Extension of valuations to the Henselization

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Abstract

We discuss the extensions of a valuation from a local domain to its henselization. There is a classical theory of henselization of valued fields but here we discuss henselization of rings. One motivation is a problem which seems essential for local uniformization of valuations on excellent equicharacteristic local domains:

Can a valuation of an excellent local domain (R, m) be extended to a valuation of a quotient of its m-adic completion *with the same value group?*

Another motivation is to show that the spaces of valuations centered at regular points of an algebraic variety are all homeomorphic.
The first part of the talk will be devoted to reminders about the geometry of valuations and of henselization.

Framework : Valuations (1)

• A valuation ring is a commutative domain V where every finitely generated ideal is generated by one element of any of its set of generators. This implies that it is a local domain.
There is a preorder on $V \setminus \{0\}$: $a \leq b$ if the ideal (a, b) is generated by a. The element $0 \in V$ is larger than any non zero element.
This induces an equivalence relation : $a \cong b$ if the ideal (a, b) can be generated by both a and b. Then $V \setminus \{0\}/\cong$ is a totally ordered commutative semigroup $\Phi_{\geq 0}$ for the multiplication, but its operation is noted additively. So we have a map

$$\nu \colon V \setminus \{0\} \to \Phi_{\geq 0}$$

satisfying $\nu(ab) = \nu(a) + \nu(b)$ and, since $a + b \in (a, b)$, the inequality $\nu(a + b) \geq \min(\nu(a), \nu(b))$, with equality if $\nu(a) \neq \nu(b)$. Note that the first equality implies that V is a domain and the inequality implies that it is a local domain since the non-invertible elements, which are those of value $> 0 \in \Phi_{\geq 0}$, form an ideal.

Framework : Valuations (2)

• The valuation ν extends to the field of fractions K of V by $\nu(\frac{a}{b}) = \nu(a) - \nu(b)$. It then takes values in a totally ordered group Φ whose semigroup of non negative elements is $\Phi_{\geq 0}$. As a map $K^* \to \Phi$, the valuation ν satisfies the same inequalities and equalities as above. The elements of K whose values are non-negative are the elements of V.

Often one completes Φ by adding an element ∞ larger than all elements of Φ, so that $0 \in K$ has a value. I shall sometimes forget the difference. But note that 0 is the only element with value ∞.

• Note that if V is noetherian its maximal ideal is principal.

When we wish to specify the ring of a given valuation ν, we shall write R_ν and not V. The residue field R_ν/m_ν is denoted by k_ν.

Framework : Valuations (3)

• We shall denote by (R, ν) a local domain R contained in a valuation ring R_ν of its field of fractions. This means only that we assume that ν takes non negative values on R, that is, $R \subset R_\nu$. If $m_\nu \cap R = m$ we say that the valuation is *centered* at R. In what follows we shall consider only valuations centered in R.

• If $K = k(X)$ is the field of rational functions on an algebraic variety X It is useful to think of a valuation on K as the order of vanishing (or of a pole) along "something" contained in X. For example the order of vanishing along a divisor $D \subset X$ give a *divisorial valuation* on $k(X)$ with value group $\mathbf{Z}$. If X is normal, its valuation ring is $\mathcal{O}_D$, the local ring of X along D. If X is a curve, we are talking about the vanishing (or pole) order of a rational function at a point $x \in X$.

Framework : Valuations (4)

Similarly, the p-adic valuation on **Q** comes from the "vanishing order" ν of an integer n at the prime $p \in \mathrm{Spec}\mathbf{Z}$ in the decomposition $n = p^\nu n'$ with $(p, n') = 1$. In higher dimension or in the singular case, the situation becomes much more rich and closely connected with resolution of singularities because the inclusion $R \subset R_\nu$ as above is birational and R_ν is "regular" although not a geometric ring.

Framework : Valuations (5)

A more instructive example is to take a power series in x with rational exponents

$$y(x) = \sum_{i=1}^{\infty} a_i x^{\gamma_i},$$

where (γ_i) is a well ordered set of rational numbers with denominators tending to infinity, with $\gamma_1 > 0$. Then, for any polynomial or power series $p(x, y)$ in $k[x, y]$ or $k[[x, y]]$, the series in x obtained as $p(x, y(x))$ is non zero by Newton-Puiseux's theorem. The smallest power of x appearing in this series is a rational number $\nu(p(x, y))$ and the map $p(x, y) \mapsto \nu(p(x, y))$ is a valuation on $k[[x, y]]$ and hence on $k[x, y]_{(x,y)}$. One can choose the series $y(x)$ to obtain any subgroup of $\mathbf{Q}$, including $\mathbf{Q}$ itself, as value group. This kind of valuation, which is the "order of vanishing" along a "curve" which does not exist in algebraic or analytic geometry, is called "infinitely singular".

Valuations (6)

Valuations are not just important because they provide a measure of size essential for finding roots of polynomials by an approximation process, as discovered by Hensel and detailed below. The completion of number fields with respect to the (discrete rank one) valuations of their rings of integers play an important role in number theory.

Valuations are also important because they are closely connected to resolution of singularities of algebraic varieties, as discovered by Zariski after the pioneering work of Dedekind-Weber for curves.

Let R be a local domain and let $\mathrm{RZ}(R)$ be the space of valuations centered in R. If K is the fraction field of R, then $\mathrm{RZ}(R)$ consist of the set of all valuation rings of K which dominate R endowed with the Zariski topology. This topology is obtained by taking as a basis of open sets the subsets $U(A)$, whose elements are the valuation rings of K dominating R and containing A, where A ranges over the family of all finite subsets of K.

If X is an algebraic variety over a field k the union of the $\mathrm{RZ}(\mathcal{O}_{X,x})$ over (scheme theoretic) points of X is the Zariski-Riemann manifold of X. Unless X is a curve, it is not an algebraic variety. However its local rings, the valuation rings R_ν, are regular in any reasonable sense.

Valuations (7)

Any valuation ν on a field *K* has extensions to its algebraic closure $\overline{K}$ and given two such extensions ν, ν', there exists a *K*–automorphism π of $\overline{K}$ such that $\tilde{\nu}' = \tilde{\nu} \circ \pi$.

Totally ordered abelian groups (1)

Given a totally ordered abelian group Φ, the first invariant one can attach to it is its rational rank, $\dim_{\mathbf{Q}}\Phi \otimes_{\mathbf{Z}} \mathbf{Q}$, the maximum number of rationally independent elements of Φ. The next invariant measures how far Φ is from being a subgroup of **R**.

• A convex subgroup of Φ is a subgroup Ψ of Φ such that if $\phi \in \Psi_{\geq 0}$ and $0 < \phi' < \phi$, then $\phi' \in \Psi$. This is equivalent to the fact that there exists on the quotient Φ/Ψ a unique total ordering such that the quotient map $\Phi \to \Phi/\Psi$ is monotonous.

Totally ordered abelian groups (2)

• Basic example: in $\mathbf{Z}^2_{lex}$, the subgroup $\{0\} \times \mathbf{Z}$ is the only convex subgroup. The *rank*, or *height*, of Φ is the *cardinal* of the totally ordered set (for inclusion) of convex subgroups of Φ different from Φ. The set of convex subgroups of Φ may not be well ordered. However, the smallest convex subgroup containing a subset of Φ exists as the intersection of such convex subgroups.

• Φ is of rank one if and only if Φ is isomorphic as ordered group to a subgroup of **R**.

• The rank $h(\Phi)$ of Φ is less than the rational rank $\mathrm{rr}(\Phi)$ of Φ.

Totally ordered abelian groups (3)

If R is a noetherian local domain dominated by the valuation ring R_ν in the sense that $m_\nu \cap R = m$, with residue field $k \subset k_\nu$, we have Abhyankar's inequality

$$\mathrm{rr}(\Phi) + \mathrm{tr}_k k_\nu \leq \dim R.$$

Where $\mathrm{tr}_k k_\nu$ is the transcendence degree. In this case the rational rank is finite, so the rank is also finite and we have a nested sequence of convex subgroups

$$(0) = \Psi_h \subset \Psi_{h-1} \subset \ldots \Psi_1 \subset \Psi_0 = \Phi,$$

where h is the rank of Φ and the quotients Ψ_j/Ψ_{j+1} are totally ordered abelian groups of rank one, and so ordered subgroups of $\mathbf{R}$.

Totally ordered abelian groups (4)

Let ν be a valuation on R with value group Φ. Let Ψ be a proper convex subgroup of Φ, $\Psi \neq (0)$. Let m_Ψ (resp. p_Ψ) be the prime ideal of R_ν (resp. R) corresponding to Ψ, that is,

$$m_\Psi = \{x \in R_\nu \mid \nu(x) \notin \Psi\} \text{ and } p_\Psi = m_\Psi \cap R.$$

The valuation ν is composed of a *residual valuation* $\bar{\nu}_\Psi$, whose valuation ring $R_{\bar{\nu}_\Psi}$ is the quotient R_ν / m_Ψ and with values in Ψ, and a valuation ν'_Ψ whose valuation ring is the localization R_{m_Ψ} and with values in Φ/Ψ.
With the usual notation, $\nu = \nu'_\Psi \circ \bar{\nu}_\Psi$.

Totally ordered abelian groups (5)

For every $x \in R_\nu \setminus m_\Psi$, we have $\bar{\nu}_\Psi(\bar{x}) = \nu(x)$, where $\bar{x}$ denotes the residue class of x in R_ν/m_Ψ. We have an injective local ring map $R/p_\Psi \hookrightarrow R_\nu/m_\Psi$ and the valuation $\bar{\nu}_\Psi$ induces by restriction a valuation centered in R/p_Ψ (with value group contained in Ψ). We denote this valuation also by $\bar{\nu}_\Psi$ and call it the *residual valuation on R/p_Ψ*. We extend its definition to the case of $\Psi = \Phi$ setting $p_\Phi = (0)$ and $\bar{\nu}_\Phi = \nu$.

Totally ordered abelian groups (6)

Example: Let $f \in R$ be such that $R/(f)$ is a domain endowed with a valuation $\overline{\nu}$ with value group Ψ, and for $x \in R$ let n be the largest integer a such that $x \in f^a R$. Write $x = f^n g$ with $g \notin fR$.Then the map $x \mapsto (n, \overline{\nu}(g mod. fR)) \in (\mathbf{Z} \oplus \Psi)_{lex} = \Phi$ is a valuation of rank equal to the rank of $\overline{\nu}$ plus one.

Framework: The associated graded ring

- For any subring R of R_ν we can define the valuation ideals

$$\mathcal{P}_\phi(R) = \{x \in R | \nu(x) \geq \phi\} \text{ and } \mathcal{P}_\phi^+(R) = \{x \in R | \nu(x) > \phi\}.$$

Implicitly, the element $0 \in R$ belongs to each of these.
If R is noetherian and so the semigroup of values $\Gamma = \nu(R \setminus \{0\}) \subset \Phi_{\geq 0}$ is well ordered, and $\phi \in \Gamma$, then $\mathcal{P}_\phi^+(R) = \mathcal{P}_{\phi^+}(R)$, where ϕ^+ is the successor of ϕ in Γ. If $\phi \notin \Gamma$, then $\mathcal{P}_\phi^+(R) = \mathcal{P}_\phi(R)$.

- For any subring R of R_ν, define the associated graded ring

$$\mathrm{gr}_\nu R = \bigoplus_{\phi \in \Phi} \mathcal{P}_\phi(R) / \mathcal{P}_\phi^+(R).$$

It is not noetherian in general, even if R is.

- The degree zero part of $\mathrm{gr}_\nu R_\nu$ is k_ν and $\mathrm{gr}_\nu R_\nu$ is isomorphic to the semigroup algebra $k_\nu[t^{\Phi_{\geq 0}}]$.

Framework: The initial form

Each element x of a valued ring (R,ν) has an initial form $\mathrm{in}_\nu x$ which is its image in the quotient $\mathcal{P}_{\nu(x)}(R)/\mathcal{P}^+_{\nu(x)}(R)$. The initial form of 0 is $0 \in \mathrm{gr}_\nu R$.

• We have $\mathrm{in}_\nu xy = \mathrm{in}_\nu x.\mathrm{in}_\nu y$ since $\nu(xy) = \nu(x)\nu(y)$: $\mathrm{gr}_\nu R$ is a domain!

• $\mathrm{in}_\nu(x+y) = \mathrm{in}_\nu x$ if $\nu(x) < \nu(y)$, and $\mathrm{in}_\nu(x+y) = \mathrm{in}_\nu x + \mathrm{in}_\nu y$ if $\nu(x) = \nu(y)$ UNLESS $\mathrm{in}_\nu x + \mathrm{in}_\nu y = 0$ and that happens if and only if $\nu(x) = \nu(y)$ and $\nu(x+y) > \min(\nu(x),\nu(y))$.

Henselization (1)

A local ring R is henselian if given a monic polynomial $p(X) \in R[X]$, any factorization of the image $\overline{p}(X) \in (R/m)[X]$ into a product of coprime monic polynomials lifts to a decomposition of $p(X)$ in $R[X]$.
It is in fact equivalent to the fact that a simple root in R/m of $\overline{p}(X) \in (R/m)[X]$ lifts to a simple root of $p(X)$.
This definition, due to Azumaya, reveals itself thank to work of Nagata and Lafon.

The henselization of a local ring (R, m) is a local ring (R^h, m^h) with a local morphism $R \to R^h$ which uniquely factorizes local maps from R to a henselian local ring.

Henselization (2)

Proposition

Let (R, m_R) be a local ring and (S, m_S) a local R-algebra. The following assertions are equivalent:

1. *S is a localization of a finite R-algebra and is flat over R, and $S/m_R S = R/m_R = S/m_S$.*
2. *S is of the form $(R[X]/(F(X)))_{\mathcal{N}}$ where F(X) is a unitary polynomial of the form*

$$X^n + a_1 X^{n-1} + \cdots + a_{n-1}X - a_n,$$
$$\text{where } a_i \in R \text{ for } 1 \leq i \leq n \text{ and } a_n \in m_R,\ a_{n-1} \notin m_R,$$

and $\mathcal{N}$ is the maximal ideal of $R[X]/(F(X))$ containing the class x of X modulo $F(X)$, which is the image of the maximal ideal (m_R, X) of $R[X]$.

Henselization (3)

The preceding are also equivalent to:

• S is a localization of a finite R-algebra, and for every local subalgebra R_0 dominated by R which is essentially of finite type over $\mathbf{Z}$ and contains the coefficients of $F(X)$ so that $F(X) \in R_0[X]$, the natural map $R_0 \to S_0 = (R_0[X]/(F(X)))_{\mathcal{N}_0}$, where $\mathcal{N}_0$ is the image of (m_{R_0}, X), induces an isomorphism of the completions.

• We note that the polynomials in (2) above, called Nagata polynomials, are simple cases of those appearing in the definition of henselian rings: modulo the maximal ideal, they are of the form $XQ(X)$ with $Q(0) \neq 0$ so that X and $Q(X)$ are coprime in $k[X]$. We adopt the convention that the constant term of a Nagata polynomial has a minus sign.

Henselization (4)

Lafon calls such extensions $R \to S$ *Nagata extensions*; they are also called standard étale extensions of R or, assuming that R is noetherian, étale R-algebras quasi-isomorphic to R.
Morphisms of Nagata extensions of R are local morphisms of local R-algebras. A morphism from a Nagata extension S to another one S' exists if and only if there is an element ξ' in the maximal ideal of S' such that $F(\xi') = 0$. There exists at most one such morphism, determined by sending the image $x \in S$ of X to $\xi' \in S'$ and then S' is a Nagata extension of S. Lafon proves that Nagata extensions of R form an inductive system and that:

The henselization R^h of R is the inductive limit of its Nagata extensions. In particular it has the same residue field as R.

Henselization (5)

So we see that the henselization of R is the smallest local ring R^h containing R and the solutions of all Nagata polynomials with coefficients in R^h. If R is noetherian its m-adic topology is separated and R^h is contained in its m-adic completion.

In order to study extensions of a valuation ν centered in R to R^h it suffices to study its extensions to Nagata extensions $(R[X]/(F(X)))_{\mathcal{N}}$. As we shall see, extending ν essentially amounts to attributing a value to $h(\sigma_\infty)$ for every polynomial $h(X) \in R[X]$, where σ_∞ is a uniquely determined root of $F(X) = 0$.

Before embarking on the study of Nagata polynomials, let us state our main result:

Henselization (6)

Given a valuation ν centered in the local domain R:

1. There exists a unique prime ideal $H(\nu)$ of R^h lying over the zero ideal of R such that ν extends to a valuation $\tilde{\nu}$ centered in $R^h/H(\nu)$ through the inclusion $R \subset R^h/H(\nu)$. In addition, the ideal $H(\nu)$ is a minimal prime and the extension $\tilde{\nu}$ is unique.
2. With the notation of (1), the valuations ν and $\tilde{\nu}$ have the same value group.

Henselization (7)

One defines a semivaluation of a ring R centered at a prime P of R as a valuation on R/P. In other words a semivaluation is just like a valuation except that $\nu(x) = \infty$ does not imply $x = 0$: the prime P is the ideal of elements with infinite value. Then we can paraphrase

A valuation centered in R extends uniquely to a semivaluation of R^h having the same value group and centered at a minimal prime.

Henselization (8)

This is essentially equivalent to a known result in the theory of valued fields:

The henselization of a valuation ring is a valuation ring with the same value group.

The difference is that instead of a Galois-theoretic approach our approach is essentially computational/constructive and has some interesting consequences.

Nagata polynomials (1)

Keeping the notations above, note that if $a_n = 0$, then S is isomorphic to R. The extension is also trivial when $n = 1$. Note also that given any element $\alpha \in m_R$, the polynomial $F_\alpha(X') = F(X' + \alpha) \in R[X']$ with $X' = X - \alpha$ satisfies the same conditions as $F(X)$. Indeed, $F_\alpha(0) = F(\alpha) \in m_R$; and the coefficient of X' in $F_\alpha(X')$ is $F'(\alpha)$, which is not in m_R since $F'(0)$ is not and $\alpha \in m_R$. Moreover, $F_\alpha(X')$ defines the same extension, that is, S is isomorphic to $S_\alpha = (R[X']/(F_\alpha(X')))_{\mathcal{N}'}$. This implies that the Nagata extension defined by the Nagata polynomial $F(X)$ is trivial if and only if $F(X)$ has a zero in the maximal ideal of R.

Nagata polynomials (2)

As a consequence of the following result, we may assume in the definition of a Nagata extension that the polynomial $F(X)$ is irreducible in $R[X]$.

Lemma

Let R be a local domain and let $F(X) \in R[X]$ be a Nagata polynomial. Let $F(X) = G(X)Q(X)$ be a factorization in $R[X]$, where up to multiplication by a unit of R we write

$$G(X) = X^s + \cdots + g_{s-1}X + g_s; \qquad Q(X) = X^t + \cdots + q_{t-1}X - q_t.$$

Then, one of the two polynomials $G(X),\ Q(X)$ must be a Nagata polynomial. It is the factor whose constant term is in m_R. If it is $Q(X)$, then $G(X) \notin (m_R, X)$.

Nagata polynomials (3)

Lemma

Let $F(X) = X^n + a_1 X^{n-1} + \cdots + a_{n-1}X - a_n \in R[X]$ *be a Nagata polynomial and note that as an element of* $R[X]$*, the polynomial* $F(X)$ *is the same as*

$$F^{(1)}(X_1) = F\left(X_1 - \frac{F(0)}{F'(0)}\right) = F\left(X_1 + \frac{a_n}{a_{n-1}}\right)$$

since $X \mapsto X_1 + \frac{a_n}{a_{n-1}}$ *is a change of variable in* $R[X]$*. Substituting* $X_1 + \frac{a_n}{a_{n-1}}$ *to* X *in* $F(X)$*, write the result* $F^{(1)}(X_1) = X_1^n + a_1^{(1)} X_1^{n-1} + \cdots + a_{n-1}^{(1)} X_1 - a_n^{(1)}$*. Then we have:*

1. *The polynomial* $F^{(1)}(X_1) \in R[X_1]$ *is a Nagata polynomial.*
2. *The coefficient* $a_i^{(1)}$ *is congruent to* a_i *modulo* $\frac{a_n}{a_{n-1}}$*.*
3. $F^{(1)}(0) = -a_n^{(1)} \in a_n^2 R$*.*

Nagata polynomials (4)

A statement equivalent to (3) above is:
Let $R \to S$ be the Nagata extension defined by $F(X)$. Denoting by x, x_1 the images in S of X, X_1, we have $x_1 \in x^2 S$. In particular, if $\tilde{\nu}$ is any semivaluation on S extending the valuation ν on R, the inequality $\tilde{\nu}(x_1) \geq 2\tilde{\nu}(x)$ holds.

Nagata polynomials (5)

As a consequence, starting from a Nagata polynomial $F(X) \in R[X]$, we can iterate the construction just described to produce:

- A sequence of generators $X_i := X_{i-1} + \frac{F^{(i-1)}(0)}{(F^{(i-1)})'(0)}$ for the polynomial ring $R[X]$, with $X_0 = X$.
- Polynomials $F^{(i)}(X_i) := F^{(i-1)}\left(X_i - \frac{F^{(i-1)}(0)}{(F^{(i-1)})'(0)}\right) \in R[X_i]$, with $F^{(0)}(X) = F(X)$.

Nagata polynomials (6)

Definition

Let ν be a valuation centered in a local domain R and let $F(X) \in R[X]$ be a Nagata polynomial. Keep the previous notations. We define the following elements of m_R:

$$\delta_k := \frac{a_n^{(k)}}{a_{n-1}^{(k)}} = -\frac{F^{(k)}(0)}{(F^{(k)})'(0)}, \text{ for } k \geq 0.$$

$$\sigma_i := \sum_{k=0}^{i-1} \delta_k, \text{ for } i \geq 1.$$

We say that $(\delta_i)_{i \in \mathbf{N}}$ and $(\sigma_i)_{i \geq 1}$ are the *Newton sequence of values* and the *sequence of partial sums* attached to $F(X)$, respectively.

Nagata polynomials (7)

The polynomials $(F^{(i)}(X_i))_{i\in\mathbf{N}}$ all define the same Nagata extension of R. If at some step $i \geq 0$ we find $F^{(i)}(0) = 0$, this implies that $F(X)$ defines a trivial extension, so we may assume that this does not happen and we shall do so.
By construction, we have $X = X_i + \sigma_i$ and $x_{i+1} = x_i - \delta_i$. We verify by induction that $F^{(i)}(X_i) = F\,(X_i + \sigma_i)$ for $i \geq 1$. Setting $X_i = 0$ in this identity, we can read the definition of δ_i as given by the equality $F'(\sigma_i)\delta_i = -F(\sigma_i)$. Observe that for all $i \geq 1$, $F(\sigma_i) \neq 0$ because the Nagata extension is not trivial, and $\nu(\delta_i) = \nu(F(\sigma_i))$.

Nagata polynomials (8)

Assuming for a moment that R is complete and separated for the m_R-adic topology, the images in S of the elements X_i converge to $x_\infty = 0$ while the polynomials $F^{(i)}(X_i)$ converge to a polynomial $F^{(\infty)}(X_\infty)$ without constant term because $a_n^{(i)} \in m_R^{2^{i-1}}$. Therefore x_∞ is a root of $F^{(\infty)}(X_\infty)$, which is simple since $a_{n-1}^{(\infty)} \notin m_R$. Since $x_\infty = x - \sum_{k=0}^{\infty} \delta_k$ and $F^{(\infty)}(X_\infty) = F(X)$ this tells us that $\sum_{k=0}^{\infty} \delta_k$ is a simple root of $F(X)$, which is contained in the maximal ideal m_R of R. Since our assumption on $F(X)$ is equivalent to the statement that the image of $F(X)$ in $k[X]$, where $k = R/m_R$, has 0 as a simple root, this is indeed a version of Hensel's lemma.

Nagata polynomials (9)

We stress the fact that by our assumption that the Nagata extension is not trivial we have $\delta_0 \in m_R \setminus \{0\}$ and δ_{i+1} is a non zero multiple of δ_i^2 for any $i \geq 0$, so that we expect to have a root of $F(X)$ which is represented as a sum $\sum_{k=0}^{\infty} \delta_k$ of elements of strictly increasing valuations.

This sum may not have a meaning in the m-adic topology if it is not separated, but since for $i' > i$ we have $\nu(\sigma_{i'} - \sigma_i) = \nu(\delta_i)$ and the $\nu(\delta_i)$ increase, the sequence of the $\nu(\sigma_i)$ is always a pseudo-convergent sequence.

Pseudo-convergent sequences (1)

A *pseudo-convergent* sequence (They are also known as pseudo-Cauchy sequences. This concept is due to Ostrowski) of elements of a valued ring (R, ν) is a sequence $(y_\tau)_{\tau \in T}$ indexed by a well ordered set T without last element, which satisfies the condition that whenever $\tau < \tau' < \tau"$ we have $\nu(y_{\tau'} - y_\tau) < \nu(y_{\tau"} - y_{\tau'})$.
An element y is said to be a *pseudo-limit*,or simply limit, of this pseudo-convergent sequence if $\nu(y_{\tau'} - y_\tau) \leq \nu(y - y_\tau)$ for $\tau, \tau' \in T,\ \tau < \tau'$.
One observes that if (y_τ) is pseudo-convergent, for each $\tau \in T$ the value $\nu(y_{\tau'} - y_\tau)$ is independent of $\tau' > \tau$ and can be denoted by w_τ.
The balls $B(y_\tau, w_\tau) = \{x \in R | \nu(x - y_\tau) \geq w_\tau\}$ then form a strictly nested sequence of balls and their intersection is the set of pseudo-limits of the sequence. In particular, if in our ring every pseudo-convergent sequence has a pseudo-limit it is said to be spherically complete.

Pseudo-convergent sequences (2)

For example, assume that our value group Φ has rank > 1 and let Ψ be a non trivial convex subgroup of Φ. Let u_i be a family of elements of R with values γ_i indexed by a totally ordered set I as above and strictly increasing. Then the sum $y = \sum_{i \in I} u_i$, assuming it exist, is a pseudo-limit of the pseudo-convergent sequence $y_\tau = \sum_{i \leq \tau} u_i$, but if we now take any element z whose value is not in Ψ, then $y + z$ is another pseudo-limit. If we consider the sequence $\tilde{y}_\tau = \sum_{i \leq \tau}(u_i + z)$ it is still pseudo-convergent and certainly cannot have a limit in our ring, but its image in the quotient R/p_Ψ has a limit.

Nagata polynomials (10)

Proposition

Let $F(X) \in R[X]$ be a Nagata polynomial. Given an extension $\tilde{\nu}$ of ν to $\overline{K}$, there exists a unique root of $F(X)$ in $\overline{K}$ with positive $\tilde{\nu}$-value. If we call σ_∞ this root of $F(X)$, then the following also holds:

1. *σ_∞ is a limit of the pseudo–convergent sequence $(\sigma_i)_{i\geq 1}$ associated to $F(X)$.*
2. *For any $z \in \overline{K} \setminus \{\sigma_\infty\}$ such that $F(z) = 0$ we have $\tilde{\nu}(z) = 0$.*
3. *σ_∞ is a simple root of $F(X)$.*

Nagata polynomials (11)

Proof.

Write $F(X) = \prod_{j=1}^{n}(X - r_j)$ in $\overline{K}[X]$. For all $i \geq 1$, we have $\nu(F(\sigma_i)) = \sum_{j=1}^{n} \tilde{\nu}(\sigma_i - r_j)$. Hence if none of the r_j is a limit of the pseudo–convergent sequence $(\sigma_i)_{i\geq 1}$ then $(\nu(F(\sigma_i)))_{i\geq 1}$ is eventually constant. However $\nu(F(\sigma_i)) = \nu(\delta_i)$ for all $i \geq 1$, so we can assume that r_1 is a limit of $(\sigma_i)_{i\geq 1}$. In particular, $\tilde{\nu}(\sigma_i - r_1) = \nu(\sigma_{i+1} - \sigma_i) = \nu(\delta_i)$ for all $i \geq 1$.

For $1 \leq j \leq n$, we have $\tilde{\nu}(r_j) \geq 0$ because r_j is integral over R. In addition, $\nu(\sigma_i) = \nu(\delta_0) = \nu(F(0)) = \sum_{j=1}^{n} \tilde{\nu}(r_j)$ for all $i \geq 1$. If $\nu(\sigma_i) > \tilde{\nu}(r_1)$ for some i, we obtain $\nu(\delta_i) = \tilde{\nu}(\sigma_i - r_1) = \tilde{\nu}(r_1) < \nu(\delta_0)$, which gives us a contradiction. We conclude that $\tilde{\nu}(r_j) = 0$ if $j \neq 1$ and $\tilde{\nu}(r_1) = \nu(\delta_0) > 0$. □

Nagata polynomials (12)

The natural homomorphism $R[X]/(F(X)) \to K(\sigma_\infty) \subset \overline{K}$ determined by $h(X) \mapsto h(\sigma_\infty)$ induces a homomorphism of R-algebras

$$E_S(\nu)\colon S = (R[X]/(F(X)))_{\mathcal{N}} \longrightarrow K(\sigma_\infty).$$

Definition

Let ν be a valuation centered in the local domain R and let S be a Nagata extension of R determined by the polynomial $F(X) \in R[X]$. The kernel of the homomorphism $E_S(\nu)\colon S \to \overline{K}$ defined above is denoted by $H_S(\nu)$.

Nagata polynomials (13)

Observe that the ideal $H_S(\nu)$ of S depends only on the valuation ν since it depends only on σ_∞. It has the following properties:

Lemma

Let ν be a valuation centered in a local domain R. Then:

1. *For any Nagata extension $R \to S$, we have $H_S(\nu) \cap R = (0)$ so that the ideal $H_S(\nu)$ is a minimal prime of S.*
2. *Given a map $f\colon S \to S'$ of Nagata extensions of R, we have $f^{-1}(H_{S'}(\nu)) = H_S(\nu)$.*

Nagata polynomials (14)

Proposition

There is a unique valuation centered in $S/H_S(\nu)$ which extends ν through the inclusion $R \subset S/H_S(\nu)$.

Proof.

Any such extension of ν can be obtained in the way explained above starting from an extension to $\overline{K}$. Therefore it suffices to take two extensions $\tilde{\nu}$ and $\tilde{\nu}'$ of ν to $\overline{K}$ and show that $\tilde{\nu}_S = \tilde{\nu}'_S$. In that situation, there exists a K–automorphism π of $\overline{K}$ such that $\tilde{\nu}' = \tilde{\nu} \circ \pi$. Let σ_∞ and σ'_∞ be the distinguished roots of $F(X)$ in $\overline{K}$ associated to $\tilde{\nu}$ and $\tilde{\nu}'$, respectively. Since $\tilde{\nu}'(\pi^{-1}(\sigma_\infty)) = \tilde{\nu}(\sigma_\infty) > 0$, the automorphism π must send σ'_∞ to σ_∞. We have $\pi_{\sigma_\infty} = \pi|_{K(\sigma'_\infty)} \circ \pi_{\sigma'_\infty}$ and $\tilde{\nu}_S = \tilde{\nu}'_S$. □

Nagata polynomials (15)

Let us prove that the valuation ν uniquely determines the support of the semivaluation which extends it to the henselization:

Proposition

Let ν be a valuation centered in a local domain R and let $R \to S$ be a Nagata extension. If $\mathfrak{p}$ is a prime ideal of S such that $\mathfrak{p} \cap R = (0)$ and ν extends to a valuation centered in $S/\mathfrak{p}$ through the inclusion $R \subset S/\mathfrak{p}$, then $\mathfrak{p} = H_S(\nu)$.

Nagata polynomials (16)

Now by passing to the inductive limit on Nagata extensions, we prove almost immediately that given a valuation ν on R there is a uniquely determined minimal prime H of R^h, the limit over Nagata extensions S of R of the $H_S(\nu)$ and a unique extension of ν to R^h/H.

Same group (1)

Now we have to prove that the value groups are the same. Our method is to fix a presentation of the previous quotient as a local R–algebra $R[\sigma_\infty]_{(m_R,\sigma_\infty)} \subset \overline{K}$ and investigate the way in which ν determines the value of the extended valuation $\tilde{\nu}$ on each element $h(\sigma_\infty)$ with $h(X) \in R[X]$. We describe the behavior of the valuations $\nu(h(\sigma_i))$, $i \geq 1$. Indeed, if these valuations form an eventually constant sequence, then their stationary value is $\tilde{\nu}(h(\sigma_\infty))$; and otherwise, they are cofinal in a certain convex subgroup of Φ. Except in some particular cases (for instance, if the valuation ν is of rank one, in which case the cofinality implies that $h(\sigma_\infty) = 0$), this is not sufficient to obtain the desired result.

Same group (2)

Let us consider the sequences $(\delta_i)_{i\in\mathbf{N}}$ and $(\sigma_i)_{i\geq 1}$ attached to the Nagata polynomial $F(X) \in R[X]$. For $i \geq 1$, set

$$\eta_i := \sigma_\infty - \sigma_i \in R[\sigma_\infty]_*.$$

As we saw, we have that $\tilde{\nu}(\eta_i) = \nu(\delta_i)$ for all $i \geq 1$.
Let $h(X)$ be a polynomial in $R[X]$ of degree $s \geq 0$. We note that $h(\sigma_i) \in R$ for all $i \geq 1$ and we are going to study the behavior of the $\nu(h(\sigma_i))$ as i increases. Since the Nagata extension is non trivial, the σ_i are all different and $h(\sigma_i) \neq 0$ for all i large enough.

Same group (3)

Consider the usual expansion

$$h(X+\alpha)=\sum_{m=0}^{s} h_m(X)\alpha^m$$

of $h(X+\alpha)$ as a polynomial in X and α. If the polynomial $h_m(X)$ is not zero, its degree is $s-m$.

The maps $\partial_m\colon h(X)\mapsto h_m(X)$ are Hasse–Schmidt derivations satisfying the identities $\partial_m\circ\partial_{m'}=\partial_{m'}\circ\partial_m=\binom{m+m'}{m}\partial_{m+m'}$. Some use the mnemonic notation $\partial_m=\frac{1}{m!}\frac{\partial^m}{\partial X^m}$.

Same group (4)

We have the following identities in $K(\sigma_\infty)$:

$$h(\sigma_\infty) = h(\sigma_i) + \sum_{m=1}^{s} h_m(\sigma_i)\eta_i^m, \quad (*)$$

$$h(\sigma_i) = h(\sigma_\infty) + \sum_{m=1}^{s} h_m(\sigma_\infty)(-1)^m\eta_i^m, \quad (**)$$

Since $\sigma_{i+1} = \sigma_i + \delta_i$, we also have the identity:

$$h(\sigma_{i+1}) = h(\sigma_i) + \sum_{m=1}^{s} h_m(\sigma_i)\delta_i^m. \quad (***)$$

Same group (5)

A consequence of these identities is:

Lemma

The subgroup of the value group Φ of ν generated by the values of the δ_i is finitely generated and therefore of finite rational rank.

Same group (6)

Proposition

(Ostrowski-Kaplansky) *Let* Φ *be a totally ordered abelian group. Let* $\beta_1, \ldots, \beta_s \in \Phi$ *and distinct integers* $t_1, \ldots, t_s \in \mathbf{N} \setminus \{0\}$ *be given. Let* $(\gamma_\tau)_{\tau \in T}$ *be a strictly increasing family of elements of* Φ *indexed by a well ordered set* T *without last element. There exist an element* $\iota \in T$ *and a permutation* $(k_1, \ldots, k_s)$ *of* $(1, \ldots, s)$ *such that for all* $\tau \geq \iota$ *we have the inequalities*

$$\beta_{k_1} + t_{k_1}\gamma_\tau < \beta_{k_2} + t_{k_2}\gamma_\tau < \cdots < \beta_{k_s} + t_{k_s}\gamma_\tau.$$

Same group (7)

Proposition

Let $h(X) \in R[X]$ be a polynomial of degree $s > 0$. There exist $i_0 \in \mathbf{N}$ and $k \in \{1, \ldots, s\}$ such that for $i \geq i_0$ we have

$$\mathrm{in}_{\tilde{\nu}}(h(\sigma_\infty) - h(\sigma_i)) = -\mathrm{in}_{\tilde{\nu}}(h_k(\sigma_\infty)(-1)^k \eta_i^k).$$

In particular, $\tilde{\nu}(h(\sigma_\infty) - h(\sigma_i)) = \tilde{\nu}(h_k(\sigma_\infty)(-1)^k \eta_i^k)$ for $i \geq i_0$ and $h(\sigma_\infty)$ is a limit for the valuation $\tilde{\nu}$ of the pseudo–convergent sequence $(h(\sigma_i))_{i \geq i_0}$.

Same group (8)

The polynomial $h_s(X)$ is a nonzero constant polynomial. If $s = 1$ then the first statement is trivial. In the general case it is enough to apply the theorem of Ostrowski-Kaplansky to the $\beta_m = \tilde{\nu}(h_m(\sigma_\infty))$ in the value group $\tilde{\Phi}$ of $\tilde{\nu}$, with $t_m = m$, $\gamma_i = \nu(\delta_i)$, and $T = \mathbf{N}$, recalling that $\tilde{\nu}(\eta_i) = \nu(\delta_i)$ and $\nu(\delta_{i+1}) \geq 2\nu(\delta_i)$.

Same group (9)

This Proposition is essentially a slightly more precise version of a result of Ostrowski to the effect that the values taken by a polynomial with coefficients in R on a pseudo–convergent sequence of elements of R (in this case the σ_i) form themselves a pseudo–convergent sequence and therefore their valuations are eventually either constant or strictly increasing. Ostrowski's result, proved for rank one valuations, is more general in that it applies to all pseudo–convergent sequences.

Same group (10)

We can now prove our result in the case where the value group Φ of ν is of rank one: In this case the $\nu(\delta_i)$ are cofinal in Φ since $\nu(\delta_{i+1}) \geq 2\nu(\delta_i)$ and **R** is archimedian. So either the pseudo-convergent sequence $\nu(h(\sigma_i))$ is eventually constant, and then its stationary value is $\nu(h(\sigma_\infty))$ which is the value of $\tilde{\nu}$ on $h(X) \in S$, or we have $h(\sigma_\infty) = 0$ which means that $h(X) \in H_S(\nu)$. This means that either the value of $h(X)$ is infinite, and its image in in $S/H_S(\nu)$ is zero, or this value is in Φ.

Same group (11)

In the general case, the first idea is to consider the smallest convex subgroup of Φ containing all the $\nu(\delta_i)$, but unfortunately, things are not so simple. We must begin with the

Definition

Let ν be a valuation centered in a local domain R and let Φ be its value group. Let $F(X) \in R[X]$ be a Nagata polynomial defining a non trivial Nagata extension of R. The *intrinsic convex subgroup* Ψ *of* Φ *associated to* $F(X)$ is the smallest convex subgroup of Φ containing all the $\nu(F(a))$ with $a \in m_R$.

Then we can prove that if the rank of Φ is finite, we can always make a change of variable $X \mapsto X - a$, with $a \in m_R$ so that for the new Nagata polynomial, the group Ψ is the smallest convex subgroup of Φ containing all the (new) $\nu(\delta_i)$ and they are cofinal in it.

Same group (12)

However, even after passing this first hurdle, it turns out that things work as we wish only if $F(X)$ is not only irreducible, but ν-residually irreducible in the following sense:

Definition

Let ν be a valuation centered in a local domain R and let $F(X) \in R[X]$ be a Nagata polynomial. Let Ψ_F be the convex subgroup of the value group of ν attached to $F(X)$ as we saw above and let p_{Ψ_F} be the corresponding prime ideal of R. We say that $F(X)$ is *ν-residually irreducible* if the image $\overline{F}(X) \in R/p_{\Psi_F}[X]$ of $F(X)$ is irreducible in $L_F[X]$, where L_F denotes the fraction field of R/p_{Ψ_F}.

Same group (13)

This irreducibility implies that $\overline{F}(X)$ defines a non trivial Nagata extension of R/p_{Ψ_F} and that Ψ_F is the intrinsic convex subgroup of the Nagata extension defined by $F(X)$. It also implies that for any polynomial $h(X) \in R[X]$ such that $0 \leq \deg h(X) < \deg F^*(X)$, we have for large i the equality $\mathrm{in}_\nu(h(\sigma_i)) = \mathrm{in}_{\tilde{\nu}}(h(\sigma_\infty))$.
Here $F^*(X)$ is the minimal polynomial of σ_∞ which, up to normalization, we may assume to be in $R[X]$.

Same group (14)

Kaplansky showed that a pseudo-convergent sequence $(y_\tau)_{\tau\in T}$ of elements of a valued field (K, ν) which is of algebraic type and has no limit in K defines an algebraic extension with the same value group and residue field by the adjunction of a root of a polynomial $Q(X) \in K[X]$ of minimal degree among those polynomials $h(X) \in K[X]$ for which $\nu(h(y_\tau))$ is not eventually stationary.

Same group (15)

In our situation the issue is that a Nagata polynomial $F(X)$ is a polynomial of minimal degree attached to the pseudo-convergent sequence $(\sigma_i)_{i\in\mathbf{N}}$ *only if it is ν-residually irreducible*. To overcome this difficulty we build a sequence of nested Nagata extensions $R_j = R_\nu[\sigma_\infty, \tau_\infty^{(0)}, \tau_\infty^{(1)}, \tau_\infty^{(2)}, \cdots, \tau_\infty^{(j)}]_{m_j}$ of R_ν by Nagata polynomials which do satisfy the ν-residual irreductiblity condition and thus provide well defined extensions $(R_j, \nu^{(j)})$ with the same value group.

Same group (16)

The construction stops only when $F_j(X) = F(X + \tau_\infty^{(0)} + \tau_\infty^{(1)} + \tau_\infty^{(2)} + \cdots + \tau_\infty^{(j)})$ is $\nu^{(j)}$-residually irreducible in $R_{\nu^{(j)}}[X]$, and we prove that it has to stop because the constant terms $F_j(0)$ belong to a strictly increasing sequence of convex subgroups of the value group of ν. Then, the pseudo-convergent sequence attached to the Nagata polynomial $F_j(X) \in R_j[X]$ does define an immediate and unique extension of R_j in the manner described by Kaplansky. Since by construction the extension of ν to the valuation $\nu^{(j)}$ on R_j is unique and has the same value group, the result follows.

Same group (17)

Finally there is the issue that the polynomial $h(X) \in R[X]$ to which we want to attribute a value may be zero in $R/p_{\Psi_F}[X]$.
This is easily overcome by blowing-up in $\mathrm{Spec} R$ the ideal generatd by the coefficients of $h(X)$ and replacing R by the local ring R' at the point picked by the valuation; then $R' \subset R_\nu$. In $R'[X]$ we have $h(X) = h_t h'(X)$ where $h'(X)$ has a coefficient equal to one so that its image in $R'/p'_\Psi(F)$ cannot be zero, and of course the value of h_t is in Φ.
Because the semigroup of values $\nu(R' \setminus \{0\})$ is in general larger that $\nu(R \setminus \{0\})$, this suggests that the semigroup of values $\nu(R^h \setminus \{0\})$ will in general be larger than that of $\nu(R \setminus \{0\})$, a phenomenon discovered by Cutkosky.

Applications (1)

Corollary

Let R be a local domain and let $\{H_\iota\}_{\iota\in I}$ be the set of minimal primes of R^h. Let $\varphi : \mathrm{RZ}(R) \to \bigsqcup_{\iota\in I} \mathrm{RZ}(R^h/H_\iota)$ be the map which to a valuation ring $R_\nu \in \mathrm{RZ}(R)$ associates the minimal prime $H(\nu)$ of R^h and the valuation ring $R_{\tilde{\nu}} \in \mathrm{RZ}(R^h/H(\nu))$ of the extension $\tilde{\nu}$ of ν to $R^h/H(\nu)$. Then, the map φ satisfies the following:

1. *It is a homeomorphism.*
2. *It induces a bijection between the set of connected components of $\mathrm{RZ}(R)$ and $\{H_\iota\}_{\iota\in I}$.*

Applications (2)

Here we study the extension of the valuation of a valuation ring to its henselization. This result concerns the approximation of elements of the henselization $(K^h, \tilde{\nu})$ of a valued field (K, ν) by elements of K and we can state it as follows since we know that $R_\nu^h = R_{\tilde{\nu}}$ and the value groups are equal.

Theorem

(Kuhlmann) *Let K be a field endowed with a valuation ν determined by the valuation ring R_ν and let Φ be the value group of ν. Let K^h be the field of fractions of the henselization $R_\nu^h = R_{\tilde{\nu}}$ of R_ν. For every element $z \in K^h \setminus K$ there exist a convex subgroup Ψ of Φ and an element $\varphi \in \Phi$ such that $\varphi + \Psi$ is cofinal in the ordered set*

$$\tilde{\nu}(z - K) = \{\tilde{\nu}(z - c) \mid c \in K\} \subset \Phi.$$

Applications (3.1)

After Ostrowski and Kaplansky, one says that a pseudo–convergent sequence $(y_\tau)_{\tau\in T}$ of elements of a valued field (K,ν) is of *algebraic type* if there exist polynomials $h(X)\in K[X]$ such that $(\nu(h(y_\tau)))_{\tau\in T}$ is not eventually constant. We propose the following, where as usual $\tau+1$ designates the successor of τ in the well ordered set T:

Definition

Let ν be a valuation centered in a local domain R. A pseudo–convergent sequence $(y_\tau)_{\tau\in T}$ of elements of the maximal ideal m_R of R is *of étale type* if there exist polynomials $h(X)\in R[X]$ such that one has the equality $\nu(h(y_\tau))=\nu(y_{\tau+1}-y_\tau)$ for $\tau\geq\tau_0\in T$, where τ_0 may depend on the polynomial $h(X)$.

Applications (3.2)

Proposition

Let R be a local domain with maximal ideal m_R, and let ν be a valuation of finite rank centered in R. The local domain R is henselian if and only if every pseudo–convergent sequence of elements of m_R which is of étale type has a limit in m_R.

A. de Felipe, B. Teissier, Valuations and henselization, Math. Annalen 377 (2020) No. 3-4, 935-967.