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# INTRODUCTION TO EQUISINGULARITY PROBLEMS

B. TEISSIER

*"Better a house without roof than a house without view"*
*Hunza saying*

ABSTRACT

A short journey into the range of equisingularity. In § 1 classical material is presented, in § 2 equisingularity is studied from a numerical view point, by associating to a hypersurface with isolated singularity a generalized multiplicity which has an algebraic definition, but can also be defined topologically with the Milnor numbers of the generic plane sections of X in all dimensions. The results lead to a general conjecture concerning the behaviour of equisingularity with respect to projections and plane sections. In § 3, the general problem of relating the topology of a hypersurface to that of a generic plane section is studied, and new invariants are introduced, which seem to link together many numerical invariants of singularities : e.g. for hypersurfaces with isolated singularities, the Milnor number of the hypersurface and of its generic hyperplane section, the smallest possible exponent $\theta$ in the Łojasiewicz inequality $|\text{grad } f(z)| \geq C|f(z)|^{\theta}$, and the vanishing rates of certain gradient cells in the nearby non singular fiber.

## § 1

1.1 In these notes I shall present some features of the theory of equisingularity in characteristic zero which is gradually emerging after the fundamental work of Zariski (see $[Z_i]$, $i = 1,2,\ldots$). The basic question is to find algebraic criteria to decide when a space X, algebraic over $\mathbb{C}$, or complex analytic, can be said to be equisingular along a non singular subspace $Y \subset X$ at a point $0 \in Y$, the idea being that the singularities of X at all points of Y near 0 are alike in some sense. Roughly speaking, we would like equisingularity to be a condition as strong as possible, subject to the requirement of openness :

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(O.E.) The set of points $y \in Y$ such that $X$ is equisingular along $Y$ at $y$ is the complement of a nowhere dense closed subspace of $Y$.

This sort of question arises immediately when one tries to give a precise formulation to the intuitive idea that if you take a germ of reduced complex analytic hypersurface $(X_0, 0) \subset (\mathbb{C}^{n+1}, 0)$, or a germ of curve $(\Gamma, 0) \subset (\mathbb{C}^{n+1}, 0)$, then although it will not be true in general that "almost all" sections of $(X_0, 0)$ by hyperplanes through the origin in $\mathbb{C}^{n+1}$, or "almost all" images of $\Gamma$ by linear projections $(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^2, 0)$ are analytically isomorphic, still, almost all these sections or images must "look alike" in some way, near 0.
As an example, take the cone $x^4 + y^4 + zx^2y = 0$ in $\mathbb{C}^3$, cut it by hyperplanes $z = ax + by$, and see how the cross-ratio of the four lines you get (except for special values of a and b) varies with a and b. Now the cross-ratio is an invariant of the analytic type.
In both cases, we can reduce to the basic question : for hyperplane sections, we choose an open $U \subset \mathbb{C}^{n+1}$ in which an equation $f(z_0, \dots, z_n) = 0$ defining $X_0$ is convergent, homogeneous coordinates $(a_0 : \dots : a_n)$ on the $\mathbb{P}^n$ of hyperplanes, and consider the subspace $X$ of $U \times \mathbb{P}^n$ defined by the ideal $(f(z_0, \dots, z_n), \sum_0^n a_i z_i)$. The fiber of the induced projection $\pi : X \underset{\sigma}{\leftrightarrows} \mathbb{P}^n$ above a hypersurface $(a_0 : \dots : a_n)$ is precisely $X_0 \cap H$, and $\pi$ has a section $\sigma : \mathbb{P}^n \to X$ picking the origin in each fiber. So if we set $Y = \sigma(\mathbb{P}^n) = 0 \times \mathbb{P}^n \subset X$ we see that what we really want to do is to study equisingularity of $X$ along $Y$, i.e. we would like to say that there is a Zariski open dense $V \subset \mathbb{P}^n$ such that $X$ is equisingular along $0 \times \mathbb{P}^n$ at every point of $0 \times V$, and understand what this means for the fibers of $\pi$.
There is a similar construction for projections.
Now suppose that we are given in general a situation $\pi : X \underset{\sigma}{\leftrightarrows} Y$ such that the fibers of $\pi$, $(X_y, \sigma(y))$ are reduced hypersurfaces (and $\pi$ is flat).
There is a simple way of saying that two fibers of $\pi$ "look alike" :

<u>Definition</u> : Two germs of reduced complex analytic hypersurfaces $(X_1, x_1)$ and $(X_2, x_2)$ of the same dimension n are said to be of the same topological type if there exist representatives $(X_i, x_i) \subset (U_i, x_i)$ where $U_i$ is open in $\mathbb{C}^{n+1}$, $i = 1,2$, and a homeomorphism of pairs $(U_1, x_1) \approx (U_2, x_2)$ mapping $X_1$ to $X_2$.

In fact, given $\pi : X \underset{\sigma}{\leftrightarrows} Y$ such that $Y$ is reduced and the fibers of $\pi$ are reduced hypersurfaces one can show with the stratification arguments of Thom-Whitney and Mather, to which I will come back below (see [Ma ]), that there exists a nowhere dense closed analytic subset $F \subset Y$ such that for all $y \in Y-F$ the fibers $(X_y, \sigma(y))$ have the same topological type, so that the condition "topological type of the fibers locally constant" almost satisfies (O.E.). It is however a priori rather far from being algebraic !

Now it seems that Zariski's algebraic approach to equisingularity did not quite arise from such problems, but rather from his work on resolution of singularities.
The inductive procedure proposed by Zariski to resolve singularities of a (pure-dimensional) variety X (say, a hypersurface in some non-singular projective variety) was to look at a generic map from X to a non-singular projective variety of the same dimension say $\pi : X \to S$, $\pi$ a finite map, then look at the branch locus $D \subset S$ of $\pi$, and by the inductive assumption, (embedded resolutions ; see Lipman's lectures), obtain a map $h : S' \to S$ with S' non-singular and $h^{-1}(D)$ a divisor with normal crossings in S'. (See Lipman's lectures in this volume). By pull-back we get a finite map $\pi' : X' \to S'$, the branch locus of which has only normal crossings, and therefore one can hope that the singularities of X' are easier to resolve. (On the other hand the map $X' \to X$ is not too hard to analyze). Therefore, the first thing to look at is a finite map such as $\pi' : X' \to S'$, the branch locus of which is non singular (I shall say smooth). Here several beautiful things happen simultaneously, which I shall try to condensate, at least in the special case of hypersurfaces.

- In fact, from now on, I shall examine the basic problem only in the case where X is a complex analytic hypersurface (and of course in a neighborhood of a given point $0 \in X$) ; hypersurface because in the general case I know too little, and complex analytic because there will be transcendental avatars of the algebraic problems occasionnally.

<u>Warning</u> : While not writing the structure sheaves, the inverse images will always be meant ideal-theoretically, and $Z_{red}$ will mean the space Z with reduced structure ; furthermore, everything being local, (Z,z) will usually stand for a small enough representative of a germ. I think "small enough" will have a clear meaning in each situation.

1.2 Now for the properties of hypersurfaces with smooth branch locus.

<u>Theorem 1</u> (Zariski $[Z_{1,2}]$) : Let $(X,0) \subset (\mathbb{C}^{N+1},0)$ be a reduced hypersurface, and $\pi : (\mathbb{C}^{N+1},0) \to (\mathbb{C}^N,0)$ a holomorphic projection such that $\pi^{-1}(0) \not\subset X$.

By the Weierstrass preparation theorem we can describe this situation by choosing as equation for X :

$$F = z_{N+1}^{\nu} + A_1(z_1,\ldots,z_N)z_{N+1}^{\nu+1} + \ldots + A_\nu(z_1,\ldots,z_N) = 0. \quad A_i \in \mathbb{C}\{z_1,\ldots,z_N\}.$$

Let $R \in \mathbb{C}\{z_1,\ldots,z_N\}$ be the $z_{N+1}$-discriminant of F. The following conditions are equivalent :

i) The branch locus $Y_\pi$ of $\pi|X$ is smooth at 0 (i.e., by an analytic change of the coordinates $z_1,\ldots,z_N$, we can write in $\mathbb{C}\{z_1,\ldots,z_N\}$:

$$R(z_1,\ldots,z_N) = z_1^{\Delta}.\varepsilon(z_1,\ldots,z_N) \ ; \ \varepsilon(0,\ldots,0) \neq 0 \ ; \ \Delta \in \mathbb{N}).$$

ii) $Y = (\pi^{-1}(Y_\pi) \cap X)_{red}$ is smooth at 0, $\pi$ induces an isomorphism $(Y,0) \xrightarrow{\sim} (Y_\pi,0)$, and for any retraction $r : (\mathbb{C}^{N+1},0) \to (Y,0)$ there exists a neighborhood U of 0 in Y such that for all $y \in U$, the germs $(r^{-1}(y) \cap X, y) \subset (r^{-1}(y),y)$ are germs of reduced (plane) curves, and have the same topological type. (In particular, if X is singular at 0, Y is the entire singular locus of X.)

iii) For every projection $\pi' : (\mathbb{C}^{N+1},0) \to (\mathbb{C}^N,0)$ which is transversal to X,(i.e. not only $\pi'^{-1}(0) \not\subset X$, but the tangent space $\operatorname{Ker} d\pi' = T_{\pi'^{-1}(0),0} \not\subset C_{X,0}$, the tangent cone of X at 0) the branch locus of $\pi'|X$ is smooth at 0.
(Note that transversality implies $\operatorname{Ker} d\pi \cap T_{Y,0} = (0)$ since $Y \subset X$.)

iv) The normalization $n : \bar{X} \to X$ of X is such that

α) $\bar{X}$ is smooth at every point of $n^{-1}(0)$.

β) The Ideal $\sum_1^{N+1} \frac{\partial F}{\partial z_i}.\mathcal{O}_{\bar{X}}$ is invertible in a neighborhood of $n^{-1}(0)$ in $\bar{X}$). ($n^{-1}(0)$ is a finite set of points, one for each irreductible component of X at 0.)

γ) n is the composition of a finite number of permissible blowing ups [in the sense of Hironaka, i.e. $\bar{X} = X^{(k)} \to X^{(k-1)} \to \ldots \to X^{(0)} = X$ and $X^{(i+1)} \xrightarrow{b^{(i)}} X^{(i)}$ is the blowing up of a non singular subspace $Y^{(i)}$ of $X^{(i)}$ such that $X^{(i)}$, which is actually a hypersurface in a non singular space, is locally equimultiple along $Y^{(i)}$], $Y^{(0)}$ is the singular locus of X, the blowing ups $X^{(i+1)} \to X^{(i)}$ induce local isomorphisms $Y^{(i+1)} \to Y^{(i)}$, (i.e. $Y^{(i+1)} \to Y^{(i)}$ is etale) ; and $(b^{(i)^{-1}}(Y^{(i)}))_{red} = Y^{(i+1)}$. (If X is reductible, the $Y^{(i)}$ are not necessarily connected.)

Comments : (I shall give some hints for a proof later.)

1) The first part of condition ii) is the very important nonsplitting phenomenon which was emphasized by Abhyankar in his talk at the Woods Hole conference (see [$Ab_1$]) (but in characteristic $p>0$, where it is extremely delicate). It says in view of iv), $\gamma$) that if the branch locus is smooth, the singular locus Y of X itself is smooth and the multiplicity of X is locally constant on Y, which is the same as saying that the blowing up $X^{(1)} \to X$ of Y is a finite map, and iv) tells us that $X^{(1)}$ has along the reduced inverse image of Y the same properties as X along Y, and we can go on until we reach $\bar{X}$ which is smooth.
I will show below a generalization of the nonsplitting phenomenon.

2) If we view X as a family of plane curves parametrized by Y, with the help of a retraction as in ii), what iv) tells us is that the process of resolution of singularities for all the curves $(r^{-1}(y) \cap X, y)$ is the same for $y \in V$, a neighborhood of 0 in Y (see [$Z_1$]).

3) The equivalence of i) and iii) is very important for the following reason : in the inductive approach to resolution mentionned above, even if we start with a transversal projection $\pi : X \to S$, after simplification of the branch locus, the map $\pi' : X' \to S'$ obtained will in general no longer be transversal. But often the assumption of transversality makes proofs easier. i) $\Leftrightarrow$ iii) shows that in fact we lose nothing.

1.3 Zariski proposed a general algebraic definition of equisingularity for hypersurfaces, by induction on the codimension, as follows :

Definition (Zariski [$Z_4$]) :

(Z) Let $(X,0) \subset (\mathbb{C}^{N+1},0)$ be a reduced hypersurface and $(Y,0) \subset (X,0)$ a non singular susbspace of X. X is equisingular along Y at 0 if there exists a projection $\pi : (\mathbb{C}^{N+1},0) \to (\mathbb{C}^N,0)$ such that :

1) $\operatorname{Ker} d\pi \cap T_{Y,0} = (0)$, i.e. $\pi$ maps isomorphically $(Y,0)$ to a non singular subspace $(Y_\pi,0) \subset (\mathbb{C}^N,0)$, and $\pi^{-1}(0) \not\subset X$.
2) The branch locus $(B_\pi,0) \subset (\mathbb{C}^N,0)$ of $\pi|X$ (which is a reduced hypersurface) is equisingular along $Y_\pi$ at 0 (in particular $(Y_\pi,0) \subset (B_\pi,0)$).

When Y is of codimension 0 in X, equisingularity is just $Y = X$, and the Theorem 1 above is a list of characteristic properties of equisingularity in codimension 1, some of them independent of the mention of any projection.

Remark also that it follows inductively from this definition that the condition (O.E.) is satisfied.

Now we have of course :

<u>Problem</u> : Compare (Z) with the other (possibly non-algebraic) definitions of equisingularity, for example :

1) <u>Topological equisingularity</u> (see [$Z_5$])

(T.E.) : In the same situation as in (Z), X is topologically equisingular along Y at 0 if for any retraction $r : (\mathbb{C}^{N+1},0) \to (Y,0)$ there exists a neighborhood U of 0 in Y such that for all $y \in U$, the germs $(r^{-1}(y) \cap X, y) \subset (r^{-1}(y), y)$ are germs of reduced hypersurfaces with the same topological type. In other words, the topological type of a section of X by the non singular subspaces in $\mathbb{C}^{N+1}$ $(r^{-1}(y),y)$ transversal to Y is constant along Y.

We can add the extra condition that the topological type thus obtained is also independent of the choice of the retraction r.

(T.E.) + extra condition will be noted (S.T.E.).

2) <u>Topological triviality</u>

(T.T.) X is topologically trivial along Y at 0 if there exist a retraction $r : (\mathbb{C}^{N+1},0) \to (Y,0)$ and a germ at 0 of homeomorphism of pairs

$$(\mathbb{C}^{N+1}, X) \approx (r^{-1}(0) \times Y,\ \ (r^{-1}(0) \cap X) \times Y))$$

compatible with r i.e. such that

$$\begin{array}{ccc} \mathbb{C}^{N+1} & \approx & r^{-1}(0) \times Y \\ & {\scriptstyle r}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & Y & \end{array}$$

commutes.

3) <u>Whitney conditions</u>

Whitney's stratification theorem (see [ Ma ], [$W_1$]) implies the existence of a (locally) finite partition of X into non-singular locally closed subspaces $Y_\alpha$ with the properties that :

i) $\bar{Y}_\alpha$ and $\bar{Y}_\alpha - Y_\alpha$ are closed subspaces of X, $\dim(\bar{Y}_\alpha - Y_\alpha) < \dim Y_\alpha$, $\bar{Y}_\alpha \cap Y_\beta \neq \emptyset \Rightarrow Y_\beta \subset \bar{Y}_\alpha$.

ii) If $Y_\beta \subset \bar{Y}_\alpha$, the following conditions of incidence (Whitney conditions) hold :

a) For any sequence of points $y_i \in Y_\alpha$ converging to a point

$y \in Y_\beta$ and such that the directions of the tangent spaces $T_{Y_\alpha, y_i}$ converge (in the grassmannian of $\dim Y_\alpha$-planes in $\mathbb{C}^{N+1}$) to a plane T, we have : $T_{Y_\beta, y} \subset T$.

b) Given a sequence of points $y_i \in Y_\alpha$ converging to y and a sequence of points $u_i \in Y_\beta$, also converging to y, and such that again $T_{Y_\alpha, y_i}$ converges to a plane T, and also the direction of the secant $\overline{u_i y_i}$ in $\mathbb{C}^{N+1}$ converges (in $\mathbb{P}^n$) to $\ell$, we have : $\ell \subset T$.

(If y is fixed we will speak of "Whitney conditions at y".)

iii) The set $X^o$ of smooth points of X is a stratum (we do not ask strata to be connected) and so of course all other strata are in its closure.

Define Whitney equisingularity of X along Y at 0 to be :

(W) Y is a stratum of some Whitney stratification of X (in a neighborhood of 0).

Now the situation in general is this : Varchenko ([V]) gave a topological proof of (Z) ⇒ (T.T.). Thom and Mather ([Ma ], [$Th_1$]) gave differential-geometric proofs of (W) ⇒ (T.T.), and of course (T.T.) ⇒ (T.E.). Speder [Sp] modified Zariski's definition by requiring that the projection $\pi$ appearing in (Z) should satisfy in addition a certain condition, which is satisfied by "almost all" projections, and implies transversality. With this different definition (Sp), he proved inductively that if X satisfies (Sp) along Y at 0, then the pair $(X^o, Y)$ satisfies Whitney conditions at every point of a neighborhood of 0 in Y.
(Sp) is unfortunately lost when we apply imbedded resolution to the discriminant.
This brings us to a question which is anyway crucial when one tries to compare (Z) with other definitions.
Here, in first analysis, I shall assume for the sake of simplicity that the coordinates are so chosen that Y is a linear subspace in $\mathbb{C}^{N+1}$ and consider only linear projections $\pi : \mathbb{C}^{N+1} \to \mathbb{C}^N$.

<u>Question A</u> : (This is essentially Zariski's question I in [$Z_5$]).
Assuming that X is equisingular along Y at 0 (in some sense, and in particular of course (Z)), is the set of those linear projections $\pi : \mathbb{C}^{N+1} \to \mathbb{C}^N$ satisfying :

1) $\mathrm{Ker}\,\pi \cap Y = (0)$

2) The branch locus $B_\pi$ of $\pi|X$ is equisingular along $Y_\pi = \pi(Y)$ at 0, a dense (Zariski)-constructible subset of the space of linear projections ?

Here I will venture the following :

<u>Conjecture 1</u> : In the situation of question A, for a linear projection $\pi : \mathbb{C}^{N+1} \to \mathbb{C}^N$ such that $\operatorname{Ker}\pi \cap Y = (0)$ the following conditions are equivalent :

$\alpha$) $B_\pi$ is equisingular along $Y_\pi$ at 0

$\beta$) For all i, $\dim Y \leq i \leq N$, there exists a (Zariski) open dense subset $U^{(i)}$ of the Grassmannian of i-planes of $\mathbb{C}^N$ containing $Y_\pi$, such that for all $H \in U^{(i)}$, the hypersurface $(\pi^{-1}(H) \cap X)_{red} \subset \pi^{-1}(H)$ is equisingular along Y at 0.

Notice that for $i = \dim Y$, $\beta$) is just the nonsplitting of $\pi^{-1}(Y_\pi) \cap X$. This conjecture is proved in the codimension 1 case, where we have :

<u>Theorem 2</u> (Zariski $[Z_4]$) : Assume $X \subset \mathbb{C}^{N+1}$ (Z)-equisingular along Y at $0 \in Y$. If $\dim Y = N-1$, then for a linear projection $\pi : \mathbb{C}^{N+1} \to \mathbb{C}^N$ with $\operatorname{Ker} d\pi \cap T_{Y,0} = (0)$ the following are equivalent :

$\alpha$) The branch locus $B_\pi$ is equisingular along $Y_\pi$ at 0 (i.e. $(B_\pi, 0) = (Y_\pi, 0)$).

$\beta'$) Setting $\ell_y = \pi^{-1}(\pi(y))$, for $y \in Y$, the intersection number $(\ell_y . X)_y$ of the line $\ell_y$ with X at y, is independent of $y \in Y$ (locally around 0).

Now it is clearly equivalent to say that $(\ell_y, X)_y$ is independent of $y \in Y$, and to say that $(\pi^{-1}(Y_\pi) \cap X)_{red} = Y$, and if $\dim Y = N-1$, the only dimension we have to look at to check $\beta$) is $i = \dim Y = N-1$.

Even if one can prove conjecture 1, it does not help much to answer question A unless one can also answer :

<u>Question B</u> : Assume $(X,0) \subset (\mathbb{C}^{N+1},0)$ is equisingular at 0 along $(Y,0) \subset (X,0)$ (here again for simplicity we assume Y linear in $\mathbb{C}^{N+1}$). Is it true that for each i, $\dim Y \leq i \leq N+1$, there exists a dense Zariski-open subset $U^{(i)}$ of the grassmannian of i-planes of $\mathbb{C}^{N+1}$ containing Y, such that $H \in U^{(i)} \Rightarrow X \cap H$ is equisingular along Y at 0 (or, better, a constructible $C^{(i)}$ such that $H \in C^{(i)} \Leftrightarrow X \cap H$ equisingular along Y) ?

A priori, question B looks much simpler that question A, and even rather simple-minded. However, except of course when Y is of codimension

1 in X, there is no answer, for any notion of equisingularity. Zariski proved, however, that (Z) implies equimultiplicity, and Hironaka [$H_6$] proved that in the most general case, Whitney conditions implie equimultiplicity. Therefore there are affirmative answers to question B for $i = \dim Y+1$. But the point in question B is of course the case $i = N$, the opposite extreme.

Anyway, I think an affirmative answer to question B and conjecture 1 would settle question A.

Now if we want to study equisingularity in codimension > 1, the first thing to look at is the case where Y is the singular locus of X, but not necessarily of codimension 1. Then we can view X as a family of hypersurfaces with isolated singularities, parametrized by Y. So we turn to :

## § 2. ISOLATED SINGULARITIES OF HYPERSURFACES AND THE NUMERICAL APPROACH

2.1 Let me first recall some facts from asymptotic algebra in the sense of Samuel [Sa]. For details I refer to [L.T.], [Se], [P.T.], [$T_2$].
Let $\mathcal{O}$ be the local analytic algebra of a reduced complex analytic space Z at a point z. Given an ideal I in $\mathcal{O}$ an element $h \in \mathcal{O}$ is said to be integral over I if it satisfies an integral dependence relation

$$h^k + a_1 h^{k-1} + \dots + a_k = 0 \quad \text{with } a_i \in I^i \quad .$$

The set of such elements is an ideal $\bar{I}$ in $\mathcal{O}$, the integral closure of I in $\mathcal{O}$.
This purely algebraic notion due to Prüfer [P] has the following transcendental interpretation, the idea of which is due to Hironaka : If we choose generators $g_1, \dots, g_\ell$ for I, then $h \in \bar{I}$ if and only if there exists a neighborhood U of z in Z and a constant $C \in \mathbb{R}_+$ such that for all $z' \in U$ $|h(z')| \leq C.\mathrm{Sup}|g_i(z')|$.
In the same vein, let me define the Łojasiewicz exponent $\theta_I(h)$ of h with respect to I as the greatest lower bound of those $\theta \in \mathbb{R}_+$ such that there exists a neighborhood U of z in Z and a $C \in \mathbb{R}_+$ (depending on U) such that

$$|h(z')|^{\theta} \leq C.\mathrm{Sup}|g_i(z')| \quad \text{for all } z' \in U.$$

Then (see [L.T.]) we have the following :

<u>Proposition</u> : $\theta_I(h) = \frac{1}{\bar{\nu}_I(h)}$ where $\bar{\nu}_I(h) = \lim_{k\to\infty} \frac{\nu_I(h^k)}{k}$

with $\nu_I(g) = \max\{\nu \in \mathbb{N} / g \in I^\nu\}$, (and $\theta_I(h)$ is attained for some U and $C \in \mathbb{R}_+$). If $h \notin \sqrt{I}$, we set $\theta_I(h) = +\infty$. ($\bar{\nu}$ was introduced in [Sa], and Nagata [N] proved $\bar{\nu}_I(h) \in \mathbb{Q}$, so that Łojasiewicz exponents are always rational numbers.) We see that integral dependence will be very useful to study algebraically geometrical incidence (or limit) relations, such as the Whitney conditions, a viewpoint in fact pioneered by Hironaka in [$H_1$]. Another aspect of integral dependence will be useful for our numerical viewpoint : suppose now that I is primary for the maximal ideal $\mathfrak{M}$ of $\mathcal{O}$. Then the application $H_I : \mathbb{N} \to \mathbb{N}$ defined by $H_I(\nu) = \dim_{\mathbb{C}} \mathcal{O}/I^{\nu+1}$ coincides when $\nu$ is large enough with a polynomial in $\nu$ of degree $d = \dim \mathcal{O}$ and the highest degree term of this polynomial can be written $e(I)\frac{\nu^d}{d!}$, where $e(I) \in \mathbb{N}$ is by definition the multiplicity of the primary ideal I in $\mathcal{O}$. The result I will use is : $e(I) = e(\bar{I})$.

2.2 For example, let $f \in \mathcal{O}_{n+1} = \mathbb{C}\{z_o,\dots,z_n\}$ be such that $f = 0$ defines a germ of hypersurface with isolated singularity $(X,0) \subset (\mathbb{C}^{n+1},0)$. By the nullstellensatz, this means that the ideal in $\mathcal{O}_{n+1}$ generated by $(f, \frac{\partial f}{\partial z_o}, \dots, \frac{\partial f}{\partial z_n})$ is primary for the maximal ideal. Define $\mu(X,0)$ to be the multiplicity of this primary ideal, and remember the very useful <u>Fact</u> (see [$T_2$]) : For any choice of the coordinates $(z_o,\dots,z_n)$, f is integral in $\mathcal{O}_{n+1}$ over the ideal generated by $(z_o.\frac{\partial f}{\partial z_o}, \dots, z_n.\frac{\partial f}{\partial z_n})$ (which depends on the choice of coordinates). In particular, f is integral over the product ideal $\mathfrak{M}.j(f)$ where $\mathfrak{M}$ is the maximal ideal of $\mathcal{O}_{n+1}$ and j(f) the ideal generated by $(\frac{\partial f}{\partial z_o}, \dots, \frac{\partial f}{\partial z_n})$. ($\mathfrak{M}.j(f)$ does not depend upon the choice of coordinates).And a fortiori, f is integral over j(f.) (All this is true in general, i.e. for any $f \in \mathfrak{M}$.)
So the multiplicity of (f,j(f)) is equal to that of j(f), which must be also primary, whence $(\frac{\partial f}{\partial z_o}, \dots, \frac{\partial f}{\partial z_n})$ is a regular sequence in $\mathcal{O}_{n+1}$, and then by a theorem of Samuel, $\mu(X,0) = \dim_{\mathbb{C}} \mathcal{O}_{n+1}/j(f)$, the Milnor number of the singularity.
Note that the ideal $(f, \frac{\partial f}{\partial z_o}, \dots, \frac{\partial f}{\partial z_n})$ depends only on the quotient $\mathcal{O}_{n+1}/(f)$, so $\mu(X,0)$ is an analytic invariant, but better still, an attentive reader of Milnor's book [Mi] will remark that :
<u>If $(X_1,0)$ and $(X_2,0)$ are two germs of hypersurfaces with isolated singularity of the same topological type, we have $\mu(X_1,0) = \mu(X_2,0)$.</u>
This follows from the interpretation of $\mu(X,0)$ as "number of vanishing

cycles" of the singularity $(X,0)$. See [Mi] and Brieskorn's lectures in this volume. (For a "proof of the remark", see [$T_2$], and [$L_2$] for generalization to non-isolated singularities.)

2.3 Now we have a topological invariant defined algebraically, so we should be in good position to study some of the questions of § 1 in the special case considered here. First, a few remarks on the semi-continuity properties of the Milnor number. Given a family of complex hypersurfaces, i.e. a commutative diagram

$$\begin{array}{ccc} (X,0) & \hookrightarrow & (\mathbb{C}^k \times \mathbb{C}^{n+1},0) \\ & \pi \searrow \quad \swarrow p_1 & \\ & (\mathbb{C}^k,0) & \end{array}$$

where $(X,0) \hookrightarrow (\mathbb{C}^k \times \mathbb{C}^{n+1},0)$ is defined by $F=0$, $F \in \mathbb{C}\{y_1,\dots,y_k, z_o,\dots,z_n\}$. Assume $F(0,\dots,0, z_o,\dots,z_n)=0$ is a hypersurface with isolated singularity. Then the subspace $P$ of $(\mathbb{C}^{N+1},0)=(\mathbb{C}^{n+k+1},0)$ defined by the ideal $J$ generated by $(\frac{\partial F}{\partial z_o},\dots,\frac{\partial F}{\partial z_n})$ is finite over $(\mathbb{C}^k,0)$ (by the Weierstrass preparation theorem) and $(\frac{\partial F}{\partial z_o},\dots,\frac{\partial F}{\partial z_n})$ is a regular sequence, so that in fact $\mathbb{C}\{y_1,\dots,y_k, z_o,\dots,z_n\}/J$ is a free $\mathbb{C}\{y_1,\dots,y_k\}$-module. This means also that if we take any $y \in \mathbb{C}^k$, $p_1^{-1}(y) \cap P$ is a finite set of points $x_i$, the local rings $\mathcal{O}_{P,x_i}$ are artinian, i.e. finite dimensional vectorspaces and $\sum_{x_i \in p_1^{-1}(y) \cap P} \dim_{\mathbb{C}} (\mathcal{O}_{P,x_i})$ is independent of $y \in (\mathbb{C}^k,0)$, and is therefore equal to the Milnor number of the hypersurface $(X_o,0)=(\pi^{-1}(0),0) \subset (\mathbb{C}^{n+1},0)$ defined by $F(0,\dots,0, z_o,\dots,z_n)=0$. The singular points of $X_y \in \pi^{-1}(y)$ are the points of $\pi^{-1}(y) \cap P$, which are some of the $x_i \in p_1^{-1}(y) \cap P$, and if $x_i \in \pi^{-1}(y) \cap P$, $\dim_{\mathbb{C}} \mathcal{O}_{P,x_i}$ is just the Milnor number of the isolated singularity of hypersurface $(\pi^{-1}(y),x_i) \subset (p_1^{-1}(y),x_i)$. So we have :

2.3.1 For any $y \in (\mathbb{C}^k,0)$ if the $x_i$ are the singular points of the fiber $X_y$ of $\pi$

$$\mu(X_o,0) \geq \sum_{x_i \in X_y} \mu(X_y,x_i)$$

and equality holds for all $y$, if and only if $P_{red} \subset X$, i.e. : there exists an integer $r$ such that $F^r \in J$.

Assume now that we are given a section $\sigma : (\mathbb{C}^k,0) \to (X,0)$ of $\pi$. By a change of the coordinates $(z_o,\dots,z_n)$, we may assume that the image of $\sigma$ is $(\mathbb{C}^k \times \{0\},0)$ i.e. is defined by the ideal $S = (z_o,\dots,z_n)$. Then, by the semi-continuity property of the fibers of coherent sheaves, we have :

2.3.2 1) For all $y \in (\mathbb{C}^k,0)$, $\mu(X_o,0) \geq \mu(X_y,y)$ .

2) In view of the preceding remark, equality holds for all $y \in (\mathbb{C}^k,0)$ if and only if $(P_{red},0) = \sigma(\mathbb{C}^k,0) = (\mathbb{C}^k \times \{0\})$ i.e. iff there exists an integer t such that $S^t \subset J$.

3) Given $(Y,0) \subset (X,0) \subset (\mathbb{C}^{N+1},0)$ with $(Y,0)$ smooth, and a retraction $r : (\mathbb{C}^{N+1},0) \to (Y,0)$ such that $(X_o,0) = (r^{-1}(0) \cap X,0) \subset (r^{-1}(0),0)$ is a hypersurface with isolated singularity, the condition

<u>($\mu$ constant)</u> : the Milnor number of $(r^{-1}(y) \cap X,y) = (X_y,y)$ is locally constant on Y,

satisfies the condition (O.E.) of § 1.

2.4 Applying this last result to the family of sections of a given hypersurface $(X_o,0) \subset (\mathbb{C}^{n+1},0)$, with isolated singularity, we obtain, for each i $(0 \leq i \leq n+1)$ an integer $\mu^{(i)}(X_o,0)$ and a Zariski-open dense $U^{(i)} \subset G^{(i)}$ (= the grassmannian of i-planes) such that $H \in U^{(i)} \Rightarrow \mu(X_o \cap H,0) = \mu^{(i)}(X_o,0)$. $\mu^{(n+1)}(X_o,0)$ is $\mu(X_o,0)$, $\mu^{(1)}(X_o,0) = m(X_o,0) - 1$ where $m(X_o,0)$ is the multiplicity of $X_o$ at 0, and $\mu^{(o)}(X_o,0) = 1$. Set $\mu^*(X_o,0) = (\mu^{(n+1)}(X_o,0),\dots,\mu^{(1)}(X_o,0),\mu^{(o)}(X_o,0))$. Again, for a situation as in 2.3.2, the condition

<u>($\mu^*$ constant)</u> : $\mu^*(X_y,y)$ is locally constant.

satisfies (O.E.).

2.5 In order to state the results which motivate the next questions, I need to introduce still another notion of equisingularity, to add to our collection of § 1. This section owes much to reminiscences of (unpublished) lectures of Hironaka.

Let $(Y,0) \subset (X,0) \subset (\mathbb{C}^{N+1},0)$ be a smooth space in a hypersurface in $\mathbb{C}^{N+1}$ as usual. Choose coordinates $(y_1,\dots,y_k , z_o,\dots,z_n)$ $(N = n+k)$ so that Y is defined by the ideal $S = (z_o,\dots,z_n)\mathcal{O}_{N+1}$ $(\mathcal{O}_{N+1} = \mathbb{C}\{y_1,\dots,y_k , z_o,\dots,z_n\})$. let $F(y_1,\dots,y_k , z_o,\dots,z_n) = 0$ be an equation for X, and J be the ideal $(\frac{\partial F}{\partial z_o},\dots,\frac{\partial F}{\partial z_n})\mathcal{O}_{N+1}$ . The new condition of equisingularity is :

<u>(c)</u> $$\frac{\partial F}{\partial y_i} \in \overline{S.J} \quad (\text{in } \mathcal{O}_{N+1}) \quad 1 \leq j \leq k \ .$$

To throw some light on it, let us see that it implies that the smooth part $X^o$ of X satisfies Whitney's conditions a) and b) along Y, near O. In fact, condition (c) implies that if we take any sequence of points $p_i \to y \in Y$ in $\mathbb{C}^{N+1}$, not necessarily in X, such that the level hypersurface $F(y_1,\ldots,y_k, z_o,\ldots,z_n) = F(p_i)$ is non singular at $p_i$, the limit position of the tangent hyperplanes, of homogeneous coordinates $(\frac{\partial F}{\partial y_1}(p_i):\ldots:\frac{\partial F}{\partial y_1}(p_i) : \frac{\partial F}{\partial z_o}(p_i):\ldots:\frac{\partial F}{\partial z_n}(p_i))$ is of the form $(0:\ldots:0:a_o:\ldots:a_n)$, i.e. contains Y (or $T_{Y,y}$), because condition c) implies : $|\frac{\partial F}{\partial y_j}(p_i)| \leq C \,.\, \mathrm{dist}(p_i,Y) \,.\, \mathrm{Sup}_k |\frac{\partial F}{\partial z_k}(p_i)|$, in view of 2.2.
The proof for condition b) is similar, if more delicate. One has to use the important fact, due to Whitney, [$W_1$] that the pair of strata $(X^o,(0))$ satisfies condition b). Remark finally that in choosing coordinates as we have done, we have in fact chosen a retraction $r : (\mathbb{C}^{N+1},0) \to (Y,0)$, so the following makes sense :

2.6 <u>Theorem 3</u> : Given $(Y,0) \subset (X,0) \subset (\mathbb{C}^{N+1},0)$ as above, and a retraction $r : (\mathbb{C}^{N+1},0) \to (Y,0)$ such that

1) $(X_o,0) = (r^{-1}(0) \cap X,0)$ is a hypersurface with isolated singularity ( and Y is the entire singular locus of X)
2) $\dim Y = 1$ .

Then : ($\mu^*$ constant) $\Leftrightarrow$ condition (c).
(And the $\mu^*$ is actually independent of the choice of the retraction.)

The proof of ($\Rightarrow$) goes as follows : first, condition (c) satisfies (O.E.) of § 1. (The proof of the lemma in 2.7 of [$T_2$] extends easily.) Second, ($\mu^*$ constant) is actually a condition of equimultiplicity for the family of ideals induced by S.J in the fibers of r : with the notations of 2.2, given an isolated singularity of hypersurface, we consider the application $K : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $K(\nu_1,\nu_2) = \dim_{\mathbb{C}} \mathcal{O}_{n+1}/\mathfrak{M}^{\nu_2}.j(f)^{\nu_1}$ .
When $\nu_1$ and $\nu_2$ are both sufficiently large, K takes the same values as a polynomial in $\nu_1$, $\nu_2$, of degree n+1, the highest degree part of which is :

$$\overline{K}(\nu_1,\nu_2) = \frac{1}{(n+1)!} (\mu^{(n+1)}\nu_1^{n+1} + \ldots + \binom{n+1}{i}\mu^{(n+1-i)}.\nu_1^{n+1-i}.\nu_2^{i} + \ldots + \mu^{(0)}\nu_2^{n+1}),$$

and in particular, $e(\mathfrak{M}.j(f)) = \sum_{0}^{n+1} \binom{n+1}{i} \mu^{(i)}$. ($\mu^{(i)} = \mu^{(i)}(X,0)$). So the family of ideals induced by S.J on the fibers of r is equimultiple if ($\mu^*$ constant) is realized.
In that case, provided $\dim Y = 1$ (this is the only place where this unnatural assumption is needed) one can prove ([$T_2$], prop. 3.1) that "$\bar{\varphi}$ can

be computed at the generic point of Y''. But condition c) is :
$\bar{\nu}_{SJ}(\frac{\partial F}{\partial y_i}) \geq 1$ $(1 \leq i \leq k)$ and as mentionned in the beginning, it is realized at the generic point of Y. Equimultiplicity implies that it is also realized at 0. The proof of $(\Leftarrow)$ does not use $\dim Y = 1$. One shows that $(c) \Rightarrow \mu^{(n+1)}(X_y, 0)$ constant $(n = N - \dim Y)$ : since Y is the singular locus of X, there exists an integer r such that
$S^r \subset (\frac{\partial F}{\partial y_1}, \ldots, \frac{\partial F}{\partial y_k}, \frac{\partial F}{\partial z_0}, \ldots, \frac{\partial F}{\partial z_n})\mathcal{O}_{N+1}$. But (c) implies that a power of this last ideal is contained in J. So a power of S in contained in J, which means $\mu^{(n+1)}$ constant (2.3.2). One concludes by proving :

<u>Proposition</u> : For (c)-equisingularity, the answer to question B of § 1 is yes.

I will not give the proof here, but only remark that (c)-equisingularity is in fact an ad-hoc definition for this proposition to hold.

2.7 So if we could prove that $\mu^*(X_0, 0)$ is in fact also an invariant of the topological type as $\mu(X_0, 0)$ is, we would have, at least in the special case where Y is the singular locus of X, (and $\dim Y = 1$) a completely algebraic interpretation of topological equisingularity (and in fact (S.T.E.)) thus extending Theorem 1. Of course, from the algebraic viewpoint, it is enough to prove : $\mu^{(n+1)}(X_y, y)$ constant $\Rightarrow \mu^{(n)}(X_y, y)$ constant. Then, we can take a hyperplane H in $\mathbb{C}^{N+1}$ containing Y and such that $\mu^{(n)}(X_0 \cap H, 0) = \mu^{(n)}(X_0, 0)$ and by the semi-continuity properties :

$$\begin{array}{ccc} \mu^{(n)}(X_0 \cap H, 0) & = \mu^{(n)}(X_0, 0) & \text{by our choice of H} \\ \bigvee\!| & \| & \text{by assumption} \\ \mu^{(n)}(X_y \cap H, y) & \geq \mu^{(n)}(X_y, y) & \end{array}$$

So that $X \cap H$ again satisfies ($\mu$ constant) along Y, and we can go down the staircase of dimensions to prove ($\mu$ constant) $\Rightarrow$ ($\mu^*$ constant) which is enough to prove the equivalence of all definitions of equisingularity given in § 1 (except Zariski's) with condition (c), in the special case we are considering here. So let me state the :

<u>Conjecture $B_\mu$</u> : $\mu^{(n+1)}$ constant $\Rightarrow \mu^{(n)}$ constant.

which, as we have just seen, is only question B for ($\mu$-constant)-equisingularity.

2.8 It might seem that we have moved very far away from Zariski's discriminant (branch locus) condition.
Let me show that this is not the case : suppose we want to compare theorem 3 with

Theorem 4 (Lê Dũng Tráng and C.P. Ramanujam [L.R ]) : In the situation $(Y,0) \subset (X,0) \subset (\mathbb{C}^{N+1},0)$, with a retraction $r : (\mathbb{C}^{N+1},0) \to (Y,0)$ assume :

1) $(X_0,0)$ is a hypersurface with isolated singularity.
2) The codimension of Y in X is $\neq 2$.
3) X satisfies the condition ($\mu$ constant) along Y at 0, with respect to r.

Then, X is topologically equisingular along Y at 0, with respect to r.

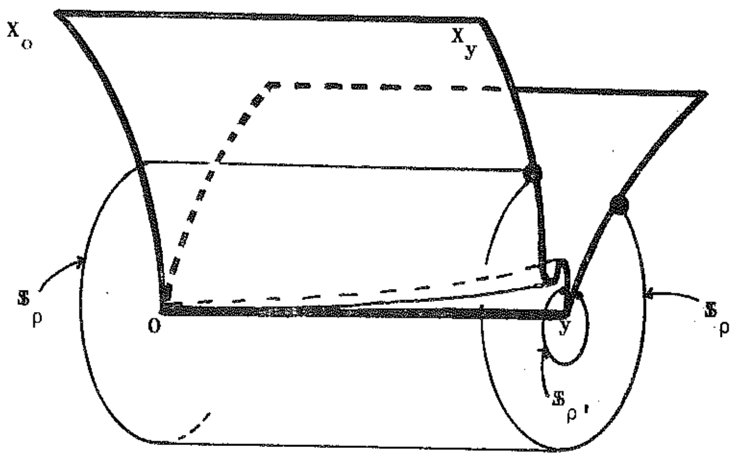
Using this result, and a deep theorem of topology, Timourian [Ti] proved :

Theorem 4' (Timourian) : Under the same assumptions, topological triviality holds.

2.8.1 After theorems 3 and 4, we find ourselves in an embarrassing situation. Since we know that the Milnor number is an invariant of the topological type ; theorems 4 and 4' do provide us with an algebraic interpretation of topological equisingularity and topological triviality when Y is smooth and is the entire singular locus of X :
($\mu$ constant) $\Leftrightarrow$ (topological equisingularity) $\Leftrightarrow$ (topological triviality), except when Y is of codimension 2 in X.
On the other hand, if we could prove conjecture $B_\mu$ and remove the $\dim Y = 1$ assumption, we would have a complete equivalence :
($\mu^*$ constant) $\Leftrightarrow$ (condition c) $\Leftrightarrow$ (Whitney conditions) $\Leftrightarrow$ (topological triviality) $\Leftrightarrow$ (topological equisingularity) $\Leftrightarrow$ ($\mu$ constant).

The moment is well chosen to analyze the difference between Whitney conditions and topological equisingularity.
Let us, by a choice of retraction, view (X,0) as a hypersurface in $(Y \times \mathbb{C}^{n+1},0)$. As usual, $(X_y,y)$ will be the intersection of X with $(y \times \mathbb{C}^{n+1}, y \times 0)$.
One of the main points of Whitney conditions (see $[H_1]$, and Lemma 5.1 of [Ma ]) is that if they are satisfied by $(X^0,Y)$, then we can find a real number $\rho_0$ (permissible radius) such that for any $y \in (Y,0)$ (i.e. y is

sufficiently close to 0) the spheres $\mathbb{S}_\rho \subset \{y\} \times \mathbb{C}^{n+1}$ of radius $\rho$ (real-analytic manifolds of dimension 2n+1) are all transversal to the smooth part of $X_y$, for $0 < \rho \leq \rho_o$. Roughly speaking, this means that if we consider the cylinder $C_\rho = \mathbb{S}_\rho \times Y$, and the tube $T_\rho = \mathbb{B}_\rho \times Y$ in the ambiant space, $X \cap T_\rho$ is "cone like over its boundary $X \cap C_\rho$" (a cone with vertex Y). So theorem 3 implies that if $\mu^*$ is constant, we cannot have the situation described by the following picture :




And the gist of theorem 4 is that even if we did have such a situation, if ($\mu$ constant) holds, we can, by using the h-cobordism theorem, (this is the reason of the restriction on codimension) and the properties of Milnor fibrations ([Mi ] and Brieskorn's lectures), build a homeomorphism $(\mathring{\mathbb{B}}_\rho, X_o) \approx (\mathring{\mathbb{B}}_{\rho'}, X_y)$ where $\rho'$ is a radius permissible for the fiber $X_y$, and $\rho$ is permissible for $X_o$.

2.8.2 A few words on discriminants. In what follows we will often consider the following situation :

$$\begin{array}{ccc} (X,0) & \subset & (\mathbb{C}^{N+1},0) \\ & \pi \searrow \quad \circlearrowleft \quad \swarrow p & \\ & (\mathbb{C}^k,0) & \end{array}$$

where (X,0) is a hypersurface in $(\mathbb{C}^{N+1},0)$, and $\pi$ is flat. If we choose coordinates $(y_1,\dots,y_k, z_o,\dots,z_n)$ on $\mathbb{C}^{N+1}$ $(N = n+k)$ such that p is described by $(y_1,\dots,y_k, z_o,\dots,z_n) \mapsto (y_1,\dots,y_k)$ and an equation

$F(y_1,\dots,y_k, z_o,\dots,z_n) = 0$ for $X$, the critical subspace $C_\pi$ of $\pi$ is defined by $(\frac{\partial F}{\partial z_o},\dots,\frac{\partial F}{\partial z_n})\mathcal{O}_X$. If $C_\pi$ is proper over $\mathbb{C}^k$, which in our local situation means it is finite, i.e. $(\pi^{-1}(0),0) = (X_o,0)$ has an isolated singularity, we can define the discriminant $(D_\pi,0) \subset (\mathbb{C}^k,0)$ of $\pi$ as the analytic direct image of $C_\pi$, the underlying set of which shall be the set of points of $(\mathbb{C}^k,0)$ such that the fiber has singularities. To be brief, I will just recall the following : (see [$T_2$] or Brieskorn's lectures). Since $(X_o,0)$ has an isolated singularity, $\pi : (X,0) \to (\mathbb{C}^k,0)$ comes by base change from the miniversal deformation of $(X_o,0)$ :

$$\begin{array}{ccc} (X,0) & \longrightarrow & (U,0) \\ \pi\downarrow & & \downarrow G \\ (\mathbb{C}^k,0) & \xrightarrow{h} & (S,0) \end{array}$$

Now $G$ has a discriminant $D_G$ which is a reduced hypersurface in the non-singular space $(S,0)$. The formation of the discriminant commutes with base extension, so the discriminant $D_\pi$ of $\pi$ is $(h^{-1}(D_G),0)$, which is defined by a principal ideal in $\mathcal{O}_k$. The branch locus $B_\pi$ of $\pi$ is $D_{\pi,red}$. We will want to speak of the multiplicity at 0 of the discriminant : If $B_\pi \neq \mathbb{C}^k$, it is just the multiplicity at 0 of the hypersurface $(D_\pi,0)$. By convention, if $B_\pi = \mathbb{C}^k$, "the multiplicity of the discriminant" will be the Milnor number of the fiber $(X_o,0)$. For example, in this case we will say that "the discriminant is equimultiple along $(\mathbb{C}^k,0)$" if we are in the equality case of 2.3.1.

Let me also remark that when $D_\pi$ is a hypersurface, given $(Y,0) \subset (D_\pi,0)$, it is equivalent to say that $D_\pi$ is equimultiple along $Y$, or to say that $B_\pi$ is so, provided $Y$ is smooth.

2.8.3 The connection between theorems 3 and 4, conjecture $B_\mu$ and Zariski's discriminant condition is due to the following :

Proposition ([$T_2$]) : Let $(X,0) \subset (\mathbb{C}^{n+1},0)$ be a hypersurface with isolated singularity, and $p : (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ a projection such that the fiber $(X_o,0)$ of $\pi = p|X$ again has an isolated singularity. Then, the multiplicity of the discriminant $D_\pi$ of $\pi$ is $\Delta = \mu^{(n+1)}(X,0) + \mu^{(n)}(X_o,0)$. Note that the assumptions imply that $B_\pi = (0)$, i.e. if we take a coordinate $z_o$ in $(\mathbb{C},0)$, an equation for $D_\pi$ is $z_o^\Delta = 0$.

There are generalizations of this formula to complete intersections (see Brieskorn's lectures), and Lê [$L_2$] generalized the "vanishing cycles" aspect of it to arbitrary singularities of hypersurfaces. We will see more about this proposition in § 3, but what I need here is the

<u>Corollary</u> : Let

$$\begin{array}{ccc} (X,0) & \subset & (\mathbb{C}^{N+1},0) \\ & \pi \searrow \quad \swarrow p & \\ & (\mathbb{C}^k,0) & \end{array}$$

be as in 2.8.2, and assume that $(X_o,0)$ has an isolated singularity and $D_\pi$ is a hypersurface in $(\mathbb{C}^k,0)$.
Let $(L,0)$ by any smooth curve in $(\mathbb{C}^k,0)$ transversal to $(D_\pi,0)$ (i.e. $(D_\pi.L)_o = m_o(D_\pi)$, the multiplicity of $D_\pi$ at 0). Then

$$m_o(D_\pi) = \mu(X_o,0) + \mu(X_L,0) \quad \text{where } X_L = p^{-1}(L)\cap X \quad .$$

<u>Proof</u> : The formation of discriminant commutes with base change, so $D_\pi \cap L$ is the discriminant of the induced map $X_L \to L$, to which we can apply the proposition.
Since L is transversal to $D_\pi$, the multiplicity of $D_\pi \cap L$ is the multiplicity of $D_\pi$ at 0.
Notice that when $k = N+1$, the discriminant is X itself, $X_o$ is a point, L a line transversal to X and we recover : $m_o(X) = 1 + \mu^{(1)}(X)$.
At the other extreme, if $k = 0$, there is no L, so we are tempted to set the multiplicity of the discriminant equal to $\mu(X_o,0)$, as I did in 2.8.2. (There are, however, more serious reasons to do that !)

2.8.4 Now, back to $B_\mu$. We take the usual situation, choose a retraction, etc., so we are looking at a family of hypersurfaces $X = \bigcup_{y\in Y} X_y$, $(X,0) \subset (Y \times \mathbb{C}^{n+1},0)$, defined by $F(y_1,\dots,y_n\, , z_o,\dots,z_n) = 0$, and we assume $\mu^{(n+1)}$ constant along $0\times Y$. Let us look at the projection $p : (Y\times \mathbb{C}^{n+1},0) \to (Y \times \mathbb{C},0)$ defined by $(z_o,\dots,z_n\, , \underline{y}) \mapsto (z_o,\underline{y})$.
$p|X = \pi : (X,0) \to (Y\times\mathbb{C},0)$ has a discriminant $(D_\pi,0) \subset (Y\times\mathbb{C},0)$, containing $Y_\pi = 0\times Y$. By the corollary above, the multiplicity of $D_\pi$ at a point $y\in Y_\pi$ is $m_y(D_\pi) = \mu^{(n+1)}(X_y) + \mu^{(n)}(X_y \cap H)$, where H is the hyperplane $z_o = 0$.
So to prove $B_\mu$ is just the same as to prove that for a sufficiently general hyperplane H containing Y, the discriminant $D_\pi$ of the corresponding projection is equimultiple along $Y_\pi$, i.e. the branch locus $B_\pi$ coincides with $Y_\pi$. And thus we find something very similar to Zariski's discriminant condition ! In fact, if Y is of codimension 1 in X, it is exactly Zariski's discriminant condition.

Remark that if X satisfies ($\mu$ constant) along Y, the singular locus of X is Y, and if there are points of $D_\pi$ outside $Y_\pi$, they are the images by $\pi$ of points of X outside Y where $\frac{\partial F}{\partial z_1} = \dots = \frac{\partial F}{\partial z_n} = 0$, i.e., where the tangent hyperplane to X is parallel to the hyperplane $z_0 = 0$.

If $\mu^{(n)}(X_y \cap H, y)$ is not constant, we have in X a curve of such points, which I like to call the curve of vanishing folds of X along Y with respect to H. The picture above can also be deemed to represent vanishing folds. If we agree to say that "X has no vanishing fold along Y" if it has no vanishing folds with respect to a generic hyperplane H containing Y, $B_\mu$ becomes the slogan : "($\mu$ constant) $\Rightarrow$ no vanishing folds".

2.8.5 As an application of the numerical viewpoint, let us stop a while to sketch the proof of theorems 1 and 2, and in fact of the equivalence of all definitions of equisingularity, in the case where (Y,0) is (smooth) of codimension 1 in X, and also of dimension 1 since we will use theorems 3 and 4: (The results here are due to Zariski, and although what follows does offer some alternative proofs in the special case $\dim Y = 1$, it is given mostly as an informal introduction to the reading of $[Z_2]$, $[Z_4]$).

We have therefore, after choice of retraction, a 1-parameter family of reduced plane curves, described by $F(y_1, z_0, z_1) = 0$, $F \in \mathbb{C}\{y_1, z_0, z_1\}$, $F(y_1, 0, 0) \equiv 0$. The first point is that when (Y,0) is of codimension 1 in (X,0), conjecture $B_\mu$ is proved, thanks to theorem 4 and to the well known fact that the multiplicity of a reduced plane curve is an invariant of the topological type : (see the appendix at the end of this section). Indeed, here, $n = 1$, so $X_y$ is the curve $F(y, z_0, z_1) = 0$, $(y = (y_1, 0, 0))$ and $\mu^{(1)}(X_y, y) = m_y(X_y) - 1$ : by theorem 4, all the $(X_y, y)$ have the same topological type if $\mu^{(2)}(X_y, y)$ is constant, hence $m_y(X_y)$ is constant. (See appendix at the end of this section).

Let us now take a projection $p : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ such that $p^{-1}(0) \not\subset X$, and $\operatorname{Ker} dp \cap T_{Y,0} = (0)$. Choose coordinates so that p is written $(y_1, z_0, z_1) \to (y_1, z_0)$. $(Y_\pi, 0)$ is defined by $z_0 = 0$. Then, the following are equivalent :

1) $(D_\pi, 0)$ is equimultiple along $(Y_\pi, 0)$, i.e. $(B_\pi, 0) = (Y_\pi, 0)$.

2) $\mu^{(2)}(X_y, y)$ and the intersection number $(\ell_y, X)_y$ are independant of $y \in (Y, 0)$, where $\ell_y$ is the line $p^{-1}(p(y))$.

Proof : Using 2.8.3, and the fact that if there are several critical points in the same fiber, the multiplicity of the discriminant is the sum of the multiplicities corresponding to each critical point, we find :

$$m_y(D_\pi) = \sum_{x\in\ell_y\cap X} (\mu^{(2)}(X_y,x) + \mu^{(1)}(X_y \cap H,x)) \qquad (*)$$

where H is the plane $z_o = 0$. (Of course, for a small enough representative of $(X,0)$, and in particular we consider only the points in $\ell_y \cap X$ which tend to 0 as $y \to 0$.)
Remark that $\mu^{(1)}(X_y \cap H,x) = \mu^{(1)}(X_y \cap \ell_y,x) = (\ell_y.X)_x - 1$. So by the semi-continuity properties of 2.3, if $m_y(D_\pi) = m_o(D_\pi)$ for all $y \in (Y,0)$, we must have :

$$\mu^{(2)}(X_o,0) = \sum_{x\in\ell_y\cap X} \mu^{(2)}(X_y,x) \qquad \text{for all } y \in (Y,0)$$

$$(\ell_o.X)_o - 1 = \sum_{x\in\ell_y\cap X} ((\ell_y.X)_x - 1) \quad .$$

On the other hand, the basic properties of intersection numbers imply $(\ell_o.X)_o = \sum_{x\in\ell_y\cap X} (\ell_y.X)_x$ , so 1) implies $\ell_y \cap X = \{y\}$ for all $y \in (Y,0)$, and therefore 2) by the two equalities above. Conversely, if we have 2) the equality (*) shows that $D_\pi$ is equimultiple along $Y_\pi$ at 0.
At this point, we have proved theorem 2, and the fact that (Z)-equisingularity ⇒ (μ constant). But as mentionned above, we know in this case that (μ constant) ⇒ ($\mu^*$ constant) and I have already said that ($\mu^*$ constant) ⇒ (c) ⇒ (Whitney conditions) ⇒ (Topological triviality) ⇒ (Topological equisingularity) ⇒ (μ constant). So all that remains to prove is that ($\mu^*$ constant) ⇒ (Z)-equisingularity, and that ($\mu^*$ constant) ⇒ assertion iv) of theorem 1. The proof of the first implication will also prove i) ⇒ iii) of theorem 1 if we do it as follows : ($\mu^*$ constant) ⇒ for any projection $\pi : X \to \mathbb{C}^2$ transversal to X, $(D_\pi,0) = (Y_\pi,0)$. But to say that $\pi$ is transversal means $(\ell_o.X)_o = m_o(X)$ (with the notations introduced above), and if $\mu^*$ is constant, $(X,0)$ is equimultiple along $(Y,0)$, and $m_o(X) = m_o(X_o)$, so that $(\ell_y.X)_y \geq m_y(X) = m_o(X) = m_o(X_o) = (\ell_o.X)_o$ which implies $(\ell_y.X)_y = (\ell_o.X)_o$ by semi-continuity, and then the equality (*) implies that $(D_\pi,0)$ is equimultiple along $(Y_\pi,0)$. Q.E.D.
To see that ($\mu^*$ constant) ⇒ assertion iv) of theorem 1, we can first reduce to the case where the $(X_y,y)$ are irreducible plane curves : indeed, since we have topological equisingularity,

the number of irreducible components of the fibers $(X_y,y)$ is independent of y, and in fact each irreducible component of X is a ($\mu^*$ constant)-family of irreducible plane curves (see the appendix).
The proof of the blowing-up part of iv) follows from the following fact : if you blow up the origin in an irreducible curve $(X_o,0)$, there is only

one point lying over 0 in the blown-up curve $(X'_o,0') \to (X_o,0)$, and $\mu^{(2)}(X'_o,0') = \mu(X_o,0) - m(m-1)$ (see [$Z_4$], prop. 5.1 or [$T_2$], lemma 5.16.1) where $m = m_o(X_o)$. So if X is irreductible, $\mu^*$ constant, when we blow up Y in X, say $b : X' \to X$, we find that since X is equimultiple along Y and irreducible, $b^{-1}(Y)_{red} = Y'$ is smooth over Y, and $b^{-1}(X_y,y)$ is just the curve obtained by blowing up y in $X_y$ : therefore X' is again a family of irreducible curves with Milnor number constant along Y' and equal to $\mu^{(2)}(X_o) - m(m-1)$. So we can go on, blow up Y' in X', etc. but as the Milnor number strictly decreases at each step, eventually we will reach an $\bar{X}$ which is smooth. By equimultiplicity, each blowing up in the sequence is finite (and bimeromorphic) so $\bar{X} \xrightarrow{n} X$ is finite and bimeromorphic, and since $\bar{X}$ is smooth, n is the normalization of X. Of course this sketch of proof can be read backwards to show that iv) $\Rightarrow$ ($\mu^*$ constant).

From the same numerical view point it is possible to prove that if (Y,0) is the entire singular locus, ($\mu^*$ constant) $\Leftrightarrow$ (iv,$\beta$) of theorem 1 using [$T_2$], chap. II, prop. 3.1) and the fact that ($\mu^*$ constant) is also in this case a condition of equimultiplicity for the ideals generated by $(\frac{\partial F}{\partial z_o}, \frac{\partial F}{\partial z_1})$ in the local rings of the fibers $(X_y,y)$, since we have in general :

<u>Proposition</u> ([$T_2$], chap. II, cor. 1.5) : Let $(X_o,0) \subset (\mathbb{C}^{n+1},0)$ be a germ of hypersurface with isolated singularity, say with equation $f(z_o,\dots,z_n) = 0$. Then the multiplicity in $\mathcal{O}_{X_o,0}$ of the jacobian ideal j' $(= j(f).\mathcal{O}_{X_o,0})$ generated by the images of $(\frac{\partial f}{\partial z_i})$ $0 \le i \le n$, is $\mu^{(n+1)}(X_o,0) + \mu^{(n)}(X_o,0)$.

In our case this gives precisely $\mu^{(2)}(X_y,y) + \mu^{(1)}(X_y,y)$.

I emphasize however that it is proved in [$Z_2$] that when Y is of codimension 1 in X, and is the entire singular locus of X, condition (iv), $\beta$) of theorem 1 is in itself a necessary and sufficient condition of (Z)-equisingularity.

<u>Appendix</u> : Numerical invariants of reduced plane curves.

Attached to a germ of reduced and irreducible plane curve $(X_o,0)$, there is a sequence of integers, the characteristic of the curve $(m,\beta_1,\dots,\beta_g) = C(X_o,0)$, m is the multiplicity of $X_o$ at 0, and the $\beta_i$ are the characteristic exponents of the curve (see [$Z_3$], [$P_2$]) this characteristic can be quickly, if not very informatively, described by saying that $(X_o,0)$ has the same topological type as the curve given parametrically by

$$z_o = t^m$$

$$z_1 = t^{\beta_1} + t^{\beta_2} + \dots + t^{\beta_g} \qquad (m < \beta_1 < \dots < \beta_g)$$

and does not have the same topological type as any curve given similarly $z_o = t^{m'}$, $z_1 = t^{\beta'_1} + \dots + t^{\beta'_{g'}}$ with $m' < \beta'_1 < \dots < \beta'_{g'}$, and $g' < g$. So the characteristic is a complete set of topological invariants for irreducible plane curves (see [$P_2$], [$Z_3$]).
The situation for reducible plane curves is described by :

Theorem (Zariski) : Two germs of reduced plane curves $(X_1,0)$ and $(X_2,0)$ have the same topological type if and only if there exists a bijection B from the set of irreducible components of $X_1$ to the set of irreducible components of $X_2$, which respects topological types and intersection multiplicities, i.e. $B(X_{1,i},0)$ has the same topological type (or characteristic) as $(X_{1,i},0)$ and $(X_{1,i}.X_{1,j})_o = (B(X_{1,i}).B(X_{1,j}))_o$.

Zariski proved the "if" part using his theory of saturation (see [$Z_4$], theorems 2.1 and 6.1). I cannot, however, find a reference for the "only if" part, except perhaps [$H_5$]. (The only problem, of course, concerns intersection multiplicities, and they can be defined topologically as linking number of the knots obtained by intersecting $X_{i,o}$ and $X_{j,o}$ with a sufficiently small sphere $\mathbb{S}^3$ in $\mathbb{C}^2$ centered at 0, see Brieskorn's lectures and [$H_5$].)
Anyhow, the multiplicity of a reduced plane curve, sum of the multiplicities of its irreducible components, depends only on its topological type!

2.9 I want now to explain what looks to me as a set of "axioms" for equisingularity from the view point of plane sections, discriminants, and the nonsplitting phenomenon.
In what follows equisingular will mean : any definition, and when I write discriminants I assume they exist, i.e. the corresponding fiber has an isolated singularity. First a definition : $(X,0) \subset (\mathbb{C}^{N+1},0)$, a hypersurface, is said to be (i)-equisingular along a smooth $(Y,0) \subset (X,0)$ $(i \geq \dim Y)$ if, assuming as usual Y linear in $\mathbb{C}^{N+1}$, there exists a dense Zariski open $U^{(i)}$ in the grassmannian $G^{(i)}$ of i-planes containing Y such that $H \in U^{(i)} \Rightarrow (X \cap H,0)$ is equisingular along $(Y,0)$.
Now, I think a good theory of equisingularity should satisfy the following :

1) Going up : Given a projection (linear, for simplicity)

$$\begin{array}{ccc} (X,0) & \subset & (\mathbb{C}^{N+1},0) \\ & \pi \searrow \quad \swarrow p & \\ & (\mathbb{C}^k,0) & \end{array}$$

and a $(Y_\pi,0) \subset (B_\pi,0)$ where $B_\pi$ is the branch locus of $\pi$, if $(Y_\pi,0)$ is a linear subspace of $(\mathbb{C}^k,0)$ and $(B_\pi,0)$ is $(i)$-equisingular along $(Y_\pi,0)$ $(\dim Y_\pi + 1 \leq i \leq k)$, then $Y = (p^{-1}(Y_\pi) \cap C_\pi)_{red}$ is smooth (where $C_\pi$ is the critical locus of $\pi$), p induces an isomorphism $(Y,0) \simeq (Y_\pi,0)$, and $(X,0)$ is $(N+1-k+i)$-equisingular along $(Y,0)$.

2) Going down : Assume $(X,0)$ is $(j)$-equisingular along a linear $(Y,0) \subset (X,0)$ $(\dim Y \leq j \leq N+1)$.
Then, for any linear projection $p : (\mathbb{C}^{N+1},0) \to (\mathbb{C}^k,0)$ such that

α) $\operatorname{Ker} p \cap Y = (0)$

β) Setting $(Y_\pi,0) = (p(Y),0)$, for every i-plane H in some dense Zariski open subset of the grassmannian of i-planes in $(\mathbb{C}^k,0)$ containing $Y_\pi$, $(i + N + 1 - k \leq j)$, $X \cap p^{-1}(H)$ is equisingular along $(Y,0)$.

we have that $B_\pi$ is $(j+k-N-1)$-equisingular along $Y_\pi$ at 0.
We can alternatively ask :

2') Generic going down : There exists a dense Zariski-open subset of the set of linear projections $p : (\mathbb{C}^{N+1},0) \to (\mathbb{C}^k,0)$ such that this conclusion holds.

Remark : If $i = \dim Y + 1$, X $(i)$-equisingular along Y means X is equimultiple along Y. (Of course I ask all definitions of equisingularity to give $Y = X$ in codimension 0.) It may happen that $B_\pi = \mathbb{C}^k$. In this case we will say that $(B_\pi,0)$ is $(k)$-equisingular along $(\mathbb{C}^k,0)$ if we are in the equality case of 2.3.1.
Remark also that any definition of equisingularity which goes up and down has to be equivalent to Zariski's definition. (Indeed, it needs only to go up and down for $k = N$.) To show that this is not just a formality, let me at least prove the nonsplitting part of the going up, assuming $D_\pi$ is $(\dim Y_\pi + 1)$-equisingular along $Y_\pi$, that is, equimultiple. Here the basic result is

Theorem 5 (see [$L_5$], [La ], [$T_2$] chap. III) : Suppose that we have a projection

$$\begin{array}{ccc} (X,0) & \hookrightarrow & (\mathbb{C}^{N+1},0) \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle p} & \\ & (\mathbb{C}^k,0) & \end{array}$$

such that $(B_\pi,0) = (\mathbb{C}^k,0)$ and $B_\pi$ is (k)-equisingular along $(\mathbb{C}^k,0)$, i.e. for any $y \in (\mathbb{C}^k,0)$, $\sum_{x\in\pi^{-1}(y)} \mu(X_y,x) = \mu(X_o,0)$. Then the induced map $\pi : (C_{\pi,red},0) \to (\mathbb{C}^k,0)$ is an isomorphism (and $X_y$ has only one singular point).

This non splitting result has many algebraic translations, for example going back to the notations of 2.3.1 and 2.3.2 : $\exists\, r : F^r \subset J \Rightarrow \exists t : S^t \subset J$, which is a priori rather odd (see also [$T_2$] chap. III), but not algebraic proof. I think it is a very good problem to seek an algebraic proof of it. Anyway, it has the

Corollary : Given a projection as in 2.9, 1), suppose there is a linear subspace $(Y_\pi,0) \subset (B_\pi,0)$ such that $(B_\pi,0)$ is equimultiple along $(Y_\pi,0)$. Then if $(C_\pi,0) \subset (X,0)$ is the critical susbpace, $(\pi^{-1}(Y_\pi) \cap C_\pi)_{red}$ is smooth and isomorphic to $Y_\pi$ by $\pi$. (Compare with 2.8.5. This generalizes unpublished results of Zariski, and was also noticed by Lê.)

Proof : As we have already remarked, $B_\pi$ is equimultiple along Y if and only if $D_\pi$ is so. Then (this is $\dim Y_\pi + 1$-equisingularity) we can find a $(\dim Y_\pi + 1)$-plane H containing $Y_\pi$ in $(\mathbb{C}^k,0)$ such that $(D_\pi \cap H)_{red} = Y_\pi$. We make the base change of $\pi$ by the inclusion $(H,0) \to (\mathbb{C}^k,0)$, i.e. restrict everything over H and thus we obtain a situation

$$\begin{array}{ccc} (X_H,0) & \hookrightarrow & (p^{-1}(H),0) \\ & {\scriptstyle\pi_H}\searrow \quad \swarrow{\scriptstyle p_H} & \\ & (H,0) & \end{array}$$

where now $B_{\pi_H} = Y_\pi$, $D_{\pi_H}$ is equimultiple along $Y_\pi$. We can choose a retraction $\rho : (H,0) \to (Y_{\pi_H},0)$ and by 2.8.3 we have

$$m_o(D_{\pi_H}) = \mu(X_o,0) + \mu(\widetilde{X}_o,0) \qquad \text{where } \widetilde{X}_o = \pi^{-1}(\rho^{-1}(0))$$

$$= m_y(D_{\pi_H}) = \sum_x \mu(X_y,x) + \mu(\widetilde{X}_y,x) \qquad \widetilde{X}_y = \pi^{-1}(\rho^{-1}(y))$$

the sum being over the points of $\pi_H^{-1}(y) \cap C_{\pi_H}$ .

By 2.3.1, we know

$$\mu(X_o,0) \geq \Sigma\, \mu(X_y,x)$$

$$\mu(\tilde{X}_o,0) \geq \Sigma\, \mu(\tilde{X}_y,x)$$

so both must be equalities, which implies that if we now restrict over $Y_\pi$, $(X_{Y_\pi},0) \to (Y_\pi,0)$ satisfies the assumptions of theorem 5, from which we deduce the desired result.

Notice that we have proved much more, namely that $X_H$ satisfies ($\mu$ constant) along $Y = (\pi_H^{-1}(Y_\pi) \cap C_{\pi_H})_{red}$ and even has a hyperplane section containing Y which also satisfies ($\mu$ constant), which gives some hope for the going up.

Problem : Take any of the definitions of equisingularity given in § 1 (or (c)) and prove it goes up and down.

All we know in general is that topological triviality goes up (Varchenko).

2.10 With the idea that any good definition of equisingularity has to go up and down, one can give a purely topological equisingularity problem, as follows :

Given a reduced hypersurface $(X,0) \subset (\mathbb{C}^{N+1},0)$, take a non singular subspace $(Y,0) \subset (X,0)$, and for simplicity, I will assume $\dim Y = 1$. Then we can define the "vanishing cycles" of X along Y at 0 with respect to a retraction $r : (\mathbb{C}^{N+1},0) \to (Y,0)$ like this (see [H5]) : If as usual $X_y = r^{-1}(y) \cap X$, one can prove that there exists $\varepsilon_o \in \mathbb{R}^+$, and a monotone increasing function $\lambda : [0,\varepsilon_o] \to \mathbb{R}^+$ (depending on (X,0), (Y,0) and r) such that, if we call $y_1$ a coordinate on (Y,0), the topological type of $X_y \cap \mathring{B}_\varepsilon$ is independent of $\varepsilon$ and y provided $0 < \varepsilon < \varepsilon_o$, $0 < |y_1| < \lambda(\varepsilon)$, and independent of the choice of r, provided r is "sufficiently general". The reduced cohomology groups $H^i(X_y \cap \mathring{B}_\varepsilon, \mathbb{Z})$ will be called the "groups of vanishing (co-)cycles" of X along Y at 0. We will say that X "has no vanishing cycles" along Y at 0 if they are all 0.

Problem : Show that the condition "no vanishing cycle along Y" goes up and down, that is :

1) Given a linear projection $p : (\mathbb{C}^{N+1},0) \to (\mathbb{C}^N,0)$ with $\operatorname{Ker} d\pi \cap T_{Y,0} = (0)$, set $\pi = p|X$, and $Y_\pi = p(Y)$. If the branch locus $B_\pi$ has no vanishing cycles along $Y_\pi$, X has no vanishing cycles along Y.

2) If X has no vanishing cycles along Y, show that there exists a

(Zariski) open dense subset U in the space of linear projections such that if $p \in U$, $D_\pi$ has no vanishing cycles along $Y_\pi$.

This problem is settled by what we have seen above at least in the case where Y is of codimension 1 in X, since we have (using theorem 5) :

Proposition (Hironaka [$H_5$]) : If Y is the entire singular locus of X, "X has no vanishing cycles along Y" $\Leftrightarrow$ ($\mu$ constant) holds, and of course, if Y is of codimension 0 in X, no vanishing cycles means $Y = X$.
By the way, $B_\mu$ becomes thus a better slogan : "No vanishing cycles $\Rightarrow$ no vanishing folds". It seems to me that this problem is very attractive for the following reason : its solution in the large requires the construction of what I like to call "the ultimate Morse theory in the complex domain", namely a theory which tells us that under suitable genericity assumptions on $\pi$, we can lift (vanishing) cycles from the branch locus to X itself . Of course, we can put a similar problem for projections to $\mathbb{C}^k$ $k \leq N$, and for $k = \dim Y + 1$, it would prove $B_\mu$ (see 2.8.4).

2.11 I'll now try to add a little to our experience with the numerical view point.

Certainly, from a differential-geometric view point, the aim of a theory of equisingularity is to obtain a partition of a given complex analytic space X, $X = \bigcup X_\alpha$ with the following properties :

1) each point $x \in X$ has a neighborhood which meets only finitely many $X_\alpha$.

2) each $X_\alpha$ is a smooth and locally closed subspace in X, and $\bar{X}_\alpha - X_\alpha$ and $\bar{X}_\alpha$ are closed subspaces of X, and $\dim(\bar{X}_\alpha - X_\alpha) < \dim X_\alpha$ .

3) $\bar{X}_\alpha \cap X_\beta \neq \emptyset \Rightarrow X_\beta \subset \bar{X}_\alpha$ and for any $x \in X_\beta$ and any imbedding of a small enough neighborhood U of X in an affine $\mathbb{C}^{N+1}$, $\bar{X}_\alpha \cap U$ is "cone-like over its boundary, with respect to $X_\beta$" in the sense of 2.8.1.

These conditions are precisely what was achieved by Whitney [$W_1$], and what allows one to do some differential geometry on singular varieties. Apart form the motivations given in § 1, the construction of Whitney offers a challenge to algebraic geometers, namely :

Challenge : To describe in many algebraic (i.e. complex analytic !) ways a partition of a singular space which has the properties listed above.

The first thing to do is to describe algebraically what "cone like" can mean. Let me take the case where $X = \bar{X}_\alpha$ is a hypersurface in $\mathbb{C}^{N+1}$, and set

$Y = X_\beta$.
Then we can describe this as usual by $F(y_1,\dots,y_k, z_o,\dots,z_n) = 0$ $(n+k = N)$. Then, an algebraic geometer has a normal cone of X along Y obtained as follows : expand F into homogeneous polynomials in $z_o,\dots,z_n$ with coefficients in $\mathbb{C}\{y_1,\dots,y_n\}$. $F = F_m + F_{m+1} + \dots$ , $F_i$ homogeneous of degree i in $(z_o,\dots,z_n)$. Then the normal cone is defined by $F_m(y_1,\dots,y_k ; z_o,\dots,z_n) = 0$, which can be viewed as a family of cones parametrized by Y : $C_{X,Y} \to Y$, (see [$H_3$]). It turns out that it is hopeless to look only at the normal cone to see if X is cone-like over its boundary with respect to Y, simply because you can find two hypersurfaces with the same normal cone, one of which is cone-like, and the other not. e.g. in $\mathbb{C}^3(y_1,z_o,z_1)$ $z_1^2 - z_o^7 = 0$ (a product of a curve by the $y_1$-axis) and $z_1^2 - z_o^7 + y_1 z_o^3 = 0$. So to understand what "cone like" might mean algebraically, we have to look further into the expansion of F. Now there are two ways of doing this indirectly.

1) Blow up Y in X and look again (compare theorem 1, iv)).
2) Look at the ideal $J = (\frac{\partial F}{\partial z_o},\dots,\frac{\partial F}{\partial z_n})\, \mathbb{C}\{y_1,\dots,y_k, z_o,\dots,z_n\}$ .

Now J is the ideal that comes into Whitney conditions and condition (c), and the first point I want to make here is the following : What theorem 3 says, (going back to 2.8.1) is that to ensure that X is "cone-like over its boundary" we can replace the (non-complex analytic) tube $\mathbb{B}_\rho \times Y$ by the (complex analytic) "generic flag of plane sections through Y" which comes into ($\mu^*$ constant), for the study of J (or of limit positions of tangent hyperplanes at smooth points of X as they tend to Y, which is the same as the study of J).
The second point is something akin to the problems considered in [$H_1$] and which throws some new light on what "cone-like" may mean algebraically :

<u>Definition</u> : Given $F(y_1,\dots,y_k, z_o,\dots,z_n)$ as above, let me say that the t-jet of F along Y is strongly sufficient if the following holds : For any $t' \geq t$ and any $G \in S^{t'}$ $(S = (z_o,\dots,z_n)\mathbb{C}\{y_1,\dots,y_k, z_o,\dots,z_n\})$ there exists $\varepsilon_i \in S^{\nu(t')}$ (depending on G) such that

1) $F(y_1,\dots,y_k, z_o,\dots,z_n) + G(y_1,\dots,y_k, z_o,\dots,z_n) \equiv F(y_1,\dots,y_k, z_o + \varepsilon_o,\dots,z_n + \varepsilon_n)$.

2) $\frac{\nu(t')}{t'} > \frac{1}{2}$ .

Then

<u>Theorem 6</u> (with J.P.G. Henry, see [H.T ]) : For F as above, such that $J \subset S$, the following are equivalent :

1) there exists a t such that the t-jet of F along Y is strongly sufficient.

2) $F = 0$ satisfies ($\mu$ constant) along Y, near 0.

So

So the existence of a strongly sufficient jet along Y may be a subtle way to say algebraically in this case, that X is "cone like" with respect to Y (at least if we assume $B_\mu$ is true).

## § 3. POLAR VARIETIES

3.1 This is a presentation of some algebraic ways to investigate the relationship between a hypersurface, say, and its generic hyperplane section. (The details will appear in $[T_3]$.) We have many motivations to do this from § 1 and 2, but let me add a few more.

First, remark that the topological type of a generic hyperplane section of a given hypersurface is well defined, as can be seen by applying the Whitney stratification theorem (1.1) to the family of hyperplane sections constructed in § 1, and then using $(W) \Rightarrow (T.E.)$. (This works for plane sections of any dimension).

Question $B_{top.}$ : If two germs of complex analytic hypersurfaces have the same topological type, does it imply that their generic hyperplane sections also have the same topological type ?

(By convention, for a plane curve, the "topological type" of the generic hyperplane section is the multiplicity of the curve. The answer to the question above is known only for curves, see 2.8 App.) An affirmative answer to this question would imply in particular that if two hypersurfaces with isolated singularities have the same topological type, they have the same $\mu^*$, and in general, it would imply that same topological type $\Rightarrow$ same multiplicity, thus answering a question of Zariski $[Z_5]$ (by descending induction).

I want to relate $B_{top.}$ to the following, inspired by a question of Thom in $[Th_3]$.

Question C : Is it true that any germ of complex analytic hypersurface has the same topological type as a germ of algebraic hypersurface ?

Of course, if (X,0) has an isolated singularity, it is known that (X,0) is even analytically isomorphic to a germ of algebraic hypersurfa-

ce ; in fact (see e.g. [Mi]) given an equation $f(z_o,\dots,z_n)=0$ for a germ of hypersurface with isolated singularity, there exists an integer t depending only on j(f) such that if $g\in\mathfrak{M}^t$, then we can find a change of coordinates $z_i\to u(z_i)=z_i+\varepsilon_i(z_o,\dots,z_n)$ $\varepsilon_i\in\mathfrak{M}^2$, such that $f(\underline{z})+g(\underline{z})=f(\underline{u}(\underline{z}))$, and in practice, if s is the smallest integer such that $j(f)\supset\mathfrak{M}^s$, which exists since $f=0$ has an isolated singularity, we can take $t=2s+1$. Thus $f=0$ is analytically isomorphic to the hypersurface obtained by forgetting all the terms of degree $\geq t$ in the Taylor expansion of f. This method will not work, at least directly, for the non-isolated singularities, even if we want only to preserve the topological type. Look at the Whitney example ([$W_1$]) of a hypersurface in $\mathbb{C}^3$ which is not locally analytically isomorphic to an algebraic hypersurface : it is defined by $F=z_o.z_1.(z_o-z_1)(z_o-(3+y_1)z_1)(z_o-\gamma(y_1)z_1)=0$ where $\gamma(y_1)$ is any transcendental function with $\gamma(0)=4$ e.g. $\gamma=4.e^{y_1}$. This is an equisingular family of curves, each curve consisting of 5 lines. From the fact that equisingularity implies topological triviality, we know that it has the same topological type as an algebraic hypersurface, namely $\tilde{F}=z_o z_1(z_o-z_1)(z_o-3z_1)(z_o-4z_1)=0$. However, for any $t\geq 1$, $F+y_1^t=0$ has an isolated singularity at 0 and therefore does not have the same topological type as $F=0$, as one can check by using for example the first theorem in [$L_2$]. Please compare all this with Theorem 6.
So we set ourselves the following :

Problem : Define algebraically a complete set of invariants for the topological type of a germ of complex hypersurface.

Ideally, this set of invariants should enable me to play the following :

Game : Given a hypersurface $(X,0)\subset(\mathbb{C}^{N+1},0)$, I fill a bag with invariants arranged in layers according to dimensions $1\leq i\leq N+1$, the top layer corresponding to $i=1$. Then I give somebody the first layer of invariants, and he constructs a germ of plane curve, the invariants of which are precisely those in the first layer, and which has the same topological type as the section of my hypersurface by a generic 2-plane. I then give him the second layer, and from this and the curve he constructed before he builds a surface, which has the same topological type as the section of (X,0) by a generic 3-plane, and a generic plane section of which has the same topological type as the curve constructed before, etc. When the bag is empty he has built a hypersurface (X',0) with the same topological type as (X,0). Of course, for sheer economy, at each step he will build an algebraic hypersurface !

Now the invariants constructed below from polar curves should, I think, be at least part of what one needs to put in the bag.

3.2 Where to look for invariants ? an answer is suggested by the following, which is an improvement on a theorem of Lê and Ramanujam ([L.R. ]) :

<u>Theorem 6</u> (see [$T_3$]) : Let $(X_o,0) \subset (\mathbb{C}^{N+1},0)$ and $(X_1,0)\subset(\mathbb{C}^{N+1},0)$ be two germs of hypersurfaces with isolated singularity, defined respectively by $f(z_o,\dots,z_n)=0$, $g(z_o,\dots,z_n)=0$. If $\mathcal{O}_{n+1}/\overline{j(f)} \simeq \mathcal{O}_{n+1}/\overline{j(g)}$ are isomorphic as analytic algebras (notations of 2.1 and 2.2), then there exists a one parameter family of complex hypersurfaces $X \overset{\sigma}{\leftrightarrows} \mathbb{D}$ where say $\mathbb{D}=\{y\in\mathbb{C} \,/\, |y|<2\}$ such that $(X_o,\sigma(0))=(X_o,0)$, $(X_1,\sigma(1))=(X_1,0)$ and X is Whitney equisingular (even (c)-equisingular) along $\sigma(\mathbb{D})=Y$ at every point. In such a case, we will say that $(X_o,0)$ and $(X_1,0)$ are (c)-cosecant. In particular, they have the same topological type. Since $\overline{j(f)}$ is determined by $(f,j(f))$ as we saw in 2.2, and $(f,j(f))$ determines the singular subspace of $(X,0)\subset(\mathbb{C}^{n+1},0)$ we have for isolated singularities of hypersurfaces, the following :

"The analytic type of the singular subspace determines the topological type". (We will see below how false the converse is.)

Therefore, we look for topological invariants in the jacobian ideal $j(f)$, even for non isolated singularities, where a result similar to the one above is not known. In the isolated singularity case, we have already the sequence $\mu^*(X,0)$ of the Milnor numbers of generic plane sections, which can easily be shown, by its algebraic definition as generalized multiplicity (2.6), to depend only on $\overline{j(f)}$.(Perhaps it is time to point out the difference between $j(f)$ and $\overline{j(f)}$. For example the only isolated singularities of hypersurfaces such that $\overline{j(f)}$ is a power of the maximal ideal are those which have the same topological type as their tangent cone ([$T_2$]). On the other hand, the only isolated singularities of hypersurfaces such that $j(f)=\overline{j(f)}$ are those of type $A_k$, i.e. which can be defined in suitable coordinates by $z_o^{k+1}+z_1^2+\dots+z_N^2=0$.) There are however other geometric ways of studying $j(f)$, in the general case :

3.3 <u>Proposition-Definition</u> : Let $(X,0)\subset(\mathbb{C}^{N+1},0)$ be a germ of complex hypersurface, defined by $f(z_o,\dots,z_N)=0$. Let $H\in G^{(i)}$ be a direction of i-plane in $\mathbb{C}^{N+1}$. Consider the set $P_H$ of points $p\in(\mathbb{C}^{N+1},0)$ such that

1) $f(z_o,\dots,z_N)=f(p)$ ($X_p$, the level hypersurface of f through p) is smooth at p

2) $T_{X_p,p} \supset H$ (i.e. $T_{X_p,p}$ contains a i-plane parallel to H).
The closure $Q_H$ of this set in $(\mathbb{C}^{N+1},0)$ is a closed complex analytic subset of $(\mathbb{C}^{N+1},0)$. In fact, $Q_H$ can be defined as an analytic subspace of $(\mathbb{C}^{N+1},0)$, as follows : choosing coordinates so that H is defined by $z_o = \dots = z_{N-i} = 0$, $Q_H$ is the subspace defined by the ideal $\{(\frac{\partial f}{\partial z_k})\mathcal{O}_{N+1}\ ;\ N-i+1 \le k \le N\}$. The union of those irreducible components of $(Q_H,0)$ which are of dimension N+1-i will be called the polar subspace of $f=0$ with respect to H, and noted $S_H$.

__Theorem 7__ ([$T_3$]) : If (X,0) has an isolated singularity, for each $0 < i < N+1$, there exists a Zariski open subset $V^{(i)}$ of the Grassmannian $G^{(i)}$ of i-planes through 0 in $(\mathbb{C}^{N+1},0)$, such that if $H \in V^{(i)}$, $S_H$ is reduced, $m_o(S_H) = \mu^{(i)}$ and the plane (H,0) is transversal to $(S_H,0)$ at 0 in the sense that $\mu^{(i)}$ is also the intersection multiplicity $(S_H.H)_o$. For $i=0$, we set $S_H = \mathbb{C}^{N+1}$, and for $i = N+1$, $S_H$ is the subspace defined by j(f), which is of multiplicity $\mu^{(N+1)}(X,0)$. Hence $\mu^*(X,0)$ is also the sequence of multiplicities at 0 of the generic polar varieties.

3.4 For $i=N$, $S_H$ is a curve, called the polar curve with respect to the hyperplane direction H, and its consideration has been advocated by Thom for the study of monodromy problems. It also comes into the proof of the proposition in 2.8.3., as follows : (please go back to 2.8.3) : by the projection formula for multiplicities ([Se ]), $\Delta$ is the multiplicity of the critical subspace of $\pi : (X,0) \to (\mathbb{C},0)$. We can always suppose that the projection $p : (\mathbb{C}^{N+1},0) \to (\mathbb{C},0)$ inducing $\pi$ is a coordinate function $z_o$. Then $\Delta$ is the multiplicity of the ideal $(\frac{\partial F}{\partial z_1},\dots,\frac{\partial F}{\partial z_n})\mathcal{O}_{X,0}$ if X is defined by $F(z_o,\dots,z_N)=0$. But that is precisely ([Se ]) the intersection multiplicity of the polar curve $S_H : \left(\frac{\partial F}{\partial z_1},\dots,\frac{\partial F}{\partial z_N}\right)$ in $(\mathbb{C}^{N+1},0)$ with our hypersurface (X,0) at 0 . Since we assume (X,0) to be with isolated singularity, $(\frac{\partial F}{\partial z_o},\dots,\frac{\partial F}{\partial z_N})$ is a regular sequence, and hence all the components of $\frac{\partial F}{\partial z_1} = \dots = \frac{\partial F}{\partial z_N} = 0$ are of dimension 1. Let us decompose $S_H$ in irreducible curves ( not necessarily reduced, since we do __not__ assume the hyperplane $H : z_o = 0$ is in the open $V^{(N)}$ of theorem 7, but only that $(H \cap X,0)$ has an isolated singularity). Let this decomposition be written $S_H = \cup \Gamma_q$ . Now to each $\Gamma_q$ is attached an integer $n(\Gamma_q)$ such that for any hypersurface $(X',0) \subset (\mathbb{C}^{N+1},0)$ we have $(\Gamma_q.X')_o = n(\Gamma_q).(\tilde{\Gamma}_q.X')_o$ where $\tilde{\Gamma}_q = \Gamma_{q,red}$. Now we remember that $\tilde{\Gamma}_q$ can be parametrized, i.e. the normalization $(\bar{\Gamma}_q,0) \to (\tilde{\Gamma}_q,0) \subset (\mathbb{C}^{N+1},0)$ is such that $(\bar{\Gamma}_q,0) \simeq (\mathbb{C},0)$ since $\tilde{\Gamma}_q$ is reduced and irreducible. Let us choose a local coordinate $u_q$ on $(\bar{\Gamma}_q,0)$. The composed map above $(\bar{\Gamma}_q,0) \xrightarrow{h_q} (\mathbb{C}^{N+1},0)$ can be described by N+1 holomorphic functions $z_k = z_k(u_q)$. Then the intersection number of (X,0) with $(\tilde{\Gamma}_q,0)$ is the order in $u_q$ of $f(z_o(u_q),\dots,z_{N+1}(u_q)) = f\circ h_q$. But we have $\frac{d}{du_q} f\circ h = \frac{\partial f}{\partial z_o}\circ h_q \cdot \frac{d(z_o\circ h_q)}{du_q}$ since on $\tilde{\Gamma}_q$ all $\frac{\partial f}{\partial z_i}$ $(1\le i\le N+1)$ are 0.

Taking orders gives

$$(\tilde{\Gamma}_q.X)_o - 1 = (\tilde{\Gamma}_q.X'_o) + (\tilde{\Gamma}_q.H)_o - 1$$

and adding after multiplying by $n(\Gamma_q)$ gives $\Delta = (S_H.X)_o = (S_H.X')_o + (S_H.H)_o$ where X' is $\frac{\partial f}{\partial z_o} = 0$. But it follows from classical results on intersection theory ([Se]) that $(S_H,X')_o = \mu^{(N+1)}(X,0)$ and it is easy to see that $(S_H.H)_o = \mu^{(N)}(X\cap H,0)$.

3.5 All this was done for a given hyperplane H such that $X\cap H$ has an isolated singularity. Now we have

3.5.1 <u>Proposition</u> ($[T_3]$) : Given a hypersurface $(X,0)\subset(\mathbb{C}^{N+1},0)$ with equation $f=0$, there is a Zariski-open dense $W^{(N)}\subset \mathbb{P}^N$ such that : the polar curve $S_H$ is reduced, the number $\ell$ of its irreducible components, $\left(S_H = \bigcup_{q=1}^{\ell}\Gamma_q\right)$ and the sets of integers $\{m_q = (\Gamma_q.H)_o\}$, $\{e_q = (\Gamma_q.X)_o - (\Gamma_q.H)_o\}$ are independent of H, provided $H\in W^{(N)}$. Notice $e_q\ge 0$ by the computation in 3.4. If (X,0) has an isolated singularity, we have furthermore :

$$\sum_1^{\ell} e_q = \mu^{(N+1)}(X,0) \quad ; \quad \sum_1^{\ell} m_q = \mu^{(N)}(X,0) \quad \text{by 3.4} \ .$$

<u>Remark</u> : In the general case, it can happen that $(\Gamma_q,0)\subset(X,0)$, in which case we set $e_q = +\infty$.
Then we have :

3.5.2 <u>Proposition</u> : The set of quotients $\left\{\frac{e_q}{m_q}\right\}_{red}$, where "red" means that each rational number, or $+\infty$, appears only once, depends only upon the integral closure $\overline{j(f)}$ (i.e. can be computed from the ideal $\overline{j(f)}$), and therefore only on the local algebra $\mathcal{O}_{X,0}$. We will denote this set by $F^{(N+1)}(X,0)$.

Let me give an example showing that we have to reduce the sequence if we hope to get topological invariants in this way : consider the family of plane curves : $z_1^4 + z_o^9 + y_1 z_o^5 z_1^2 = 0$. For $y_1 = 0$ the generic polar curve has only one irreducible component, giving $e = 24$, $m = 3$. For $y_1 \neq 0$, it has two irreducible components giving $e_1 = 8$, $m_1 = 1$ and $e_2 = 16$, $m_2 = 2$ respectively ! (This is related to the phenomenon discovered by Pham, that equisingularity does not imply "jacobian equisingularity" see [$P_1$], [$P_3$]. In this example, the analytic type of $\mathcal{O}_2/\overline{j(f)}$ varies although the family is equisingular, as can be quickly checked with Zariski's discriminant criterion.)

At least when the singularity is isolated, I can prove that all the invariants $F^{(i)}(X,0)$ of generic i-plane sections of X are well defined and depend only on $\overline{j(f)}$ hence on $\mathcal{O}_{X,0}$ $(0 \le i \le N+1)$. ($F^{(1)} = \{\mu^{(1)}\}$, $F^{(o)} = \{1\}$.)

3.5.3 Problem : Show that the $F^{(i)}(X,0)$ are always well defined, and are invariants of the topological type, of $(X,0) \subset (\mathbb{C}^{N+1},0)$.

The answer is now known for irreducible plane curves :

Theorem 8 (M. Merle [Me]) : For an irreducible plane curve (X,0), the datum of $(F^{(1)}(X,0);F^{(2)}(X,0)) = \left(m-1,\frac{e_1}{m_1},\ldots,\frac{e_g}{m_g}\right)$ is equivalent by a universal algorithm to the datum of the characteristic $(m,\beta_1,\ldots,\beta_g)$ of the curve.

So in this case, $F^{(1)}(X,0)$ and $F^{(2)}(X,0)$ are enough to fill the bag of invariants.

3.6 I want to quote some results in support of the claim that the $F^{(i)}(X,0)$ should go into the bag, the proofs are in [$T_3$].

3.6.1 Proposition : Given a hypersurface $(X,0) \subset (\mathbb{C}^{N+1},0)$ with equation $f(z_o,\ldots,z_N) = 0$ and isolated singularity. A necessary and sufficient condition for (X,0) to be (c)-cosecant with a hypersurface $(X',0): g(z_1,\ldots,z_N) + z_o^{a+1} = 0$ such that $H : z_o = 0$ is a generic section in the sense that $\mu(X' \cap H,0) = \mu^{(N)}(X',0)$ is that $F^{(N+1)}(X,0) = \{a\}$, $a \in \mathbb{N}$, i.e. all $\frac{e_q}{m_q}$ are equal to the same integer a. In particular this gives inductively a purely algebraic sufficient condition for a hypersurface to have the same topological type as a hypersurface of the Pham-Brieskorn type, i.e. $z_o^{a_o} + \ldots + z_N^{a_N} = 0$.

3.6.2 Proposition : Given a hypersurface $(X,0) \subset (\mathbb{C}^{N+1},0)$. Assume the singular locus Y of X is of dimension 1 and smooth. A necessary and sufficient condition for (X,0) to satisfy "$\mu$ constant" along (Y,0) is : $F^{(N+1)}(X,0) = \{+\infty\}$. (This follows from 2.3.2.)

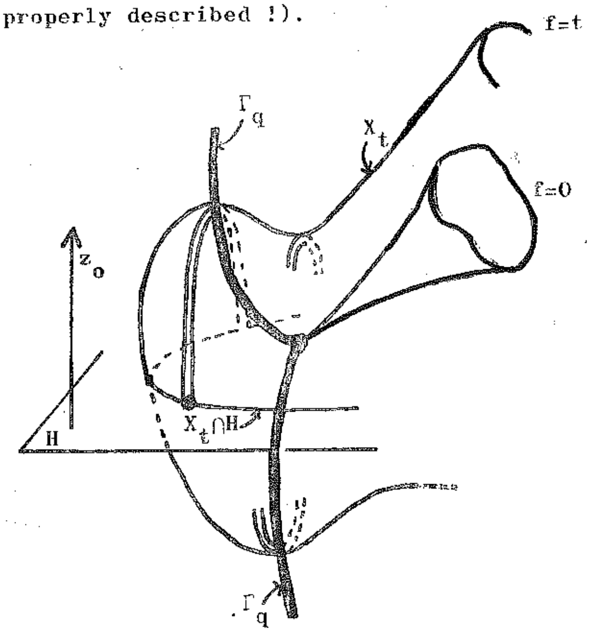
3.6.3 Proposition :

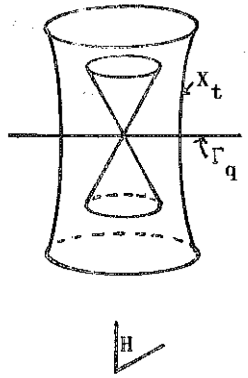
For a hypersurface (X,0) $(f=0)$ with isolated singularity, set $\eta = \operatorname{Sup}\left(\frac{e_q}{m_q}\right)$. Then the smallest possible exponents for the Łojasiewicz inequalities $|\text{grad } f(z)| \geq c_1 |f(z)|^{\theta_1}$, $|\text{grad } f(z)| \geq c_2 |z|^{\theta_2}$ to hold in some neighborhood of the origin in the ambiant space are : $\theta_1 = \frac{\eta}{\eta+1}$, $\theta_2 = \eta$. (What one actually shows is $\bar{\nu}_j(f) = \frac{\eta+1}{\eta}$, $\bar{\nu}_j(\mathfrak{M}) = \frac{1}{\eta}$, where $j = j(f)$, see 2.1). So a positive answer to 3.5.3 would in particular show that these Łojasiewicz exponents are in fact topological invariants.

3.6.4 If we go back to the geometric definition of the polar curve, we obtain the following transcendantal interpretation of $\left(\frac{e_q}{m_q}\right)$, which I explain only in the isolated singularity case, for simplicity : let us choose a general hyperplane H (see theorem 7,(3.3)) and look at the polar curve $S_H = \bigcup \Gamma_q$ ; by theorem 7,(3.3) we can parametrize $\Gamma_q$ as follows : $z_o = u_q^{m_q}$, $z_i = a_i u_q^{k_{i,q}} + \ldots$, where $k_{i,q} \geq m_q$, $1 \leq i \leq N+1$, if $H : z_o = 0$. In a sufficiently small neighborhood of 0 in $\mathbb{C}^{N+1}$, the set of points where the tangent space to a level hypersurface $X_t : f(z_o,\ldots,z_N) = t$ $(t \neq 0$, sufficiently small) is parallel to H is precisely $S_H \cap X_t$. (Recall that $X_t$ is smooth for $t \neq 0$.) Furthermore, each $\Gamma_q$ is smooth outside 0 and meets $X_t$ transversally. The number of these points is $(S_H . X)_o = \mu^{(N+1)}(X,0) + \mu^{(N)}(X,0)$ by Proposition 3.5.1.
Now we can also view these points as the critical points of the function $|z_o|$ ("distance" to H) restricted to $X_t$, so that we can attach to each of these points a gradient cell of dimension N, which originates at the critical point and ends in $X_t \cap H$. (See [M], [$Th_2$] and [$L_4$] where Lê independently introduces rationnal numbers which are actually the $\left(\frac{e_q}{m_q}\right)$, from a topological viewpoint.) In the isolated singularity case the exact sequence of relative homology reduces to : (see Brieskorn's lectures)

$$0 \to H_N(X_t,\mathbb{C}) \to H_N(X_t, X_t \cap H, \mathbb{C}) \to H_{N-1}(X_t \cap H, \mathbb{C}) \to 0$$

and $\dim H_N(X_t,\mathbb{C}) = \mu^{(N+1)}(X,0)$, $\dim H_{N-1}(X_t \cap H,\mathbb{C}) = \mu^{(N)}(X,0)$ since H is general. Hence, our $\mu^{(N+1)}(X,0) + \mu^{(N)}(X,0)$ gradient cells, which generate $H_N(X_t, X_t \cap H)$, are in fact a basis. The idea is in the following picture, (which is misleading because the indices of the critical points are not properly described !).




(Less misleading picture)

The double lines picture gradient cells, and we can see them generating the relative homology $H_N(X_t, X_t \cap H, \mathbb{C})$

Now these gradient cells do not all vanish at the same rate as $t \to 0$. To see this let us compute the value of $f$ on $\Gamma_q$. By the definition of intersection numbers, we have on $\Gamma_q$ an expansion : $f|\Gamma_q = c_q u_q^{e_q+m_q} + \ldots$ ($c_q \in \mathbb{C}^*$). Now we want to compute the distance to H of a point of $\Gamma_q \cap X_t$, i.e. the "height" of the corresponding gradient cell, as a function of t, to see how fast it vanishes when $t \to 0$. Since $z_o|\Gamma_q = u_q^{m_q}$, we find a Puiseux expansion

$$z_o|\Gamma_q = \left(\frac{t}{c_q}\right)^{\frac{m_q}{e_q+m_q}} + \ldots$$

which represents the $z_o$ coordinates of the $e_q + m_q$ intersection points $X_t \cap \Gamma_q$. Thus we have, with the obvious definition of "vanishing rate" ;

Proposition : The vanishing rate of the height of a gradient cell attached to a point of $\Gamma_q \cap X_t$ is equal to $\left(\frac{e_q}{m_q} + 1\right)^{-1}$.

So our sequence $\left(\frac{e_q}{m_q}\right)_{red}$ in fact corresponds to all vanishing rates of the gradient cells, or if you prefer, it indexes a certain filtration on the vanishing homology group $H_n(X_t, X_t \cap H, \mathbb{C})$ (by the "rate of vanishing") which a priori depends on H, but can be seen to be in fact independent of the choice of a sufficiently general H. This gives further reasons to put $\left(\frac{e_q}{m_q}\right)_{red}$ in our bag of invariants : it describes an important feature of the "vanishing Morse theory" which builds $X_t$ from $X_t \cap H$ by attaching gradient cells.

<u>Problem</u> : Show that $F^{(i)}(X,0)$ are constant in a μ-constant family of hypersurfaces.

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<u>Post-scriptum</u> : On the real analytic case : it is proved in 2.8.5 that a family of reduced plane complex curves topological equisingularity ⇔ Whitney conditions. Although Whitney conditions imply topological equisingularity also in the real-analytic (and even differentiable) case, see [Ma], the converse is not true in general. Here is an example : the surface $X : x_o^3 - y_1 x_1^2 - x_1^5 = 0$ in $\mathbb{R}^3$ is easily seen to be topologically trivial along $Y : x_1 = x_o = 0$, but does not even satisfy Whitney's condition a) along Y. Also, after conversations with Merle and Risler, I became convinced that if $f(x_o, \ldots, x_n) \in \mathbb{R}\{x_o, \ldots, x_n\}$, the Łojasiewicz exponents I compute in 3.6.3 for the complexification of f (assuming it has an isolated singularity) are in general larger than the smallest exponents for the corresponding inequalities in the real-analytic case.

<u>Post-sriptum 2</u> : Important : See after the references for recent developments.

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Centre de Mathématiques
de l'Ecole Polytechnique
~~17, rue Descartes~~
~~75230 Paris Cedex 05~~
Plateau de Palaiseau
91128 Palaiseau.
(France)

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(Added in February, 1975). It has turned out that many problems were too optimistic. Zariski has communicated to me a counterexample to conjecture 1, p. 600 : X is defined in $\mathbb{C}^4$ by $Z^3 - (X_2 - X_1 Z)^2 X_3 = 0$ and Y is the line $X_2 = X_3 = Z$, along which X is even analytically trivial. The branch locus of the projection of X to the $(X_1, X_2, X_3)$ space is not equisingular, although condition β) is satisfied. Briançon and Speder of Nice, Compte rendu notes, Feb. (1975) have given a counterexample to Conjecture $B_\mu$ (p. 606 ) : X is defined in $\mathbb{C}^4$ by $z_o^5 + yz_o z_1^6 + z_1^7 z_2 + z_2^{15} = 0$, and Y is $z_o = z_1 = z_2 = 0$. X is a family of quasi-homogeneous (weights $(z_o, z_1, z_2) = (\frac{3}{15}, \frac{2}{15}, \frac{1}{15})$) hypersurfaces with isolated singularity $(X_y, 0)$, all with the same topological type, and $\mu(X_y, 0)$ is constant (= 364). However, the section of X by a general hyperplane $z_2 = az_o + bz_1$ does not satisfy (μ constant), so X does not satisfy (μ* constant). Furthermore, (X-Y,Y) does not satisfy Whitney conditions, and worse still, as noted by Henry, X is (Z)-equisingular along Y (the branch locus of the projection to $(z_o, z_1, y)$ is equisingular along Y . This example and [Sp], or other examples of Briancon-Speder, show that the answers to question A (p.599) question I of [Z5], question $B_{top}$ (p. 620) and the second problem of p. 6?? are negative. In the last problem, also, p.628 replace (μ constant) by (μ* constant).

On the positive side, I have been able to remove the dim Y = 1 assumption in theorem 3 (p. 605), and Briançon-Speder (to appear) have adapted the proof of the proposition p. 606 to show the answer to question B is yes for Whitney-equisingularity when Y the singular locus of X, i.e. in this case (μ* constant) ⇒ (Whitney equisingularity) ⇔ condition (C). Also I have learnt that Mr. A. Nobile (State U. of Louisiana, Baton Rouge , Louisiana) has given, in his 1970 M.I.T. Thesis, a positive answer to question (C) p. 620 in dimension 2. Zariski communicated to me that the reference needed p. 614 is [Z6, II, proof of lemme 7.1, page 942-43]