PSEUDO-RATIONAL LOCAL RINGS AND A THEOREM OF BRIANÇON-SKODA ABOUT INTEGRAL CLOSURES OF IDEALS

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INTRODUCTION

In the notes [23] of C. T. C. Wall, one finds that Mather raised the problem of computing for each n the smallest integer k such that, for any non-unit f in the ring $\mathbf{O}_n = \mathbf{C}\{z_1,\dots,z_n\}$ (convergent complex power series in n variables), one has $f^k \in j(f)$ where

$$j(f) = (\partial f/\partial z_1,\dots,\partial f/\partial z_n)\mathbf{O}_n$$

is the jacobian ideal of f. It was known then that f always belongs to the integral closure $\overline{j(f)}$ of $j(f)$ (see Section 1 below), a fact which implies the existence for a given f of an integer k such that $f^k \in j(f)$. Shortly afterwards, Saito proved (see [20]) that if one assumes that the origin is an isolated critical point of f, then the inclusion $f \in j(f)$ holds if and only if f is a quasi-homogeneous polynomial in some coordinate system, and therefore ([15, Section 9]) the monodromy of the fibration over $\mathbf{D} - \{0\}$ (where $\mathbf{D} = \{t \in \mathbf{C} \mid |t| < \eta\}$) defined by f in a neighborhood of $0 \in \mathbf{C}^n$ is finite. More generally, in [21] J. Scherk has shown that the smallest k such that $f^k \in j(f)$ is greater than or equal to the exponent of nilpotence of this monodromy. The first problem, however, is to find a bound on k valid for *any* non-unit f. This problem was solved by Briançon and Skoda, who proved:

THEOREM (see [2] for a statement which is slightly weaker, but whose proof can be modified to give:) If a non-zero ideal I in $\mathbf{O}_n$ can be generated by d elements, then for every integer $\lambda \geq 1$ we have

$$\overline{I^{\lambda+d-1}} \subseteq I^{\lambda}$$

where "$\overline{}$" denotes "integral closure" of an ideal, cf. Section 1.

In particular since $j(f)$ is generated by n elements this gives $\overline{j(f)^n} \subseteq j(f)$ and therefore $f^n \in j(f)$.

The proof given by Briançon and Skoda of this completely algebraic statement is based on a quite transcendental deep result of Skoda in [22]. The absence of an algebraic proof has been for algebraists something of a scandal—perhaps even an insult—and certainly a challenge.

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The first remark about the theorem is that if J is a *reduction of I* (that is, $J \subseteq I$ and $\bar{J} = \bar{I}$, see [16]) then $J^{\lambda} \subseteq I^{\lambda}$ and for any $\delta > 0$ we have $\overline{J^{\lambda+\delta-1}} = \overline{I^{\lambda+\delta-1}}$, and so if $\overline{J^{\lambda+\delta-1}} \subseteq J^{\lambda}$ then $\overline{I^{\lambda+\delta-1}} \subseteq I^{\lambda}$. Thus in the statement of the theorem we may replace d by the "analytic spread" $d(I)$, that is, the least number of elements which can generate a reduction of I. Clearly $d(I) \leq d$; and also $d(I) \leq n$ because $d(I) - 1$ is the dimension of the closed fibre of the birational morphism $X \to \operatorname{Spec}(\mathbf{O}_n)$ obtained by blowing up I [*ibid,* p. 149, Definition 3].

We present here an algebraic proof of the theorem in the case where the ideal I has a reduction J which is generated by a regular sequence (consisting necessarily of $d(I)$ elements). In fact the proof of this result (and of its generalization Corollary (2.2)) is valid in an arbitrary regular local ring, and even more generally in any "reasonable" *pseudo-rational* local ring R (cf. Section 2), where "reasonable" means that the localization R_p is also pseudo-rational for any prime ideal p in R. [If a pseudo-rational R has a residual complex (e.g. if R is essentially of finite type over a Gorenstein local ring) then R is reasonable (cf. Corollary of (iii) in Section 4).] As in Corollary (2.2), one reduces easily to the crucial case where R is pseudo-rational and $\sqrt{I} = M$, the maximal ideal of R; this case is treated in Section 2. (Note that when $\sqrt{I} = M$, any reduction of I is generated by a system of parameters (cf. [16, p. 154]) so we may assume that $d = n$, the dimension of R, and that $I = (f_1, \ldots, f_n)R$ where $(f_1, \ldots, f_n)$ is a regular sequence.) The algebraic proof in Section 2 was inspired by the transcendental argument given at the end of Section 1, which shows that when $R = \mathbf{O}_n$ and $\sqrt{I} = M$, the theorem is a truly simple consequence of the theory of residues and local duality as explained in [3].

In both Sections 1 and 2 we take $\lambda = 1$; but in Section 3 we reproduce an ingenious argument of Melvin Hochster which reduces the theorem for any λ to the case $\lambda = 1$ (always with $\sqrt{I} = M$).

In Section 4 we show that every regular local ring is pseudo-rational. Though the use of Grothendieck duality theory in this general context may make the proof appear to be rather elaborate, at least the underlying idea is quite simple, as can be seen by considering any geometric situation where dualizing sheaves can be described concretely by means of differential forms.

In Section 5 we show that in the special case $n = 2$ much more is true: among two-dimensional normal local rings R, the pseudo-rational ones are exactly those such that *for every ideal I primary for the maximal ideal in R, $I\bar{I}$ (the product of I and $\bar{I}$) is integrally closed.* Moreover in a two-dimensional pseudo-rational local ring, for *every* ideal I and every integer $\lambda \geq 1$, we have

$$\overline{I^{\lambda+1}} = I^{\lambda}\bar{I}.$$

(For non-principal ideals I, the theorem above gives only $\overline{I^{\lambda+1}} \subseteq I^{\lambda}$). It may be remarked here that in the two-dimensional case $\{R$ pseudo-rational and analytically normal$\} \Leftrightarrow \{R$ rational$\}$ (cf. Example (a) in Section 2, and [14, p. 157]).

The main remaining problems are to see to what extent these remarkable facts in the two-dimensional case can be extended to higher dimensions; and to determine whether the theorem holds for an arbitrary ideal in a reasonable pseudo-rational local ring (possibly by reduction to the case treated here). [In equicharacteristic "F-pure" local rings, using Corollary (5.3) (ii) in Section 5 and "reduction to characteristic p," Hochster has shown (unpublished) that for any d-generator ideal I, we have $\overline{I^{d+1}} \subseteq I$. *Added in proof:* The theorem has been proved for arbitrary ideals in regular local rings [25].]

Another problem of interest in this direction is, given $f \in \bar{I}$, to evaluate the minimum degree of an integral dependence equation for f over I.

We wish to express our gratitude to Melvin Hochster for a number of stimulating conversations.

1. INTEGRAL DEPENDENCE

In this section we recall some of the basic facts concerning integral dependence on ideals in commutative algebra and in analytic geometry; our sources are [24, Appendix 4] (algebraic), and [9], [11] (analytic). We then show how the transcendental interpretation of integral dependence ties in with local duality to give a proof of the theorem of Briançon-Skoda in the special case where $\sqrt{I} = M$, the maximal ideal of $\mathbf{O}_n$.

Definition 1.1. ([18]. Let R be a commutative ring and let I be an ideal of R. An element $h \in R$ is said to be integrally dependent on I if it satisfies a relation

$$h^k + a_1 h^{k-1} + \dots + a_k = 0 \qquad (a_i \in I^i,\ 1 \leq i \leq k).$$

If we consider the graded subring $P(I) = \sum_{n \geq 0} I^n T^n$ of the polynomial ring $R[T]$, we check that h is integrally dependent on I if and only if the element $hT \in R[T]$ is integrally dependent on the ring $P(I)$ in the classical sense (that is, $F(hT) = 0$, where F is a monic polynomial with coefficients in $P(I)$). From this it follows at once that the set $\bar{I}$ consisting of all elements in R which are integral over I is an *ideal* in R. $\bar{I}$ is called the *integral closure* of I in R. I is *integrally closed* in R if $I = \bar{I}$. For any ideal I, $\bar{I}$ is the smallest integrally closed ideal in R containing I.

(1.2) This notion of integral dependence can be globalized: if X is a topological space with a sheaf of rings $\mathcal{O}_X$, and $\mathcal{I}$ is an $\mathcal{O}_X$-ideal sheaf, then there is an $\mathcal{O}_X$-ideal sheaf $\bar{\mathcal{I}}$ such that for each $x \in X$, the stalk $\overline{\mathcal{I}}_x$ is the integral closure $\overline{\mathcal{I}_x}$ of $\mathcal{I}_x$ in $\mathcal{O}_{X,x}$. To see this, one notes that if U is an open neighborhood of x and $h \in \Gamma(U, \mathcal{O}_X)$ is such that the germ $h_x \in \overline{\mathcal{I}_x}$, then an equation of integral dependence of h_x on $\mathcal{I}_x$ "spreads out" to an open neighborhood $U' \subseteq U$, and hence $h_y \in \overline{\mathcal{I}_y}$ for each $y \in U'$.

If $(X, \mathcal{O}_X)$ is a locally noetherian scheme and $\mathcal{I}$ is coherent then $\bar{\mathcal{I}}$ is coherent (because "integral closure commutes with localization"; that is, if I is an ideal in a commutative ring R, and S is any multiplicatively closed subset of R, then, as is easily checked, the integral closure in $S^{-1}R$ of the ideal $S^{-1}I$ is $S^{-1}\bar{I}$). A similar statement holds in the complex-analytic framework, that is, if

$(X, \mathcal{O}_X)$ is a complex-analytic space. (The proof is of course less elementary.) In either case, given $h \in \Gamma(X, \mathcal{O}_X)$, the set $Y = \{y \in X \mid h_y \notin \overline{\mathscr{I}_y}\}$ is a closed subspace of X, namely the support of the coherent $\mathcal{O}_X$-module $(h\,\mathcal{O}_X + \bar{\mathscr{I}})/\bar{\mathscr{I}}$.

(1.3) **(Valuative criterion of integral dependence.)** With notation as in (1.1) assume that R is noetherian. Then an element $h \in R$ is integrally dependent on I if and only if, for every homomorphism $\varphi : R \to V$, where V is a discrete valuation ring, we have $\varphi(h) \in \varphi(I)V$, or equivalently: $v(h) \geq v(I)$ where v is the order function on R obtained from the valuation of V. (Cf. [24, p. 353, Thm. 3], which treats the case of domains, which is all we need and to which anyhow the general case is easily reduced.)

The complex-analytic avatar is this:

Let $\mathscr{I}$ be a coherent sheaf of ideals on a complex space X, and $h \in \Gamma(X, \mathscr{I})$. Then $h \in \Gamma(X, \bar{\mathscr{I}})$ if and only if for every morphism $\varphi : \mathbf{D} \to X$ ($\mathbf{D}$ is the unit disc in $\mathbf{C}^1$) we have $h \circ \varphi \in \Gamma(\mathbf{D}, \mathscr{I}\mathcal{O}_{\mathbf{D}})$. In fact as a consequence of the coherence of $\bar{\mathscr{I}}$, given $x \in X$, then $h_x \in \overline{\mathscr{I}_x}$ if and only if for every map-germ $\varphi : (\mathbf{D}, 0) \to (X, x)$ we have $h_x \circ \varphi \in \mathscr{I}_x\,\mathcal{O}_{\mathbf{D},0}$.

(1.4) Assume that R is noetherian and normal, and let $Z \to \operatorname{Spec}(R)$ be a proper birational map with Z normal and $I\mathcal{O}_Z$ invertible. Then we have

$$\bar{I} = H^0(Z, I\mathcal{O}_Z) \subseteq H^0(Z, \mathcal{O}_Z) = R.$$

With the valuative criterion for properness, this can be proved similarly to the Lemma on p. 354 of [24]. For a proof without valuations, note that $I\mathcal{O}_Z = \bar{I}\mathcal{O}_Z$ (because Z is normal and $I\mathcal{O}_Z$ is invertible), and that $\bar{I} = H^0(Z, \bar{I}\mathcal{O}_Z)$ [12, Prop. 6.2].

The avatar of this in complex-analytic geometry is as follows:

Let $\mathscr{I}$ be a coherent sheaf of ideals on a normal complex space X and let $Z \to X$ be a proper bimeromorphic map such that Z is normal and $\mathscr{I}\mathcal{O}_Z$ is invertible. Then

$$\Gamma(X, \bar{\mathscr{I}}) = \Gamma(Z, \mathscr{I}\mathcal{O}_Z).$$

A consequence of this fact is the following *transcendental criterion for integral dependence.*

Assume that the subspace of X defined by $\mathscr{I}$ is nowhere dense in a neighborhood of some point $x \in X$. Set $R = \mathcal{O}_{X,x}$ and $I = \mathscr{I}_x \subset R$. Let $(f_{1,x}, \ldots, f_{m,x})$ be generators for I, where the f_i generate $\Gamma(U, \mathscr{I})$ for some open neighborhood U of x. Then given $h \in \Gamma(U, \mathcal{O}_X)$, $h_x \in \bar{I}$ *if and only if there exists a neighborhood* $U' \subseteq U$ of x and a real constant $C > 0$ such that for every $y \in U'$ we have

$$(*) \qquad |h(y)| \leq C \cdot \sup_i |f_i(y)|.$$

Indeed, denoting now by X a suitable neighborhood of x, consider the normalized blow up $Z \to X$ of $\mathscr{I}$ (blow up $\mathscr{I}$, then normalize), or more generally any proper bimeromorphic map $\pi : Z \to X$ with Z normal and $\mathscr{I}\mathcal{O}_Z$ invertible. Then the inequalities (*) can be lifted to Z. Since Z is normal and $\mathscr{I}\mathcal{O}_Z$ is locally principal, it follows from the Riemann extension theorem applied to $(\mathscr{I}\mathcal{O}_Z)^{-1} \cdot (h \circ \pi)$ on Z that inequalities such as (*) imply that $h \circ \pi \in \mathscr{I}\mathcal{O}_Z$, hence $h \in \bar{\mathscr{I}}$. Conversely, if $h \in \bar{\mathscr{I}}$, $h \circ \pi \in \mathscr{I}\mathcal{O}_Z$ and we therefore have locally on Z inequalities similar to (*), which imply the inequalities (*) on X since π is proper and surjective.

(1.5) **Local duality** ([3, p. 659]). Take $R = \mathbf{O}_n = \mathbf{C}\{z_1, \ldots, z_n\}$ and let $\mathbf{f} = (f_1, \ldots, f_n)$ be a regular sequence of elements of the maximal ideal M of R. Denoting by I the ideal of R generated by the f_i, we have $\sqrt{I} = M$ and there exists a *residue bilinear pairing* associated with $\mathbf{f}$:

(1.5.1) $$\mathrm{Res}_{\mathbf{f}} : R/I \otimes_{\mathbf{C}} R/I \to \mathbf{C}$$

induced by the bilinear pairing $R \otimes_{\mathbf{C}} R \to \mathbf{C}$ given by

$$\mathrm{Res}_{\mathbf{f}}(g,h) = \int_{|f_i(z)| = \varepsilon} gh \frac{dz_1 \wedge \ldots \wedge dz_n}{f_1 \cdots f_n}$$

the orientation on the real n-cycle $|f_i(z)| = \varepsilon$ $(1 \le i \le n)$ being that given by the condition $d(\arg f_1) \wedge \ldots \wedge d(\arg f_n) \ge 0$. Note that the integral is independent of ε for all sufficiently small ε because of Stokes' theorem.

(1.5.2) The *local duality theorem* is the assertion that the pairing (1.5.1) is non-degenerate; that is if $\mathrm{Res}_{\mathbf{f}}(g,h) = 0$ for all $g \in \mathbf{O}_n$, then $h \in I$.

(1.6) **A new transcendental proof of the theorem of Briançon-Skoda in the special case where $\sqrt{I} = M$ and $\lambda = 1$.**

As already observed in the introduction, we may replace I by one of its reductions, and therefore assume that I is generated by a regular sequence $(f_1, \ldots, f_n)$. We are going to use local duality to prove that $\overline{I^n} \subseteq I$; it is enough, then, to show for any $h \in \overline{I^n}$ and $g \in \mathbf{O}_n$ that $\mathrm{Res}_{\mathbf{f}}(g,h) = 0$.

Since $gh \in \overline{I^n}$, the transcendental criterion for integral dependence gives an $\varepsilon_0 > 0$ and a $C > 0$ such that for any ε with $0 < \varepsilon < \varepsilon_0$ and $z \in \mathbf{C}^n$ with $|f_i(z)| = \varepsilon$ $(1 \le i \le n)$, we have

$$|(gh)(z)| \le C \cdot \sup_{\alpha} |f_\alpha(z)| = C\varepsilon^n$$

where f_α runs through all the monomials $f_1^{\alpha_1} \ldots f_n^{\alpha_n}$ with $\alpha_i \ge 0$ and $\sum \alpha_i = n$. (Note that the map $\mathbf{C}^n \to \mathbf{C}^n$ with coordinate functions f_i is finite near 0, so that the cycle $|f_i(z)| = \varepsilon$ $(1 \le i \le n)$ shrinks to the origin as $\varepsilon \to 0$.) In particular

$$|(gh)(z)/(f_1 \cdots f_n)(z)| \le C$$

and hence

$$|\mathrm{Res}_{\mathbf{f}}(g,h)| \le C \int_{|f_i(z)| = \varepsilon} |dz_1 \wedge \ldots \wedge dz_n|.$$

The integral on the right tends to 0 with ε, so indeed $\mathrm{Res}_f(g,h) = 0$.

(1.7) *Remark.* In the two-dimensional local ring

$$R = \mathbf{C}\{z_1, z_2, z_3\}/z_1^m + z_2^m + z_3^m = \mathbf{C}\{\bar{z}_1, \bar{z}_2, \bar{z}_3\} \qquad (m \geq 1)$$

the integral closure of the ideal $I = (\bar{z}_1, \bar{z}_2)R$ is the maximal ideal $(\bar{z}_1, \bar{z}_2, \bar{z}_3)R$; but $\bar{z}_3^{m-1} \notin I$. So the Briançon-Skoda result cannot hold with an *arbitrary* analytic local ring in place of $\mathbf{O}_n$.

2. PSEUDO-RATIONAL LOCAL RINGS

In this section we define pseudo-rational local rings, give some examples, and prove in such local rings the Briançon-Skoda result for ideals which are primary for the maximal ideal.

Let R be an n-dimensional local ring (noetherian and commutative), with maximal ideal M. We say that R is **pseudo-rational** if it satisfies all of the following four conditions:

(i) R is normal.

(ii) R is Cohen-Macaulay.

(iii) The completion $\hat{R}$ is reduced (i.e. has no non-zero nilpotents).

(iv) For any proper birational map $f: W \to X = \mathrm{Spec}(R)$ with W normal, if $E = f^{-1}\{M\}$ is the closed fibre, then the canonical map (an edge-homorphism in the Leray spectral sequence for cohomology with supports [7, p. 73, Prop. 5.5])

$$\delta_f : H^n_M(R) = H^n_{\{M\}}(\mathscr{O}_X) = H^n_{\{M\}}(f_* \mathscr{O}_W) \to H^n_E(\mathscr{O}_W)$$

is *injective.*

Remarks. (a). Let $f: W \to \mathrm{Spec}(R)$ be as in (iv), let $g: W' \to W$ be a proper birational map, and let $E' = g^{-1}(E) = (fg)^{-1}\{M\}$. Then the spectral sequence for g gives a map

$$\delta_g : H^n_E(\mathscr{O}_W) \to H^n_{E'}(\mathscr{O}_{W'})$$

and one checks that

$$\delta_{fg} = \delta_g \circ \delta_f.$$

So if δ_{fg} is injective, then δ_f is too. As in [14, p. 157], using Chow's lemma and Rees' characterization of analytically unramified local rings [19], we see then that *an n-dimensional normal Cohen-Macaulay local ring R is pseudo-rational if and only if it satisfies one of the following equivalent conditions:*

(iv)′: For any projective birational map $f: W \to \mathrm{Spec}(R)$ there exists a proper birational map $g: W' \to W$ such that W' is normal and δ_{fg} is injective. [Whenever we speak of birational maps, it is to be understood that the schemes involved are reduced and irreducible.]

(iv)″: For any proper birational map $f: W \to \mathrm{Spec}\,(R)$, the *normalization* $g: W' \to W$ is finite, and δ_{fg} is injective.

(b) (Not used elsewhere.) If R is *any* n-dimensional normal local ring and $f: W \to \mathrm{Spec}\,(R)$ is proper and birational, then for $q > 0$ the support of $R^q f_* \mathcal{O}_W$ has dimension less than or equal to $n-1-q$ (use [5, (4.2.2)]). So $H^p_{\{M\}}(R^q f_* \mathcal{O}_W) = 0$ for $p+q=n$, $q>0$; and hence δ_f is always *surjective.*

Examples. **(a)**. Let R be a two-dimensional normal local ring, and let $f: W \to \mathrm{Spec}\,(R)$ be proper and birational, with W normal. Then $R^1 f_* \mathcal{O}_W$ is supported in the closed point, and the Leray spectral sequence gives an exact sequence

$$H^1_E(\mathcal{O}_W) \to H^0_{\{M\}}(R^1 f_* \mathcal{O}_W) \to H^2_M(R) \xrightarrow{\delta_f} H^2_E(\mathcal{O}_W).$$
$$\Vert$$
$$H^1(W, \mathcal{O}_W)$$

But $H^1_E(\mathcal{O}_W) = 0$ [14, p. 177, Thm. 2.4]. Hence δ_f *is injective if and only if* $H^1(W, \mathcal{O}_W) = 0$. Thus for $n=2$, our definition agrees with the one on p. 156 of [14]. In particular any two-dimensional rational singularity is pseudo-rational.

More information about the case $n=2$ is given in Section 5 below.

(b). Let R be an excellent regular local ring containing a field of characteristic zero. Then by Hironaka's resolution of singularities, R satisfies (iv)′ of Remark (a), and so R is pseudo-rational. (In fact W' can be obtained from $\mathrm{Spec}(R)$ by a succession of blow-ups with non-singular centers; then $R^q(fg)_* \mathcal{O}_{W'} = 0$ for all $q>0$, and the Leray spectral sequence for fg degenerates, so that δ_{fg} is bijective.)

More generally, in Section 4 we will show that *any regular local ring is pseudo-rational.* For this result, resolution of singularities is not available in the required generality; instead we mount a massive attack with Grothendieck's duality theory. A simpler proof would of course be desirable.

(c). (For getting rid of finite residue fields). Let R be a pseudo-rational local ring, with maximal ideal M. Let T be an indeterminate, and let R_1 be the localization of the polynomial ring $R[T]$ at the prime ideal $MR[T]$. *Then R_1 is pseudo-rational.*

Sketch of proof. Certainly R_1 is normal and Cohen-Macaulay. [6, (6.3.6).] Let us check condition (iv)′ of Remark (a). Any projective birational map $f_1: W_1 \to \mathrm{Spec}\,(R_1)$ is obtained by blowing up an ideal I_1 in R_1, and we may assume that I_1 is generated by finitely many polynomials in $R[T]$. Let I be the ideal in R generated by all the coefficients of these polynomials, and let $h: W' \to \mathrm{Spec}\,(R)$ be obtained by blowing up I and normalizing. Setting $W'_1 = W' \otimes_R R_1$, we see that $I_1 \mathcal{O}_{W'_1}$ is invertible, so there is a proper birational map $g_1: W'_1 \to W_1$. Now $f_1 g_1 = h \otimes_R R_1$; and one checks that δ commutes with $\otimes_R R_1$. Thus $\delta_{f_1 g_1} = \delta_h \otimes_R R_1$ is injective.

THEOREM 2.1. *Let R be a pseudo-rational n-dimensional local ring, and let I be an ideal in R. Then for any integer $\lambda \geq 1$, we have*

$$\overline{I^{\lambda+n-1}} \subseteq I^{\lambda}$$

where "$\overline{\quad}$" denotes "integral closure."

COROLLARY 2.2. *Assume that the localization R_p is pseudo-rational for every prime ideal p in R. (This is automatically true if R has a residual complex, cf. Corollary of (iii) in Section 4.) Suppose that I has a reduction J such that* $\dim R_p \leq \delta$ *for every associated prime ideal p of J^{λ} (δ-some integer, which may be assumed less than or equal to n). Then*

$$\overline{I^{\lambda+\delta-1}} \subseteq I^{\lambda}. \tag{2.3}$$

In particular, if J can be generated by a regular sequence $(f_1,\ldots,f_\delta)$, then (2.3) *holds for any integer* $\lambda \geq 1$.

Proof of 2.2. The first assertion follows easily from (2.1) (replace I by J and localize at the associated primes of J^{λ}). The second assertion holds then because in a Cohen-Macaulay ring every power of an ideal generated by a regular sequence is unmixed [24, p. 401., Lemma 5.1].

Proof of 2.1. To begin with we argue as in [2]: suppose (2.1) holds whenever $\sqrt{I} = M$, the maximal ideal of R; then for *any* ideal I, and any integer $s > 0$, we have

$$\overline{I^{\lambda+n-1}} \subseteq \overline{(I+M^s)^{\lambda+n-1}} \subseteq (I+M^s)^{\lambda} \subseteq I^{\lambda} + M^s$$

and therefore

$$\overline{I^{\lambda+n-1}} \subseteq \bigcap_{s>0} (I^{\lambda} + M^s) = I^{\lambda}.$$

So we assume henceforth that $\sqrt{I} = M$.

We first treat the case $\lambda = 1$; the general case is reduced to this one in Section 3 below. By Example (c) above, we may assume that R/M is infinite; and then (as in the Introduction) replacing I by a reduction allows us to assume that $I = (f_1,\ldots,f_n)R$, where $(f_1,\ldots,f_n)$ is a regular sequence in R.

Let $h \in \overline{I^n}$. We want to show that $h \in I$. There is a natural *injective* map

$$\phi : R/I \to H^n_M(R)$$

(cf. [10, p. 49, Lemma 5.10]). With $U = \operatorname{Spec}(R) - \{M\}$, we also have a natural surjective (bijective, if $n > 1$) map $H^{n-1}(U, \mathcal{O}_U) \to H^n_M(R)$. Since U is covered by the affine open sets $U_i = \operatorname{Spec}(R_{f_i})$, we see, using Čech cohomology, that $H^{n-1}(U, \mathcal{O}_U)$—and hence $H^n_M(R)$—is a homomorphic image of

$$H^0(U_1 \cap \ldots \cap U_n, \mathcal{O}_U) = R_{f_1 f_2 \cdots f_n}$$

that is, we have a surjective map

$$\psi : R_{f_1 f_2 \cdots f_n} \to H^n_M(R).$$

Following through definitions, we find that

$$\phi(h \bmod I) = \psi(h/f_1 f_2 \cdots f_n).$$

Since ϕ is injective, it will suffice to show that $\psi(h/f_1 f_2 \ldots f_n) = 0$.

Now let $f\colon W \to \operatorname{Spec}(R)$ be any proper birational map with W normal and $I\mathcal{O}_W$ invertible (for instance the map obtained by blowing up I and normalizing). We will show that $\psi(h/f_1 f_2 \ldots f_n)$ is in the kernel of $\delta_f\colon H^n_m(R) \to H^n_E(\mathcal{O}_W)$, which, be definition, is zero when R is pseudo-rational.

With $U' = f^{-1}(U) = W - E$, we have a canonical commutative diagram

$$\begin{array}{ccccc} H^{n-1}(W, \mathcal{O}_W) & \to & H^{n-1}(U', \mathcal{O}_{U'}) & \to & H^n_E(\mathcal{O}_W) \\ & & \uparrow & & \uparrow \delta_f \\ & & H^{n-1}(U, \mathcal{O}_U) & \to & H^n_M(R) \end{array}$$

whose top row is exact. Let $V_i \subset W$ be the open set defined by

$$V_i = \{w \in W \mid I\mathcal{O}_{W,w} = f_i\,\mathcal{O}_{W,w}\}.$$

Then $W = V_1 \cup V_2 \cup \ldots \cup V_n$, and $V_i \cap U' = f^{-1}(U_i)$. To say that $\delta_f \psi(h/f_1 f_2 \ldots f_n) = 0$ is to say that the Čech n-cocycle

$$h/f_1 f_2 \ldots f_n \in H^0(f^{-1}(U_1) \cap \ldots \cap f^{-1}(U_n), \mathcal{O}_{U'})$$

determines an element h^* in $H^{n-1}(U', \mathcal{O}_{U'})$ whose image in $H^n_E(\mathcal{O}_W)$ is zero; and for this to be so, it suffices that the cocycle lift to an n-cocycle of the covering $\{V_i\}$ of W (which implies that h^* is the image of an element in $H^{n-1}(W, \mathcal{O}_W)$). But such a lifting certainly exists, because on $V_1 \cap \ldots \cap V_n$, $I^n\mathcal{O}_W$ is generated by $f_1 f_2 \ldots f_n$, and

$$h \in \overline{I^n} = H^0(W, I^n\,\mathcal{O}_W)$$

(cf. 1.4).

3. REDUCTION TO THE CASE $\lambda = 1$

Having proved Theorem (2.1) for $\lambda = 1$, we shall now deduce the result for any $\lambda \geq 1$. The following argument is due essentially to Melvin Hochster, and is given here with his permission.

As before, we may assume that $I = (f_1, \ldots, f_n)R$ with a regular sequence $(f_1, \ldots, f_n)$. The basic observation is that

$$I^\lambda = \bigcap_{\lambda_1, \ldots, \lambda_n} I_{\lambda_1 \lambda_2 \ldots \lambda_n}$$

with

$$I_{\lambda_1\lambda_2\ldots\lambda_n} = (f_1^{\lambda_1},\ldots,f_n^{\lambda_n})R$$

where $(\lambda_1,\lambda_2,\ldots,\lambda_n)$ runs through all n-tuples of strictly positive integers with $\lambda_1 + \lambda_2 + \ldots + \lambda_n = \lambda + n - 1$. Here the non-obvious inclusion, namely $I^\lambda \supseteq \cap I_{\lambda_1\ldots\lambda_n}$ can be proved by induction on λ; to get from λ to $\lambda + 1$ one needs the following fact: *if* $\alpha \in R$ *is such that*

$$\alpha f_1^{\nu_1} f_2^{\nu_2} \ldots f_n^{\nu_n} \in (f_1^{\nu_1+1}, f_2^{\nu_2+1}, \ldots, f_n^{\nu_n+1})R \qquad (\nu_i \geq 0) \tag{3.1}$$

then $\alpha \in I$. But (3.1) can be rewritten as

$$(\alpha f_1^{\nu_1} \ldots f_{n-1}^{\nu_{n-1}} - \beta f_n) f_n^{\nu_n} \in (f_1^{\nu_1+1}, \ldots, f_{n-1}^{\nu_{n-1}+1})R \qquad (\beta \in R)$$

and $(f_1^{\nu_1+1}, f_2^{\nu_2+1}, \ldots, f_n^{\nu_n})$ is a system of parameters, hence a regular sequence, so that

$$\alpha f_1^{\nu_1} \ldots f_{n-1}^{\nu_{n-1}} \in (f_1^{\nu_1+1}, \ldots, f_{n-1}^{\nu_{n-1}+1})R \qquad (\operatorname{mod} f_n).$$

Since $(f_1,\ldots,f_{n-1})$ is a regular sequence mod f_n, we can proceed by induction on n to conclude that indeed $\alpha \in I$.

It will therefore suffice to show, for $\lambda \geq 1$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_n = \lambda + n - 1$, that $\overline{I^{\lambda+n-1}} \subseteq I_{\lambda_1\ldots\lambda_n}$.

Let $\mu = \max(\lambda_1,\ldots,\lambda_n)$. If $h \in \overline{I^{\lambda+n-1}}$, then

$$f_1^{\mu-\lambda_1} f_2^{\mu-\lambda_2} \ldots f_n^{\mu-\lambda_n} h \in \overline{I^{n\mu}}.$$

By the valuative criterion (1.3), one sees that, with $I_\mu = (f_1^\mu,\ldots,f_n^\mu)R$ we have $\overline{(I_\mu)^n} = \overline{I^{n\mu}}$; so by the case $\lambda = 1$ of Theorem (2.1) we get

$$\overline{I^{n\mu}} \subseteq I_\mu = (f_1^\mu,\ldots,f_n^\mu)R.$$

Thus

$$h \in (f_1^\mu,\ldots,f_n^\mu)R : f_1^{\mu-\lambda_1} \ldots f_n^{\mu-\lambda_n} = I_{\lambda_1\ldots\lambda_n}$$

(the last equality is proved by reasoning as above).

This completes the proof of Theorem (2.1).

4. REGULAR LOCAL RINGS ARE PSEUDO-RATIONAL

We begin with an outline of the proof. Let R be an n-dimensional normal local ring with maximal ideal M and fraction field K. We assume that R has a residual complex. (This assumption holds, for example, if R is regular; or more generally if R is essentially of finite type over a Gorenstein local ring [8, bottom of p. 306].) We consider proper birational maps $f: Y \to X = \operatorname{Spec}(R)$ (with Y a reduced scheme). Without harm we may assume that f induces the *identity map* from the local ring of the generic point of X to the local ring of the generic point of Y, both local rings being equal to K; then f is uniquely determined by Y. For the most part what we will be doing is to explicate a portion of Grothendieck duality theory for such f by defining, for each Y, a "dualizing sheaf" ω_Y, which is a coherent $\mathscr{O}_Y$-submodule of the constant sheaf K on Y, with the following properties (i), (ii), (iii), (iv):

(i) *For any point $y \in Y$, the stalk $\omega_{Y,y}$ depends only on the local ring $\mathscr{O}_{Y,y}$ (and not on Y).*

(Note that $\mathscr{O}_{Y,y}$ is an R-subalgebra of K.)

(ii) *For any $y \in Y$ as in* (i), *we have*

$$\omega_{Y,y} = \bigcap_{p \in P_y} (\omega_{Y,y})_p$$

where P_y is the set of all height one prime ideals in $\mathscr{O}_{Y,y}$.

Given $f: Y \to X$ as above, since R is normal there is a non-empty open set $U \subset X$, with $X - U$ of codimension greater than or equal to 2, such that f induces an isomorphism $f^{-1}(U) \xrightarrow{\sim} U$. For any $y \in f^{-1}(U)$ the corresponding local homomorphism $\mathscr{O}_{X,f(y)} \hookrightarrow \mathscr{O}_{Y,y}$ is the identity map. So, in view of (i), the sheaves $f_*(\omega_Y)$ and ω_X have the same stalk at every point of U, that is, their restrictions to U coincide (as subsheaves of K). Thus there is an *inclusion* map

$$\tau_Y : H^0(Y,\omega_Y) \hookrightarrow H^0(f^{-1}(U),\omega_Y) = H^0(U, f_*\omega_Y) = H^0(U,\omega_X) = H^0(X,\omega_X)$$

(the last equality because of (ii), since $X - U$ has codimension greater than or equal to 2).

We set $\omega_R = H^0(X, \omega_X)$.

(iii). (Duality). *Let I be an injective hull of the R-module R/M, and for any R-module H set $H' = \operatorname{Hom}_R(H,I)$. There exist R-isomorphisms*

$$H^n_M(R) \xrightarrow{\sim} (\omega_R)'$$
$$H^n_E(\mathscr{O}_Y) \xrightarrow{\sim} H^0(Y,\omega_Y)'$$

via which the map δ_f (Section 2) is dual to τ_Y.

COROLLARY (of (iii)). *If the normal local ring R is Cohen-Macaulay and $\hat{R}$ is reduced, then R is pseudo-rational if and only if for every proper birational $f: Y \to X = \operatorname{Spec}(R)$ with Y normal, τ_Y is surjective; that is,*

$$H^0(Y,\omega_Y) = \omega_R$$

(equivalently: $f_*\omega_Y = \omega_X$).

Finally, when R is regular, we will have $\omega_R = R$, and furthermore:

(iv) *If R is regular and $y \in Y$ is such that $\mathcal{O}_{Y,y}$ is a discrete valuation ring, then* $\omega_{Y,y} \supseteq \mathcal{O}_{Y,y}$.

From (ii) and (iv) we deduce that if R is regular and Y is normal, then $\omega_Y \supseteq \mathcal{O}_Y$, whence

$$H^0(Y, \omega_Y) \supseteq H^0(Y, \mathcal{O}_Y) = R = \omega_R \supseteq H^0(Y, \omega_Y)$$

that is, $H^0(Y, \omega_Y) = \omega_R$; and so by the preceding corollary, any regular R is indeed pseudo-rational.

Let us then define ω_Y and prove (i), (ii), (iii) and (iv).

Let $\mathscr{R}^{\cdot}_X$ be a normalized residual complex on $X = \operatorname{Spec}(R)$ [8, p. 276]. Let $\mathscr{R}^{\cdot}_Y = f^{\Delta}(\mathscr{R}^{\cdot}_X)$ [8, p. 318], and set

$$\omega_Y^* = H^{-n}(\mathscr{R}^{\cdot}_Y).$$

This ω_Y^* is defined only up to isomorphism. Since residual complexes have (by definition) coherent cohomology, and since $\mathscr{R}^j_Y = 0$ for $j < -n$ and $\mathscr{R}^{-n}_Y$ is a sheafified injective hull of the residue field at the generic point of Y, that is, $\mathscr{R}^{-n}_Y$ is isomorphic to the constant sheaf K, therefore ω_Y^* is isomorphic to a coherent $\mathcal{O}_Y$-submodule of K. ω_Y, defined below, will be a particular $\mathcal{O}_Y$-submodule of K isomorphic to ω_Y^*.

Furthermore, localized at a point $y \in Y$, the differential $d^{-n}: \mathscr{R}^{-n}_Y \to \mathscr{R}^{1-n}_Y$ looks—up to isomorphism—like

$$d_y^{-n}: K \to \bigoplus_{p \in P_y} J(p)$$

where P_y is as in (ii) above, and $J(p)$ is the injective hull of the fraction field of $\mathcal{O}_{Y,y}/p$ considered as an $\mathcal{O}_{Y,y}$-module. (Here we need know no more about d_y^{-n} than its source and target.) So, first of all, the generic stalk of ω_Y^* is isomorphic to K (i.e. $\omega_Y^* \neq 0$). Secondly

$$\omega^*_{Y,y} \cong \text{kernel of } d_y^{-n} = \bigcap_{p \in P_y} (\text{kernel of } \pi_p \circ d_y^{-n})$$

where $\pi_p : (\bigoplus_{q \in P_y} J(q)) \to J(p)$ is the projection. But $\pi_p \circ d_y^{-n}$ is just the localization of d_y^{-n} at p; so after identifying ω_Y^* with the submodule ω_Y of K, we will have

$$\omega_{Y,y} = \bigcap_{p \in P_y} (\omega_{Y,y})_p$$

which is (ii).

Now we specify ω_Y. First fix an $\mathcal{O}_X$-submodule ω_X of K such that ω_X is isomorphic to ω_X^*. Any choice of ω_X will do; however, when R is *regular*—or, more generally, Gorenstein—we make the natural choice $\omega_X = \mathcal{O}_X$ [8 p. 229], so that in this case, (cf. sentence preceding (iv))

$$\omega_R = H^0(X, \omega_X) = H^0(X, \mathcal{O}_X) = R.$$

Next, choose a non-empty open set $U \subseteq X$ with $X - U$ of codimension greater than or equal to 2 such that f maps $f^{-1}(U)$ isomorphically to U. Then ω_Y^* and $f^*\omega_X$ are isomorphic over $f^{-1}(U)$; and any isomorphism extends to an embedding $i_Y : \omega_Y^* \to K$ over all of Y. We set $\omega_Y = i_Y(\omega_Y^*)$.

Note that i_Y is determined only up to automorphisms of $\omega_X|U$, that is, up to multiplication by units in $H^0(U, \mathcal{O}_U) = R$; but ω_Y *is* uniquely determined. (Actually there is a natural choice for i_Y, cf. proof of (iii) below).

Having thus fixed ω_Y, we can prove (i) as follows.

The problem comes down to this: given two proper birational maps $f_1 : Y_1 \to X$, $f : Y \to X$, open sets $V_1 \subseteq Y_1$, $V \subseteq Y$, and an X-isomorphism $g : V_1 \to V$, show that

$$g_*(\omega_{Y_1}|V_1) = \omega_Y|V \tag{4.1}$$

Enlarging (V_1, V, g) as much as possible, *we may assume that there is a non-empty open set U in X such that $X - U$ has codimension greater than or equal to 2, $f^{-1}(U) \subseteq V$, and f maps $f^{-1}(U)$ isomorphically to U. Now* $g_*(\omega_{Y_1}|V_1)$ is isomorphic to $\omega_Y|V$; and moreover, by the construction of ω_Y, these two sheaves coincide on $f^{-1}(U)$ with $f^*(\omega_X)$; since R is normal and $X - U$ has codimension greater than or equal to 2, we see then that any isomorphism from $g_*(\omega_{Y_1}|V_1)$ onto $\omega_Y|V$ is given by multiplication by an element r of K such that r is in fact a unit in R; this gives (4.1), and so (i) is proved.

Next we prove (iii).

There is a *trace map* $f_*(\mathcal{R}^\cdot_Y) \to \mathcal{R}^\cdot_X$ which is a homomorphism of complexes [8, p. 369]. Taking kernels in degree $-n$, we get a homomorphism of sheaves

$$tr^{-n} : f_*\omega_Y \to \omega_X.$$

Letting $U \subseteq X$ be as usual, and taking global sections over X and U, we get from tr^{-n} a commutative diagram

$$\begin{array}{ccccc}
H^0(Y, \omega_Y) & = H^0(X, f_*\omega_Y) & \xrightarrow{\tau'_Y} & H^0(X, \omega_X) & = \omega_R \\
\tau_Y \downarrow & & & \| & \\
H^0(f^{-1}(U), \omega_Y) & = H^0(U, f_*\omega_Y) & \xrightarrow[\nu_Y]{} & H^0(U, \omega_X) & = \omega_R
\end{array}$$

ν_Y is an *isomorphism,* because the trace map is compatible with the base change $U \hookrightarrow X$, and $f^{-1}(U) \hookrightarrow U$ is an isomorphism. [There is a unique choice of the above embedding $i_Y : \omega_Y^* \to K$ making ν_Y = identity.] So in proving (iii), we may replace τ_Y by τ'_Y.

The *duality theorem,* in the form given in [14, p. 188], yields an isomorphism:

$$\mathbf{R}\Gamma_E\,\mathcal{O}_Y = \mathbf{R}\Gamma_M(\mathbf{R}f_*\,\mathcal{O}_Y) \xrightarrow{\text{local duality}} \operatorname{Hom}(\mathbf{R}f_*\,\mathcal{O}_Y, \mathcal{R}^{\cdot}_X)'$$
$$\xrightarrow{\text{trace}} \mathbf{R}\operatorname{Hom}(\mathbf{R}f_*\,\mathcal{O}_Y, f_*\mathcal{R}^{\cdot}_Y)' \xrightarrow{\text{natural}} \mathbf{R}\operatorname{Hom}(\mathcal{O}_Y, \mathcal{R}^{\cdot}_Y)' = (\Gamma_Y\mathcal{R}^{\cdot}_Y)'$$

Taking cohomology in degree n, we get one isomorphism $H^n_E(\mathcal{O}_Y) \xrightarrow{\sim} H^0(Y,\omega_Y)'$ of (iii), and in particular, for $Y = X$, the other isomorphism $H^n_M(R) \xrightarrow{\sim} (\omega_R)'$.

The map δ_f of Section 2 is an "edge homomorphism," gotten by taking cohomology in degree n in the obvious map of complexes

$$\mathbf{R}\Gamma_M(\mathcal{O}_X) = \mathbf{R}\Gamma_M(f_*\,\mathcal{O}_Y) \to \mathbf{R}\Gamma_M(\mathbf{R}f_*\,\mathcal{O}_Y)$$

The above map τ'_Y is gotten by taking cohomology in degree $-n$ in the map of complexes

$$\begin{array}{ccc} \Gamma_Y\mathcal{R}^{\cdot}_Y = \Gamma_X f_*\mathcal{R}^{\cdot}_Y & \xrightarrow{\quad\text{trace}\quad} & \Gamma_X\mathcal{R}^{\cdot}_X \\ \| & & \| \\ \mathbf{R}\operatorname{Hom}(f_*\,\mathcal{O}_Y, f_*\mathcal{R}^{\cdot}_Y) & & \mathbf{R}\operatorname{Hom}(f_*\,\mathcal{O}_Y, \mathcal{R}^{\cdot}_X) \end{array}$$

With everything thus made explicit, it is straightforward to check that δ_f is indeed dual to τ'_Y, as asserted in (iii).

It remains to prove (iv).

So assume that R is regular, and let $y \in Y$ be such that $\mathcal{O}_{Y,y}$ is a discrete valuation ring. We proceed by induction on n, the case $n = 1$ being trivial. Let $f(y) = p$, a prime ideal in R. From the definition of "residual complex," it is immediate that $\mathcal{R}^{\cdot}_X \otimes_R R_p$ is a residual complex on Spec(R_p) (not, however, normalized unless $p = M$); and by compatibility of f^{Δ} with flat base change (which can be verified easily for the particular base change Spec$(R_p) \to$ Spec(R)), setting $f_p = f \otimes_R R_p$, we have an isomorphism

$$f_p^{\Delta}(\mathcal{R}^{\cdot}_X \otimes_R R_p) \cong R^{\cdot}_Y \otimes_R R_p.$$

Hence our definition of ω_Y "commutes with localization on R"; and so, by the inductive hypothesis, we may assume $p = M$.

Since R is universally catenary, the residue field of $\mathcal{O}_{Y,y}$ has transcendence degree $n - 1$ over the residue field of R. Hence by [1, p. 77, Prop. 4.4. (where the world "algebraic" can be omitted)], if we set $X_0 = X$, and (for $i > 0$) X_i = the blowup of X_{i-1} along the closure of the image of y in X_{i-1}, then for some m, $\mathcal{O}_{Y,y}$ is the local ring of a point on X_m. (*Note.* Since $\mathcal{O}_{Y,y}$ is a discrete valuation ring, there is a neighborhood V_i of y in Y and a birational map $V \to X_{\iota-1}$ for each $i > 0$.) Because of (i), we may assume that $Y = X_m$.

We proceed by induction on m. Assume $m > 0$ (the case $m = 0$ is trivial). Suppose we can show that

$$\omega_1 = \omega_{X_1} \supseteq \mathcal{O}_{X_1}.$$

Since X_1 is obtained by blowing up M, therefore X_1 is regular and so ω_1 is *invertible.* Let $f_1 : Y \to X_1$ be the obvious map, and set $x_1 = f_1(y)$. Again by localizing at x_1, and by the inductive hypothesis (on m), we see that

$$\omega_{Y,y} \otimes (f_1^* \omega_1)_y^{-1} \supseteq \mathcal{O}_{Y,y}$$

so that

$$\omega_{Y,y} \supseteq (f_1^* \omega_1)_y \supseteq (f_1^* \mathcal{O}_{X_1})_y = \mathcal{O}_{Y,y}.$$

Thus we need only consider the case $m = 1$.

So assume that $Y = X_1$, and $\omega_1 = \omega_Y$. ω_Y is then an invertible sheaf whose restriction to $U = Y - E$ is $\mathcal{O}_U$. (Recall that $\omega_X = \mathcal{O}_X$, and note that $U = \text{Spec}(R) - \{M\}$.) Here E is a divisor,

$$E = f^{-1}\{M\} = \mathbf{P}^{n-1}$$

(projective $(n-1)$-space over the residue field of R), so $\omega_Y = \mathcal{O}_Y(aE)$ for some integer a. By the "adjunction formula" (cf. [8, p. 190, 2) and 3)]), if $i : E \hookrightarrow Y$ is the inclusion, then $i^*(\omega_Y \otimes \mathcal{O}_Y(E))$ is a dualizing sheaf on $\mathbf{P}^{n-1}$, that is,

$$i^*(\mathcal{O}_Y((a+1)E)) \cong \mathcal{O}_E(-n).$$

But

$$i^*(\mathcal{O}_Y(E)) = \mathcal{O}_E(E) = \mathcal{O}_E(-1).$$

Hence $a = n - 1$, that is,

$$\omega_Y = \mathcal{O}_Y((n-1)E) \supseteq \mathcal{O}_Y.$$

5. TWO-DIMENSIONAL PSEUDO-RATIONAL LOCAL RINGS

We begin with a technical lemma. Let Y be a scheme, let F be a locally free $\mathcal{O}_Y$-module of rank d, and let $\sigma : F \to \mathcal{O}_Y$ be an $\mathcal{O}_Y$-homomorphism. From these data we obtain a complex of $\mathcal{O}_Y$-modules (exterior powers of F)

$$K(F,\sigma) : 0 \to \Lambda^d F \to \Lambda^{d-1} F \to \ldots \to \Lambda^1 F = F \xrightarrow{\sigma} \mathcal{O} \to 0$$

where the maps $\sigma_i : \Lambda^i F \to \Lambda^{i-1} F$ are given locally by

$$\sigma_i(e_1 \wedge e_2 \ldots \wedge e_i) = \sum_{j=1}^{i} (-1)^{j-1} \sigma(e_j)\, e_1 \wedge \ldots \wedge \hat{e}_j \wedge \ldots \wedge e_i$$

(as usual "$\hat{e}_j$" signifies "omit e_j"). Locally $K(F,\sigma)$ is a Koszul complex. In particular, if σ is surjective then $K(F,\sigma)$ is exact.

Now let L be an *invertible* $\mathcal{O}_Y$-module generated by finitely many global sections $(f_1, \ldots, f_d)$. Let

$$F = L^{-1} \oplus L^{-1} \oplus \ldots \oplus L^{-1} \qquad (d \text{ times})$$

and let $\sigma : F \to \mathcal{O}_Y$ be the *sum* of the homomorphisms $f_i : L^{-1} \to \mathcal{O}_Y$ $(i = 1,2,\ldots,d)$. Then σ is *surjective,* and so we have an *exact* complex $K(F,\sigma)$. Note that in this case $\Lambda^i F$ is a direct sum of $\binom{d}{i}$ copies of L^{-i}. [In fact $K(F,\sigma)$ is the tensor product of the d two-term complexes $L^{-1} \xrightarrow{f_i} \mathcal{O}_Y$ $(i = 1,2,\ldots,d)$.]

Tensoring this $K(F,\sigma)$ with L^s, we get an exact sequence

$$K(F,\sigma)(s) : 0 \to L_{ds} \to L_{d-1,s} \to \ldots \to L_{1s} \to L_{0s} = L^s \to 0$$

where L_{is} is a direct sum of $\binom{d}{i}$ copies of L^{s-i}.

LEMMA 5.1. *With preceding notation, assume that* $L \subseteq \mathcal{O}_Y$, *and let* I *be the* $H^0(\mathcal{O}_Y)$*-ideal generated by* $(f_1, \ldots, f_d)$. *Suppose that*

$$H^1(L^{s-2}) = H^2(L^{s-3}) = \ldots = H^{d-1}(L^{s-d}) = 0.$$

Then

$$H^0(L^s) = IH^0(L^{s-1}).$$

Proof. From $K(F,\sigma)(s)$ we get exact sequences

$$0 \to K_i \to L_{is} \to K_{i-1} \to 0 \qquad (i = 1,2,\ldots,d-1)$$

with

$$K_0 = L_{0s} = L^s, \quad K_{d-1} = L_{ds} = L^{s-d}.$$

Since L_{is} is a direct sum of copies of L^{s-i}, we see that the resulting connecting homomorphisms

$$H^1(K_1) \to H^2(K_2) \to \ldots \to H^{d-2}(K_{d-2}) \to H^{d-1}(K_{d-1})$$

are all *injective.* Since $H^{d-1}(K_{d-1}) = H^{d-1}(L^{s-d}) = 0$, therefore $H^1(K_1) = 0$, and consequently

$$H^0(L_{1s}) \to H^0(K_0) = H^0(L^s)$$

is surjective, which gives the stated result.

Remark 5.2. By a similar argument, if we assume that

$$H^1(L^{s-2}) = H^2(L^{s-3}) = \ldots = H^{d-2}(L^{s-d+1}) = 0$$

and that

$$H^1(L^{s-1}) = H^2(L^{s-2}) = \ldots = H^{d-1}(L^{s-d+1}) = 0$$

then we get an *isomorphism.*

$$H^{d-1}(L_{ds}) = H^{d-1}(L^{s-d}) \cong H^0(L^s)/IH^0(L^{s-1}).$$

COROLLARY 5.3. *Let R be a normal local domain whose completion $\hat{R}$ is reduced* (i.e. *has no nonzero nilpotent elements*). *Let $f_1, \ldots, f_d \in R$, and for each integer $t > 0$ let I_t be the ideal $(f_1^t, \ldots, f_d^t)R$. Let $Y \to$ Spec(R) be obtained by blowing up I and normalizing. Then, for all sufficiently large t we have (with $I = I_1$):*

(i) $H^{d-1}(\mathcal{O}_Y) = \overline{I_t^d}/I_t\overline{I_t^{d-1}} = \overline{I^{td}}/I_t\overline{I^{t(d-1)}}$

and

(ii) $\overline{I_t^{d+1}} = \overline{I^{t(d+1)}} = I_t\overline{I^{td}} = I_t\overline{I_t^d}.$

Proof. Since $\hat{R}$ is reduced, $Y \to$ Spec(R) is a finite-type morphism [19]. $L_1 = I\mathcal{O}_Y$ is an *ample* invertible $\mathcal{O}_Y$-module [4, (4.6.6), (8.1.7), and 5, (2.6.2)], and for any $t > 0$, L_1^t is generated by its global sections $f_1^t, \ldots, f_d^t$. Since L_1 is ample, the hypotheses of remark (5.2) will be satisfied if $L = L_1^t$ for t sufficiently large and $s = d$; so in view of (1.3) and (1.4), (5.2) gives (a). (b) is obtained similarly; or it can be deduced from (a) by using the sequence $f_1, \ldots, f_d, f_{d+1}$ with $f_{d+1} = f_d$, and noting that $H^d(\mathcal{O}_Y) = 0$ (since Y is covered by d affine open subsets).

COROLLARY 5.4. *Let R be a two-dimensional pseudo-rational local ring. Then for every ideal I in R and every integer $\lambda > 0$ we have*

$$\overline{I^{\lambda+1}} = I\,\overline{I^{\lambda}} = I^{\lambda}\bar{I}.$$

Proof. We assume that R has an infinite residue field (the reduction to this case is left to the reader; use Example (c) of Section 2 and [13, p. 660, (e)]). Then I has a reduction generated by two elements, and we may replace I by this reduction, that is, we may assume that I itself is generated by two elements (f_1, f_2). Let $f\colon Y \to$ Spec(R) be obtained by blowing up I and normalizing. Since R is pseudo-rational, $H^1(\mathcal{O}_Y) = 0$ (Example (b), Section 2). Setting $L = I\mathcal{O}_Y$, we have that, for any $t \geq 0$, L^t is generated by its global sections, that is, L^t is a homomorphic image of $\mathcal{O}_Y^N$ for some $N > 0$, and hence $H^1(L^t) = 0$. (Note that $H^2(K) = 0$, where K is the kernel of the surjection $\mathcal{O}_Y^N \to L^t$, since Y is covered by two affine open subsets.) The first equality in (5.4) follows now from Lemma (5.1), with $d = 2$, $s = \lambda + 1$. The second equality is then obtained by induction on λ.

PROPOSITION 5.5. *Let R be a two-dimensional normal local ring, whose completion $\hat{R}$ is reduced. Then R is pseudo-rational if and only if, for every ideal I primary for the maximal ideal of R, $I\bar{I}$ is integrally closed* (i.e. $I\bar{I} = \overline{I^2}$).

Proof. We have just seen that if R is pseudo-rational then $\overline{I^2} = I\bar{I}$ for *every* ideal I in R.

For the converse, take $d = 2$ and t large in (i) of (5.3) to see that $H^1(\mathcal{O}_Y) = 0$ for every $Y \to \mathrm{Spec}(R)$ obtained by blowing up and normalizing an ideal in R generated by a system of parameters (f_1, f_2). It will therefore suffice to show that every proper birational $f: W \to \mathrm{Spec}(R)$ is "dominated" by such a $Y \to \mathrm{Spec}(R)$, that is, there exists a commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & W \\ & \searrow \quad \swarrow f & \\ & \mathrm{Spec}(R) & \end{array}$$

(cf. Section 2, Remark (a) and Example (b)).

Let $v_1, \ldots, v_m$ be the discrete valuations centered along components of the closed fibre $f^{-1}\{M\}$ (M = maximal ideal of R). For each i, there exist non-zero elements a_i, b_i in R such that $v_i(a_i/b_i) = 0$ and the image of a_i/b_i in the residue field of the valuation ring R_{v_i} is transcendental over the residue field of R. Choose a non-unit c_i in R not lying in any height one prime containing $(a_i, b_i)R$, so that $J_i = (a_i, b_i, c_i)R$ is M-primary. Replacing c_i by some power c_i^N, we may assume that $v_i(c_i) \geq v_i(b_i)$. Now

$$R_{v_i} \supseteq R[a_i/b_i, c_i/b_i]$$

and hence v_i has a one-dimensional center on the scheme obtained by blowing up J_i. Thus if $Y \to \mathrm{Spec}(R)$ is obtained by blowing up the product $J_1 J_2 \ldots J_m$ and normalizing, then *all* of $v_1, \ldots, v_m$ have one-dimensional centers on Y, and so by Zariski's main theorem, $Y \to \mathrm{Spec}(R)$ dominates f. Since J is M-primary, some power of J has a reduction generated by a system of parameters (f_1, f_2) [17, p. 356, Theorem 4]; blowing up $(f_1, f_2)R$ and normalizing gives the same $Y \to \mathrm{Spec}(R)$.

Added in proof:

1. The theorem mentioned at the end of Introduction has been proved for arbitrary ideals in regular local rings [25].

2. The equality $\phi(h \bmod I) = \psi(h/f, f_2 \ldots f_n)$ in the proof of 2.1 can also be taken as the definition of ϕ, and then the injectivity is an early consequence of the first fact established in Section 3.

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Added in proof:

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