Foreword

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In the general scientific culture Mathematics can appear as quite disconnected. One knows about calculus, complex numbers, Fermat's last theorem, convex optimization, fractals, vector fields and dynamical systems, the law of large numbers, projective geometry, vector bundles, the Fourier transform and wavelets, the stationary phase method, numerical solutions of PDE's, etc., but no connection between them is readily apparent. For the mathematician, however, all these and many others are lineaments of a single landscape. Although he or she may spend most of his or her time studying one area of this landscape, the mathematician is conscious of the possibility of travelling to other places, perhaps at the price of much effort, and bringing back fertile ideas. Some of the results or proofs most appreciated by mathematicians are the result of such fertilizations.

I claim that Singularity Theory sits inside Mathematics much as Mathematics sits inside the general scientific culture. The general mathematical culture knows about the existence of Morse theory, parametrizations of curves, Bézout's theorem for plane projective curves, zeroes of vector fields and the Poincaré-Hopf theorem, catastrophe theory, sometimes a version of resolution of singularities, the existence of an entire world of commutative algebra, etc. But again, for the singularist, these and many others are lineaments of a single landscape and he or she is aware of its connectedness. Moreover, just as Mathematics does with science in general, singularity theory interacts energetically with the rest of Mathematics, if only because the closures of non singular varieties in some ambient space or their projections to smaller dimensional spaces tend to present singularities, smooth functions on a compact manifold must have critical points, etc. But singularity theory is also, again in a role played by Mathematics in general science, a crucible where different types of mathematical problems interact and surprising connections are born.

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- Who would have thought in the 1950's that there was a close connection between the classification of differentiable structures on topological spheres and the boundaries of certain isolated singularities of complex hypersurfaces?
- or that Thom's study of singularities of differentiable mappings would give birth to a geometric vision of bifurcation phenomena and of fundamental concepts such as structural stability?
- Who would have thought in the 1970's that there was a relation between the work of Lefschetz comparing the topological invariants of a complex projective variety with those of a general hyperplane section and the characterization of the sequences of integers counting the numbers of faces of all dimensions of simple polytopes?
- Or that one could produce real projective plane curves with a prescribed topology by deforming piece-wise linear curves in the real plane?
- Or in the 2000's that properties of the intersections of two curves on a complex surface would lead to the solution of a problem connected with the colouring of graphs?
- Or that the algebraic study of the space of arcs on the simplest singularities ($z^n = 0$ in $\mathbf{C}$, $n \geq 2$) would provide new proofs and generalizations of the Rogers-Ramanujan and Gordon identities between the generating series of certain types of partitions of integers?

These are only a few examples. But to come back to the theory of singularities, I would like to emphasize that what I like so much about it is that not only are surprising connections born there, but also very simple questions lead to ideas which resonate in other part of the field or in other fields. For example, if an analytic function has a small modulus at a point, does it have a zero at a distance from that point which is bounded in terms of that modulus? what is a general smooth function on a smooth compact manifold? A Morse function, with very mild singularities! And what happens if you replace the smooth manifold by a space with singularities? And then, given a function, can we measure how far it is from behaving like a general function? Suppose that a holomorphic function has a critical point at the origin. How can we relate the nature of the fiber of the function through this critical point with the geometry or topology of the nearby non singular fibers? How can we relate it with the geometry of the mappings resolving singularities of this singular fiber? Then again, what is a general map between smooth manifolds? and how do you deform a singular space into a non singular one in general? Well, that's more complicated. But I hope you get the idea.
The down side is that before he or she can successfully detect and try to answer such apparently simple and natural questions the student of singularities must become familiar with different subjects and their techniques, and the learning process is long.

And this is why a handbook which presents in-depth and reader-friendly surveys of topics of singularity theory, with a carefully crafted preface explaining their place within the theory, is so useful!