. Pure Appl. Algebra* 71 (1991) 175–186.
6. J. Herrera, M. A. Olalla and M. Spivakovsky, 'Valuations in algebraic field extensions', *J. Algebra* 312 (2007) 1033–1074.
7. I. Kaplansky, 'Maximal fields with valuations I', *Duke Math. J.* 9 (1942) 303–321.
8. I. Kaplansky, 'Maximal fields with valuations II', *Duke Math. J.* 12 (1945) 243–248.
9. J. Lipman, 'Desingularization of two-dimensional schemes', *Ann. of Math.* 107 (1978) 151–207.
10. H. Matsumura, *Commutative algebra* (Benjamin/Cummings Publishing Co., Reading, MA, 1970).
11. M. Nagata, *Local rings* (Interscience Publishers, New York, 1960).
12. M. Spivakovsky, 'Valuations in function fields of surfaces', *Amer. J. Math.* 112 (1990) 107–156.
13. B. Teissier, 'Valuations, deformations, and toric geometry', *Proceedings of the Saskatoon Conference and Workshop on valuation theory (second volume)*, Fields Institute Communications 33 (eds F-V. Kuhlmann, S. Kuhlmann and M. Marshall; American Mathematical Society, Providence, RI, 2003) 361–459.
14. M. Vaquié, 'Valuations', *Resolution of singularities, a research textbook in tribute to Oscar Zariski*, Progress in Mathematics 181 (Birkhäuser, Basel, 2000).
15. M. Vaquié, 'Famille admissible de valuations et défaut d'une extension', *J. Algebra* 311 (2007) 859–876.
16. O. Zariski, 'Local uniformization theorem on algebraic varieties', *Ann. of Math.* 41 (1940) 852–896.
17. O. Zariski and P. Samuel, *Commutative algebra*, Vol. II (Springer, Berlin, 1960).

F. J. Herrera Govantes and M. A. Olalla Acosta
Departamento de Álgebra
Facultad de Matemáticas
Apdo. de correos 1160
41080 Sevilla
Spain

jherrera@algebra.us.es
miguelolalla@algebra.us.es

M. Spivakovsky
Institut de Mathématiques de Toulouse
UMR 5219 du CNRS,
Université Paul Sabatier
118, route de Narbonne
31062 Toulouse cedex 9
France

mark.spivakovsky@math.univ-toulouse.fr

B. Teissier
Equipe "Géométrie et Dynamique"
Institut Mathématique de Jussieu
UMR 7586 du CNRS
175 Rue du Chevaleret
F-75013 Paris
France

teissier@math.jussieu.fr