Lipschitz fractions of a complex analytic algebra and Zariski saturation

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Introduction

While seeking to define a good notion of equisingularity (see [Zar65a], [Zar65b]) , Zariski was led to define in [Zar68] what he calls the *saturation* of a local ring: the saturated ring $\widetilde{A}$ of a ring A contains A and is contained in its normalization $\overline{A}$, and for a complete integral ring of dimension 1, the datum of the saturated ring is equivalent to the datum of the set of Puiseux characteristic exponents of the corresponding algebroid curve.

In the case of complex analytic algebras, it is well known that the normalization $\overline{A}$ coincides with the set of germs of meromorphic functions with bounded module; among the intermediate algebras between A and $\overline{A}$, there is one which can be introduced quite naturally: it is the algebra of the germs of Lipschitz meromorphic functions. We propose to study this algebra, first formally (Section 1), then geometrically (Section 2), and to prove (Sections 3 and 5) that at least in the case of hypersurfaces, it coincides with the Zariski saturation. In Section 4, in the case of a reduced but not necessarily irreducible curve, we show how the constructions of Sections 1 and 2 provide a sequence of rational exponents (defined intrinsically, without reference to any coordinates system), which generalizes the sequence of characteristic Puiseux exponents of an irreducible curve. Finally, in Section 6, we recover in a very simple way the result of Zariski which states that the equisaturation of a family of hypersurfaces implies their topological equisingularity (we even obtain the Lipschitz equisingularity, realized by a Lipschitz deformation of the ambient space).

All the arguments are based on the techniques of normalized blow-ups (recalled in the Preliminary Section; see also [Hir64b]), and we thank Professor H. Hironaka who taught it to us.[1]

[1] We are also very grateful to Mr. Naoufal Bouchareb who brilliantly and expertly translated our 1969 manuscript from French to English and from typewriting to LaTeX.

Preliminaries

(Reminders on the techniques of normalized blow-ups and majorations of analytic functions)

Conventions

In what follows the rings are commutative, unitary and noetherian. A ring A is said to be *normal* if it is integrally closed in its total ring of fractions $\operatorname{tot}(A)$. An analytic space $(X, \mathcal{O}_X)$ is said to be *normal* if at every point $x \in X$, $\mathcal{O}_{X,x}$ is normal. We will denote by $\overline{A}$ the integral closure of a ring A in $\operatorname{tot}(A)$.

0.1 Universal property of the normalisation

Let $n : \overline{X} \to X$ be the *normalisation* of an analytic space $\overline{X}$, i.e., $\overline{X} = \operatorname{specan}_X \overline{\mathcal{O}_X}$, where $\overline{\mathcal{O}_X}$ is a finite $\mathcal{O}_X$-algebra satisfying $(\overline{\mathcal{O}_X})_x = \overline{\mathcal{O}_{X,x}}$.

Definition 0.1 For every normal analytic space $Y \xrightarrow{f} X$ above X such that the f-image of any irreducible component of Y is not contained in $N = \operatorname{supp} \overline{\mathcal{O}_X}/\mathcal{O}_X$ (the analytic subspace of points in X where $\mathcal{O}_{X,x}$ is not normal), there exists a unique factorization:

$$\begin{array}{ccc} Y & \xrightarrow{\;\overline{f}\;} & \overline{X} \\ & \searrow^{f} & \big\downarrow n \\ & & X \end{array}$$

Proof (a) Algebraic version:

Let $\varphi : A \to B$ be a homomorphism of rings. Let $(\mathfrak{p}_i)_{i=1,\dots,k}$ be the prime ideals of 0 in B.

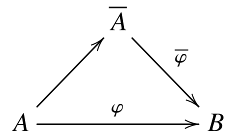
We suppose that:

(i) B is normal;

(ii) for every $i = 1, \dots, k$, $C_{\overline{A}}(A)$ is not included in $\varphi^{-1}(\mathfrak{p}_i)$, where $C_{\overline{A}}(A)$ denotes the conductor of A in $\overline{A}$:

$$C_{\overline{A}}(A) = \{g \in A / g\overline{A} \subset A\}.$$

Then, there is a unique factorization:



Indeed, by the Prime Avoidance Lemma (see [Bou61, §1]), there exists $g \in C_{\overline{A}}(A)$ such that $g \notin \varphi^{-1}(\mathfrak{p}_i)$ for all $i = 1, \ldots, k$. This implies that $\varphi(g)$ is not a divisor of 0 in B. For every $h \in \overline{A}$, set:

$$\overline{\varphi}(h) = \frac{\varphi(g.h)}{\varphi(g)} \in \operatorname{tot}(B)$$

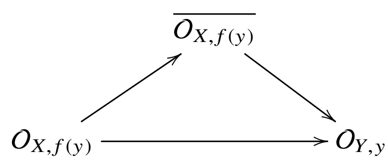
Since h is integral on A, $\overline{\varphi}(h)$ is integral on $\varphi(A)$, and thus also on B. Hence, $\overline{\varphi}(h) \in B$ and $\overline{\varphi}$ is the desired factorization. The uniqueness is obvious.

(b) Geometric version:

Let $Y \xrightarrow{f} X$ satisfy the conditions of the statement. The conditions of the statement remain true locally at $y \in Y$ since if $\varphi^{-1}(N)$ contains locally an irreducible component of Y, it contains it globally. We deduce from this that the local homomorphism:

$$\mathcal{O}_{X,f(y)} \longrightarrow \mathcal{O}_{Y,y}$$

satisfies the conditions of the algebraic version. We then have the unique factorization:



and by the coherence of $\overline{\mathcal{O}}_X$, the existence and uniqueness of the searched morphism. □

0.2 Universal property of the blowing-up (see [Hir64a])

Proposition 0.2 *Let $Y \hookrightarrow X$ be two analytic spaces and let $\mathcal{I}$ be the ideal of Y in X. There exists a unique analytic space $Z \xrightarrow{\pi} X$ over X such that:*

i) $\pi^{-1}(Y)$ is a divisor of Z, i.e., $\mathcal{I}.\mathcal{O}_Z$ is invertible.

ii) for every morphism $T \xrightarrow{\varphi} X$ such that $\mathcal{I}.\mathcal{O}_T$ is invertible, there is a unique factorization:

$$\begin{array}{ccc} T & \xrightarrow{\mathrm{Bl}(\varphi)} & Z \\ & \overset{\varphi}{\searrow} & \downarrow{\scriptstyle \pi} \\ & & X \end{array}$$

The morphism $Z \xrightarrow{\pi} X$ is called the blow-up of X along Y. Recall that π is bimeromorphic, proper and surjective and that $\pi|Z \setminus \pi^{-1}(Y)$ is an isomorphism on $X \setminus Y$.

0.3 Universal property of the normalized blow-up

Proposition 0.3 *Let $Y \hookrightarrow X$ such that X is normal outside of Y. Then, for every morphism $T \xrightarrow{\varphi} X$ such that:*

i) T is normal;
ii) $\mathcal{I}.\mathcal{O}_T$ is invertible,

there exists a unique factorization.

$$\begin{array}{ccc} T & \xrightarrow{\overline{\mathrm{Bl}(\varphi)}} & \overline{Z} \\ & \overset{\varphi}{\searrow} & \downarrow{\scriptstyle n\circ\pi} \\ & & X \end{array}$$

Proof It is sufficient to check that the factorization $T \xrightarrow{\mathrm{Bl}(\varphi)} Z$ satisfies the conditions of Subsection 0.2. Since $\pi|Z \setminus \pi^{-1}(Y)$ is an isomorphism, $Z \setminus \pi^{-1}(Y)$ is normal and it is sufficient to verify that the image of each irreducible component of T meets $Z \setminus \pi^{-1}(Y)$. But the inverse image of $\pi^{-1}(Y)$ by $\mathrm{Bl}(\varphi)$ is a divisor by assumption. Since T is normal, this divisor cannot contain any irreducible component. □

0.4 Normalized blow-up and integral closure of an ideal

(See also [Lip69, Chap. II].)

Let A be the analytic algebra of an analytic space germ $(X, 0)$, let I be an ideal of A and let $Y \hookrightarrow X$ be the corresponding sub-germ. It is known that the blow-up of the germ Y in the germ X[2] is the projective object $Z = \mathrm{Proj}_A\, E$ over X associated with the graded algebra $E = \bigoplus_{n\geqslant 0} I^n$. The normalization of Z can be written $\overline{Z} = \mathrm{Proj}_A\, \overline{E}$, with $\overline{E} = \bigoplus_{n\geqslant 0} \overline{I^n}$ (where, for an ideal J of A, we define:

[2] Here, as in other places, we abuse language to identify the germ $(X, 0)$ with one of its representatives.

$$\overline{J} = \left\{ h \in \mathrm{tot}(A) \;\mid\; \exists j_1 \in J, j_2 \in J^2, \ldots, j_k \in J^k : \;\; h^k + j_1 h^{k-1} + \cdots + j_k = 0 \right\},$$

which is the ideal of $\overline{A}$ called the *integral closure of the ideal J in $\overline{A}$*).

As an object over $\overline{X}$, the space $\overline{Z}$ equals $\mathrm{Proj}_{\overline{A}}\,\overline{E}$. But since $\overline{E}$ is a graded $\overline{A}$-algebra of finite type, there exists a positive integer s such that the graded algebra

$$\overline{E}^{(s)} = \bigoplus_{n \geqslant 0} \overline{I^{n.s}}$$

is generated by its degree 1 elements: $\overline{E}_1^{(s)} = \overline{I^s}$. But then, $\overline{E}_n^{(s)} = (\overline{I^s})^n$, and as we know that there is a canonical isomorphism $\overline{Z} = \mathrm{Proj}_{\overline{A}}\,\overline{E}^{(s)}$, we see that the normalized blow-up $\overline{Z}$ of I in A, with its canonical morphism to $\overline{X}$, coincides with the blow-up of $\overline{I^s}$ in $\overline{A}$.

Proposition 0.4 *I and $\overline{I}$ generate the same ideal of $O_{\overline{Z}}$, i.e., $IO_{\overline{Z}} = \overline{I}O_{\overline{Z}}$.*

Proof $\overline{E}$ is a finite type E-module, so for N big enough, $I.\overline{I^N} = \overline{I^{N+1}}$. But $\overline{I}.\overline{I^N} \subset \overline{I^{N+1}}$, therefore:

$$\overline{I}O_{\overline{Z}}\,.\,\overline{I^N}O_{\overline{Z}} \subset IO_{\overline{Z}}\,.\,\overline{I^N}O_{\overline{Z}}. \tag{1}$$

But if $N = k.s$, then $\overline{I^N}\,.\,O_{\overline{Z}} = (\overline{I^s})^k\,.\,O_{\overline{Z}}$. The latter ideal being invertible, we can simplify by $\overline{I^N}O_{\overline{Z}}$ in the inclusion (1). Then $\overline{I}.O_{\overline{Z}} \subset I.O_{\overline{Z}}$. The reverse inclusion is obvious. □

Proposition 0.5 *$\overline{I}$ coincides with the set of elements of $\overline{A}$ which define a section of $I.O_{\overline{Z}}$.*

Proof If $f \in \overline{I}$, then f obviously defines a section of $\overline{I}O_{\overline{Z}}$. But $\overline{I}O_{\overline{Z}} = IO_{\overline{Z}}$ according to Proposition 0.4. Conversely, suppose that $f \in \overline{A}$ defines a section of $IO_{\overline{Z}}$; by writing what this means in some affine open sets $\overline{Z}_{(g_k)} \subset \overline{Z}$, where $g_k \in \overline{I^s}$, one finds that there must exist some integers μ_k such that $f.g_k^{\mu_k} \in I.(\overline{I^s})^{\mu_k}$.

Let (g_k) be a finite family of generators of $\overline{I^s}$. For N large enough, every monomial of degree N in the g_k's will contain one of the $g_k^{\mu_k}$ as a factor, so:

$$f.\,(\overline{I^s})^N \subset I.\,(\overline{I^s})^N,$$

i. e., by choosing a base (e_i) of $(\overline{I^s})^N$,

$$f.\,e_i = \sum_j a_{ij} e_j, \qquad a_{ij} \in I.$$

Since $\overline{A}$ can be supposed to be integral, we deduce from this that

$$\det(f.\,1 - \left\| a_{ij} \right\|) = 0,$$

which is an equation of integral dependence for f on I. □

0.5 Majoration theorems

Theorem 0.6 (well known, see for example [Abh64])

Let A be a reduced complex analytic algebra and let $(X, 0)$ be the associated germ. For every $h \in \text{tot}(A)$, the following properties are equivalent:

i) $h \in \overline{A}$
ii) h defines on X^{red} a function germ with bounded module.

Theorem 0.7 *Let A be a complex analytic algebra, let $(X, 0)$ be the associated germ, let $I = (x_1, \dots, x_p)$ be an ideal of A and let $\overline{Z}$ be the normalized blow-up of I in X. For every $h \in \text{tot}(A)$, the following properties are equivalent:*

i) $h \in I.\mathcal{O}_{\overline{Z}}$
ii) h defines on X^{red} a germ of function with module bounded by $\sup|x_i|$ (up to multiplication by a constant).

Proof Let A be a noetherian local ring and let $I = (x_1, \dots, x_p)$ be a principal ideal of A. Then I is generated by one of the x_i's (easy consequence of Nakayama's lemma). Thus $\overline{Z}$ is covered by a finite number of open sets such that in each of them, one of the x_i's generates $I.\mathcal{O}_{\overline{Z}}$.

To show that $\frac{|h|}{\sup|x_i|}$ is bounded on X, we just have to prove that it is bounded on each of these open-sets, since $\overline{Z} \to X$ is proper and surjective. In the open set where x_i generates $I.\mathcal{O}_{\overline{Z}}$, $\frac{|h|}{\sup|x_i|}$ is bounded if and only if $\frac{|h|}{|x_i|}$ is bounded and we are back to theorem 0.6.

Corollary 0.8 *(from Preliminary 0.4)*

For every $h \in \overline{A}$, the following properties are equivalent:

i) $h \in \overline{I}$
ii) h defines on X^{red} a germ of function with module bounded by $\sup|x_i|$ (up to multiplication by a constant).

1 Algebraic characterization of Lipschitz fractions

Let A be a reduced complex analytic algebra and let $\overline{A}$ be its normalization ($\overline{A}$ is a direct sum of normal analytic algebras, each being therefore an integral domain, one per irreducible component of the germ associated to A). Consider the ideal:

$$I_A = \ker(\overline{A} \mathop{\widehat{\otimes}}_{\mathbf{C}} \overline{A} \to \overline{A} \mathop{\otimes}_{A} \overline{A}),$$

where $\widehat{\otimes}$ means the operation on the algebras that corresponds to the cartesian product of the analytic spaces.

Definition 1.1 We will call *Lipschitz saturation* of A the algebra:

$$\widetilde{A} = \{ f \in \overline{A} \mid \widehat{f\otimes 1} - \widehat{1\otimes f} \in \overline{I_A} \}$$

where $\overline{I_A}$ denotes the integral closure of the ideal I_A (in the sense of Subsection 0.4).

Theorem 1.2 *$\widetilde{A}$ is the set of fractions of A that define Lipschitz function germs on the analytic space X, a small enough representative of the germ $(X, 0)$ associated to A.*

Proof Firstly, let us remark that that every Lipschitz function is locally bounded and that the set of bounded fractions of A constitutes the normalization $\overline{A}$ (Theorem 0.6). However, denoting by $\overline{X}$ the disjoint sum of germs of normal analytic spaces associated to the algebra $\overline{A}$, the Lipschitz condition $|f(x) - f(x')| \leqslant C \sup |z_i - z_i'|$ for an element $f \in \overline{A}$ is equivalent to say that on $\overline{X} \times \overline{X}$, the function $\widehat{f\otimes 1} - \widehat{1\otimes f}$ has its module bounded by the supremum of the modules of the $\widehat{z_i\otimes 1} - \widehat{1\otimes z_i}$, where $z_1, \dots, z_r$ denotes a system of generators of the maximal ideal of A. But the ideal generated by $\widehat{z_i\otimes 1} - \widehat{1\otimes z_i}, i = 1 \dots, r$ is nothing but the ideal I_A defined above. Theorem 1.2 is therefore a simple application of Corollary 0.8. ☐

Corollary 1.3 *$\widetilde{A}$ is a local algebra (and thus an analytic algebra).*

Proof Since the algebra $\widetilde{A}$ is intermediate between A and $\overline{A}$, it is a direct sum of analytic algebras. If this sum had more than one term, the element $1 \oplus 0 \oplus \cdots \oplus 0$ of $\widetilde{A}$ would define on X a germ of function equal to 1 on at least one of the irreducible components of X, and to 0 on another of these components. But such a function could not be continuous on X and a fortiori not Lipschitz. ☐

The following geometric construction, which comes from Subsection 0.4, will play a fundamental role in the sequel. We will associate the following commutative diagram to the analytic space germ X:

$$\begin{array}{ccc} D_X & \hookrightarrow & E_X \\ \downarrow & & \downarrow \\ \overline{X} \underset{X}{\times} \overline{X} & \hookrightarrow & \overline{X} \times \overline{X} \end{array}$$

where E denotes the projective object over $\overline{X} \times \overline{X}$ obtained by the blow-up with center $\overline{X} \underset{X}{\times} \overline{X}$ followed by the normalization (i.e., E_X is the normalized blow-up of the ideal I_A which defines $\overline{X} \underset{X}{\times} \overline{X}$ in $\overline{X} \times \overline{X}$); the space D_X is the *exceptional divisor*, inverse image of $\overline{X} \underset{X}{\times} \overline{X}$ in E_X. According to Subsection 0.4, the condition:

$$\widehat{f\otimes 1} - \widehat{1\otimes f} \in \overline{I_A},$$

which defines $\widetilde{A}$, is equivalent to:

$$(f\widehat{\otimes}1 - 1\widehat{\otimes}f)|D_X = 0.$$

In other words, the germ $\widetilde{X}$ associated with the analytic algebra $\widetilde{A}$ is nothing but the coequalizer[3] of the canonical double arrow

$$D_X \rightrightarrows \overline{X}$$

obtained by composing the natural map $D_X \to \overline{X} \times \overline{X}$ with the two projections to $\overline{X}$. This germ of analytic space $\widetilde{X}$ will be called the *the Lipschitz saturation* of the germ X.

It is easy to see that the above local construction can be globalized: it is well known for the objects E_X and D_X, which come from blow-ups and normalizations. Likewise for $\widetilde{X}$: it is easy to define, on an analytic space $X = (|X|, \mathcal{O}_X)$, the sheaf $\widetilde{\mathcal{O}}_X$ of germs of Lipschitz fractions, and to verify that it is a coherent sheaf of $\mathcal{O}_X$-modules (as a subsheaf of the coherent sheaf $\overline{\mathcal{O}}_X$); we thus define an analytic space $\widetilde{X} = (|X|, \widetilde{\mathcal{O}}_X)$ called *the Lipschitz saturation* of $X = (|X|, \mathcal{O}_X)$, whose underlying topological space $|X|$ coincides with that of X (in fact, the canonical morphism is bimeromorphic and with Lipschitz inverse, so it is a homeomorphism).

Question 1. The inclusion $\widetilde{A} \subset \overline{A}$ was obvious in the transcendental interpretation: "every Lipschitz fraction is bounded".

But if one is interested in objects other than analytic algebras, for example in algebras of formal series, there is no longer any reason for $\overline{A}$ to play a particular role in the definition of $\widetilde{A}$. For example, we can define, for any extension B of A in its total fractions ring, the *Lipschitz saturation of A in B*:

$$\widetilde{A}(B) = \left\{f \in B | f \otimes 1 - 1 \otimes f \in \overline{I_{A(B)}}\right\}$$

with

$$I_{A(B)} = \ker(B \underset{\mathbf{C}}{\otimes} B \to B \underset{A}{\otimes} B).$$

The question then arises whether we still have the inclusion $\widetilde{A}(B) \subset \overline{A}$.

2 Geometric interpretation of the exceptional divisor $\mathbf{D_X}$: pairs of infinitely near points on X

Each point of D_X^{red} (the reduced space of the exceptional divisor D_X) will be interpreted as a pair of infinitely near points on X. The different irreducible components ${}^\tau D_X^{\text{red}}$ of D_X^{red}, labelled by the index τ, will correspond to different *types* of infinitely near points. The image of ${}^\tau D_X^{\text{red}}$ in X (by the canonical map ${}^\tau D_X^{\text{red}} \hookrightarrow D_X \to \widetilde{X} \to X$) is an irreducible analytic subset germ ${}^\tau X \subset X$, which we can call *confluence locus of the infinitely near points of type* τ. Among the types of infinitely near points, it is

[3] So we have a canonical morphism of analytic spaces $D_X \to \widetilde{X}$.

necessary to distinguish the *trivial types* whose confluence points are the irreducible components of X: the *generic* point of a *trivial* ${}^{\tau}D_X^{\mathrm{red}}$ will be a pair obtained by making two points of X tend towards the same smooth point of X. All the other (*non-trivial*) types have their confluence locus consisting of singular points of X: for example, we will see later that every hypersurface has as non-trivial confluence locus the components of codimension 1 of its singular locus.

What do the Lipschitz fractions become in this context? We have seen in Section 1 that a Lipschitz fraction is an element $f \in \overline{A}$ such that $(f \otimes 1 - 1 \otimes f)\,|D_X = 0$. But, since D_X is a divisor of the normal space E_X, this condition will be satisfied everywhere if it is only satisfied in a neighbourhood of a point of each irreducible component of this divisor; or, in intuitive language: "to verify the Lipschitz condition it is enough to verify it for a pair of infinitely near points of each type". Notice that we do not need to worry about trivial types, for which the condition is trivially satisfied for all $f \in \overline{A}$ (note also that the trivial ${}^{\tau}D_X$ are reduced).

We deduce from this the following result.

Theorem 2.1 *A meromorphic function which is locally bounded on the complex analytic space X is locally Lipschitz at every point if only it is locally Lipschitz at one point in each confluence locus ${}^{\tau}X$.*

To give a first (very rough) idea of the shape of the ${}^{\tau}D_X^{\mathrm{red}}$, let us look at their images in the space $\widehat{E}_X$ defined by blowing-up the ideal I_A in $\overline{X} \times \overline{X}$. The space E_X that we are interested in is the normalization of $\widehat{E}_X$. But $\widehat{E}_X$ has a simpler geometric interpretation: it is the closure in $\overline{X} \times \overline{X} \times \mathbf{P}^{N-1}$ of the graph Γ of the map

$$(\overline{X} \times \overline{X} - \overline{X} \underset{X}{\times} \overline{X}) \longrightarrow \mathbf{P}^{N-1}$$

which maps each pair $(\overline{x}, \overline{x}')$ outside of the diagonal to the line defined, in homogeneous coordinates, by:

$$(z_1 - z_1' : z_2 - z_2' : \ldots : z_N - z_N')\,,$$

where $(z_1, z_2, \ldots, z_N)$ denotes a system of generators of the maximal ideal of $\mathcal{O}_{X,x}$.

We will denote by $\hat{z}\colon \widehat{E}_X \to \mathbf{P}^{N-1}$ the underlying morphism and by $\hat{z}'\colon \widehat{D}_X \to \mathbf{P}^{N-1}$ the restriction of $\hat{z}$ over $\overline{X} \underset{X}{\times} \overline{X}$ (these morphisms depend on the choice of the generators $(z_1, z_2, \ldots, z_N)$). The fiber $\widehat{D}_X(x)$ of the exceptional divisor $\widehat{D}_X$ over a point $x \in X$ is the disjoint sum of a finite number of algebraic varieties (as many as $\overline{X} \underset{X}{\times} \overline{X}$ has points over x) that are embedded in $\mathbf{P}^{N-1}$ by the map $\hat{z}'|\widehat{D}_X(x)$. In particular, if x is a smooth point, $\widehat{D}_X(x)$ is nothing but the projective space $\mathbf{P}^{n-1}$ associated with the tangent space to X at x.

By composition with the finite morphisms $E_X \to \widehat{E}_X$ (normalization) and $D_X \to \widehat{D}_X$, we deduce from $\hat{z}$ and $\hat{z}'$ two morphisms

$$\tilde{z}\colon E_X \to \mathbf{P}^{N-1}$$

$$\tilde{z}' = \tilde{z}|\, D_X : D_X \to \mathbf{P}^{N-1}$$

where $\tilde{z}'$ has the following property: its restriction to the fiber $D_X(x)$ of D_X over $x \in X$ is a finite morphism.

Corollary 2.2 *If $X \subset \mathbf{C}^N$ is of pure dimension n, the confluence loci ${}^{\tau}X$ are of dimension at least equal to* $2n - N$.

Proof According to the finiteness of the above morphism, $\dim D_X(x) \leqslant N - 1$, so each irreducible component ${}^{\tau}D_X^{\mathrm{red}}$ of D_X will have an image ${}^{\tau}X$ in X of dimension:

$$\dim {}^{\tau}X \geq \dim {}^{\tau}D_X^{\mathrm{red}} - (N-1) = (2n-1) - (N-1) = 2n - N.$$

The special case of hypersurfaces. In this case, $N = n + 1$, so the confluence loci are of dimension at least equal to $n - 1$. The only non-trivial confluence loci are the codimension 1 components of the singular locus of X. Furthermore, the fibres ${}^{\tau}D_X(x)$ of the non-trivial ${}^{\tau}D_X$ are sent onto $\mathbf{P}^{N-1}$ by finite morphisms (which are surjective by a dimension argument). In the special case of hypersurfaces, Theorem 2.1 is thus formulated as follows:

Theorem 2.3 *A meromorphic function on a complex analytic hypersurface X is locally Lipschitz at every point if only it is locally Lipschitz at one point of each irreducible component (of codimension* 1*) of its polar locus.*

Definition 2.4 At a generic point of the divisor ${}^{\tau}D_X^{\mathrm{red}}$, this divisor is a smooth divisor of the smooth space E_X. Let s be its irreducible local equation. The ideal of the non reduced divisor ${}^{\tau}D_X$ is then locally of the form $(s^{\mu(\tau)})$, where $\mu(\tau)$ is a positive integer, the *multiplicity of the divisor* ${}^{\tau}D_X$".

3 Lipschitz fractions relative to a parametrization

Let $R \subset A$ be an analytic subalgebra of A and let S be the associated analytic space germ. By considering X as a relative analytic space over S, we are going to proceed to a construction analogous to that of Section 1, where the product $\bar{X} \times \bar{X}$ is replaced by the fiber product on S. This gives a diagram:

$$\begin{array}{ccc} D_{X/S} & \hookrightarrow & E_{X/S} \\ \downarrow & & \downarrow \\ \overline{X} \underset{X}{\times} \overline{X} & \hookrightarrow & \overline{X} \underset{S}{\times} \overline{X} \end{array}$$

which enables one to define the *algebra of Lipschitz fractions relative to S*:

$$\widetilde{A}^R = \left\{ f \in \overline{A} \mid (\widehat{f \otimes 1} - \widehat{1 \otimes f})|D_{X/S} = 0 \right\},$$

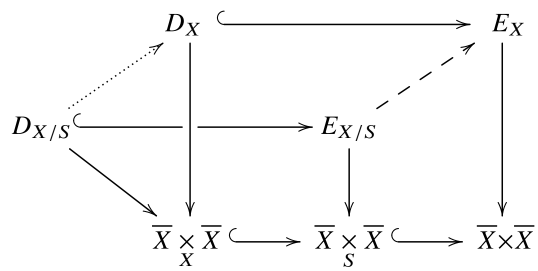
whose geometric interpretation is given by the “relative” analog to Theorem 1.2:

Theorem 3.1 (Relative Theorem 1.2) *$\widetilde{A}^R$ is the set of fractions of A that satisfy a Lipschitz condition:*

$$|f(x) - f(x')| \leq C \sup_i \left|z_i - z_i'\right|$$

for every pair of points (x, x') taken in the same fiber of X/S (with the same constant C for all fibers).

Notice the inclusion $\widetilde{A} \subset \widetilde{A}^R$, which is evident in the geometric interpretation. Formally, this inclusion can also be deduced from the existence of a "morphism" from the above relative diagram to the absolute diagram of Section 1:



where the dotted arrow $- - - - - \succ$ is defined by the universal property of the normalized blow-up (see Subsection 0.3, noting that $\overline{X} \times \overline{X}$ is normal).

We will now assume that X is of pure dimension n and we will be interested in the case where R is a *parametrization* of A, i.e., the regular algebra $\mathbf{C}\{z_1, z_2, ..., z_n\}$ generated by a system of parameters of A (an n-uple of elements of A such that the ideal generated in A contains a power of the maximal ideal). In other words, $X \to S$ is a finite morphism from X to a Euclidean space of dimension equal to that of X. Let $z = (z_1, z_2, ..., z_N)$ be a system of generators of the maximal ideal of A, and let us consider n linear combinations of them:

$$\begin{aligned}(az)_1 &= a_{11}z_1 + a_{12}z_2 + \cdots + a_{1N}z_N\\(az)_2 &= a_{21}z_1 + a_{22}z_2 + \cdots + a_{2N}z_N\\(az)_n &= a_{n1}z_1 + a_{n2}z_2 + \cdots + a_{nN}z_N\end{aligned}$$

$$\left(a_{i,j} \in \mathbf{C}\right)$$

The set of the $a = (a_{ij})$ for which $\mathbf{C}\{(az)_1, (az)_2, ..., (az)_n\}$ is a parametrization of A forms, obviously, a dense open set of the space $M_{N\times n}(\mathbf{C})$ of all the $N \times n$ matrices . We will say more generally that a family $\mathcal{P}$ of parametrizations is *generic* if for every system $z = (z_1, z_2, ..., z_n)$ of generators of the maximal ideal of A, the set of matrices a for which $\mathbf{C}\{(az)_1, (az)_2, ..., (az)_n\} \in \mathcal{P}$ contains a dense open set of $M_{N\times n}(\mathbf{C})$.

We propose to prove the:

Theorem 3.2 *For any generic family $\mathcal{P}$ of parametrizations,*

$$\widetilde{A} = \bigcap_{R\in\mathcal{P}} \widetilde{A}^R$$

It follows from this theorem that the following two questions admit identical answers:

Question 2. Is the equality $\widetilde{A} = \widetilde{A}^R$ generically true (i.e., for a generic family R of parametrizations)?

Question 2'. Is $\widetilde{A}^R$ generically independent of R?

We will see that at least in the case of hypersurfaces the answer to these two questions is yes.

Proof (of Theorem 3.2) We have already seen that $\widetilde{A} \subset \widetilde{A}^R$ for every R. Conversely, consider a function $f \in \bigcap_{R\in\mathcal{P}} \widetilde{A}^R$; does it belong to $\widetilde{A}$?

Let us consider the family of irreducible divisors in E_X consisting of the ${}^\tau D_X^{\mathrm{red}}$ and of the irreducible components of $\{\widehat{f\otimes 1} - \widehat{1\otimes f} = 0\}$. Let us denote by ${}^\tau\Delta_f$ the set of points of ${}^\tau D_X^{\mathrm{red}}$ which:

1. do not belong to any other irreducible divisor of the family;
2. are smooth points of ${}^\tau D_X^{\mathrm{red}}$ and of E_X.

Since E_X is normal, hence non-singular in codimension 1, ${}^\tau\Delta_f$ is a Zariski dense open set of ${}^\tau D_X^{\mathrm{red}}$. At every point $w \in {}^\tau D_X^{\mathrm{red}}$, the local ideal of ${}^\tau D_X$ in E_X is of the form $\left(s^{\mu(\tau)}\right)$, where s is a coordinate function of a local chart of E_X, and $\mu(\tau)$ an integer ≥ 1 (the multiplicity of the divisor ${}^\tau D_X$). Moreover, the function $\widehat{f\otimes 1} - \widehat{1\otimes f}$ is of the form $us^{\nu(\tau)}$, where u is a unit of the local ring of E_X at the point w and $\nu(\tau)$ is an integer ≥ 0.

Then, it remains to prove that $\nu(\tau) \geq \mu(\tau)$ for every τ (see Section 2).

Let S be the germ associated with a parametrization $R \in \mathcal{P}$ and let us denote by $E^*_{X/S}$ (resp. $D^*_{X/S}$) the image of $E_{X/S}$ (resp. $D_{X/S}$) in E_X by the canonical map $E_{X/S} \dashrightarrow E_X$ defined at the beginning of the section. By definition, $D^*_{X/S} = E^*_{X/S} \cap D_X$, so that if $E^*_{X/S}$ contains a point $w \in {}^\tau\Delta_f$, the divisor $D^*_{X/S}$ will be given in $E^*_{X/S}$, in a neighbourhood of this point, by the ideal $(s^{\mu(\tau)})$. If this ideal is not zero, i.e., if $E^*_{X/S}$ is not included in D_X, the relative Lipschitz condition:

$$(\widehat{f\otimes 1} - \widehat{1\otimes f})|\ D_{X/S} = 0$$

implies that the function $(\widehat{f\otimes 1} - \widehat{1\otimes f})|\ E^*_{X/S}$ is divisible by $s^{\mu(\tau)}$ in a neighbourhood of w.

By writing $\widehat{f\otimes 1} - \widehat{1\otimes f} = us^{\nu(\tau)}$ and by remarking that u, which is a unit of E_X, remains a unit after restriction to $E^*_{X/S}$, we deduce from this that $\nu(\tau) \geq \mu(\tau)$.

On the way, we had to admit that there exists an $R \in \mathcal{P}$ such that, for every non-trivial type τ, $E^*_{X/S}$ meets ${}^\tau\Delta_f$ and is not locally included in ${}^\tau\Delta_f$. To make sure of this, and thus to complete the proof of Theorem 3.2, it suffices to prove:

Lemma 3.3 *For every Zariski dense open set* ${}^\tau\Delta \subset {}^\tau {D_X}^{\mathrm{red}}$ *(τ non-trivial) consisting of smooth points of* ${}^\tau D_X^{\mathrm{red}}$ *which are also smooth points of* E_X*, there exists a generic family of parametrizations* R *for which the map* $E_{X/S} \to E_X$ *intersects* ${}^\tau\Delta$ *in at least one point* w *and is an embedding transversal to* ${}^\tau\Delta$ *at this point.* □

(The condition of "transversal embedding" is obviously stronger than what we asked, but will be more manageable).

Let $z = (z_1, z_2, ..., z_n)$ be a system of generators of the maximal ideal of $\mathcal{O}_{X,0}$, and denote by ${}^\tau\tilde{z} : {}^\tau D_X^{\text{red}} \to \mathbf{P}^{N-1}$ the restriction of the morphism $\tilde{z} : E_X \to \mathbf{P}^{N-1}$ of Section 2. To every parametrization $R(a) = \mathbf{C}\{(az)_1, (az)_2, ..., (az)_n\}$ defined by a matrix $a \in M_{N\times n}(\mathbf{C})$, let us associate the $(N-n-1)$-plane $\mathbf{P}^{N-n-1}(a) \subset \mathbf{P}^{N-1}$ defined as the projective subspace associated to the kernel of the matrix a.

Lemma 3.4 *If the map ${}^\tau\tilde{z} : {}^\tau\Delta \to \mathbf{P}^{N-1}$ is effectively transversal* [4] *to $\mathbf{P}^{N-1}(a)$ at the point $w \in {}^\tau\Delta$, then the map $E_{X/S(a)} \to E_X$ is an embedding effectively transversal to ${}^\tau\Delta$ at this point.* □

Proof (of Lemma 3.4) Since ${}^\tau\tilde{z}$ is the restriction of $\tilde{z} : E_X \to \mathbf{P}^{N-1}$, the transversality of ${}^\tau\tilde{z}$ implies the transversality of $\tilde{z}$.

Hence, $\tilde{z}^{-1}(\mathbf{P}^{N-n-1}(a))$ is a smooth subvariety of E_X of dimension n, which intersects ${}^\tau\Delta$ transversely along the smooth subvariety ${}^\tau\tilde{z}^{-1}(\mathbf{P}^{N-n-1}(a))$ of dimension $n-1$. In particular, $\tilde{z}^{-1}(\mathbf{P}^{N-n-1}(a))$ is the closure of the complement of ${}^\tau\tilde{z}^{-1}(\mathbf{P}^{N-n-1}(a))$, i.e., the closure of its part located outside of the exceptional divisor. But outside of the exceptional divisor, the right vertical arrow of the following commutative diagram is an isomorphism (since $\bar{X} \times \bar{X}$ is normal), while the left vertical arrow is surjective.

$$\begin{array}{ccc} E_{X/S(a)} & \longrightarrow & E_X \\ \downarrow & & \downarrow \\ \overline{X} \underset{S(a)}{\times} \overline{X} & \hookrightarrow & \overline{X} \times \overline{X} \end{array}$$

Therefore, the image $E^*_{X/S(a)}$ of the upper arrow is identified with $\overline{X} \underset{S(a)}{\times} \overline{X}$, i.e., with $\tilde{z}^{-1}(\mathbf{P}^{N-n-1}(a))$. Since the equality

$$E^*_{X/S(a)} = \tilde{z}^{-1}(\mathbf{P}^{N-n-1}(a))$$

is true outside of the exceptional divisor, it is true everywhere, by taking the closure.

It remains to prove that $E_{X/S(a)} \to E^*_{X/S(a)}$ is an isomorphism (in a neighbourhood of w), but it is obvious. Indeed, it is the germ of a morphism between two smooth varieties of the same dimension which sends a smooth divisor of one onto a smooth divisor of the other and which is an isomorphism outside of these divisors.□

[4] The sentence: *the map is transversal at w* expresses one of the following two possible cases:

1. w sends itself outside of the subvariety into consideration;
2. w is sent into the subvariety into consideration, and the image of the tangent map to the point w is a vector subspace transversal to the tangent space of this subvariety.

In the second case, we will say that the map is *effectively transversal in w*.

By Lemma 3.4, we can consider Lemma 3.3 as a simple consequence of:

Lemma 3.5 *There exists a dense open set of matrices* $a \in M_{N\times n}(\mathbf{C})$ *for which the map* ${}^{\tau}\tilde{z} : {}^{\tau}\Delta \to \mathbf{P}^{N-1}$ *is effectively transversal to* $\mathbf{P}^{N-n-1}(a)$ *in at least one point* $w \in {}^{\tau}\Delta$. □

Proof Let us construct a *stratification* of ${}^{\tau}D_X{}^{\mathrm{red}}$ such that each of the following analytic sets is a union of strata:

1. the reduced fiber ${}^{\tau}D_X^{\mathrm{red}}(0)$ of ${}^{\tau}D_X^{\mathrm{red}}$ over the origin $0 \in X$;
2. the complement of the Zariski open set ${}^{\tau}\Delta$.

Let us denote by W be the maximal stratum of this stratification (obviously $W \subset {}^{\tau}\Delta$) and by W_0 the maximal stratum of one (arbitrarily chosen) of the irreducible components of ${}^{\tau}D_X^{\mathrm{red}}$. We will assume that the stratification has been chosen sufficiently fine so that every pair of strata (W_0, V) satisfies Whitney (a)-Condition [Whi65], where V belongs to the *star* of W_0 (see the appendix in the present paper). In these conditions, it follows from the appendix that if the map ${}^{\tau}\tilde{z} : {}^{\tau}D_X^{\mathrm{red}} \to \mathbf{P}^{N-1}$ has its restriction to W_0 effectively transversal to $\mathbf{P}^{N-n-1}(a)$ at a point $w_0 \in W_0$, then its restriction to W will be effectively transversal to $\mathbf{P}^{N-n-1}(a)$ in at least one point $w \in W$ close to w_0. □

But, we will now prove the:

Lemma 3.6 *There exists a dense open set of matrices* $a \in M_{N\times n}(\mathbf{C})$ *for which the map* ${}^{\tau}\tilde{z}_{\,|W_0} : W_0 \to \mathbf{P}^{N-1}$ *is effectively transversal to* $\mathbf{P}^{N-n-1}(a)$ *in at least one point* $w_0 \in W_0$. □

Proof Since τ is a non-trivial type, the image of the projection ${}^{\tau}D_X^{\mathrm{red}} \to X$ is of dimension $\leqslant n-1$, so that the dimension of the fiber ${}^{\tau}D_X^{\mathrm{red}}(0)$ must be at least n (as $\dim {}^{\tau}D_X^{\mathrm{red}} = 2n-1$). Now, we know (Section 2) that the map ${}^{\tau}\tilde{z}$ restricted to ${}^{\tau}D_X^{\mathrm{red}}(0)$ is a finite morphism. By considering the algebraic variety of dimension $\geqslant n$ in $\mathbf{P}^{N-1}$ defined as the image of a component of ${}^{\tau}D_X^{\mathrm{red}}(0)$, and the Zariski dense open set of this variety defined as the image of the set of points of W_0 where the morphism is a local isomorphism, we see that Lemma 3.6 is reduced to:

Lemma 3.7 *Consider an algebraic variety of dimension* $\geqslant n$ *in the projective space* $\mathbf{P}^{N-1}$ *and a Zariski dense open set in this variety. The set of* $(N-n-1)$*-planes of* $\mathbf{P}^{N-1}$ *which intersect transversely this open set in at least one smooth point contains a dense open set of the Grassmann manifold.* □

The proof of this lemma is left to the reader. This completes the proof of Lemma 3.6. □

To summarize:

$$\left.\begin{array}{r}\text{Lemma 3.7} \Longrightarrow \text{Lemma 3.6} \Longrightarrow \text{Lemma 3.5}\\ \text{Lemma 3.4}\end{array}\right\} \Longrightarrow \text{Lemma 3.3} \Longrightarrow \text{Theorem 3.2}$$

This completes the proof of Theorem 3.2. □

Remark 3.8 The arguments of Section 2 generalize without difficulty to the relative case. Thus, for every analytic subalgebra $R \subset A$, we have the notion of *confluence locus relative to* R and the relative analog of Theorem 2.1. If R is a parametrization of A, we can see, by an argument similar to that of Section 2, that the dimension of the relative confluence locus admits the same lower bound $2n-N$ as in the absolute case; in particular, the confluence locus of a hypersurface X relative to a parametrization are the codimension 1 components of the *relative singular locus* of X, i.e., the set of points of X where the finite morphism $X \to S$ is not a submersion of smooth varieties.

We deduce from this:

Theorem 3.9 (relative version of Theorem 2.3)

Let $X \to S$ be a finite morphism of a complex analytic hypersurface to a smooth variety of the same dimension. Then, a meromorphic function on X is locally Lipschitz relatively to S at every point of X if and only if it is locally Lipschitz relatively to S at one point of each irreducible component (of codimension 1*) of its polar locus.*

4 The particular case of plane curves

Let $X \overset{(x,y)}{\hookrightarrow} \mathbf{C}^2$ be a germ of reduced analytic plane curve and let $\tilde{z} : E_X \to \mathbf{P}^1$ be the morphism corresponding to the germ of embedding (x, y) (Section 2).

Let U be the dense open set of $\mathbf{P}^1$ defined as the complement of the tangent directions of X.

Let $u \in U$. By performing a linear change of coordinates if necessary, we can assume that u corresponds to the direction of $\{x = 0\}$. In a neighbourhood of u, we take as local coordinate v in $\mathbf{P}^1$ the inverse of the slope in these coordinates.

Proposition 4.1 *In a neighbourhood of every point $w \in D_X \cap \tilde{z}^{-1}(u)$, $\tilde{z}|D_X^{\mathrm{red}}$ is an isomorphism, E_X is smooth, and $E_X \cong E_{X/S(u)} \times D_X^{\mathrm{red}}$.*

Proof Firstly, let us remark that for every $|v|$ and $|t|$ (and obviously every $|x|$ and $|y|$) small enough, the line $x - vy = t$ remains non-tangent to X and therefore, intersects X transversally at simple points if t is non-zero.

Let $\Gamma \subset (\overline{X} \times \overline{X} - \overline{X} \underset{X}{\times} \overline{X}) \times \mathbf{P}^1$ be the graph of the map defined in Section 2. We consider the map $\Psi_0 \colon \Gamma \to \mathbf{C} \times \mathbf{P}^1$ defined by $(P, P', v) \mapsto (x(P) - vy(P), v)$ (by noticing that, by definition, $x(P) - vy(P) = x(P') - vy(P')$). The map Ψ_0 extends to a meromorphic map $\widehat{E}_X \overset{\Psi_1}{\longrightarrow} \mathbf{C} \times \mathbf{P}^1$ which is obviously bounded, and so extends locally to a unique morphism $E_X \overset{\Psi}{\longrightarrow} \mathbf{C} \times \mathbf{P}^1$ (all this is done in a neighbourhood of a point w of $\tilde{z}^{-1}(u)$ on E_X).

It is easy to check, and moreover it is geometrically obvious, that Ψ has finite fibers. In addition, by the remark of the beginning of the proof, it is clear that Ψ is unramified outside of $\{0\} \times \mathbf{P}^1$. Therefore, the ramification locus is $\{0\} \times \mathbf{P}^1$ (unless it is empty).

Hence, the vector field $\frac{\partial}{\partial v}$ of $\mathbf{C} \times \mathbf{P}^1$ is tangent to the ramification locus of Ψ. Therefore, it lifts by Ψ to a holomorphic vector field on the normal space E_X (see [Zar65a, Theorem 2]).[5] At every point $w \in D_X \cap \tilde{z}^{-1}(u)$, the integration of this vector field in a neighbourhood of w endows locally E_X with a product structure $E_X \simeq \tilde{z}^{-1}(u) \times \Psi^{-1}(\{0\} \times \mathbf{P}^1)$.

But, on the one hand, we can now apply Lemma 3.4 to prove that $\tilde{z}^{-1}(u) \simeq E_{X/S(u)}$ in a neighbourhood of w, and on other hand, again by the above remark, $\tilde{z}^{-1}(u)$ does not meet any ${}^{\tau}D_X$ with trivial type τ.

We conclude by noticing that since the origin, which is the only possible singularity of the germ X, is the support of all non trivial confluence loci ${}^{\tau}X$, we have:

$$\Psi^{-1}(\{0\} \times \mathbf{P}^1) = \bigcup_{\tau \text{ non trivial}} {}^{\tau}D_X .$$

Corollary 4.2 *(See Section 2) In this situation, the equation of $D_{X/S(u)}$ in $E_{X/S(u)}$ is the equation of D_X in E_X.*

We will now study the relative situation:

$$\begin{array}{ccc} X & \xhookrightarrow{(x,y)} & \mathbf{C}^2 \\ & {\scriptstyle (x)}\searrow & \downarrow {\scriptstyle pr_1} \\ & & S = \mathbf{C} \end{array}$$

by assuming that $\{x = 0\}$ is not tangent to X at 0.

We will denote by X_α the irreducible components of X and by n_α their multiplicities.

For a local ring of dimension 1, the normalized blow-up of an ideal is a regular ring which is nothing but the normalized ring. Hence, $E_{X/S} = \overline{\overline{X} \times_S \overline{X}}$. We can easily determine the irreducible components of $E_{X/S}$ and the morphism $E_{X/S} \to S$ by using the following lemmas, after having noticed that an irreducible component of $E_{X/S}$ projects onto a pair of irreducible components of $\overline{X}$.

Lemma 4.3 *Set $m_{\alpha,\alpha'} = \operatorname{lcm}(n_\alpha, n_{\alpha'})$ and let $\varphi : \mathbf{C}\{x\} \to \mathbf{C}\{s\}$ be given by $\varphi(x) = s^{m_{\alpha,\alpha'}}$. The set B of $\mathbf{C}\{x\}$-homomorphisms*

$$\mathbf{C}\{x^{1/n_\alpha}\} \underset{\mathbf{C}\{x\}}{\otimes} \mathbf{C}\{x^{1/n_{\alpha'}}\} \longrightarrow \mathbf{C}\{x\}$$

can be identified with the set of pairs $\{(\beta, \beta') \in \mathbf{C}^2 : (\beta^{n_\alpha}, \beta'^{n_{\alpha'}}) = (1, 1)\}$ by the correspondance:

$$\begin{cases} x^{1/n_\alpha} \otimes 1 \longmapsto \beta s^{m_{\alpha\alpha'}/n_\alpha} \\ 1 \otimes x^{1/n_{\alpha'}} \longmapsto \beta' x^{m_{\alpha\alpha'}/n_{\alpha'}} \end{cases}$$

[5] We can also see this by an argument similar to that of Lemma 6.6 below.

(the pairs (β, β') *correspond to the pairs of determinations of* $(x^{1/n_\alpha}, x^{1/n_{\alpha'}})$*).*

If we endow B *with the equivalence relation:* $b_1 \sim b_2$ *if* $b_1 - b_2$ *is a* $\mathbf{C}\{x\}$*-automorphism of* $\mathbf{C}\{s\}$ *(it is the equivalence of pairs of determinations "modulo the monodromy"), then, the set* $B/_\sim$ *has* $(n_\alpha, n_{\alpha'})$ *elements.*

Lemma 4.4
$$\overline{\mathbf{C}\{x^{1/n_\alpha}\} \underset{\mathbf{C}\{x\}}{\otimes} \mathbf{C}\{x^{1/n_{\alpha'}}\}} = \underset{B/_\sim}{\oplus} \mathbf{C}\{x^{1/m_{\alpha\alpha'}}\}$$

with the obvious arrows.

Lemma 4.4 can be proved by using Lemma 4.3 and the universal property of the normalization. The proof of Lemma 4.3 is left to the reader.

We can now determine the equation of $D_{X/S}$ in $E_{X/S}$. At a point of an irreducible component of $E_{X/S}$, the ideal of ${}^\tau D_{X/S}$ is generated by $y \otimes 1 - 1 \otimes y = a_\tau s^{\mu(\tau)}$ (where a_τ is a unit of $\mathbf{C}\{s\}$), which can be interpreted as the difference of $y_{\alpha\beta}(x) - y_{\alpha'\beta'}(x)$ of the Puiseux expansions of y_α and y'_α computed for the "determinations" (β, β') of $(x^{1/n_\alpha}, x^{1/n_{\alpha'}})$ corresponding to the chosen irreducible component:

$$y_{\alpha\beta}(x) - y_{\alpha'\beta'}(x) = a_{\beta\beta'} x^{\mu(\beta,\beta')/m_{\alpha\alpha'}}$$

where $a_{\beta\beta'}$ is a unit of $\mathbf{C}\{x^{1/m_{\alpha\alpha'}}\}$, $a_{\beta\beta'} = a_\tau$ and $\mu(\beta, \beta') = \mu(\tau)$.

In the particular case where X is irreducible of multiplicity n at the origin, we deduce from this that the sequence of the distinct $\mu(\tau)$ (for τ non trivial), indexed in increasing order, coincides with the sequence:

$$\left\{\frac{m_1}{n_1}n, \frac{m_2}{n_1 n_2}n, \ldots, \frac{m_g}{n_1 \ldots n_g}n\right\}$$

where the $\frac{m_i}{n_1 \ldots n_i}n$ are the characteristic Puiseux exponents.

Now, we return back to E_X and D_X. If ${}^\tau D_{X/S}$ is an irreducible component of D_X, we know from Section 2 that $\tilde{z}|{}^\tau D_X$ is a finite morphism, and it follows from Proposition 4.1 that its ramification locus is contained in the set of directions of tangent lines to the irreducible components X_α and $X_{\alpha'}$ corresponding to ${}^\tau D_X$.

Proposition 4.5 *(i) If* X_α *and* $X_{\alpha'}$ *have the same tangent line, then* $\deg \tilde{z}|{}^\tau D_X = 1$*, so the number of types* τ *corresponding to the pair* (α, α') *equals* $(n_\alpha, n_{\alpha'})$*.*

(ii) If X_α *and* $X_{\alpha'}$ *have distinct tangent lines, then* $\deg \tilde{z}|{}^\tau D_X = (n_\alpha, n_{\alpha'})$ *and there is a unique type* τ*.*

Proof In Case (i), let $r \in \mathbf{P}^1$ be the direction of the common tangent line. Since $\mathbf{P}^1 \setminus \{r\}$ is contractible, ${}^\tau D_X \setminus \tilde{z}^{-1}(r)$ is a trivial fiber bundle on $\mathbf{P} \setminus \{r\}$. This fiber bundle is connected since ${}^\tau D_X$ is irreducible, therefore, it is a covering space of degree 1.

Case (ii) is more delicate. Let r_1 and r_2 be the two tangent directions and let $u \in \mathbf{P}^1 \setminus \{r_1, r_2\}$. We have to prove that we can join any two points of $E_{X/S(u)}$ by a path contained in ${}^\tau D_X$ and avoiding $\tilde{z}^{-1}(r_1) \cup \tilde{z}^{-1}(r_2)$. We can do this by looking at two pairs of points $(P_\alpha, P_{\alpha'})$ and $(Q_\alpha, Q_{\alpha'})$, where $P_\alpha, Q_\alpha \in X_\alpha \setminus \{0\}$ and $P_{\alpha'}, Q_{\alpha'} \in X_{\alpha'} \setminus \{0\}$ are close to the origin and located on the same line with slope

u. It is possible to pass continuously from the pair $(P_\alpha, P_{\alpha'})$ to the pair $(Q_\alpha, Q_{\alpha'})$ in such a way that the slopes of the lines joining the intermediate pairs stay at bounded distance from r_1 and r_2. We then conclude by taking the limit.

5 Lipschitz saturation and Zariski saturation

Let $R \subset A$ be a parametrization of a complex analytic algebra A, and let $X \to S$ be the associated germ of morphism of analytic spaces. Zariski defines a *domination* relation between fractions of A which, translated into transcendental terms, can be formulated as follows:

Definition 5.1 *f dominates g over R ($f \underset{R}{>} g$) if and only if, for every pair $g_\beta(x), g_{\beta'}(x)$ of distinct determinations of g, considered as a multivalued function of $x \in S$, the quotient*

$$\frac{f_\beta(x) - f_{\beta'}(x)}{g_\beta(x) - g_{\beta'}(x)}$$

has bounded module, where $f_\beta(x)$ and $f_{\beta'}(x)$ denote the corresponding determinations of f.

An extension B of A in its total ring of fractions is said *saturated over R* if every fraction which dominates an element of B belongs to B.

The *saturated algebra of A* (with respect to R) is defined as the smallest saturated algebra containing A.

Question 3. Is there a relation between the saturated algebra in the sense of Zariski and the algebra $\widetilde{A}^R$ defined in Section 3?

In the particular case of hypersurfaces, $A = R[y]$, we can easily see that the Zariski saturation coincides with the set of fractions which dominate y, i.e., in this case, with the algebra $\widetilde{A}^R$ of Lipschitz fractions relative to the parametrization R.

In the general case of an arbitrary codimension, $A = R[y_1, \ldots, y_k]$, the Zariski saturation and the Lipschitz saturation are both more complicated to define, and answering Question 3 does not seem easy to us.

In some cases, including the case of hypersurfaces, Zariski can prove that his saturation is independant of the chosen parametrization as long as the latter is generic. Therefore, we obtain, in the case of hypersurfaces, a positive answer to Questions 2 and 2' of Section 3. More precisely, we have:

Theorem 5.2 [6] *Let A be the complex analytic algebra of a hypersurface germ X, and consider the (generic) family $\mathcal{P}$ of the parametrizations defined by a direction of projection transversal to X (i.e., not belonging to the tangent cone) at a generic point of each irreducible component of codimension 1 of the singular locus. Then, for every $R \in \mathcal{P}$, $\widetilde{A} = \widetilde{A}^R$, which equals the Zariski saturation.*

[6] (Added in 2020) For a more algebraic approach, see [Li75a] and Li75b. For a more general result without the hypersurface assumption, see [Bog74] and [Bog75].

Indeed, this family of parametrizations $\mathcal{P}$ is the one for which Zariski proves the invariance of his saturation ([Zar68, Theorem 8.2]).

6 Equisaturation and Lipschitz equisingularity

The notion of saturation used in this section is the Lipschitz saturation which, as we have just seen, coincides with the Zariski saturation in the case of hypersurfaces.

Let $r : X \to T$ be an analytic retraction of a reduced complex analytic space germ X on a germ of smooth subvariety $T \hookrightarrow X$. Denote by $X_0 = r^{-1}(0)$ the fiber of this retraction over the origin $0 \in T$.

Definition 6.1 We say that (X, r) is *equisaturated along* T if the saturated germ $\widetilde{X}$ admits a product structure:

$$\widetilde{X} = \widetilde{X}_0 \times T$$

compatible with the retraction r (i.e., such that the second projection is $\widetilde{X} \to X \xrightarrow{r} T$).

Theorem 6.2 *If (X, r) is equisaturated along T, then (X, r) is topologically (and even Lipschitz) trivial along T.*

By *topological triviality*, we mean the following property: for every embedding

$$\begin{array}{ccc} X & \hookrightarrow & \mathbf{C}^N \\ & {\scriptstyle r}\searrow & \downarrow {\scriptstyle r^N} \\ & & T \end{array}$$

of the retraction r in a retraction r^N of a euclidean space, the pair $(\mathbf{C}^N, X)$ is homeomorphic to the product $(\mathbf{C}^{N-p} \times T, X_0 \times T)$, in a compatible way with the retraction r^N.

By *Lipschitz triviality*, we mean that the above homeomorphism is Lipschitz as well as its inverse.

Proof Let $(t_1, \ldots, t_p)$ be a local coordinates system on T. By using the product structure $\widetilde{X} = \widetilde{X_0} \times T$, let us denote by ∇_i the vector field on $\widetilde{X}$ whose first projection is zero and whose second one equals $\frac{\partial}{\partial t_i}$. Let A be the algebra of X; ∇_i is a derivation from $\widetilde{A}$ to $\widetilde{A}$. Then, by restriction, it defines a derivation from A to $\widetilde{A}$. Let us consider an embedding $X \hookrightarrow \mathbf{C}^N$, i.e., a system of N generators of the maximal ideal of A:

$$(z_1, z_2, \ldots, z_{N-p}, t_1 \circ r, t_2 \circ r, \ldots, t_p \circ r)$$

(by a change of coordinates, all the systems can be reduced to this form). The functions $\nabla_i z_1, \nabla_i z_2 \ldots, \nabla_i z_{N-p}$ are Lipschitz functions on X. Then, they can extend to Lipschitz functions $g_{i,1}, g_{i,2}, \ldots, g_{i,N-p}$ on all $\mathbf{C}^N$. Hence, for all $i = 1, 2, \ldots, p$, we have a Lipschitz vector field on $\mathbf{C}^N$:

$$g_{i,1}\frac{\partial}{\partial z_1} + g_{i,2}\frac{\partial}{\partial z_2} + \cdots + g_{i,N-p}\frac{\partial}{\partial z_{N-p}} + \frac{\partial}{\partial t_i}$$

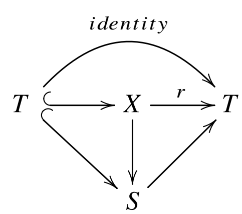
which is tangent to X and projects onto the vector field $\frac{\partial}{\partial t_i}$ of T. Since they are Lipschitz, these vector fields are locally integrable and their integration realizes the topological triviality of X. □

Relative equisaturation

We will now define a relative notion of equisaturation. Let X/S be a germ of analytic space *relative to a parametrization*, consisting of the data of a reduced analytic germ X of pure dimension n and of the germ of a finite morphism $X \to S$ on a germ of smooth variety of dimension n. Let

$$r : X/S \to T$$

be an *analytic retraction of the relative analytic space X/S on a smooth subvariety T*. By this, we mean the datum of a commutative diagram:



Denote by X_0/S_0 the relative analytic space defined as the inverse image of the point $0 \in T$ by this retraction.

Definition 6.3 (Relative Definition 6.1) We say that $(X/S, r)$ is *equisaturated along T*, if the germ of relative saturated space $\widetilde{X}^S/S$ admits a product structure:

$$\begin{array}{ccccc} \widetilde{X}^S & = & \widetilde{X_0}^{S_0} & \times & T \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle id} \\ S & = & S_0 & \times & T \end{array}$$

which is compatible with the retraction r.

In the case where X is a hypersurface, it results immediately from Theorem 5.2 that if S is a generic parametrization, the equisaturation of X/S (*relative equisaturation*) implies the equisaturation of X (*absolute equisaturation*).

Question 4. Conversely, does the equisaturation of X imply the existence of a generic parametrization S such that X/S is equisaturated?

It would be interesting to know the answer to this question because the work of Zariski gives a lot of informations on the relative notion of equisaturation.

We assume in the sequel that X is a hypersurface. Let $R = \mathbf{C}\{z_1, \dots, z_n\}$ be a parametrization of A. We can write:

$$A = R[y] = R[Y]/(f),$$

where f is a reduced monic polynomial in Y with coefficients in R and where $y = Y + (f)$ is the residue class of Y modulo f. The reduced discriminant of this polynomial generates an ideal in R which depends only of A and R; we will call it the *ramification ideal* of the parametrization R. We will denote by $\Sigma \subset S$ the subspace defined by this ideal; this subspace will be called the *ramification locus* of X/S.

Definition 6.4 We say that $(X/S, r)$ has *trivial ramification locus along* T if the pair (S, Σ) admits a product structure:

$$\begin{array}{ccccc} \Sigma & = & \Sigma_0 & \times & T \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle id} \\ S & = & S_0 & \times & T \end{array}$$

which is compatible with the retraction r.

Theorem 6.5 (Zariski [Zar68])

Let X be a hypersurface. The following two properties are equivalent:

(i) $(X/S, r)$ *is equisaturated along* T*;*
(ii) $(X/S, r)$ *has trivial ramification locus.*

Moreover, these two properties imply the topological triviality along T of the hypersurface X.

Notice that in the case of a generic parametrization, where the relative equisaturation implies the absolute equisaturation, the last part of Theorem 6.5 is a simple Corollary of our Theorem 5.2. But Zariski proves Theorem 6.5 for any parametrization.

We will limit ourselves to the proof of the implication $(ii) \Rightarrow (i)$ and we refer to [Zar68] for the rest.

Lemma 6.6 *Every derivation of the ring R in itself which leaves stable the ideal of ramification extends canonically into a derivation of the relative saturation $\widetilde{A}^R$ in itself.*

Proof Since A is finite over R, every derivation $\nabla : R \to R$ admits a canonical extension to the ring of fractions of A. Explicitly, we have:

$$\nabla y = -\left(\sum_{i=1}^{n} \frac{\partial f}{\partial z_i} \nabla z_i\right) / \frac{\partial f}{\partial y}.$$

We have to prove that under the hypothesis of the lemma, $\nabla g \in \widetilde{A}^R$ for every $g \in \widetilde{A}^R$. But the polar locus of every $g \in \overline{A}$ is obviously included in the singular locus of X, so, in the zero locus of $\frac{\partial f}{\partial y}$. By writing

$$\nabla g = \frac{\partial g}{\partial y} \nabla y + \sum_{i=1}^{n} \frac{\partial g}{\partial z_i} \nabla z_i,$$

we deduce from this that the polar locus of ∇g is included in the zero locus of $\frac{\partial f}{\partial y}$.

In order to check that $g \in \widetilde{A}^R$, it is then sufficient (by Theorem 3.9) to check it at a generic point of each irreducible component (of codimension 1, of course) of the zero locus of $\frac{\partial f}{\partial y}$.

Let ${}^S X$ be such an irreducible component, restricted to a small neighbourhood of one of its points. For a generic choice of the point, we can assume that:

(1) ${}^S X$ is smooth and the restriction to ${}^S X$ of the morphism: $X \to S$ is an embedding;
(2) ${}^S X = (X|\Sigma)^{\text{red}}$, where $\Sigma \subset S$ denotes the image of ${}^S X$, i.e., the ramification locus of the morphism $X \to S$;
(3) the finite cover $D^{\text{red}}_{X/S} \to {}^S X$ is étale, i.e., $D^{\text{red}}_{X/S}$ is a disjoint union of[7] components ${}^\tau D^{\text{red}}_{X/S}$ isomorphic to ${}^S X$.

By (1), Σ is a smooth divisor of S and we can choose local coordinates $(x, t_1, t_2, \dots, t_{n-1})$ in S so that $x = 0$ is a local equation of this divisor. (2) means that x does not vanish outside of ${}^S X$.

Locally in S, the submodule of the derivations which leave stable the ramification ideal (x) is generated by $x\frac{\partial}{\partial x}$ and the $\frac{\partial}{\partial t_i}$'s.

Therefore, it suffices to prove that the functions $x\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial t_i}$ are Lipschitz relatively to S.

Consider the space $E_{X/S}$, that we can assume to be smooth, in a neighbourhood of one of the components ${}^\tau D^{\text{red}}_{X/S}$ of the étale cover of (3). The function x is well defined on $E_{X/S}$ (by composition with the canonical morphism $E_{X/S} \to \overline{X} \times_S \overline{X} \to S$) and by (2), it does not vanish outside of $D^{\text{red}}_{X/S}$. Therefore it is of the form $x = s^{m(\tau)}$, where $s = 0$ is an irreducible equation of the smooth divisor ${}^\tau D^{\text{red}}_{X/S}$. On the other hand, the ideal of the non-reduced divisor ${}^\tau D_{X/S}$ is generated by:

$${}^\tau \Delta y = y \otimes 1 - 1 \otimes y = a_\tau(t) s^{\mu(\tau)} + \cdots,$$

where a_τ must be a unit of the ring $\mathbf{C}\{t_1, t_2, \dots, t_{n-1}\}$ since ${}^\tau \Delta y$ vanishes only on ${}^\tau D^{\text{red}}_X$.

[7] (added in 2020) ... open subsets of components ${}^\tau D^{\text{red}}_{X/S}$, isomorphic to their image in ...

Thus, for every τ, we have a series expansion whose terms are increasing powers of $x^{1/m(\tau)}$ (compare to Section 4):

$$^{\tau}\Delta y = a_{\tau}(t)x^{\frac{\mu(\tau)}{m(\tau)}} + \cdots ,$$

and a function g will be Lipschitz relatively to S if and only if for every τ, the series expansion of $^{\tau}\Delta g$ into rational powers of x has no terms with exponents less than $\mu(\tau)/m(\tau)$. Let g be such a function:

$$^{\tau}\Delta g = b_{\tau}(t)x^{\frac{\mu(\tau)}{m(\tau)}} + \cdots$$

We have:

$$^{\tau}\Delta(x\frac{\partial g}{\partial x}) = x\frac{\partial}{\partial x}(^{\tau}\Delta g) = x\Big(\frac{\mu(\tau)}{m(\tau)}b_{\tau}(t)x^{\frac{\mu(\tau)}{m(\tau)}-1} + \cdots\Big)$$

and:

$$^{\tau}\Delta(\frac{\partial g}{\partial t_i}) = \frac{\partial}{\partial t_i}(^{\tau}\Delta g) = \frac{\partial b_{\tau}(t)}{\partial t_i}x^{\frac{\mu(\tau)}{m(\tau)}} + \cdots ,$$

so that $x\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial t_i}$ are still functions of the same type, i.e., Lipschitz functions relative to S. This completes the proof of Lemma 6.6 □

***Proof* of (ii) ⇒ (i) of theorem 6.5.**
Let us choose local coordinates $(x_1, x_2, \dots, x_{n-p}, t_1, t_2, \dots, t_p)$ in S compatible with the product structure $S_0 \times T$. The vector field $\frac{\partial}{\partial t_i}$ is tangent to the ramification locus $\Sigma = \Sigma_0 \times T$. Therefore, by Lemma 6.6, it lifts to a holomorphic vector field ∇_i on $\widetilde{X}^S$. The integration of these p vector fields $\nabla_1, \dots, \nabla_p$ realizes the desired product structure on $\widetilde{X}^S$. □

Speculation on equisingularity

We would like to find a "good" definition of equisingularity of X along T, satisfying if possible the two following properties:

(TT) the equisingularity implies the topological triviality;
(OZ) the set of points of T where X is equisingular forms a dense Zariski open set.

Equisaturation satisfies (TT) (Theorem 5.2 above), but satisfies (OZ) only in the case where $\mathrm{codim}_X T = 1$ (equisaturation of a family of curves coincides with equisingularity). In the general case, one can find some $X \to T$ such that X is not equisaturated at any point of T[8].

[8] Here we are thinking about the relative equisaturation characterized (Theorem 6.5) by the triviality of the ramification locus. But likely, the notion of absolute equisaturation leads to about the same thing - don't we want to answer yes to Question 4?
Added in 2020: The approaches of E. Böger in [Bog75] and Lipman in [Li75b] would probably lead to a positive answer.

Zariski proposed a definition to the equisingularity of hypersurfaces that generalizes the idea of trivialization of the ramification locus ([Zar37], [Zar64]): the hypersurface X is *equisingular* along T if, for a generic parametrization, the ramification locus Σ is equisingular along (the projection of) T. Since the codimension of T in Σ is smaller than its codimension in X minus one, we therefore obtain a definition of the equisingularity by induction on the codimension.[9]

This definition satisfies (OZ), but we do not know how to prove (TT)[10].

In the case where T coincides with the singular locus of X (family of analytic spaces with isolated singularities), Hironaka found a criterion of equisingularity which satisfies (TT) and (OZ) at the same time. This criterion is defined [Hir64] in terms of the normalized blow-up of an ideal (i.e., the product of the ideal of T by the Jacobian ideal of X). The topological triviality is proved by integrating a vector field [11], but:

1. instead of being holomorphic on $\overline{X}$, this vector field is differentiable (i.e., C^∞) on the blown-up space $\widehat{X}$ of X (the normalized blow-up of the ideal mentioned above).
2. instead of being Lipschitz on X, i.e., satisfying a Lipschitz inequality for every ordered pair of points in $X \times X$, this vector field satisfies a Lipschitz inequality only for the ordered pair of points in $T \times X$.

The general solution to the problem of equisingularity will maybe use some rings of this type of functions (C^∞ in a blown-up space and “weakly Lipschitz" below).[12]

Appendix: stratification, Whitney's (a)-property and transversality

A *stratification* [13] of an analytic (reduced) space X is a locally finite partition of X in smooth varieties called *strata*, such that:

1. the closure $\overline{W}$ of every stratum W is an (irreducible) analytic space;
2. the boundary $\partial W = \overline{W} \setminus W$ of every stratum W is a union of strata.

We call *star* of a stratum W the set of strata which have W in their boundary.

---

[9] (Added in 2020) This theory was described by Zariski in [Zar79], [Zar80]. The reason why equisaturation does not satisfy (OZ) in general is that it corresponds to a condition of analytical triviality of the discriminant, which of course does not satisfy (OZ) in general. See also [LiT79].

[10] (added in 2020) There are now several results where Zariski equisingularity implies topological triviality sometimes via the Whitney conditions. See [Var73], [Spe75].

[11] Cf. H. Hironaka (not published but see [Hir64b]).

[12] (Added in 2020) The idea of considering vector fields which are differentiable on some blown-up space was used by Pham in [Pha71a] and, in real analytic geometry by Kuo who introduced *blow-analytic* equivalence of singularities; see [Kuo85].

[13] See also David Trotman's article "Stratifications, Equisingularity and Triangulation" in this volume.

Let (W_0, W) be an ordered pair of strata, with $W_0 \subset \partial W$. We say that this ordered pair satisfies the *property (a) of Whitney* at a point $x_0 \in W_0$ if for every sequence of points $x_i \in W$ tending to x_0 in such a way that the tangent space $T_{x_i}(W)$ admits a limit, this limit contains the tangent space $T_{x_0}(W)$ (we suppose that X is locally embedded in a Euclidean space, in such a way that the tangent spaces are realized as subspaces of the same vector space; the property (a) of Whitney is independent of the chosen embedding). For every ordered pair of strata (W_0, W) of a stratification, there exists a Zariski dense open set of points of W_0 where the property (a) of Whitney is satisfied [Whi65]. We can then refine every stratification into a stratification such that the property (a) of Whitney is satisfied at every point for every ordered pair of strata.

Proposition 6.7 *Let (X, x_0) be a stratified germ of complex analytic space such that the ordered pairs of strata (W_0, W) satisfy the property (a) of Whitney, where W_0 denotes the stratum which contains x_0 and where W is any stratum of the star of W_0. Let $\varphi : X \to \mathbf{C}^m$ be a morphism germ such that $\varphi|W_0$ is effectively transversal to the value 0 at the point x_0. Then, for every stratum W, $\varphi|W$ is effectively transversal to the value 0 at (at least) one point of W arbitrarily close to x_0.*

Proof The transversality of $\varphi|W$ at every point close to x_0 is an obvious consequence of the property (a) of Whitney for the ordered pair (W_0, W). It remains to prove the effective transversality i.e., to prove that $(\varphi|W)^{-1}(0)$ is not empty. But $(\varphi|\overline{W})^{-1}(0)$ is a close analytic subset of $\overline{W}$, non empty (because it contains the point x_0) and defined by m equations. Therefore its codimension is at most m. If $(\varphi|W)^{-1}(0)$ were empty, then ∂W would contain at least one stratum W' such that $(\varphi|W')^{-1}(0)$ is non empty and of dimension $\geqslant \dim W - m$. But on the other hand, the transversality of $\varphi|W'$ implies that $(\varphi|W')^{-1}(0)$ is a smooth variety of dimension $< \dim W' - m$, and then of dimension $< \dim W - m$. We then get a contradiction. □

Remark 6.8 Of course, in the statement of Proposition 6.7, we could replace the transversality relative to the value of 0 by the transversality relative to a smooth variety of $\mathbf{C}^m$.

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