On objectivity and subjectivity in Mathematics.

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Plan

The moving power of mathematical invention is not reasoning but imagination. (W.R. Hamilton, 1805-1865)

Part 1: Mathematics is a human science

- The vestibular line
- The visual line

Part 2: Bourbaki and Thom

- Bourbaki
- Thom
- Conclusion

It has been suggested that I talk to you about René Thom and the Bourbaki group, with both of whom I have had scientific contact. It is a very interesting suggestion because they correspond to different modes of mathematical activity and yet have something in common which does not appears at first glance. But before I can describe that difference I think it is useful to reflect on the forces which animate the search for objectivity and its allies solid foundations and rigorous reasoning, and the search for meaning, an attribute indispensable for creativity and strongly associated to subjectivity.

My title is provocative because Mathematics is supposed to be, by its very nature, objective in the sense that its meaning is independent of the subject who studies it or makes use of it. This is interpreted by some philosophers as meaning that Mathematics exists independently of human understanding in a world of ideas of which we humans perceive only projections, or shadows. It is a simplified but all too common version of platonism with which I am not even sure Plato would agree.

Anyway, I hope to convince you that Mathematics is a human science, whose origin and growth are governed by an extremely complex and mostly unconscious perception of the world and by pulsions which are just as unconscious and force us to organize those perceptions.

The impression of objectivity is due to the fact that humans share this perceptual system and those pulsions. It is indeed independent of any one individual, but that does not make it superhuman.

What is new is that the cognitive sciences allow us to begin to get an idea of how the perceptual system works to create what one can call ”protomathematical objects”. I claim that such objects are actually the foundation upon which -with much elaboration- the meaning we give to mathematics is built.

Poincaré had the intuition of this:

> In summary, for each attitude of my body, my first finger determines a point and it is that, and only that, which defines a point in space.
> Henri Poincaré, *La Science et l'hypothèse*, Flammarion.

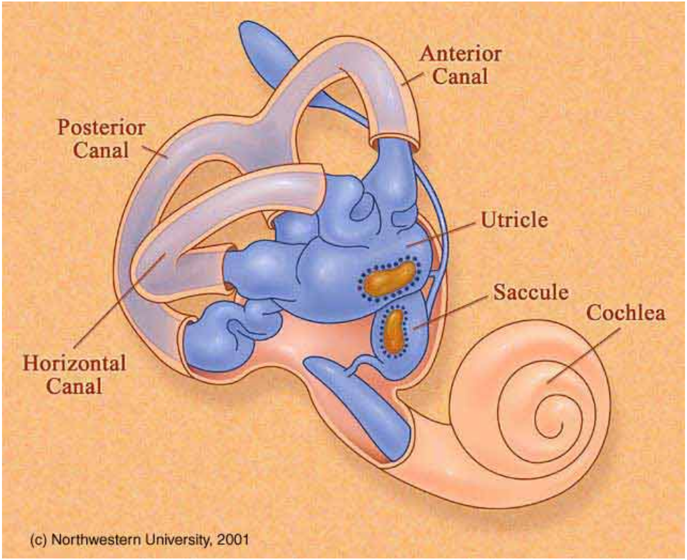
In other words, the (unconscious) tensions of the muscles which position the finger are a system of coordinates for (our perception of) space.

My favourite example is the protomathematical continuum which results from the extremely strong interaction in our perception/action system of the vestibular line, the visual line and the motor system.

The vestibular line

Our vestibular system (in the ear) is an inertial navigation system which detects accelerations, rotations, etc. with great accuracy. It is strongly connected to the muscular system in order, for example, for a biped to be able to react very quickly when it stumbles, thus creating a large acceleration of our head.

Moving at constant speed in a constant direction corresponds to a particular state of the assembly of neurons which manages the vestibular information.



The inner ear.

According to galilean relativity, the vestibular system sends no signal during the inertial motion. Let us call this particular state the vestibular line: Of course the head goes up and down, but that is compensated for. There are only two ways to measure progress along this line: the time elapsed during the motion, assuming we know the speed, and the number of steps.

The visual line

The optic nerve transmits the electric impulses from the retina to the visual areas of the brain at the back of the skull. The retina cells are already specialized and then the impulses are subjected to a filtering. There is a quite specific and extremely complex architecture of the neurons in visual area V1 which implies in particular that if a neuron detects in a certain direction of sight a segment with a given orientation, it excites the neurons in its neighborhood to reinforce the detection of a segment with a similar orientation.

This is a gross oversimplification but the end result is that our visual system can detect curves, contours, and especially lines, which correspond again to a special state of an assembly of neurons in the visual cortex. Note that the visual line has no orientation and by itself no measure of progress. There are also neurons which detect movement and speed.

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The Poincaré-Berthoz isomorphism

Professor Alain Berthoz conducted many experiments in his lab at the Collège de France to study the integration of visual, vestibular and muscular perceptions. It makes a very strong case for the idea that our unconscious perceptual system almost identifies the visual line and the vestibular line. In addition, visual perception is strongly coupled with muscular tension, which supports Poincaré's intuition. I call this the Poincaré-Berthoz isomorphism.

Of course it is not an isomorphism in the strict mathematical sense, but it permits transporting structure from one to the other. For example the natural orientation of the vestibular line, and its notion of distance, are carried over to the visual line, which is our model for the real line, and this allows us to think of it as parametrizing time.

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One cannot overestimate the consequences of the fact that the Poincaré-Berthoz isomorphism transports the continuity of motion to the continuity of the visual line.

Einstein said that one of his basic intuitions was to think of himself as moving along a ray of light.

The construction of the real line from the integers, then the rationals, then Cauchy completion, does not provide us with such a vision although it does provide a construction from set theory, considered as being objective and providing a foundation of truth for the statements concerning for example convergence of sequences, continuous functions (intermediate value theorem), etc. But does it provide a foundation of meaning?

For example, we can imagine without difficulty walking indefinitely on the vestibular line from where we are, but it much more difficult to imagine having walked indefinitely to arrive where we are. I think it took the invention of -1 as an operator one can iterate to imagine this. And it is probably also the origin of the notion of well ordered set, where there are no indefinitely decreasing sequences.

The vestibular line has its notion of boundary which when it is parametrized by time, is the instant and when it is parametrized by walking, is the end of motion, and the visual line too has an obvious notion of boundary. It is a fundamental intuition of Dedekind that the difference between the real line and a set of points given in bulk is that it is totally ordered, is divisible (one can cut intervals into parts), and is *made of boundaries* (Dedekind cuts).

In conclusion, the Poincaré-Berthoz isomorphism creates a protomathematical object blending the attributes of the visual and the vestibular line, which will serve as a source of meaning for many statements concerning sequences, arithmetic operations, parametrized curves, etc.
Of course the mathematicians learns by usage to give meaning to much more eleborate objects and statements, but I claim that at the bottom of it there are similar protomathematical objects and amazing properties of adaptation of our perceptual system to the world around us.
In short, Euclidean geometry would not exist if our visual system could not detect lines, angles and parallelism.

For example as mentioned above the ”parallel transport” of differential geometry is already cabled in the architecture of our visual system. We detect it in the world around us and this gives us the meaning of the abstract notion.

But this is not at all a reductionist discourse because I very strongly believe that the complexity of the physiological structures and dynamics which we would use to explicitly provide meaning to mathematical objects and statements are beyond human comprehension. But any progress in that direction would be fascinating.

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Let me now come to some of the largely unconscious pulsions which move us to create Mathematics and other sciences. I call it ”proto-thought”. I believe it is a very long list, of which I have only a few items, but here it is.

- Detecting structures, for example periodicities and repetitions. Most importantly, detecting boundaries.
- Capacity of creating mental images, to simplify and abstract independently of language.
- An obstinate search for causes and origins. Many problems are formulations of an unconscious : if A implies B, does B imply A?
- Comparing comparable objects *without consciously asking the question*. Which leads to consciously measuring and comparing lengths, surfaces, etc.

Our basic intuitions of space based on this kind of subconscious interiorization, through evolution, of our experience of the world, finds its limits, especially when some infinite process comes into play: think of Zeno's paradox, of the discussions about the actual or virtual existence of the infinite, and of Cantor discovering that there are as many points in a square or a cube as in a segment on the line: "I see it but I cannot believe it". A lot of Mathematics are born of such questionings and astonishments.

For example, the Greeks measured the relation between two lengths by a method called *Antiphairesis*: count how many times the smaller one goes into the larger one. This is an integer a_0. In general, there is a remainder r_0 which is smaller than the smaller length. Then count how many times the remainder goes into the smaller length and get a second integer a_1. In general, there is a remainder r_1, which is less than r_0, and continue like this: the ratio of the two lengths is expressed as a possibly infinite sequence of integers.

There is a lemma in Euclid which implies that for the diagonal of a square and its side, this sequence is $1, 2, 2, 2 \ldots$. This is a geometric lemma, using constructions of Euclidean geometry.

Hint: if s and d are the lengths of side and diagonal of your square, build a larger square of side $S = d + s$ with a side extending the diagonal of your square. Then compute the ratio of the lengths $\frac{D+S}{S}$ where D is the length of the large diagonal. You find that the antiphairesis keeps giving 2.

The fact that the sequence is infinite is equivalent to the irrationality of $\sqrt{2}$.

I believe that Mathematics develop in large part to compensate the failures of our perceptual grasp of the world, and because of the overwhelming desire to find causes. As I tried to explain above, it is much closer to our human nature than we usually believe.

For example, we have a good perceptual (or intuitive) feeling for distances, but our intuition of areas and volumes is poor. I believe the invention of the concept of area was a great mathematical moment, and a good part of Euclid's elements is devoted to considerations on area and volume, the hot mathematical topics of the time.

In Homer, the measure of the city of Troy is 10200 steps. Proclus (411-485) reports court cases of members of Greek communities who, in the first century A.D. decided to divide land equitably according to perimeter and had surprises at the time of harvest.

We note that the perimeter measures the length of the boundary of a plane domain. . .

24 centuries before the physicist Sokal derided the (mis)use of modern mathematical or physical concepts by some philosophers, Plato was making fun of the followers of Pythagoras, who put numbers everywhere, by giving in *The Republic* a farcical proof of the following statement:
The measure of the area of the image of the tyrant's pleasure is a perfect square.
Plato "proves" that it is 9.

More seriously, the astonishment of Cantor discovering that there are as many points in a square or a cube, etc, as in their side was obviously a booster, because of the more or less conscious need to find causes, for the developement of the axiomatic method and set theory.

We have a strong need to find a firm basis, undisputable axioms and undisputable rules of logical reasoning to establish facts, and especially those that defy intuition.

What I just described is *the axiomatic method.*

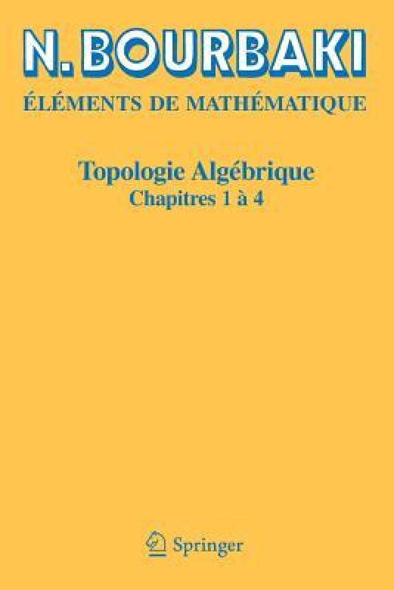
Let us now turn to Bourbaki and Thom.

Bourbaki started in the middle 1930's with a group of young mathematicians dissatisfied with the manuals from which they had to teach. They were especially dissatisfied with the proofs of Stokes' formula, which is an *n*-dimensional generalization of the formula

$$\int_a^b f'(t)dt = f(b) - f(a)$$

which computes the difference of the values of a function $f(t)$ on the boundary of an interval $[a.b]$ on the line from the values of its derivative on the interval (boundaries again!)

They were influenced by Hilbert's ideas on axiomatization. Youth and talent being what they are, and were, this dissatisfaction turned into an enterprise to rewrite most of Mathematics in a particular frame of mind, the exposition of the underlying *structures*.
This enterprise continues to this day, at its own pace.



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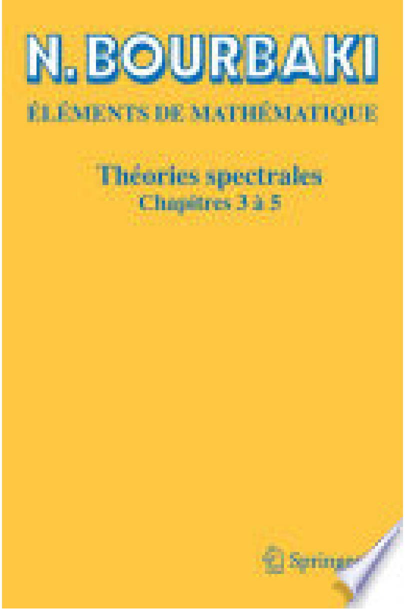

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Bourbaki at work, ca. 1982, perhaps discussing one of the preceding books.

Bourbaki adopted the axiomatic method of constructing Mathematics but went much further in taking as basic objects not the sets, to which which earlier mathematicians had painfully tried to give a firm axiomatic status, meeting paradoxes all too often, but the *structures* which, according to Bourbaki, exist independently of set theory.

The method of exposition we have chosen is axiomatic and abstract, and normally proceeds from the general to the particular. This choice has been dictated by by the main purpose of the treatise, which is to provide a solid foundation for the whole body of modern mathematics.

Nicolas Bourbaki, in the foreword to his books.

Bourbaki's goal is not to prove theorems but to provide mathematicians with a toolbox of clear and well founded definitions and results which help mathematicians attack difficult problems by relating them to well studied *structures*.

The historic example of such structure is that of group, which appears in so many fields of Mathematics, Physics, Chemistry, etc. Then there is the notion of topological space, then topological vector space, Lie group, and so on.

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Bourbaki's idea is that this will produce a considerable *economy of thought* because instead of trying to adapt some interesting ideas from one theory to the other it will suffice to recognize that the two theories share a common structure!

The common trait of these notions which we have designated under this name (of structures) is that they apply to sets of elements *whose nature is not specified*.

Nicolas Bourbaki in "L'architecture des Mathématiques" *Les grands courants de la pensée mathématique*, F. Le Lionnais éditeur, Paris 1948.

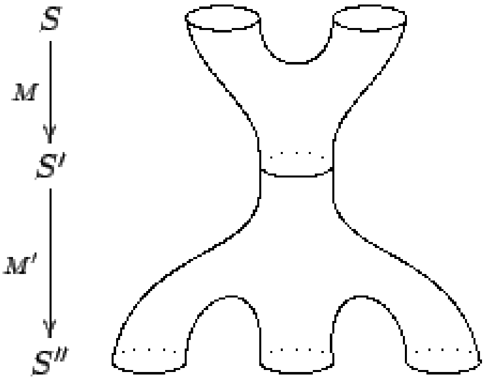
But Bourbaki is conscious that the tools do not suffice to produce Mathematics.

> ...one cannot insist too much on the role played in the mathematician's research by a particular intuition, which is not the vulgar intuition of the senses.
> It is rather a sort of direct divination (prior to any reasoning) of the normal behavior which he can expect from mathematical beings which a long interaction has made almost as familiar to him as the beimgs of the real world.

Nicolas Bourbaki in "L'architecture des Mathématiques" *Les grands courants de la pensée mathématique*, F. Le Lionnais éditeur, Paris 1948.

If the founders of Bourbaki were dissidents in their time, René Thom is also a dissident but in an entirely different manner and at a later age.

René Thom was born in 1923 and received the Fields medal in 1958 for his work on cobordism. This notion is based on the fact that for a variety (manifold) or a union of manifolds, being the boundary of some manifold of one more dimension imposes strong conditions.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After his Fields medal, Thom embarked on an extremely ambitious project of providing qualitative models for the discontinuous behavior which dynamical systems [1] depending on parameters can exhibit. This includes for example the (relatively) sudden changes in the shape of an embryo during its growth.

[1] system evolving under the action of some force, often the gradient of a potential which the system seeks to minimize

Thom's goal is very wide. He wants to provide geometric models (for him, to understand is to visualize) which help to understand how discontinuous changes in shape, in behavior occur. A basic intuition for him is that of a boundary.

Discontinuous changes (hence the name "Catastrophe theory") in the shape (hence the word "morphogenesis") or the behavior of a system depending on parameters occur when the parameters cross a certain boundary in the parameter space and he wants to provide "universal boundary shapes" which will appear in any parameter space depending of course on the nature of the system.

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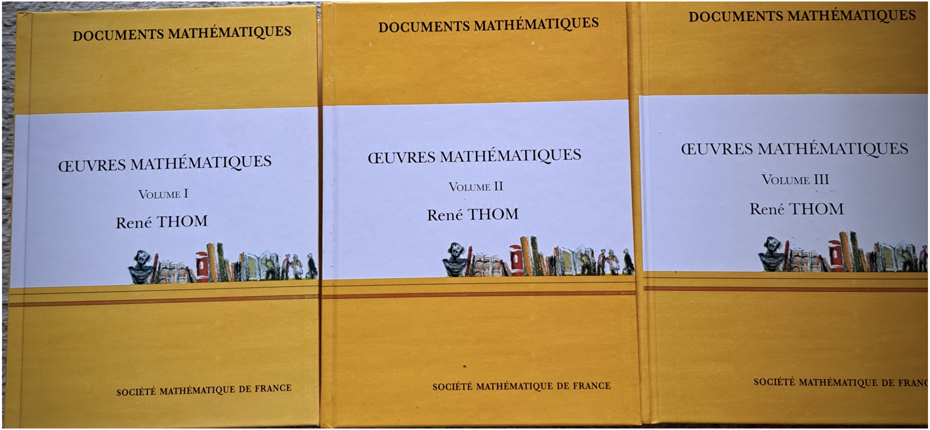
Another fundamental idea of Thom is that these discontinuous changes, or bifurcations, must appear in a *stable* manner in order to be observable. Here stable is a technical term.

This allowed him to give a classification of the stable families of gradient dynamical systems depending on at most four parameters and the corresponding bifurcation sets, or boundaries of the domains of parameters where no brutal change occurs.

One could say that for Thom what gives meaning to the behavior of a family of dynamical systems is the geometry of this boundary.



Volume III published in 2022.

René Thom

The essential idea of our theory that a certain understanding of the morphogenetic processes is possible without having recourse to special properties of the substrate of the shapes, or to the nature of the acting forces, may seem difficult to admit. . .

René Thom, in Chapter 1 of ”Mathematical models of morphogenesis” plus changes of scale in x, b, α allow us to rewrite this in the form

$$V = x^4 - bx^2 + \alpha x \quad (+ \text{const.})$$

(This is not exactly right : the new α, b are not quite the same as the old ones. But no harm will be done by abusing notation in this way.)

Equilibrium positions of the arch are given by

$$0 = \frac{\partial V}{\partial x} = 4x^3 - 2bx + \alpha .$$

Depending on (α, b) this cubic equation for x has 1 or 3 real roots. The graph M of x against (α, b) is a pleated surface, shown in figure 2, known as the *catastrophe manifold* .

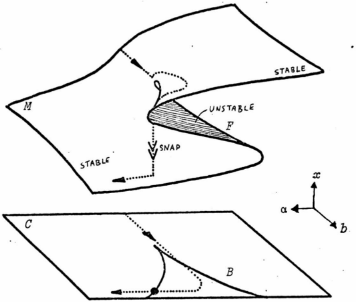


Figure 2.

The fold curve F occurs when

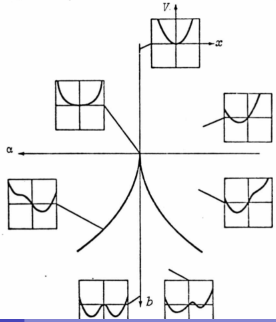
so is given parametrically by $(x, \alpha, b) = (\xi, 8\xi^3, 6\xi^2)$. This is the geometer's beloved *twisted cubic*. Its projection B on to (α, b)-space C is found by eliminating ξ, which yields

$$27\alpha^2 = 8b^3$$

a semicubical parabola with a cusp at the origin. The curve B is called the *bifurcation set*.

If (α, b) lies inside B then x can take three values of which the middle one represents *unstable* equilibrium; if outside, x can take only one. If we vary (α, b) along the path shown the value of x at first stays near 0, then increases sharply, decreases slightly, and falls in a dynamic snap to a negative value.

Considering this in terms of the potential V we see that the dynamic snap occurs when a minimum of V coalesces with a maximum and both disappear (figure 3).



A word on the relations between Thom and Bourbaki:

The collaborators of Bourbaki retire at age 50. Bourbaki recruits new members by inviting some promising young mathematician to one of his meetings, where redactions of new chapters are read and criticized.

The invitee is known as ”guinea pig” because the redactions are tested on him or her. Depending on the level of interest exhibited, the guinea pig is invited to become a member, or not. Thom was invited to such a meeting and . . . fell asleep during the reading.

Thom has sometimes criticized not the Bourbaki enterprise itself, but the abuses, for which Bourbaki bears little to no direct responsibility, which the ”structuralist” approach has generated in the teaching of Mathematics.
Jean-Pierre Serre, a Bourbaki member, helped Thom to put his thesis in shape. But the Bourbaki goal is definitely not Thom's cup of tea.

If one must choose between rigour and meaning, I shall unhesitatingly choose the latter.

René Thom

Conclusion

Bourbaki wants to define and study the abstract ”structures” which exist independently of individual mathematical theories which they organize, while Thom wants to study abstract ”morphologies” and morphogenetic processes independent of the substrates and the dynamics which they structure.

Is there not a common pulsion at work?

THANK YOU FOR YOUR ATTENTION