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Semigroups of Valuations on Local Rings

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Dedicated to Mel Hochster on the occasion of his sixty-fifth birthday

Introduction

In recent years the progress and applications of valuation theory have brought to light the importance of understanding the semigroups of values that a Krull valuation ν of some field takes on a Noetherian local ring (R, m) contained in the ring (R_ν, m_ν) of the valuation.

Two general facts about valuations dominating a Noetherian local domain are proven by Zariski and Samuel in their book on commutative algebra [17]. The first is that these semigroups are well-ordered subsets of the positive part of the value group and of ordinal type at most ω^h, where ω is the ordinal type of the well-ordered set $\mathbf{N}$ and h is the rank of the valuation (see [17, Apx. 3, Prop. 2]). Being well-ordered, each value semigroup of a Noetherian ring has a unique minimal system of generators that is indexed by an ordinal no greater than ω^h.

The second general fact is that, if $m_\nu \cap R = m$ and if R and R_ν have the same field of fractions, then the Abhyankar inequality

$$\operatorname{rr}(\nu) + \operatorname{tr}_k k_\nu \leq \dim R$$

holds for the rational rank of the group of the valuation, the transcendence degree of the residue field of R_ν as an extension of the residue field of R, and the dimension of R. If equality holds then the group of the valuation is isomorphic to $\mathbf{Z}^{\operatorname{rr}(\nu)}$ (see [17, Apx. 2]). If, in addition, the rank of ν is 1, then the ν-adic and m-adic topologies of R coincide (see [14, Prop. 5-1]).

These two conditions on ordinal type and rational rank do not characterize value semigroups of Noetherian rings. In [5], a third condition is given that concerns the rate of growth of the number of generators of the semigroup of a valuation dominating an equicharacteristic Noetherian local domain. This condition implies that there exist subsemigroups of $\mathbf{Q}_+$ that are well-ordered of ordinal type ω—and so satisfy the first two conditions—but are not semigroups of valuations dominating an equicharacteristic Noetherian local domain.

Examples, starting with plane branches (see [15; 16]) and continuing with quasi-ordinary hypersurfaces (see [8]) suggest that the structure of the semigroup contains important information on the process of local uniformization.

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In this paper we shall consider mostly valued Noetherian local rings $(R, m) \subset (R_\nu, m_\nu)$ such that R and R_ν have the same field of fractions, the equality $m_\nu \cap R = m$ holds, and the residual extension $R/m \to R_\nu/m_\nu$ is trivial. We call such valuations *rational valuations* of R. In this case, the finite generation of the semigroup is equivalent to the finite generation over $k_\nu = R_\nu/m_\nu$ of the graded algebra associated to the valuation

$$\operatorname{gr}_\nu R = \bigoplus_{\gamma \in \Gamma} \mathcal{P}_\gamma / \mathcal{P}_\gamma^+,$$

where

$$\mathcal{P}_\gamma = \{f \in R \mid \nu(f) \geq \gamma\}, \qquad \mathcal{P}_\gamma^+ = \{f \in R \mid \nu(f) > \gamma\},$$

and Γ is the value group of the valuation. This equivalence is true because each homogeneous component $\mathcal{P}_\gamma/\mathcal{P}_\gamma^+$ is nonzero exactly when γ is in the value semigroup $S = \nu(R \setminus \{0\})$, and then it is a one-dimensional vector space over $R/m = R_\nu/m_\nu$ (see [15, Sec. 4]).

If the semigroup is finitely generated, then it is not difficult to find a valuation of a local Noetherian domain that includes it as a semigroup: the semigroup algebra with coefficients in a field K is then a finitely generated algebra. We can define a weight on its monomials by giving to each generator as weight the generator of the semigroup to which it corresponds. Then we define a valuation by deciding that the valuation of a polynomial is the smallest weight of its monomials.

The simplest example is that of a subsemigroup of $\mathbf{N}$. It is necessarily finitely generated (see [13, Thm. 83, p. 203]) and, after dividing by the greatest common divisor, we may assume that its generators $\gamma_1, \ldots, \gamma_g$ are coprime, so that it generates the group $\mathbf{Z}$. It is the semigroup of values of the t-adic valuation on the ring $K[t^{\gamma_1}, \ldots, t^{\gamma_g}] \subset K[t]$ of the monomial curve corresponding to the semigroup.

In Section 2 we give an example of a semigroup of values of a valuation on a polynomial ring $L[x, y, z]$ over a field that generates the group $\mathbf{Z}^2$ but is *not* finitely generated as a semigroup. Moreover, the smallest closed cone in $\mathbf{R}^2$ with vertex 0 containing the semigroup is rational.

If the semigroup is not finitely generated, very little is known about its structure. We know that all totally ordered abelian groups of finite rational rank appear as value groups of valuations of rational function fields centered in a polynomial ring, but we do not know any general condition implying that a well-ordered subsemigroup of the positive part of a totally ordered group of finite rational rank appears as a semigroup of values of a Noetherian ring. We present in Section 4 a characterization of those well-ordered subsemigroups of $\mathbf{Q}_+$ that are the semigroups of values of a K-valuation on $K[X, Y]_{(X,Y)}$; however, no general result is known for subsemigroups contained in the positive part $\mathbf{Q}^2_{\succcurlyeq 0}$ of $\mathbf{Q}^2$ for some total ordering $\succcurlyeq$.

We show in Section 3 that a subsemigroup S of the positive part of a totally ordered abelian group of finite rational rank, where S is of ordinal type $< \omega^\omega$, has no accumulation points; we also show that S has ordinal type $\leq \omega^h$, where h is the rank of the group generated by S. These are properties that are held by the semigroup of a valuation on a Noetherian ring.

Even if one could solve the problem of determining which semigroups of a totally ordered group of (real) rank 1 come from Noetherian rings, an induction on the rank would meet serious difficulties. A naturally occurring question concerns the position of elements of R whose values are the generators of the semigroup in regard to the valuation ideals of the valuations with which the given valuation is composed.

More precisely, let ν be a valuation taking nonnegative values on the Noetherian local domain R, let Ψ be the convex subgroup of real rank 1 in the group Γ of the valuation, and let P be the center in R of the valuation ν_1 with values in Γ/Ψ with which ν is composed. The valuation ν induces a residual valuation $\bar{\nu}$ on the quotient R/P. If the image $q(\gamma)$ of γ in Γ/Ψ by the canonical quotient map $q\colon \Gamma \to \Gamma/\Psi$ is γ_1, then we have inclusions of the valuation ideals corresponding to ν and ν_1:

$$\mathcal{P}^+_{\gamma_1} \subset \mathcal{P}_\gamma \subset \mathcal{P}_{\gamma_1}.$$

Because R is Noetherian, the quotients $\mathcal{P}_{\gamma_1}/\mathcal{P}^+_{\gamma_1}$ are finitely generated (R/P)-modules for all $\gamma_1 \in \Gamma/\Psi$. Each is endowed by the sequence $(\mathcal{F}_\gamma = \mathcal{P}_\gamma/\mathcal{P}^+_{\gamma_1})_{\gamma\in q^{-1}(\gamma_1)}$, which we denote by $\mathcal{F}(\gamma_1)$, with the structure of a filtered (R/P)-module with respect to the filtration $\mathcal{F}(0)$ of R/P by its valuation ideals $(\overline{\mathcal{P}_\delta})_{\delta\in\Psi}$, where $\overline{\mathcal{P}_\delta} = \mathcal{P}_\delta R/P = \mathcal{P}_\delta/P$.

One could hope (see [15]) that the associated graded module

$$\mathrm{gr}_{\mathcal{F}(\gamma_1)} \mathcal{P}_{\gamma_1}/\mathcal{P}^+_{\gamma_1} = \bigoplus_{\gamma\in q^{-1}(\gamma_1)} \mathcal{P}_\gamma/\mathcal{P}^+_\gamma$$

is finitely generated over the associated graded ring

$$\mathrm{gr}_{\bar{\nu}}\, R/P = \bigoplus_{\delta\in\Psi_+} \mathcal{P}_\delta/\mathcal{P}^+_\delta.$$

For rational valuations, this would be equivalent to the fact that, for each $\gamma_1 > 0$, only finitely many new generators of the semigroup appear in $\mathcal{P}_{\gamma_1} \setminus \mathcal{P}^+_{\gamma_1}$. This would also place a drastic restriction on the ordinal type of the minimal set of generators of the semigroup $\nu(R \setminus \{0\})$.

In Sections 7 and 8 we give examples showing that this is not at all the case and that, in fact, the cardinality of the set of new generators may vary quite a lot with the value of γ_1. One might hope that this lack of finiteness is due to some transcendence in the residual extension from R/P to R_ν/m_{ν_1} and disappears after some birational extension of R to another Noetherian local ring contained in R_ν that absorbs the transcendence. Section 8 presents an example with no residue field extensions.

1. A Criterion for Finite Generation

Given a commutative semigroup S, a set M is endowed with a structure of an S-module by an operation $S \times M \to M$ written $(s, m) \mapsto s + m$ such that

$(s+s')+m = s+(s'+m)$. Recall that M is then said to be a *finitely generated* S-module if there exist finitely many elements $m_1,\dots,m_k$ in M such that $M=\bigcup_{i=1}^{k}(S+m_i)$.

PROPOSITION 1.1. *With notation as before, suppose that K is a field and that R is a local domain with residue field K dominated by a valuation ring R_ν of the field of fractions of R. Assume that the residual extension $R/m \to R_\nu/m_\nu$ is trivial. Then, for $\gamma_1 \in \Gamma/\Psi$, the associated graded module*

$$\mathrm{gr}_{\mathcal{F}(\gamma_1)} \mathcal{P}_{\gamma_1}/\mathcal{P}^+_{\gamma_1}$$

is a finitely generated $(\mathrm{gr}_{\bar\nu} R/P)$*-module if and only if*

$$F_{\gamma_1} = \{\nu(f) \mid f \in R \text{ and } \nu_1(f)=\gamma_1\}$$

is a finitely generated module over the semigroup $\bar F = \bar\nu(R/P\setminus\{0\})$.

Proof. Let us first remark that, because $\mathcal{P}_{\gamma_1}/\mathcal{P}^+_{\gamma_1}$ is an (R/P)-module, the set F_{γ_1} is an $\bar F$-module.

We know from [15, 3.3–3.5, 4.1] that (a) the nonzero homogeneous components of the graded algebra $\mathrm{gr}_\nu R = \bigoplus_{\gamma\in\Gamma} \mathcal{P}_\gamma/\mathcal{P}^+_\gamma$ are one-dimensional K-vector spaces whose degrees are in bijection with $F=\nu(R\setminus\{0\})$ and (b) $\bar F = F\cap\Psi$.

The $(\mathrm{gr}_{\bar\nu} R/P)$-module $\mathrm{gr}_{\mathcal{F}(\gamma_1)} \mathcal{P}_{\gamma_1}/\mathcal{P}^+_{\gamma_1}$ is nothing but the sum of the components of $\mathrm{gr}_\nu R$ whose degree is in $q^{-1}(\gamma_1)$. Since we are dealing with graded modules whose homogeneous components are one-dimensional K-vector spaces, this module is finitely generated if and only if there exist finitely many homogeneous elements $e_1,\dots,e_r \in \mathrm{gr}_{\mathcal{F}(\gamma_1)} \mathcal{P}_{\gamma_1}/\mathcal{P}^+_{\gamma_1}$ such that each homogeneous element of $\mathrm{gr}_{\mathcal{F}(\gamma_1)} \mathcal{P}_{\gamma_1}/\mathcal{P}^+_{\gamma_1}$ can be written as $\bar x e_i$ with $\bar x \in R/P$; this is equivalent to an assertion on degrees, which is exactly that the $\bar F$-module F_{γ_1} is finitely generated. □

2. An Example with Value Group $\mathbf{Z}^2$ and Non–Finitely Generated Semigroup

Let K be a field. Let r be a positive natural number such that the characteristic of K does not divide r, and let a_{ij} for $0\le i\le 2$ and $1\le j\le r$ be algebraically independent over K. Let M be the field

$$M = K(\{a_{ij} \mid 0\le i\le 2 \text{ and } 1\le j\le r\}).$$

Let $G\cong \mathbf{Z}_r$ be the subgroup of the permutation group S_r generated by the cycle $(1,2,\dots,r)$. For $\sigma\in G$, define a K automorphism of M by $\sigma(a_{ij}) = a_{i,\sigma(j)}$. Let $L=M^G$ be the fixed field of G. We have an étale Galois morphism

$$\Lambda : \mathbf{P}^2_M \to \mathbf{P}^2_L \cong \mathbf{P}^2_M/G.$$

Let x_0,x_1,x_2 be homogeneous coordinates on $\mathbf{P}^2_K$. Define $p_i=(a_{0i},a_{1i},a_{2i})\in \mathbf{P}^2_M$ for $1\le i\le r$, and let $q=\Lambda(p_1)$. Since $\{p_1,\dots,p_r\}$ is an orbit of Λ under the action of G, it follows that $q=\Lambda(p_i)$ for $1\le i\le r$ and that $\mathcal{I}_q\mathcal{O}_{\mathbf{P}^2_M} = \mathcal{I}_{p_1}\cdots\mathcal{I}_{p_r}$.

Let $\bar{\nu}$ be the m-adic valuation of $\mathcal{O}_{\mathbf{P}^2_L,q}$. Then

$$\Gamma(\mathbf{P}^2_L, \mathcal{O}_{\mathbf{P}^2}(d) \otimes \mathcal{I}^n_q) = \left\{F(x_0,x_1,x_2) \in \Gamma(\mathbf{P}^2_L, \mathcal{O}_{\mathbf{P}^2}(d)) \;\middle|\; \bar{\nu}\left(F\left(1, \tfrac{x_1}{x_0}, \tfrac{x_2}{x_0}\right)\right) \geq n\right\}. \quad (1)$$

Since M is flat over L, we have

$$\Gamma(\mathbf{P}^2_L, \mathcal{O}_{\mathbf{P}^2}(d) \otimes \mathcal{I}^n_q) \otimes_L M = \Gamma(\mathbf{P}^2_M, \mathcal{O}_{\mathbf{P}^2}(d) \otimes ((\mathcal{I}_{p_1} \cdots \mathcal{I}_{p_r})^n)). \quad (2)$$

Let ν_1 be the m-adic valuation on $R = L[x_0,x_1,x_2]_{(x_0,x_1,x_2)}$. The valuation ring R_{ν_1} of ν_1 is $L\left[x_0, \frac{x_1}{x_0}, \frac{x_2}{x_0}\right]_{(x_0)}$ with residue field $R_{\nu_1}/m_{\nu_1} \cong L\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right)$. The inclusion of the valuation ring $R_{\bar{\nu}}$ of $\bar{\nu}$ into its quotient field $L\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right)$ determines a composite valuation $\nu = \nu_1 \circ \bar{\nu}$ on the field $L(x_0,x_1,x_2)$, which dominates R. The valuation ν is rational, and its value group of ν is $\mathbf{Z} \times \mathbf{Z}$ with the lex order.

For $f \in L[x_0,x_1,x_2]$, let $L_f(x_0,x_1,x_2)$ be the leading form of f. Then L_f is a homogeneous form whose degree is the order $\operatorname{ord}(f)$ of f at the homogeneous maximal ideal of $k[x_0,x_1,x_2]$. The value of f is

$$\nu(f) = \left(\operatorname{ord}(f), \bar{\nu}\left(L_f\left(1, \tfrac{x_1}{x_0}, \tfrac{x_2}{x_0}\right)\right)\right) \in \mathbf{N} \times \mathbf{N}.$$

For $(d,n) \in \mathbf{N} \times \mathbf{N}$, we have L-module isomorphisms

$$\mathcal{P}_{(d,n)} \cap L[x_0,x_1,x_2] \cong \Gamma(\mathbf{P}^2_L, \mathcal{O}_{\mathbf{P}^2}(d) \otimes \mathcal{I}^n_q) \oplus \left(\bigoplus_{m>d} \Gamma(\mathbf{P}^2_L, \mathcal{O}_{\mathbf{P}^2}(m))\right),$$

and

$$\mathcal{P}_{(d,n)}/\mathcal{P}^+_{(d,n)} \cong \Gamma(\mathbf{P}^2_L, \mathcal{O}_{\mathbf{P}^2}(d) \otimes \mathcal{I}^n_q)/\Gamma(\mathbf{P}^2_L, \mathcal{O}_{\mathbf{P}^2}(d) \otimes \mathcal{I}^{n+1}_q). \quad (3)$$

PROPOSITION 2.1. *Suppose that $r = s^2$, where $s \geq 4$ is a natural number. Then, for $n \neq 0$,*

$$\mathcal{P}_{(d,n)}/\mathcal{P}^+_{(d,n)} = 0 \quad \text{if } d \leq ns \quad (4)$$

and, if s' is a real number such that $s' > s$, then there exist natural numbers d, n such that $d < ns'$ and

$$\mathcal{P}_{(d,n)}/\mathcal{P}^+_{(d,n)} \neq 0. \quad (5)$$

Proof. We have that $\Gamma(\mathbf{P}^2_M, \mathcal{O}_{\mathbf{P}^2}(d) \otimes ((\mathcal{I}_{p_1} \cdots \mathcal{I}_{p_r})^n)) = 0$ if $d \leq ns$ by [11, Prop. 1(1)]. Thus, (4) follows from (2) and (3).

Suppose that s' is a real number greater than s. Using the Riemann–Roch theorem and Serre duality on the blow-up of $\mathbf{P}^2_L$ at q, we compute (as in the proof of [11, Prop. 1(2)]) that

$$\dim_L(\Gamma(\mathbf{P}^2_L, \mathcal{O}_{\mathbf{P}^2}(d) \otimes \mathcal{I}^n_q) \geq \frac{d(d+3)}{2} - r\frac{n(n+1)}{2} + 1 > 0$$

if

$$\frac{d(d+3)}{2} \geq r\frac{n(n+1)}{2}.$$

Fixing a rational number λ with $s' > \lambda > s$, we see that we can find natural numbers d and m with $d/m = \lambda$ and

$$\dim_L(\Gamma(\mathbf{P}^2_L, \mathcal{O}_{\mathbf{P}^2}(d) \otimes \mathcal{I}^m_q) \neq 0.$$

Let $n \geq m$ be the largest natural number such that

$$\dim_L(\Gamma(\mathbf{P}^2_L, \mathcal{O}_{\mathbf{P}^2}(d) \otimes \mathcal{I}^n_q)) \neq 0.$$

Then $d < ns'$ and $\mathcal{P}_{(d,n)}/\mathcal{P}^+_{(d,n)} \neq 0$ by (3). Thus, (5) holds. □

The following result follows from Proposition 2.1.

PROPOSITION 2.2. *Suppose that $s \geq 4$ is a natural number. Then there exist a field M and a rank 2 valuation ν, with value group $\mathbf{Z}^2$ order lexicographically, of the rational function field $M(x_0, x_1, x_2)$ in three variables, which dominates the regular local ring*

$$R = M[x_0, x_1, x_2]_{(x_0,x_1,x_2)},$$

and is such that

$$\bigoplus_{(d,n)\in\mathbf{N}\times\mathbf{N}} \mathcal{P}_{(d,n)}/\mathcal{P}^+_{(d,n)}$$

is not a finitely generated $R/m_R \cong M$ algebra.

The semigroup $\Gamma = \nu(R \setminus \{0\})$ is not finitely generated as a semigroup. Furthermore, the closed rational cone generated by Γ in $\mathbf{R}^2$ is the rational polyhedron

$$\{(d, n) \in \mathbf{R}^2 \mid n \geq 0 \text{ and } d \geq ns\}.$$

3. Semigroups of Ordinal Type ω^h

Suppose that G is an ordered abelian group of finite rank n. Then G is order isomorphic to a subgroup of $\mathbf{R}^n$, where $\mathbf{R}^n$ has the lex order (see [1, Prop. 2.10]). If G is the value group of a valuation ν dominating a Noetherian local ring R and if S is the semigroup of values attained by ν on R, then it can be shown that S has no accumulation points in $\mathbf{R}^n$.

In this section we prove that this property is held by any well-ordered semigroup S of ordinal type $\leq \omega^h$ that is contained in the nonnegative part of $\mathbf{R}^n$. The proof relies on the following lemma. The heuristic idea of the proof of this lemma is that the semigroup generated by a set with an accumulation point has many accumulation points, accumulations of accumulation points, and so on.

LEMMA 3.1. *Let A be a well-ordered set that is contained in the nonnegative part of $\mathbf{R}$. Suppose that A has an accumulation point in $\mathbf{R}$ for the Euclidean topology. For m a positive integer, let*

$$mA = \{x_1 + x_2 + \cdots + x_m \mid x_1, \ldots, x_m \in A\}.$$

Then mA contains a well-ordered subset of ordinal type ω^m.

Proof. Let $\bar{\lambda}$ be an accumulation point of A. Since A is well-ordered, there exist $\lambda_i \in A$ for $i \in \mathbf{N}$ such that $\lambda_i < \lambda_j$ for $i < j$ and $\lim_{i\mapsto\infty} \lambda_i = \bar{\lambda}$. Let $T_1 = \{\lambda_i\}_{i\in\mathbf{N}}$; then $T_1 \subset A$ is well-ordered of ordinal type ω.

For $m \geq 2$ we will construct a well-ordered subset T_m of mA that has ordinal type ω^m. For $a_1 \in \mathbf{N}$, choose $\sigma_1^m(a_1) \in \mathbf{N}$ such that

$$\lambda_{a_1} + \bar{\lambda} < \lambda_{a_1+1} + \lambda_{\sigma_1^m(a_1)}.$$

Now for $a_2 \in \mathbf{N}$, choose $\sigma_2^m(a_1, a_2) \in \mathbf{N}$ such that

$$\lambda_{\sigma_1^m(a_1)+a_2} + \bar{\lambda} < \lambda_{\sigma_1^m(a_1)+a_2+1} + \lambda_{\sigma_2^m(a_1,a_2)}.$$

We iterate for $2 \leq i \leq m$ to choose, for each $a_i \in \mathbf{N}$, $\sigma_i^m(a_1, a_2, \ldots, a_i) \in \mathbf{N}$ such that

$$\lambda^m_{\sigma_{i-1}(a_1,\ldots,a_{i-1})+a_i} + \bar{\lambda} < \lambda_{\sigma_{i-1}^m(a_1,\ldots,a_{i-1})+a_i+1} + \lambda_{\sigma_i^m(a_1,\ldots,a_i)}.$$

Let T_m be the well-ordered subset of mA that is the image of the order-preserving inclusion $\mathbf{N}^m \to mA$, where $\mathbf{N}^m$ has the reverse lex order defined by

$$(a_m, \ldots, a_1) \mapsto \lambda_{a_1+1} + \left(\sum_{i=2}^{m} \lambda_{\sigma_{i-1}^m(a_1,\ldots,a_{i-1})+a_i+1} \right).$$

By virtue of its construction, T_m has ordinal type ω^m. □

THEOREM 3.2. *Let S be a well-ordered subsemigroup of the nonnegative part of $\mathbf{R}^n$ (with the lex order) for some $n \in \mathbf{N}$. Suppose that S has ordinal type $\leq \omega^h$ for some $h \in \mathbf{N}$. Then S does not have an accumulation point in $\mathbf{R}^n$ for the Euclidean topology.*

Proof. We prove the theorem by induction on n. For $n = 1$, the theorem follows from Lemma 3.1. Suppose that the theorem is true for subsemigroups of $\mathbf{R}^{n-1}$, and suppose that $S \subset \mathbf{R}^n$ has an accumulation point α.

Let $\pi : \mathbf{R}^n \to \mathbf{R}^{n-1}$ be projection onto the first $n - 1$ factors. For $x, y \in \mathbf{R}^n$, we have that $x \leq y$ implies $\pi(x) \leq \pi(y)$. Hence the set $\pi(S)$ is a well-ordered semigroup that has ordinal type $\leq \omega^h$ and is contained in the nonnegative part of $\mathbf{R}^{n-1}$. Let $\bar{\alpha} = \pi(\alpha)$. By the induction assumption, $\pi(S)$ has no accumulation points. Thus $\bar{\alpha} \in \pi(S)$, and there exists an open neighborhood U of $\bar{\alpha}$ in $\mathbf{R}^{n-1}$ such that $U \cap \pi(S) = \{\bar{\alpha}\}$. Therefore, $\pi^{-1}(U) \cap S = \pi^{-1}(\{\bar{\alpha}\}) \cap S$, and α is an accumulation point of $\pi^{-1}(\{\bar{\alpha}\}) \cap S$.

Projection on the last factor is a natural homeomorphism of $\pi^{-1}(\{\bar{\alpha}\})$ to $\mathbf{R}$ that is order preserving. Let $A \subset \mathbf{R}$ be the image of $\pi^{-1}(\{\bar{\alpha}\}) \cap S$. Since A is a well-ordered set with an accumulation point, by Lemma 3.1 it follows that mA has a subset with ordinal type ω^m for all $m \geq 1$. Now, projection onto the last factor identifies a subset of $\pi^{-1}(\{m\bar{\alpha}\}) \cap S$ with mA for all $m \geq 1$. Thus the ordinal type of S is no less than ω^m for all $m \in \mathbf{N}$, a contradiction. □

A variation on the proof of Theorem 3.2 proves the following corollary.

COROLLARY 3.3. *Let S be a well-ordered subsemigroup of the nonnegative part of $\mathbf{R}^n$ (with the lex order) for some $n \in \mathbf{N}$. Suppose that S has ordinal type $\leq \omega^h$ for some $h \in \mathbf{N}$. Then S has ordinal type $\leq \omega^n$.*

Proof. We prove the theorem by induction on n.

We first establish the theorem for $n = 1$. Suppose that $S \subset \mathbf{R}$ has ordinal type $> \omega$. To the ordinal number ω there corresponds an element a of S. Hence there are infinitely many elements of S in the closed interval $[0, a]$, so S has an accumulation point. By Theorem 3.2, this is impossible.

Now suppose that $S \subset \mathbf{R}^n$ and that the theorem is true for subsemigroups of $\mathbf{R}^{n-1}$. By Theorem 3.2, S has no accumulation points.

Let $\pi : \mathbf{R}^n \to \mathbf{R}^{n-1}$ be projection onto the first $n-1$ factors. The set $\pi(S)$ is a well-ordered semigroup with ordinal type $\leq \omega^h$ and contained in the nonnegative part of $\mathbf{R}^{n-1}$. By the induction assumption, $\pi(S)$ has ordinal type $\leq \omega^{n-1}$. For $\bar{x} \in \pi(X)$, $\pi^{-1}(\{\bar{x}\}) \cap S$ is a well-ordered subset of $\mathbf{R}$ with no accumulation points and thus has ordinal type $\leq \omega$. Since this is true for all $\bar{x} \in \pi(S)$, the ordinal type of S cannot exceed $\omega^{n-1}\omega = \omega^n$. □

4. Subsemigroups of $\mathbf{Q}_+$

Let S be a well-ordered subsemigroup of $\mathbf{Q}_+$, and let $(\gamma_i)_{i \in I}$ be its minimal system of generators. The set of the γ_i may or may not be of ordinal type ω.

For example, choose two prime numbers p, q and consider the positive rational numbers $\gamma_i = 1 - 1/p^i$ for $1 \leq i < \omega$ and $\gamma_i = 2 - 1/q^{i-\omega+1}$ for $\omega \leq i < \omega 2 = \omega + \omega$. These numbers form a well-ordered subset of $\mathbf{Q}_+$ of ordinal type $\omega 2$ and generate a certain semigroup $S_{p,q}$, which in turn is well-ordered by a result of Neumann (see [12; 14, Thm. 3.4]). Because of the way their denominators grow with i, the γ_i are a minimal system of generators of $S_{p,q}$. Using k different prime numbers, one can in the same way build well-ordered semigroups in $\mathbf{Q}_+$ with minimal systems of generators of ordinal type ωk for any $k < \omega$.

REMARK. We assume that $S \subset \mathbf{Q}_+$, so if S is a semigroup of values for some valuation then that valuation is of rank 1. And, by the result quoted in the Introduction, if the semigroup S comes from a Noetherian ring then it is of ordinal type $\leq \omega$. The semigroup $S_{p,q}$ is therefore an example of a well-ordered subsemigroup of a totally ordered group of finite rank that cannot be realized as the semigroup of values of a Noetherian ring. Using Theorem 3.2, we see that the semigroup S_p generated by $\gamma_i = 1 - 1/p^i$ $(0 < i < \omega)$ cannot be realized either, since it has an accumulation point.

Note that, by Corollary 3.3, the ordinal type of S_p or $S_{p,q}$ is no less than ω^ω. One can ask what is the relationship between the ordinal type of the minimal set of generators of a well-ordered subsemigroup S of $\mathbf{Q}_+$ and the ordinal type of S.

For the balance of this section we consider only semigroups $S \subset \mathbf{Q}_+$ whose minimal system of generators $(\gamma_1, \dots, \gamma_i, \dots)$ is of ordinal type $\leq \omega$. If the γ_i have a common denominator, then S is isomorphic to a subsemigroup of $\mathbf{N}$ and is finitely generated (for a short proof, see [13, Thm. 83, p. 203]).

Hence we shall assume that there is no common denominator. Let us denote by S_i the semigroup generated by $\gamma_1, \dots, \gamma_i$ and by G_i the subgroup of $\mathbf{Q}$ that it generates. We have $S = \bigcup_{i=1}^{\infty} S_i$. Set also $n_i = [G_i : G_{i-1}]$ for $i \geq 2$. It is convenient to set $n_1 = 1$. The products $\prod_{i=1}^{k} n_i$ tend to infinity with k.

By definition we have $n_i\gamma_i \in G_{i-1}$, and the image of γ_i is an element of order n_i in G_i/G_{i-1}. For each $i \geq 1$, let r_i be the positive rational number such that $r_i\gamma_1, r_i\gamma_2, \dots, r_i\gamma_i$ generate the group $\mathbf{Z}$. Because the semigroup $r_{i-1}S_{i-1}$ generates $\mathbf{Z}$ as a group, it has a finite complement (see [13, Thm. 82]) and some positive multiple of $r_{i-1}n_i\gamma_i$ is contained in it. Thus, we know that there exists a smallest integer s_i such that $s_i\gamma_i \in S_{i-1}$ and that s_i is an integral multiple of n_i.

Definition 4.1. Let S be a well-ordered subsemigroup of the semigroup of positive elements of a totally ordered abelian group of finite rank. Let $(\gamma_1, \dots, \gamma_i, \dots)$ be a minimal system of generators of S indexed by some ordinal α, and for each ordinal $\beta \leq \alpha$ define S_β to be the semigroup generated by the $(\gamma_i)_{i\leq\beta}$. Given an integer d, we say that S has *stable asymptotic embedding dimension* $\leq d$ if, for each $\beta < \alpha$, the semigroup S_β is isomorphic to the semigroup of values taken by a valuation on an equicharacteristic Noetherian local domain of embedding dimension $\leq d$ whose residue field is algebraically closed. We say that S has *stable embedding dimension* $\leq d$ if S is isomorphic to the semigroup of values of a valuation on an equicharacteristic Noetherian local domain of embedding dimension $\leq d$ whose residue field is algebraically closed.

Proposition 4.2. *Let S be a well-ordered subsemigroup of $\mathbf{Q}_+$ that is not isomorphic to* $\mathbf{N}$ *and whose minimal system of generators* $(\gamma_1, \dots, \gamma_i, \dots)$ *is of ordinal type $\leq \omega$. Then the following statements are equivalent.*

(i) *For each $i \geq 2$, we have $s_i = n_i$ and $\gamma_i > s_{i-1}\gamma_{i-1}$.*
(ii) *The stable asymptotic embedding dimension of S is* 2.
(iii) *The stable embedding dimension of S is* 2.

Proof. The conditions of (i) are known to be equivalent to the fact that each semigroup S_i is the semigroup of values of the natural valuation of a plane branch, which is of embedding dimension 2 since $S \neq \mathbf{N}$ (see [16, Apx.] for the characteristic-0 case; see [2] on the extension to positive characteristic). This shows that (i) is equivalent to (ii).

Given a sequence of γ_i satisfying (i), we can associate to it a sequence of key polynomials (SKP) as in [7, Chap. 2, Def. 2.1] over any algebraically closed field K—that is, a sequence $P_0 = x, P_1 = y, \dots, P_i, \dots$ of polynomials in $K[x, y]$ such that the conditions $\nu(P_i) = \gamma_i$ for all i determine a unique valuation ν of the regular local ring $K[x, y]_{(x,y)}$ or, if the sequence of γ_i is finite, of a one-dimensional quotient $K[x, y]_{(x,y)}/(Q(x, y))$, which is of embedding dimension 2 since $S \neq \mathbf{N}$. The semigroup of values of ν is the semigroup generated by the γ_i (see [7, Thm. 2.28]). This shows that (i) implies (iii). Finally, if the semigroup S comes from a Noetherian local ring of embedding dimension 2, then we may assume—since, by [15, Sec. 5], the semigroup does not change under m-adic completion for valuations of rank 1—that this ring corresponds either to a branch or to a two-dimensional complete equicharacteristic regular local ring. If S is the semigroup of values of a plane branch then condition (i) is satisfied (as we have already seen); if it comes from a valuation ν of $K[[x, y]]$, then by [7, Thm. 2.29] there exists a SKP associated to ν and so again condition (i) is satisfied. We remark that [7] is written over the complex numbers, but the results of its Chapter 2 are valid in any characteristic. □

REMARK. One may ask whether a subsemigroup of $\mathbf{Q}_+$ of ordinal type ω is always of bounded stable asymptotic embedding dimension or of bounded stable embedding dimension.

5. The Semigroups of Noetherian Rings Are Not Always Rationally Finitely Generated

Using the results of [7, Chap. 2], one can check that for the semigroup $S = \langle\gamma_1, \dots, \gamma_i, \dots\rangle$ of a valuation of the ring $K[x, y]_{(x,y)}$ there always exists a finite set of generators $\gamma_1, \dots, \gamma_\ell$ that rationally generate the semigroup S in the sense that, for any generator γ_j, there is a positive integer s_j such that $s_j\gamma_j \in \langle\gamma_1, \dots, \gamma_\ell\rangle$. This does not hold for polynomial rings of dimension ≥ 3, as is shown by the following example (taken from [15]).

Let us give $\mathbf{Z}^2$ the lexicographic order and consider the field $K((t^{\mathbf{Z}^2_{\text{lex}}}))$ endowed with the t-adic valuation with values in $\mathbf{Z}^2_{\text{lex}}$. We denote by $K[[t^{\mathbf{Z}^2_+}]]$ the corresponding valuation ring. Choose a sequence of pairs of positive integers $(a_i, b_i)_{i\geq 3}$ and a sequence of elements $(\lambda_i \in K^*)_{i\geq 3}$ such that $b_{i+1} > b_i$, the series $\sum_{i\geq 3} \lambda_i u_2^{b_i}$ is not algebraic over $K[u_2]$, and the ratios $(a_{i+1}-a_i)/b_{i+1}$ are positive and increase strictly with i. Let R_0 be the K-subalgebra of $K[[t^{\mathbf{Z}^2_+}]]$ generated by

$$u_1 = t^{(0,1)}, \quad u_2 = t^{(1,0)}, \quad u_3 = \sum_{i\geq 3} \lambda_i u_1^{-a_i} u_2^{b_i}.$$

There cannot be an algebraic relation between u_1, u_2, and u_3, so the ring $R_0 = K[u_1, u_2, u_3]$ is the polynomial ring in three variables and inherits the t-adic valuation of $K[[t^{\mathbf{Z}^2_+}]]$. One may check that this valuation extends to the localization $R = K[u_1, u_2, u_3]_{(u_1,u_2,u_3)}$; it is a rational valuation of rank 2 and of rational rank 2. Let us compute the semigroup S of the values that ν takes on R. We have $\gamma_1 = (0, 1)$, $\gamma_2 = (1, 0)$, and $\gamma_3 = (b_3, -a_3) \in S$. Set $S_3 = \langle\gamma_1, \gamma_2, \gamma_3\rangle$. Then $u_1^{a_3} u_3 - \lambda_3 u_2^{b_3} = \sum_{i\geq 4} \lambda_i u_1^{a_3-a_i} u_2^{b_i} \in R$ and so $\gamma_4 = (b_4, a_3 - a_4)$ is in S. It is easy to deduce from our assumptions that *no multiple of* γ_4 *is in* S_3 and that γ_4 is the smallest element of S that is not in S_3. We set $u_4 = u_1^{a_3} u_3 - \lambda_3 u_2^{b_3}$ and continue in the same manner: $u_1^{a_4-a_3} u_4 - \lambda_4 u_2^{b_4} = u_5, \dots, u_1^{a_i-a_{i-1}} u_i - \lambda_i u_2^{b_i} = u_{i+1}, \dots$ with the generators $\gamma_i = \nu(u_i) = (b_i, a_{i-1} - a_i)$ for $i \geq 4$. Finally, we obtain

$$S = \langle\gamma_1, \gamma_2, \dots, \gamma_i, \dots\rangle;$$

the initial forms of the u_i constitute a minimal system of generators of the graded K-algebra $\operatorname{gr}_\nu R$, and the equations (setting $a_2 = 0$)

$$u_1^{a_i-a_{i-1}} u_i - \lambda_i u_2^{b_i} = u_{i+1}, \quad i \geq 3,$$

describe R_0 as a quotient of $K[(u_i)_{i\geq 1}]$. It is clear that from these equations we can reconstruct the value of u_3 as a function of u_1 and u_2 by (infinite) elimination. The binomial equations defining $\operatorname{gr}_\nu R = \operatorname{gr}_\nu R_0$ as a quotient of $K[(U_i)_{i\geq 1}]$ are

$$U_1^{a_i-a_{i-1}}U_i - \lambda_i U_2^{b_i} = 0, \quad i \geq 3,$$

which shows that all the U_i for $i \geq 3$ are rationally dependent on U_1, U_2. From our assumption on the growth of the ratios we see, moreover, that *no multiple of* γ_i is in $S_{i-1} = \langle \gamma_1, \ldots, \gamma_{i-1} \rangle$. In fact, γ_i is outside of the cone with vertex 0 that is generated by S_{i-1} in $\mathbf{R}^2$.

6. A Criterion for a Series $\underline{z}$ to Be Transcendental

In this section we will use the following notation. Let K be a field and Γ a totally ordered abelian group. Let $K((t^\Gamma))$ be the field of formal power series with a well-ordered set of exponents in Γ and coefficients in K. Let ν be the t-adic valuation of $K((t^\Gamma))$. Suppose $R \subset K((t^\Gamma))$ is a local ring that is essentially of finite type over K (a localization of a finitely generated K-algebra) and suppose ν dominates R. Let $d = \dim(R)$. Write $R = A_P$, where $A \subset R_\nu$ is of finite type over K and where P is the center of ν on A.

By Noether's normalization theorem (see [17, Chap. VIII, Sec. 7, Thm. 24]) there exist $x_1, \ldots, x_d \in A$, algebraically independent over K, such that A is a finite module over the polynomial ring $B = K[x_1, \ldots, x_d]$. Hence there exist $b_1, \ldots, b_r \in A$ for some finite r such that $A = Bb_1 + \cdots + Bb_r$.

Let $A[z]$ be a polynomial ring over A. For $n \in \mathbf{N}$, define a finite-dimensional K-vector space D_n by

$$D_n = \{f \in A[z] \mid f = f_1 b_1 + \cdots + f_r b_r, \\ \text{where } f_1, \ldots, f_r \in K[x_1, \ldots, x_d, z] \text{ have total degree} \leq n\}.$$

LEMMA 6.1. *Suppose that* $w \in K((t^\Gamma))$ *has positive value and* $n \in \mathbf{N}$. *Then the set of values*

$$E_n = \{\nu(f(w)) \mid f \in D_n \text{ and } f(w) \neq 0\}$$

is finite.

Proof. For $\tau \in \Gamma_+$, let

$$C_\tau = \{f \in D_n \mid \nu(f(w)) \geq \tau\};$$

C_τ is a K-subspace of D_n. Since $C_{\tau_1} \subset C_{\tau_2}$ if $\tau_2 \leq \tau_1$, it follows that E_n must be a finite set. □

LEMMA 6.2. *Suppose that* $w \in K((t^\Gamma))$ *has positive value. Let*

$$\tau = \max\{\nu(f(w)) \mid f \in D_n \text{ and } f(w) \neq 0\}.$$

Choose $\lambda \in \Gamma$ *such that* $\lambda > \tau$ *and choose* $h \in K((t^\Gamma))$ *such that* $\nu(h) = \lambda$.

Suppose that $0 \neq f \in D_n$. *Then* $f(w+h) \neq 0$.

Proof. Suppose that $0 \neq f \in D_n$. Let $m = \deg_z(f)$. We have $0 < m \leq n$ (the case $m = 0$ is trivial). Write

$$f = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0,$$

where $a_m \neq 0$ and each a_i has the expression

$$a_i = c_{i1}b_1 + \cdots + c_{ir}b_r;$$

here $c_{ij} \in K[x_1, \ldots, x_d]$ is a polynomial of degree $\leq n$ for all i, j. Substituting $w + h$ for z yields

$$f(w+h) = h^m d_m(w) + h^{m-1}d_{m-1}(w) + \cdots + d_0(w),$$

where $d_i(z) \in D_n$ for all i and $d_m(z) = a_m$, so that $h^m d_m(w) \neq 0$.

Suppose that $i < j$, $h^i d_i(w) \neq 0$, $h^j d_j(w) \neq 0$, and $\nu(h^i d_i(w)) = \nu(h^j d_j(w))$. Then

$$i\lambda + \nu(d_i(w)) = j\lambda + \nu(d_j(w)),$$

which yields

$$(j-i)\lambda = \nu(d_i(w)) - \nu(d_j(w)).$$

But

$$\nu(d_i(w)) - \nu(d_j(w)) \leq \tau < \lambda \leq (j-i)\lambda,$$

a contradiction. Thus all nonzero terms $h^i d_i(w)$ of $f(w+h)$ have distinct values. Since at least one of these terms was shown to be nonzero, it follows that $f(w+h)$ has finite value and so $f(w+h) \neq 0$. □

THEOREM 6.3. *Suppose that K is a field, Γ is a totally ordered abelian group, and $R \subset K((t^\Gamma))$ is a local ring that is essentially of finite type over K and is dominated by the t-adic valuation ν of $K((t^\Gamma))$.*

Suppose that the $z_i \in K((t^\Gamma))$ are defined as follows. Let

$$\tau_1 = \max\{\nu(f) \mid f \in D_1 \text{ and } f(0) \neq 0\}. \tag{6}$$

Choose $\alpha_1 \in \Gamma_+$ with $\alpha_1 > \tau_1$ and $h_1 \in K((t^\Gamma))$ such that $\nu(h_1) = \alpha_1$. Set $z_1 = h_1$.

Inductively define $\alpha_i \in \Gamma_+$, $h_i \in K((t^\Gamma))$, and $z_i = z_{i-1} + h_i$ with $\nu(h_i) = \alpha_i$ for $2 \leq i$ so that, if

$$\tau_i = \max\{\nu(f(z_{i-1})) \mid f \in D_i \text{ and } f(z_{i-1}) \neq 0\}, \tag{7}$$

then

$$\alpha_i > \tau_i.$$

Then $\underline{z} = \lim_{i \mapsto \infty} z_i \in K((t^\Gamma))$ is transcendental over the quotient field L of R.

Proof. Because $\{\alpha_i\}$ is an increasing sequence in Γ with $\nu(h_j - h_i) = \alpha_{i+1}$ for $j > i$, the limit $\underline{z} = \lim_{i \mapsto \infty} z_i$ exists in $K((t^\Gamma))$.

Assume that $\underline{z}$ is not transcendental over L. Then there exists a nonzero polynomial $g(z) \in A[z]$ such that $g(\underline{z}) = 0$. Let $m = \deg_z(g) \geq 1$. Expand

$$g(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_0$$

with $a_i \in A$ for all i. Each a_i has an expansion

$$a_i = c_{i1}b_1 + \cdots + c_{ir}b_r,$$

where $c_{ij} \in K[x_1, \ldots, x_d]$ for all i, j. Let $n \in \mathbf{N}$ be such that $n \geq m$ and $n \geq \deg(c_{ij})$ for $1 \leq i \leq m$ and $1 \leq j \leq r$.

By our construction, we have

$$z_n = z_{n-1} + h_n \quad \text{with} \quad \nu(h_n) = \alpha_n > \tau_n.$$

By Lemma 6.2, $g(z_n) \neq 0$. In $K((t^\Gamma))$ we compute

$$g(\underline{z}) = g(z_n + \underline{z} - z_n) = g(z_n) + (\underline{z} - z_n)e$$

with $\nu(e) \geq 0$. Then

$$\nu(\underline{z} - z_n) = \nu(h_{n+1}) = \alpha_{n+1} > \tau_{n+1} \geq \nu(g(z_n)).$$

Hence $\nu(g(\underline{z})) \leq \tau_{n+1} < \infty$ and so $g(\underline{z}) \neq 0$, a contradiction. □

COROLLARY 6.4. *Suppose that K is a field, Γ is a totally ordered abelian group, and $R \subset K((t^\Gamma))$ is a local ring that is essentially of finite type over K and is dominated by the t-adic valuation ν of $K((t^\Gamma))$. Then there exists a $\underline{z} \in K((t^\Gamma))$ such that $\underline{z}$ is transcendental over the quotient field of R.*

REMARK 6.5. The conclusions of the theorem may fail if R is Noetherian yet not essentially of finite type over K. A simple example is $\Gamma = \mathbf{Z}$ and $R = K[[t]]$, since $K((t^\Gamma))$ is the quotient field of R.

7. An Example Where All $\mathrm{gr}_{\mathcal{F}(\gamma_1)} \mathcal{P}_{\gamma_1}/\mathcal{P}^+_{\gamma_1}$ Are Not Finitely Generated ($\mathrm{gr}_{\bar{\nu}}\, R/P$)-Modules

Let K be an algebraically closed field, and let $K(x, y)$ be a two-dimensional rational function field over K. Let $A = K[x, y]_{(x,y)}$. In [17, Chap. VI, Ex. 3], a construction is given of a valuation $\bar{\nu}$ of $K(x, y)$ that dominates A and such that the value group of $\bar{\nu}$ is $\mathbf{Q}$. By [9, Thm. 6], since K is algebraically closed and $\mathbf{Q}$ is divisible, there is an embedding $K(x, y) \subset K((t^{\mathbf{Q}}))$ of K algebras, where $K((t^{\mathbf{Q}}))$ is the formal power series field with well-ordered set of exponents in $\mathbf{Q}$ and coefficients in K.

Let $\mathbf{Q}_+$ denote the positive rational numbers. Let

$$F_0 = \{\bar{\nu}(f) \mid f \in A \text{ and } f \neq 0\} \tag{8}$$

be the semigroup of the valuation $\bar{\nu}$ on A. We will use the criterion of Theorem 6.3 to construct a limit $\underline{z} = \lim_{i\to\infty} z_i$ in $K((t^{\mathbf{Q}}))$ that is transcendental over the quotient field of A.

Let z be a transcendental element over $K[x, y]$, and let

$$D_i = \{f \in K[x, y, z] \mid \text{the total degree of } f \text{ is} \leq i\}.$$

In our construction, we inductively define $z_i \in K((t^{\mathbf{Q}}))$. Then, for $n \in \mathbf{N}$ and $F_0 = \bar{\nu}(A \setminus \{0\})$, we may define F_0-modules M_i^n by

$$M_i^n = \{\nu(a_0 + a_1 z_i + \cdots + a_n z_i^n) \mid a_0, \ldots, a_n \in K[x, y]\}.$$

Let

$$A_i = A[z_i]_{(x,y,z_i)}.$$

The local ring A_i is dominated by $\bar{\nu}$, so the semigroup Γ_i of $\bar{\nu}$ on A_i is topologically discrete. It follows that there exist arbitrarily large elements of $\mathbf{Q}_+$ that are not in Γ_i.

We first choose $\lambda_1 \in \mathbf{Q}_+$ such that $\lambda_1 \notin F_0$ and $\lambda_1 > \tau_1$, where τ_1 is defined by (6). Set $\alpha_1 = \lambda_1$, and choose $f_1, g_1 \in K[x,y]$ such that

$$\bar{\nu}\left(\frac{f_1}{g_1}\right) = \alpha_1.$$

We then inductively construct $\lambda_i \in \mathbf{Q}_+$, $\alpha_i \in \mathbf{Q}_+$, $f_i, g_i \in K[x,y]$, and $\tau_i \in \mathbf{Q}_+$ such that $\lambda_i \notin \Gamma_{i-1}$, τ_i is defined by (7), and

$$\lambda_i > \max\{\lambda_{i-1} + \bar{\nu}(g_1 \cdots g_{i-1}), \tau_i + \bar{\nu}(g_1 \cdots g_{i-1})\}.$$

Define

$$\alpha_i = \lambda_i - \bar{\nu}(g_1 \cdots g_{i-1})$$

and choose $f_i, g_i \in K[x,y]$ so that

$$\bar{\nu}\left(\frac{f_i}{g_i}\right) = \alpha_i.$$

Let

$$z_i = z_{i-1} + \frac{f_i}{g_i}.$$

The resulting series $\underline{z} = \lim_{i\to\infty} z_i$ is transcendental over $K(x,y)$ by Theorem 6.3, since $\alpha_i > \tau_i$ for all $i > 1$.

Let $B = A[\underline{z}]_{(x,y,\underline{z})}$. Since $\underline{z} \in K((t^{\mathbf{Q}}))$ is transcendental over $K(x,y)$, the embedding $A \subset K((t^{\mathbf{Q}}))$ that appears at the beginning of the section extends to an embedding $B \subset K((t^{\mathbf{Q}}))$; hence the three-dimensional local ring B is dominated by the t-adic valuation $\bar{\nu}$ of $K((t^{\mathbf{Q}}))$.

LEMMA 7.1. (a) *With the preceding notation, for $n \in \mathbf{N}$ define an F_0-module*

$$T^n = \{\bar{\nu}(a_0 + a_1\underline{z} + \cdots + a_n\underline{z}^n) \mid a_0, a_1, \ldots, a_n \in K[x,y]\}.$$

Then, for all $n > 0$, T^n is not finitely generated as an F_0-module.

(b) *With the notation just introduced, let $T^\infty = \bar{\nu}(B \setminus \{0\})$ be the semigroup of the valuation $\bar{\nu}$ on B. Then T^∞ is not a finitely generated F_0-module.*

Proof. Suppose that $n \geq 1$. We will show that T^n is not finitely generated as an F_0-module.

We compute

$$\bar{\nu}(\underline{z}) = \bar{\nu}\left(\frac{f_1}{g_1}\right) = \lambda_1$$

and, for $i \geq 2$,

$$\begin{aligned}\bar{\nu}(g_1 \cdots g_{i-1}\underline{z} &- (f_1g_2 \cdots g_{i-1} + f_2g_1g_3 \cdots g_{i-1} + \cdots + f_{i-1}g_1 \cdots g_{i-2}))\\ &= \bar{\nu}\left(g_1 \cdots g_{i-1}\frac{f_i}{g_i}\right)\\ &= \bar{\nu}\left(\frac{f_i}{g_i}\right) + \bar{\nu}(g_1 \cdots g_{i-1})\\ &= \lambda_i.\end{aligned}$$

Thus $\lambda_i \in T^n$ for all i.

The F_0-modules M_i^n introduced before are subsets of $\mathbf{Q}$ for all i. We compare the intersections of the M_i^n with various intervals $[0, \sigma)$ in $\mathbf{Q}$. Since $\bar{\nu}(z_1) = \alpha_1$, it follows that

$$M_1^n \cap [0, \alpha_1) = F_0 \cap [0, \alpha_1);$$

since

$$z_i = z_{i-1} + \frac{f_i}{g_i} \quad \text{with } \bar{\nu}\left(\frac{f_i}{g_i}\right) = \alpha_i,$$

we also have

$$M_i^n \cap [0, \alpha_i) = M_{i-1}^n \cap [0, \alpha_i)$$

for all $i \geq 2$. Moreover,

$$T^n \cap [0, \alpha_i) = M_i^n \cap [0, \alpha_i) \tag{9}$$

for $i \geq 1$.

Suppose that $n \geq 1$ and that T^n is a finitely generated F_0-module. We will derive a contradiction. With these assumptions, there exist $x_1, \ldots, x_m \in T^n$ such that every element $v \in T^n$ has an expression $v = y + x_j$ for some $y \in F_0$ and some x_j with $1 \leq j \leq m$.

There exists a positive integer l such that $x_j < \alpha_l$ for $1 \leq j \leq m$. Thus $x_1, \ldots, x_m \in M_l^n$ by (9). It follows that $T^n \subset M_l^n$, since M_l^n is an F_0-module. But $\lambda_{n+1} \notin M_l^n$ by our construction, since $M_l^n \subset \Gamma_l$. This gives a contradiction, because we have already shown that $\lambda_{l+1} \in T^n$.

The proof that $T^\infty = \bigcup_{n=0}^{\infty} T^n$ is not a finitely generated F_0-module is similar. □

PROPOSITION 7.2. *Suppose K is an algebraically closed field and $K(x, y, u, v)$ is a rational function field in four variables. Then there exists a rank-2 valuation $\nu = \nu_1 \circ \bar{\nu}$ of $K[x, y, u, v]$, with value group $\mathbf{Z} \times \mathbf{Q}$ with the lex order, that dominates the regular local ring*

$$R = K[x, y, u, v]_{(x,y,u,v)}$$

with $R/P \cong K[x, y]_{(x,y)}$, where P is the center of ν_1 on R, such that the associated graded module

$$\mathrm{gr}_{\mathcal{F}(n)} \mathcal{P}_n/\mathcal{P}_n^+$$

is not a finitely generated $(\mathrm{gr}_{\bar{\nu}} R/P)$-module for all positive integers n.

Proof. Since the $\underline{z}$ we have just constructed is transcendental over $K(x, y)$, the association $z \to \underline{z}$ defines an embedding of K algebras $K(x, y, z) \to K((t^{\mathbf{Q}}))$ that extends our embedding $K(x, y) \to K((t^{\mathbf{Q}}))$. We identify $\bar{\nu}$ with the induced valuation on $K(x, y, z)$, which by our construction has value group $\mathbf{Q}$ and residue field K.

Let R be the localization

$$R = K[x, y, u, v]_{(x,y,u,v)}$$

of a polynomial ring in four variables. Let ν_1 be the (u, v)-adic valuation of R. The valuation ring of the discrete rank-1 valuation ν_1 is

$$R_{\nu_1} = K[x, y, u, v/u]_{(u)}.$$

The residue field of R_{ν_1} is the following rational function field in three variables:

$$R_{\nu_1}/m_{\nu_1} = K(x, y, z),$$

where $z = v/u$. Let ν be the composite valuation $\nu_1 \circ \bar{\nu}$ on $K(x, y, u, v)$.

For $i \in \mathbf{N}$, we have

$$F_i = \{\nu(f) \mid f \in R \text{ and } \nu_1(f) = i\}.$$

Here F_0 is the semigroup $F_0 = \bar{\nu}(R/P \setminus \{0\})$ and the F_i are F_0-modules for all i.

We have that $m_{\nu_1} \cap R = (u, v)$ and so $F_0 = \Gamma_0$, the semigroup of (8). From our construction of $\bar{\nu}$, it follows that F_n is isomorphic to T^n as a Γ_0-module. Thus, for all $n \geq 1$, F_n is not finitely generated as an F_0-module.

By Proposition 1.1,

$$\mathrm{gr}_{\mathcal{F}(n)} \mathcal{P}_n/\mathcal{P}_n^+$$

is not a finitely generated $(\mathrm{gr}_{\bar{\nu}} R/P)$-module for all positive integers n. □

REMARK 7.3. In the example of Proposition 7.2, the residue field R_{ν_1}/m_{ν_1} is transcendental over the quotient field of R/P, a fact that is used in the construction. In Proposition 8.4 (to follow), an example is given where R_{ν_1}/m_{ν_1} is equal to the quotient field of R/P.

REMARK 7.4. We can easily construct a series $\underline{z} \in K((t^{\mathbf{Q}}))$ such that the modules T^n and T^∞ of the conclusions of Lemma 7.1 are all finitely generated Γ_0-modules, and thus the modules

$$\mathrm{gr}_{\mathcal{F}(n)} \mathcal{P}_n/\mathcal{P}_n^+$$

are finitely generated $(\mathrm{gr}_{\bar{\nu}} R/P)$-modules for all positive integers n. To make the construction, just take $\underline{z} \in K[[x, y]]$ to be any transcendental series.

8. An Example with No Residue Field Extension

Suppose that K is an algebraically closed field and $A = K[x, y]_{(x,y)}$. We will define a valuation $\bar{\nu}$ on $L = K(x, y)$ that dominates A and has value group $\bigcup_{i=0}^{\infty}(1/2^i)\mathbf{Z}$.

Define $\bar{\beta}_0 = 1$ and $\bar{\beta}_{i+1} = 2\bar{\beta}_i + 1/2^{i+1}$ for $i \geq 0$. We have

$$\bar{\beta}_i = \frac{1}{3}\left(2^{i+2} - \frac{1}{2^i}\right)$$

for $i \geq 0$ and

$$2\bar{\beta}_i = \bar{\beta}_{i-1} + 2^{i+1}\bar{\beta}_0$$

for $i \geq 1$.

Define groups

$$\Gamma_i = \sum_{j=0}^{i} \mathbf{Z}\bar{\beta}_j = \frac{1}{2^i}\mathbf{Z}$$

for $i \geq 0$. For $i \geq 1$, let

$$x_i = \frac{2^{i+1}\bar{\beta}_{i-1} + 1}{2^{i-1}}$$

and let $m_i = 2$. Because $2^{i+1}\bar{\beta}_{i-1}$ is an even integer for all i, it follows that x_i has order $m_i = 2$ in $\Gamma_{i-1}/m_i\Gamma_{i-1}$ and that $\bar{\beta}_i = x_i/m_i$.

By our construction, for $i \geq 1$ we have $\bar{\beta}_{i+1} > m_i\bar{\beta}_i$. By the irreducibility criterion of [3], [6, Rem. 7.17], or [7, Thm. 2.22], there exists a valuation $\bar{\nu}$ of L dominating A and a (minimal) generating sequence $P_0, P_1, \ldots, P_i, \ldots$ for $\bar{\nu}$ in $K[x, y]$ of the form

$$\begin{aligned} P_0 &= x, \\ P_1 &= y, \\ P_2 &= y^2 - x^5, \\ P_3 &= P_2^2 - x^8 y, \\ &\vdots \\ P_{i+1} &= P_i^2 - P_0^{2^{i+1}} P_{i-1}, \\ &\vdots \end{aligned}$$

with $\bar{\beta}_i = \nu(P_i)$ for all i. The semigroup M_0 of $\bar{\nu}$ on A is

$$M_0 = \sum_{i=0}^{\infty} \mathbf{N}\bar{\beta}_i.$$

Let $z = y/x \in L$, and for $n \in \mathbf{N}$ define

$$W_n = \{a_0 + a_1 z + \cdots + a_n z^n \mid a_0, \ldots, a_n \in K[x, y]\}.$$

Define M_0-modules M_n by

$$M_n = \{\bar{\nu}(f) \mid 0 \neq f \in W_n\}.$$

LEMMA 8.1. *For all $i \geq 0$ we have the expression $P_i = x^i h_i$, where $h_i \in W_i$.*

Proof. The statement is clear for $i = 0$ and $i = 1$. Suppose, by induction, that the statement is true for $j \leq i$, so that $P_j = x^j h_j$ with $h_j \in W_j$ for $j \leq i$. Write

$$P_{i+1} = P_i^2 - P_0^{2^{i+1}} P_{i-1}.$$

We have that

$$P_i^2 = x^{i+1}\left[\left(\frac{P_i}{x^i}\right)\left(\frac{P_i}{x^i}\right)x^{i-1}\right]$$

with

$$\left(\frac{P_i}{x^i}\right)\left(\frac{P_i}{x^i}\right)x^{i-1} \in W_{2i-(i-1)} = W_{i+1}.$$

Furthermore,

$$P_0^{2^{i+1}} P_{i-1} = x^{2^{i+1}} x^{i-1} h_{i-1} \in x^{i+1} W_{i-1} \subset x^{i+1} W_{i+1}.$$

Thus $P_{i+1} = x^{i+1} h_{i+1}$ with $h_{i+1} \in W_{i+1}$. □

For $n \in \mathbf{N}$, let

$$U_n = \left\{ \lambda \in M_n \mid \lambda = \sum_{j=0}^{n-1} l_j \bar{\beta}_j \text{ for some } l_i \in \mathbf{Z} \right\}.$$

LEMMA 8.2. *For $n \geq 1$,*

$$M_n = U_n \cup \left(\bigcup_{j \geq n} ((\bar{\beta}_j - n) + M_0) \right).$$

Proof. That $P_j/x^j \in W_j$ for all $j \geq 0$ implies

$$\frac{P_j}{x^n} = \left(\frac{P_j}{x^j} \right) x^{j-n} \in W_n \quad \text{for} \quad j \geq n.$$

Thus $\bar{\beta}_j - n \in M_n$ for $j \geq n$.

Now suppose that $\lambda \in M_n$, where $\lambda = \bar{\nu}(a_0 + a_1 z + \cdots + a_n z^n)$ for some $a_0, \ldots, a_n \in K[x, y]$. Set $\tau = \bar{\nu}(a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n) \in M_0$. We have $\lambda = \tau - n$, where $\tau = \sum l_j \bar{\beta}_j = \bar{\nu}\left(\prod P_j^{l_j}\right)$ for some $l_j \in \mathbf{N}$. Suppose $l_k \neq 0$ for some $k \geq n$. Then

$$\lambda = \bar{\nu}\left(\left(\prod_{j \neq k} P_j^{l_j} \right) P_k^{l_k - 1} \frac{P_k}{x^n} \right) = (\bar{\beta}_k - n) + \bar{\nu}\left(\left(\prod_{j \neq k} P_j^{l_j} \right) P_k^{l_k - 1} \right).$$

Now suppose that $l_k = 0$ for $k < n$. Then

$$\lambda = \sum_{j=1}^{n-1} l_j \bar{\beta}_j + (l_0 - n)\bar{\beta}_0 \in U_n. \qquad \square$$

LEMMA 8.3. *Suppose that $n \geq 1$. Then M_n is not a finitely generated M_0-module.*

Let $B = K[x, y/x]_{(x, y/x)}$; then B is a regular local ring that birationally dominates $A = K[x, y]_{(x,y)}$. Let $M_\infty = \bar{\nu}(B \setminus \{0\})$. Then M_∞ is not a finitely generated module over $\Gamma_0 = \bar{\nu}(A \setminus \{0\})$.

Proof. For $i \geq 0$, define Ψ_i to be the M_0-module generated by U_n and $\{\bar{\beta}_j - n \mid i \geq j \geq n\}$. For $i \geq n$, we have

$$\Psi_i = U_n \cup \left(\bigcup_{i \geq j \geq n} ((\bar{\beta}_j - n) + M_0) \right) \subset \frac{1}{2^i}\mathbf{N}.$$

Hence $\bar{\beta}_{i+1} - 1 \notin \Psi_i$ for $i \geq n$.

We will now show that M_n is not a finitely generated M_0-module. Suppose that M_n is finitely generated as an M_0-module. Then M_n is generated by a set $a_1, \ldots, a_r, b_1, \ldots, b_s$, where

$$a_i = (\bar{\beta}_{\sigma(i)} - n) + \lambda_i$$

with $\sigma(i) \geq n$ and with $\lambda_i \in M_0$ for $1 \leq i \leq r$ and $b_i \in U_n$ for $1 \leq i \leq s$.

Let $m = \max\{\sigma(i)\}$. Then $M_1 \subset \Psi_m$, which is impossible because $\bar{\beta}_{m+1} - n \notin \Psi_m$. The proof that M_∞ is not a finitely generated Γ_0-module is similar. $\square$

PROPOSITION 8.4. *Suppose K is an algebraically closed field and $K(x, y, u, v)$ is a rational function field in four variables. Suppose Γ is a totally ordered Abelian group and suppose $\alpha \in \Gamma$ is such that 1 and α are rationally independent and $1 < \alpha$.*

Then there exists a rank 2 valuation $\nu = \nu_1 \circ \bar{\nu}$ of $K(x, y, u, v)$, with value group

$$(\mathbf{Z} + \alpha\mathbf{Z}) \times \left(\bigcup_{i=0}^{\infty} \frac{1}{2^i}\mathbf{Z}\right)$$

in the lex order, that dominates the regular local ring $R = K[x, y, u, v]_{(x,y,u,v)}$ as follows.

(i) $R_{\nu_1}/m_{\nu_1} \cong (R/P)_P \cong K(x, y)$, *where P is the center of ν_1 on R.*
(ii) *The associated graded module*

$$\mathrm{gr}_{\mathcal{F}(n)}\, \mathcal{P}_n/\mathcal{P}_n^+$$

is not a finitely generated $(\mathrm{gr}_{\bar{\nu}}\, R/P)$*-module for $n \in \mathbf{N}$.*
(iii) *The associated graded module*

$$\mathrm{gr}_{\mathcal{F}(n\alpha)}\, \mathcal{P}_{n\alpha}/\mathcal{P}_{n\alpha}^+$$

is a finitely generated $(\mathrm{gr}_{\bar{\nu}}\, R/P)$*-module for $n \in \mathbf{N}$.*

Proof. We use the notation developed earlier in this section. Define a valuation ν_1 on the rational function field $L(u, v)$ in two variables by the embedding of L algebras

$$L(u, v) \to L((t^\Gamma))$$

induced by

$$u \mapsto t, \quad v \mapsto v(t) = \frac{y}{x}t + t^\alpha.$$

If $\nu = \nu_1 \circ \bar{\nu}$ is the composite valuation on $K(x, y, u, v)$, then ν dominates $R = K[x, y, u, v]_{(x,y,u,v)}$. The center of ν_1 on R is the prime ideal $P = (u, v)$. We have $L = (R/P)_P = R_{\nu_1}/m_{\nu_1}$ and $K = R_\nu/m_\nu$, proving part (i) of the proposition.

For $i \in \mathbf{N}$,

$$F_i = \{\nu(f) \mid f \in R \text{ and } \nu_1(f) = i\}.$$

Suppose that $f \in K[x, y, u, v]$. Expand

$$f = \sum a_{ij} u^i v^j$$

with $a_{ij} \in K[x, y]$. Then

$$f(t, v(t)) = a_{00} + \left(a_{10} + a_{01}\frac{y}{x}\right)t + (\text{higher-order terms in } t).$$

We see that $\nu_1(f) = 0$ if and only if $a_{00} \neq 0$. Thus $F_0 \cong M_0$ as semigroups. We also have $\nu_1(f) = 1$ if and only if $a_{00} = 0$ and $a_{10} + a_{01}(y/x) \neq 0$. Thus $F_1 \cong M_1$ as F_0-modules, so F_1 is not finitely generated as an F_0-module.

To show that $F_n \cong M_n$ as an F_0-module for all $n \geq 0$, we expand

$$f(t, v(t)) = \sum_{k=0}^{\infty} \sum_{i+j=k} a_{ij} t^i v(t)^j$$
$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \varphi_{jk} t^{(k-j)+j\alpha},$$

where

$$\varphi_{jk} = \sum_{i=j}^{k} a_{k-i,i} \binom{i}{j} \left(\frac{y}{x}\right)^{i-j}.$$

Now $(k_1 - j_1) + j_1\alpha = (k_2 - j_2) + j_2\alpha$ implies that $j_1 = j_2$ and $k_1 = k_2$. Hence

$$\nu_1(f) = \min\{(k - j) + j\alpha \mid \varphi_{jk} \neq 0\}.$$

We know, for fixed k, that $j_1 < j_2$ implies

$$(k - j_1) + j_1\alpha < (k - j_2) + j_2\alpha. \tag{10}$$

Suppose that $\nu_1(f) = n \in \mathbf{N}$. Then

$$f(t, v(t)) = \varphi_{0n} t^n + (\text{higher-order terms in } t)$$

with

$$\varphi_{0n} = a_{0n} + a_{n-1,1}\frac{y}{x} + \cdots + a_{0n}\left(\frac{y}{x}\right)^n \neq 0.$$

Moreover, by (10) we see that, for $n \in \mathbf{N}$ and $a_{n0}, a_{n-1,1}, \ldots, a_{0n} \in K[x, y]$,

$$\nu_1(a_{n0}u^n + a_{n-1,1}u^{n-1}v + \cdots + a_{0n}v^n) = n$$

if and only if

$$a_{n0} + a_{n-1,1}\frac{y}{x} + \cdots + a_{0n}\left(\frac{y}{x}\right)^n \neq 0.$$

Thus, for $n \in \mathbf{N}$,

$$F_n \cong \{\bar{\nu}(h) \mid h \in W_n \text{ and } h \neq 0\},$$

which is isomorphic to M_n as an M_0-module. Because M_n is not finitely generated as an M_0-module, Proposition 1.1 implies part (ii).

Let us now consider the polynomials

$$\Phi_{jk}(W) = \sum_{i=j}^{k} a_{k-i,i}\binom{i}{j} W^{i-j}$$

and remark that

$$\frac{\partial^j \Phi_{0k}(W)}{\partial W^j} = j!\,\Phi_{jk}(W).$$

For the series $f(t, v(t))$ to be of order $n\alpha$, all the φ_{jk} must be zero for $(k-j)+j\alpha < n\alpha$ while $\varphi_{nn} = a_{0n}$ must be nonzero. In particular, φ_{jn} must be zero for $j < n$.

In view of the equalities we have just seen, the n conditions $\varphi_{jn} = 0$ $(0 \leq j \leq n - 1)$ are equivalent to the fact that y/x is a root of order n of the polynomial

$\Phi_{0n}(W)$, so that $\Phi_{0n}(W) = a_{0n}(W - y/x)^n$. From this it follows that the $a_{n-j,j}$ for $j < n$ are determined and the only condition on a_{0n} is that it be divisible by x^n. The elements of $F_{n\alpha}$ coincide with the values of ν on $K[x, y]$ translated by n, so for each $n \in \mathbf{N}$ we have $F_{n\alpha} = n + F_0 \cong F_0$. Thus part (iii) also follows from Proposition 1.1. □

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