ON OBJECTIVITY AND MEANING IN MATHEMATICS

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To Jean Petitot, on the occasion of his 80-th birthday

ABSTRACT. In a first part, I shall explore the consequences of distinguishing the foundations of meaning and the foundations of truth in mathematical statements, or imagination and rigor as motors of mathematical development. The foundations of meaning can be sought in our largely unconscious perception of the world, which modern cognitive science is exploring.
In a second part, I shall illustrate this by comparing two approaches to understanding mathematical problems: creating appropriate abstract structures, which is exemplified by Galois, Hilbert, Bourbaki and Grothendieck, or creating geometric models where we can use our intuition of space, of which Riemann, Poincaré and Thom were masters. This part requires some basic mathematical knowledge.

The moving power of mathematical invention is not reasoning but imagination. (W.R. Hamilton, 1805-1865)

For a recent Academic Seminar at the Bulgarian Academy of Sciences, it has been suggested that I talk about René Thom and the Bourbaki group, with both of whom I have had scientific contact. It is a very interesting suggestion because they correspond to different modes of mathematical activity and both has a strong influence. It is also very appropriate because Jean has been close to Thom for decades and has explained and greatly developed, with his extraordinarily penetrating mind, the consequences of Thom's approach to Science in many fields: Linguistics, Semiotics, Philosophy of Science, Epistemology, Esthetics, and more.

But before I can describe that difference I think it is useful to reflect on the forces which animate the search for objectivity and its ally rigorous reasoning, and the search for meaning, an obsession indispensable for creativity.

My title is provocative because Mathematics is supposed to be, by its very nature, objective in the sense that its structure is independent of the subject who studies it or makes use of it. This is interpreted by some philosophers as implying that Mathematics exists independently of human understanding in a world of ideas of which we humans perceive only projections, or shadows. It is a simplified but all too common version of platonism with which I am not sure that even Plato would agree.

Anyway, I hope to convince the reader that Mathematics is a human science whose origin and growth are the result of extremely complex interactions between our mostly unconscious perception of the world and pulsions which are just as unconscious and force us to organize those perceptions. The world of ideas is actually the product of concrete but extremely complicated processes. The impression of objectivity is due to the fact that humans share this perceptual system and those pulsions. It is indeed independent of any one individual but that does not make it superhuman.

The other option is that there is indeed an abstract universe of ideas of which we explore only the part we can understand with our limited abilities and that understanding is what

we call ”meaning”. I do not find this satisfactory but for a long time we had no way of finding a firm basis for the human alternative.
What is new in our time is that cognitive sciences allow us to begin to get an idea of how the perceptual system works to create, among other activities, what one can call ”protomathematical objects”. I claim that such objects are the foundation upon which -with much elaboration and many layers- the meaning we give to Mathematics is built.

Poincaré had the intuition of this:
In summary, for each attitude of my body, my first finger determines a point and it is that, and only that, which defines a point in space.
Henri Poincaré, in *La Science et l'hypothèse*, Flammarion.
In other words, the (unconscious) tensions of the muscles which position the finger are a system of coordinates for (our perception of) space.

My favourite example (see [8]) is the protomathematical continuum which results from the extremely strong interaction in our perception/action system of the vestibular line, the visual line, and the motor system.

The vestibular line

Our vestibular system is an inertial navigation system which detects accelerations, rotations, etc. with great accuracy. It is strongly connected to the motor system in order, for example, for a biped to be able to react very quickly when it stumbles, thus creating a high acceleration of its head.

More precicely, in our inner ear are three *semicircular canals*, each of which detects rotations of our head around one of the directions of our three dimensional space. They are semicircular tubes filled with a fluid which, in case of movement, rubs against cilia on cells of the inner surface of the tube, thus creating a signal for the brain. There are also, next to the canals, *sacculae* which are small chambers containing tiny calcium carbonate crystals, the *otoliths* which detect linear acceleration, again by rubbing against cilia.

Moving at constant speed in a constant direction (inertial motion) corresponds to a particular state of the assembly of neurons which manages the vestibular information. According to Galilean relativity, the vestibular system sends no signal to the brain during the inertial motion. Of course the head goes up and down, which does send some signal, but this is compensated for. There are only two ways to measure the distance covered during an inertial motion: the time elapsed, assuming we know the speed, and the number of steps.

The visual line

The optic nerve transmits the electric impulses from the retina to the visual areas of the brain at the back of the skull. The retina cells are already specialized, and then the impulses are subjected to a filtering. There is a quite specific and extremely complicated architecture of the neurons in the first visual area of the brain (see [5], [6]) which implies in particular that if a neuron detects in a certain direction of sight a segment with a given orientation, it excites the neurons in its neighborhood to reinforce the detection of a segment with a similar orientation.
This is a gross oversimplification but the end result is that our visual system can detect curves, contours, and especially lines, which correspond again to a special state of an assembly of neurons in the visual cortex. Note that the visual line has no orientation and by itself no measure of progress.
Professor Alain Berthoz has conducted many experiments in his lab at the Collège de France to study the integration of visual, vestibular and muscular perceptions. It makes a very strong case (see [1]) for the idea that our unconscious perceptual system almost

identifies the visual and vestibular lines In addition, visual perception is strongly coupled with muscular tension, which supports Poincaré's intuition. I call this indentification the Poincaré-Berthoz isomorphism (see [8]).
Of course it is not an isomorphism in the strict mathematical sense, but in the sense that it allows transporting structure from one to the other. For example, the natural orientation of the vestibular line, and its notion of distance, are carried over to the visual line, which is our model for the real line, and this allows us to think of it as parametrizing time. The steps measuring the vestibular line become the integers disposed on the visual line, etc. This leads also to the concept of trajectory parametrized by time, a fundamental notion,

One cannot overestimate the consequences of the fact that the Poincaré-Berthoz isomorphism carries the continuity of motion to the continuity of the visual line.

Einstein said that one of his basic intuitions was to think of himself as moving along a ray of light

The construction of the real line from the integers, then the rationals, then Cauchy completion, does not provide us with such a vision. It does provide a construction from set theory, considered as being objective and providing a foundation of truth for the statements concerning for example the convergence of sequences, continuous functions (intermediate value theorem), etc. But does it provide a foundation of meaning?

For example, we can imagine without difficulty walking indefinitely on the vestibular line from where we are. It is much more difficult to imagine having walked indefinitely to arrive where we are. I think it took the invention of -1 as an operator one can iterate to imagine this. And it is perhaps the origin of the concept of well ordered set where here are no infinitely decreasing sequences.
The vestibular line has its notion of boundary which, when it is parametrized by time, is the instant and when it is parametrized by walking is the end of the motion. The visual line has also an obvious (sic!) notion of boundary. It is a fundamental intuition of Dedekind that important characters of the difference between the real line and a set of points given in bulk are that it is totally ordered, is divisible (one can cut intervals into parts) and is *made of boundaries*, the Dedekind cuts.

In conclusion, the Poincaré-Berthoz isomorphism creates a protomathematical object blending the attributes of the visual and vestibular lines, which will serve as a *source of meaning* for many statements concerning sequences, arithmetical operations, parametrized curves, etc.
Of course mathematicians learn by usage to give meaning to much more elaborate objects and statements, but I claim that at the bottom of it there are similar protomathematical objects and amazing properties of adaptation of our perceptual system to the world aroud us.

In particular, Euclidean geometry would not exist if our perceptual system did not detect lines, angles, parallelism.

For example, as mentioned above, the ”parallel transport” of orientations of segments, which is a concept of differential geometry is already cabled in the architecture of our visual system. We detect it in the world around us and this gives us the meaning of the abstract notion.
This is not at all a reductionist discourse because I very strongly believe that the complexity of the physiological structures and dynamics which we would use to rationally provide meaning to mathematical objects and statements is literally beyond human comprehension.

Indeed, what is superhuman, or transcendental, is the complexity of the functional architecture and the dynamics of the brain. But this is the place where meaning resides, and any progress in that direction would be fascinating.

Let me now come to the unconscious pulsions which push mankind to elaborate protomathematical objects into Mathematics. They are in a sense similar to Freud's libido but quite distinct. I can name only a few of what I believe to be a long list.

- Detecting structures, for example periodicities and repetitions. Most importantly, detecting boundaries of all kinds.
- Capacity of creating mental images to simplify and abstract independently of language.
- Comparing comparable objects *without consciously asking the question*. This leads to consciously comparing and measuring lengths, areas, etc.
- Above all, an extremely strong need to find causes and origins. In particular, if A implies B, does B imply A?

Our basic intuitions of space is based on this kind of subconscious interiorization, through evolution, of our experience of the world into for example protomathematical objects. When submitted to these unconscious forces or pulsions, it produces Mathematics, in particular when faced to something contrary to intuition. This happens especially when some infinite process comes into play. Think of Zeno's paradox, or the discussions about the actual or virtual existence of the infinite. Think of Cantor discovering that there are as many points in a square or a cube as in a segment of the real line. He wrote: "I see it but I cannot believe it". A lot of Mathematics are born of such questionings (search for causes) and astonishments.
For example, the ancient Greeks had a method, called *Anthyphairesis*, to describe the exact relationship of two lengths. Count how many times the smaller one goes into the larger one. This is an integer a_0. In general, there is a remainder r_0 which is smaller than the smaller length. Then, one counts how many times the remainder goes into the smaller length. This gives a second integer a_1. Again, there is in general a remainder r_1 which is smaller than r_0, and one continues in this manner. The ratio of the two lengths is expressed as a sequence of integers. If the sequence is finite, the two lengths are commensurable, which etymologically means that they can be expressed as integral multiples of a unit length, the last non zero remainder. Their ratio is a rational number. Otherwise, the ratio is irrational. There is a Lemma in Euclid's Elements which implies that if one performs anthyphairesis for the diagonal of a square and its side, the sequence of integers obtained is $1, 2, 2, \ldots, 2 \ldots$. This is a geometric lemma, using constructions of Euclidean geometry.
Hint: If s and d are the lengths of the side and diagobal of our square, build a larger square of side $S = s + d$ with a side extending the diagonal of our square. Easy geometric considerations will give you the value $D = 2s + d$ of the length of the large diagonal. Then compute the ratio $\frac{D+S}{S} = 1 + \frac{D}{S} = 1 + \frac{d}{s} = 2 + \frac{s}{s+d}$. You see that the anthyphairesis will keep giving 2.
This Anthyphairesis of $1 + \sqrt{2}$ proves the irrationality of $\sqrt{2}$. According to Fowler in [2] this did not really cause an intellectual upheaval in Greek science, because a constant, or even periodic, sequence of integers is still amenable to reason. Anyway this proof is far superior to the proof taught in schools because it provides approximations of $\sqrt{2}$ by rational numbers as good as one wishes by computing the rational number obtains by keeping only the first k 2's and increasing k. Anthyphairesis is known nowadays as *continued fraction expansion* of a positive real number. For much more in this direction, I recommend the excellent book [2]. I believe that Mathematics develop in part to compensate the failures of

our perceptual grasp of the world, and because of our overwhelmind desire to find causes, explanations. As I tried to explain above, it is much closer to our human nature than some philosophers would have us believe.

For example, we have a good perceptual, or intuitive feeling for distances, but our intuition of areas and volumes is poor. I believe that the invention of the concept of area was agreat mathematical momemt and a good part of Euclid's elements is devoted to considerations on area and volume, a hot mathematical topic at the time.
In Homer, the size of the city of Troy is 10200 steps. At the time, the size of a city or indeed any area, was its perimeter, because that is what one could measure.
Proclus $(411 - 485)$ reports court cases of members of Greek communities in which, in the first century A.D., it was decided to divide the land equitably, according to perimeter. There were surprises at the time of harvest. We note that the perimeter measures the length of the boundary of a plane domain...

24 centuries before the physicist Sokal derided (see [7]) the (mis)use of very elaborate modern mathematical or physical concepts by some philosophers, Plato was making fun of the followers of Pythagoras who put numbers everywhere. In *The Republic, 587d* he gives a farcical proof of the following statement:
The measure of the area of the image of the tyrant's pleasure is a perfect square.
Plato "proves" that it is 9. It is probable that in those days, the concept of area was not as common as today.

More seriously, the astonishment of Cantor discovering that there are as many points in a square or a cube, etc. was obviously a booster for the development of the axionatic method and set theory. One needed to be able to provide incontrovertible proofs based on undisputable axioms and well defined logical reasoning of theses counterintuitive results.

The price to pay is that the roots of Mathematics in the world around us disappeared under this solid slab on which Mathematics was supposed to be built by the axiomatic method. A vision of Mathematics as a purely logical construction whose main quality was to ensure the truth of its statements emerged.

Thanks to the recent progress in cognitive science, we can begin to study scientifically the foundations of meaning in Mathematics, which I think must be clearly distinguished from the foundations of truth provided by axioms and logic. One does not understand a proof by following the logical arguments but rather as a succssion of links between statements which have meaning.

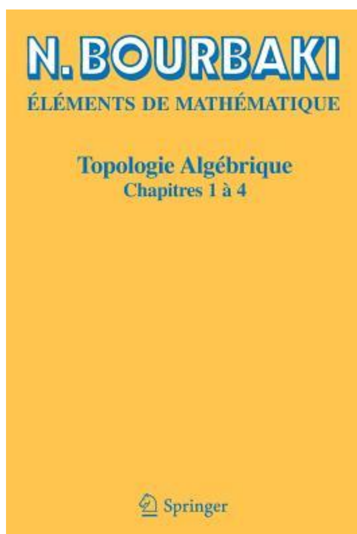
However, the part of our subconscious activities which is related to the pulsions mentioned above remains mysterious. It seems to me that a possible origin is a blending of the need to simplify and organize the enormous amount of information which the perceptual system provides (below our consciousness) to the brain and the idea proposed by cognition scientists that the brain has a Bayesian approach to this information, which means that it is constantly making conjectures about the next batch of information. One can imagine that this unconscious Bayesian interaction with the world emerges into our consciousness as curiosity and in particular curiosity about the causes.

Let us now turn to Bourbaki and Thom who, in my view, illustrate in strikingly different manners three of those pulsions.

Nicolas Bourbaki

Let me provide background with some pictures.

Bourbaki at work, ca. 1982



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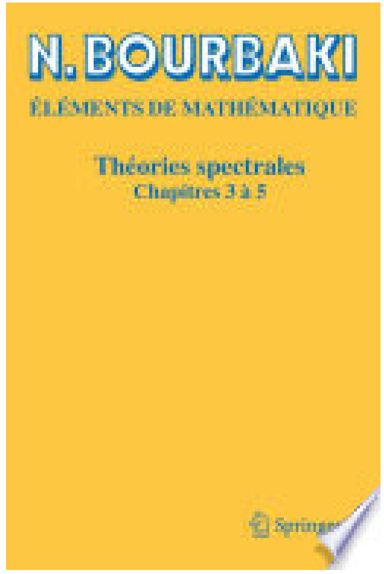

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Bourbaki started in the middle 1930's with a group of young mathematicians dissatisfied with the manuals from which they had to teach. They were especially dissatisfied with the proofs of Stokes' formula, which is an n-dimensional generalization of the formula

$$\int_a^b f'(t)dt = f(b) - f(a).$$

which computes the differences of the values of a function $f(t)$ on the boundary of an interval $[a, b]$ on the line from the values of its derivative on the interval (boundaries again!).

They were influenced by Hilbert's ideas on axiomatization. Youth and talent being what they are, and were, this dissatisfaction turned into an enterprise to rewrite most of Mathematics in a particular frame of mind, the exposition of the underlying *structures*.
This enterprise continues to this day, at its own pace (see the pictures above).

Bourbaki adopted the axiomatic method of constructing Mathematics but went much further in taking as basic objects not sets, to which earlier mathematicians had painfully tried to give a firm axiomatic status, meeting paradoxes all too often, but structures which, according to Bourbaki, exist independently of set theory.

The method of exposition we have chosen is axiomatic and abstract, and normally proceeds from the general to the particular. This choice has been dictated by the main purpose of the treatise, which is to provide a solid foundation for the whole body of modern Mathematics.
Nicolas Bourbaki, in the foreword to his books.

Bourbaki's goal is not to prove theorems but to provide mathematicians with a toolbox of clear and well founded definitions and results which help mathematicians to attack difficult problems by relating them to well studied *structures*.
The historic example of such structures is that of group, which appears in so many fields of Mathematics, Physics, Chemistry, etc. Then there is the structure of topological space, of topological vector space, Lie groups, integration, and so on. The structure of group, which is an operation with certain properties, can manifest itself in a set, for example the set of rotations around an origin in the plane, giving a group.

Bourbaki's idea is that this will produce a considerable *economy of thought* because instead of proving results in special cases adapted to special problems, it suffices to recognize that they are results in the theory of some structure.

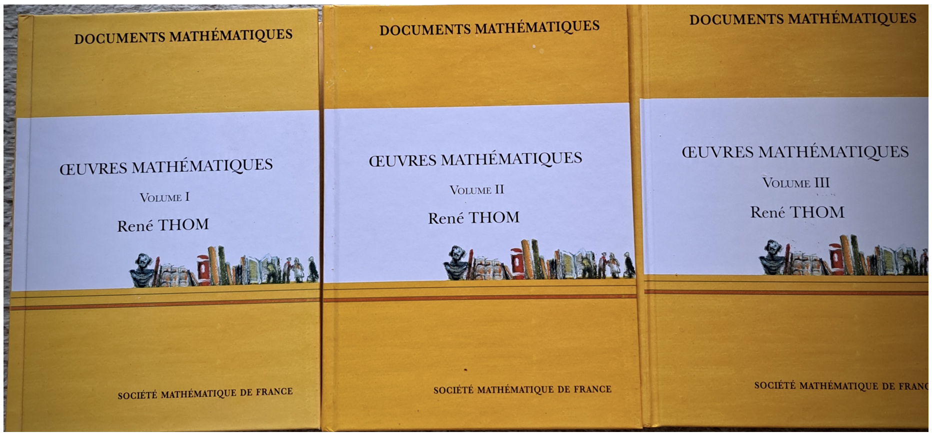
The common feature of these notions which we have designated under this name (of structures) is that they apply to sets of elements whose nature is not specified.
Nicolas Bourbaki in "L'architecture des Mathématiques", in [4].

But Bourbaki is conscious that the tools do not suffice to produce Mathematics:
One cannot insist too much on the role played in the mathematician's research by a particular intuition, which is not the vulgar intuition of the senses. It is rather a sort of direct divination (prior to any reasoning) of the normal behavior which he can expect from mathematical beings which a long interaction has made almost as familiar to him as the beings of the real world.
Nicolas Bourbaki in "L'architecture des Mathématiques", *loc. cit.*
As the reader will have guessed, I agree only partially with this formulation. But Bourbaki at that time could not guess the role of the "vulgar intuition of the senses". For a lot more on the role of structures and structuralism in Bourbaki, I recommend [3].

René Thom

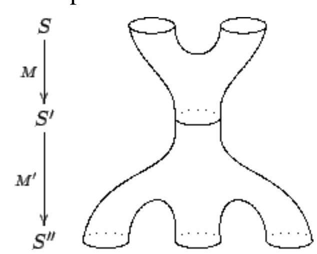
Let me provide background with a few pictures:

René Thom



Thom's mathematical works, published in 2022

If the founders of Bourbaki, who were at the beginning ostracized by some important mathematicians, were dissidents in their time, René Thom was also a dissident but in an entirely different manner and at a later age.
René Thom was born in 1923 and received the Fields medal in 1958 for his work on cobordism. This notion is based on the fact that for a manifold or a union of manifolds, being the boundary of some manifold of one more dimension imposes strong conditions. Two manifolds are cobordant if their union is the boundary a manifold. This led to a beautiful classification of manifolds up to cobordism.
The following is an example of composition of two cobordisms between sets of circles:



After his Fields medal, Thom embarked on an extremely ambitious project of providing qualitative models for the discontinuous behavior which dynamical systems depending on parameters can exhibit, This includes for example the (relatively) sudden change of shape of an embryo during its growth.

Thom's goal is very wide. He wants to provide geometric and *qualitative* models (for him, to understand is to visualize) which help to understand how discontinuous changes in shape, in behavior occur. The notion of a qualitative model was difficult to accept for scientists of the time. This qualitative aspect is also the reason why Thom's ideas apply to humanities as the work of Thom, Zeeman (see [9]), and Petitot illustrates. A basic intuition for him is that of a boundary but in a geometric, not quantitative sense.

Discontinuous changes (hence the name "Catastrophe theory") in the shape (hence the word "morphogenesis") or the behavior of a system depending on parameters occur when the parameters cross a certain boundary in the parameter space and he wants to provide "universal boundary shapes" which will appear in any parameter space depending of course on the nature of the system.

Another fundamental idea of Thom is that these discontinuous changes, or bifurcations, must appear in a *stable* manner in order to be observable, because of the inevitable ”noise” in nature. Here stable is a technical term for his mathematical models.

This allowed him to give a classification of the stable families of gradient dynamical systems depending on at most four parameters (because they are supposed to manifest themselves in space-time) and the corresponding bifurcation sets, or boundaries of the domains of parameters where no brutal change occurs.
One could say that for Thom what gives meaning to the behavior of a family of dynamical systems is the geometry of this boundary.

The essential idea of our theory that a certain understanding of the morphogenetic processes is possible without having recourse to special properties of the substrate of the shapes, or to the nature of the acting forces, may seem difficult to admit...
René Thom, in Chapter 1 of ”Mathematical models of morphogenesis”.

The following is a picture of the extrema of the potential $V = z^4 + bz^2 + az$ in one variable z depending on two parameters a, b. The gray zone corresponds to a maximum of the potential, which is an unstable extremum. It is one of Thom's 7 elementary catastrophies. The parameter space is the a, b plane below and Thom's boundary is the cusp drawn in black, although there are other conventions to define the boundary. The next picture describes how the extrema change with the parameters.

plus changes of scale in x, b, α allow us to rewrite this in the form

$$V = x^4 - bx^2 + \alpha x \quad (+\text{ const.})$$

(This is not exactly right : the new α, b are not quite the same as the old ones. But no harm will be done by abusing notation in this way.)

Equilibrium positions of the arch are given by

$$0 = \frac{\partial V}{\partial x} = 4x^3 - 2bx + \alpha \ .$$

Depending on (α, b) this cubic equation for x has 1 or 3 real roots. The graph M of x against (α, b) is a pleated surface, shown in figure 2, known as the *catastrophe manifold* .

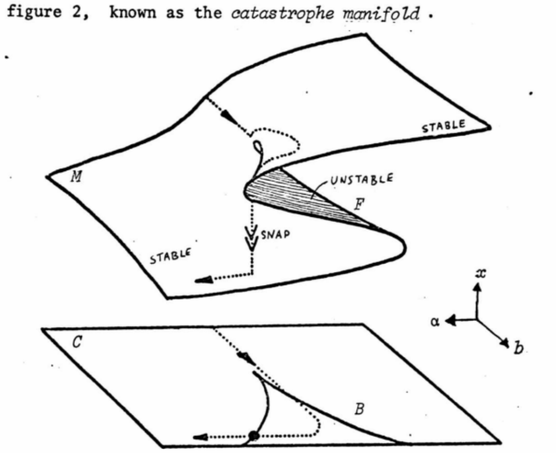


Figure 2.

The fold curve F occurs when

$$0 = \frac{\partial^2 V}{\partial x^2} = 12x^2 - 2b \ ,$$

Cusp Catastrophy

so is given parametrically by $(x,a,b) = (\xi, 8\xi^3, 6\xi^2)$. This is the geometer's beloved *twisted cubic*. Its projection B on to (a,b) - space C is found by eliminating ξ, which yields

$$27a^2 = 8b^3$$

a semicubical parabola with a cusp at the origin. The curve B is called the *bifurcation set*.

If (a,b) lies inside B then x can take three values of which the middle one represents *unstable* equilibrium; if outside, x can take only one. If we vary (a,b) along the path shown the value of x at first stays near 0, then increases sharply, decreases slightly, and falls in a dynamic snap to a negative value.

Considering this in terms of the potential V we see that the dynamic snap occurs when a minimum of V coalesces with a maximum and both disappear (figure 3).

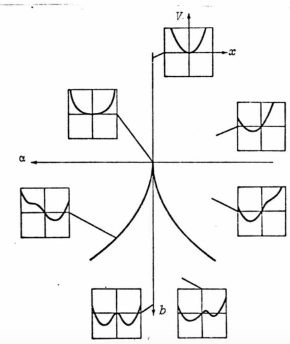


Figure 3.

competition of minima of potential

A word on the relations between Thom and Bourbaki:

The collaborators of Bourbaki retire at age 50. Bourbaki recruits new members by inviting some promising young mathematician to one of his meetings, where redactions of new chapters are read and criticized.

The invitee is known as ”guinea pig” because the redactions are tested on him or her. Depending on the level of interest exhibited, the guinea pig is invited to become a member, or not. Thom was invited to such a meeting and . . . fell asleep during the reading. Thom has sometimes criticized not the Bourbaki enterprise itself, but the abuses, for which Bourbaki bears little to no direct responsibility, which the ”structuralist” approach has generated in the teaching of Mathematics.

Jean-Pierre Serre, a Bourbaki member, helped Thom to put his thesis in shape. Thom lectured in the Bourbaki seminar. But the Bourbaki goal is definitely not Thom’s cup of tea. . .

The following quotation illustrates this.

If one must choose between rigour and meaning, I shall unhesitatingly choose the latter.

René Thom

Conclusion

Bourbaki wants to define and study the abstract ”structures” which exist independently of individual sets which they organize, while Thom wants to define and study abstract ”morphologies” and morphogenetic processes independent of the substrates and the dynamics which they structure.

Is there not a common set of pulsions at work, like pulsions 1, 2 and 4 of our list?

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