A viewpoint on resolution of singularities

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What is a resolution?

• A resolution of singularities of an algebraic variety X defined over a field k is a proper and birational map $Y \longrightarrow X$ which is defined over k and where Y is non singular.
• Often X is given as a closed subvariety of a non-singular algebraic variety W. An *embedded* resolution of singularities is then a birational and proper map $\pi\colon W' \longrightarrow W$ which is defined over k and where W' is non singular, the intersection with X of the locus W° where π is an isomorphism is dense in X and the closure in W' of $\pi^{-1}(X \cap W^\circ)$ (the *strict transform* of X) is non-singular.
This is a much stronger and much more useful result than resolution.

How is the existence of resolutions proved?

Hironaka proved in 1964 embedded resolution for algebraic varieties over a field of characteristic zero. The method is to prove by induction of the dimension the existence of sequences of blowing-ups of non singular subvarieties of successive ambient spaces, contained in the singular loci of the successive strict transforms of X, and which make an invariant closely related to the multiplicity decrease.

So far all attempts to extend this method to positive characteristic have failed, although there are remarkable results in dimension ≤ 3, which are very difficult to prove. (most recent: see Cossart-Piltant on the ArXiv).

What is our viewpoint?

Inspired by the example of complex plane branches we explore the idea that a given embedding of a singularity, say an embedding $X \subset \mathbf{A}^N(k)$ in affine space over a field k, may not be "comfortable" enough for the singularity X, and we should look for an embedding in a larger affine space where there are coordinates which are "appropriate" for X in the sense that its singularities can be resolved by one single proper and birational toric modification (i.e., essentially a birational map which is locally monomial in suitable coordinates, to be defined below) of the new affine ambient space.

Motivating example (Rebecca Goldin-B.T., 2000)

If k is algebraically closed of characteristic zero, and $(X, 0) \subset (\mathbf{A}^2(k), 0)$ is a plane branch with g characteristic Puiseux exponents, the branch can be re-embedded in $(\mathbf{A}^{g+1}(k), 0)$ in such a way that it is resolved by a single toric modification of the ambient space.

The non singular surface S in which $(X, 0)$ lies as a plane branch has a strict transform S' by this modification which is non singular, so we get an embedded resolution of $(X, 0)$ as a plane branch. The difference is that the induced map $S' \to S$ is not toric.

A toric reminder (1)

• Let $\check{\sigma} \subset \mathbf{R}^N$ be a convex cone. Let $M \simeq \mathbf{Z}^N$ denote the lattice of integral points. Then $\check{\sigma} \cap M$ is a semigroup.

A toric reminder (2)

• If the convex cone $\check{\sigma}$ is rational (generated as a convex cone by finitely many integral vectors), then $\check{\sigma} \cap M$ is finitely generated as a semigroup.

A toric reminder (3)

• So if k is a field, the semigroup algebra $k[\check{\sigma} \cap M]$ is a finitely generated algebra, quotient of a polynomial ring.

A toric reminder (4)

• This algebra is a polynomial ring in N variables if and only if the cone $\check{\sigma}$ is generated by vectors which form a basis of the integral lattice.

The algebra of blowing-up a point in the plane

Let k be a field. The polynomial ring $k[x, y]$ is the semigroup algebra over k of the semigroup $\mathbf{N}^2$ of integral points of the first quadrant of $\mathbf{R}^2$. Let $e_1 = (1, 0), e_2 = (0, 1)$. Our semigroup is $\mathbf{N}e_1 + \mathbf{N}e_2$. If we enlarge it to the semigroup $\Gamma_1 = \mathbf{N}e_1 + \mathbf{N}(e_2 - e_1)$ of integral points lying in the cone generated by e_1 and $e_2 - e_1$ we obtain a new polynomial algebra (since e_1 and $e_2 - e_1$ generate the integral lattice, $\mathbf{N}e_1 + \mathbf{N}(e_2 - e_1)$ is isomorphic to $\mathbf{N}^2$) and its semigroup algebra can be written $k[x, \frac{y}{x}]$. We could do the same thing with $\Gamma_2 = \mathbf{N}(e_1 - e_2) + \mathbf{N}e_2$ and obtain $k[\frac{x}{y}, y]$.

We set $k[\Gamma_1] = k[y_1, y_2]$, with $y_1 = x, y_2 = \frac{y}{x}$ and we see that we have defined a map $k[x, y] \to k[\Gamma_1],\ x = y_1, y = y_1 y_2$.
Similarly for the other semigroup we have
$k[x, y] \to k[\Gamma_2],\ x = z_1 z_2, y = z_1$.

Geometrically, we have two maps $\mathbf{A}^2(k) \to \mathbf{A}^2(k)$ described by monomials. If we identify the open set of the first one where $y_2 \neq 0$ with the open set of the second where $z_2 \neq 0$ by $y_2 = (z_2)^{-1}$ we obtain an algebraic variety Z with two charts isomorphic to $\mathbf{A}^2(k)$ and which is, by construction, endowed with a birational map $p\colon Z \to \mathbf{A}^2(k)$. It is the blowing up of the origin in $\mathbf{A}^2(k)$.

Note that the algebra of regular rational functions on the intersection of the two charts is $k[x, y, \frac{x}{y}, \frac{y}{x}]$ and in each chart the map is given, in suitable coordinates, by monomials.

Dual cones and fans

Given a convex cone $\sigma \subset \check{\mathbf{R}}^N$, the dual cone is the cone $\check{\sigma}$ of elements $m \in \mathbf{R}^N$ such that $m(x) \geq 0 \ \forall x \in \sigma$. Since $\check{\check{\sigma}} = \sigma$ this establishes a duality between convex cones in $\mathbf{R}^N$ and convex cones in $\check{\mathbf{R}}^N$.

Convex duality reverses inclusions and preserves rationality and regularity

- The dual of a half line is a half space.
- The dual of a cone is of maximal dimension if and only if the cone is strictly convex (does not contain any linear subspace).

The duals of the cones we have seen in the previous page are respectively the cone $\sigma = \check{\mathbf{R}}^2_{\geq 0}$, the cone $\sigma_1 = \mathbf{R}_{\geq 0}(1,1) + \mathbf{R}_{\geq 0}(0,1)$ and the cone $\sigma_2 = \mathbf{R}_{\geq 0}(1,1) + \mathbf{R}_{\geq 0}(1,0)$.
We note that they form a subdivision of the first quadrant of $\check{\mathbf{R}}^2$. The convex dual of their intersection $\mathbf{R}_{\geq 0}(1,1)$ is the half space in $\mathbf{R}^2$ which is above the line $\mathbf{R}_{\geq 0}(e_2 - e_1) + \mathbf{R}_{\geq 0}(e_1 - e_2)$. The monomials corresponding to the integral points of this half space generate the algebra $k[x, y, \frac{x}{y}, \frac{y}{x}]$, which we have seen above.

Fans (=Abanicos)

A fan in $\check{\mathbf{R}}^N$ is a finite collection of strictly convex cones σ_α, each of which is generated by integral vectors and which are such that a face of each cone of the collection is in the collection, and any two cones of the collection intersect along a common face ($\{0\}$ is a face).

We shall be interested only in fans such that $\bigcup_\alpha \sigma_\alpha = \check{\mathbf{R}}^N_{\geq 0}$, the first quadrant.

A fan is *regular* if each cone is generated by vectors which form part of a basis of the integral lattice $\check{\mathbf{Z}}^N \subset \mathbf{R}^N$.

To remember, I:

An inclusion $\sigma' \subset \sigma$ of regular cones of maximal dimension in $\check{\mathbf{R}}^N$ corresponds to a reverse inclusion $\check{\sigma} \subset \check{\sigma}'$ of their duals in $\mathbf{R}^N$. Expressing the generating vectors of $\check{\sigma}$ as positive linear combinations of those of $\check{\sigma}'$ and taking semigroup algebras gives an inclusion of polynomial algebras where each variable of the first goes to a monomial in the variables of the second, and therefore a birational monomial map of affine spaces $\mathbf{A}^N_{\sigma'} \to \mathbf{A}^N_{\sigma}$.

A fan $\Sigma = (\sigma_\alpha)$ such that $\bigcup_\alpha \sigma_\alpha = \sigma$ gives, by glueing up the monomial maps to $\mathbf{A}^N_{\sigma}$ corresponding to the σ_α of maximal dimension along the open sets corresponding to their common faces, a proper birational map $\mathbf{Z}(\Sigma) \to \mathbf{A}^N_{\sigma}$ which is locally described by monomials.

To remember, II:

Let $\sigma \subset \check{\mathbf{R}}^N_{\geq 0}$ be such an inclusion of regular cones, with

$$\sigma = \langle a^1, \ldots, a^N \rangle,\ a^i \in \check{\mathbf{Z}}^N,\ \det(a^i_j) = \pm 1.$$

It corresponds to the map of affine spaces $\mathbf{A}^N_\sigma \to \mathbf{A}^N$ given in coordinates by:

$$\begin{aligned}
u_1 = &\quad y_1^{a_1^1} y_2^{a_1^2} \ldots y_N^{a_1^N} \\
u_2 = &\quad y_1^{a_2^1} y_2^{a_2^2} \ldots y_N^{a_2^N} \\
. &\qquad\qquad\qquad . \\
. &\qquad\qquad\qquad . \\
. &\qquad\qquad\qquad . \\
u_N = &\quad y_1^{a_N^1} y_2^{a_N^2} \ldots y_N^{a_N^N}
\end{aligned}$$

NOTE: the divisors which are contracted (exceptional divisors) are exactly those $y_i = 0$ such that the vector a^i is not a vector of the canonical basis of $\mathbf{Z}^N$.

In the toric jargon, the complement of the union of coordinate hyperplanes is called "the torus" and the condition that the vectors a^i generate the integral lattice, which is equivalent to $\det(a^i_j) = \pm 1$, is equivalent to saying that our monomial map $\mathbf{A}^N_\sigma \to \mathbf{A}^N$ induces an isomorphism of the tori of the two affine spaces.

Resolution of singularities of normal toric varieties

Regular refinements of rational cones, Mumford et al., 1973

Given any finite system of rational convex cones in $\check{\mathbf{R}}^N_{\geq 0}$, there exist regular fans filling $\check{\mathbf{R}}^N_{\geq 0}$ and compatible with those cones.

Here compatibility means that each cone of the fan meets the given rational cones along a face.
This result implies that a normal toric variety, which is given combinatorially by a fan of not necessarily regular strictly convex rational cones, has a (non-embedded and non-canonical) resolution of singularities corresponding to a refinement of the given fan into a regular fan.

Embedded resolution of singularities of toric varieties (1)

Binomial ideals

Given an algebraically closed field k, an affine toric subvariety of $\mathbf{A}^N(k)$ *equipped with coordinates* $U_1, \ldots, U_N$ is a variety defined by a prime ideal of $k[U_1, \ldots, U_N]$ which is generated by binomials $(U^m - \lambda_{mn} U^n)_{(m,n)\in E}$ with $\lambda_{mn} \in k^*$.

The toric variety can be thought of as corresponding to a finitely generated semigroup Γ of $\mathbf{Z}^r$, generating $\mathbf{Z}^r$ as a group. If $k[t^\Gamma]$ is the semigroup algebra of Γ with coefficients in k, it is $\mathrm{Spec} k[t^\Gamma]$. Since the semigroup is finitely generated, this algebra is finitely presented, i.e., there is a surjection

$$k[U_1, \ldots, U_N] \longrightarrow k[t^\Gamma],\ U_i \mapsto t^{\gamma_i}$$

and it is not difficult to show that the kernel is generated by binomials $U^m - U^n$ corresponding to relations $\sum m_j \gamma_j = \sum n_k \gamma_k$ between the generators of the semigroup. Adding the constants λ_{mn} gives an isomorphic algebra.

For example, if $\Gamma = \langle \gamma_1, \ldots, \gamma_N \rangle$ is a semigroup of integers, the corresponding toric variety is the monomial curve in $\mathbf{A}^N(k)$ given parametrically by $u_i = t^{\gamma_i}$, and its equations correspond to the relations between the γ_i.

Embedded resolution of singularities of toric varieties (2)

(Pedro González Pérez and B.T., 2002)

Resolution of binomial ideals

The effect of a monomial map on a binomial is very easy to compute: If the monomial map is given by

$$
\begin{aligned}
u_1 &= y_1^{a_1^1} y_2^{a_1^2} \dots y_N^{a_1^N} \\
u_2 &= y_1^{a_2^1} y_2^{a_2^2} \dots y_N^{a_2^N} \\
&\;\vdots \\
u_N &= y_1^{a_N^1} y_2^{a_N^2} \dots y_N^{a_N^N}
\end{aligned}
$$

Embedded resolution of singularities of toric varieties (3)

If we set $\langle a^i, m\rangle = \sum_{k=1}^{N} a^i_k m_k$, the transform of the monomial u^m is

$$u^m \mapsto y_1^{\langle a^1,m\rangle} \dots y_N^{\langle a^N,m\rangle}.$$

And so

$$u^m - \lambda_{mn} u^n \mapsto y_1^{\langle a^1,m\rangle} \dots y_N^{\langle a^N,m\rangle} - \lambda_{mn} y_1^{\langle a^1,n\rangle} \dots y_N^{\langle a^N,n\rangle}.$$

Now comes the important

REMARK

If the cone σ is compatible with the hyperplane $H_{m-n} \subset \check{\mathbf{R}}^N$ which is the linear dual of the vector $m - n \in \mathbf{R}^N$, then by definition it is entirely on one side of this hyperplane so that all the $\langle a^i, m-n \rangle$ which are not zero are of the same sign.
Up to exchanging m and n and reordering the a^i we may assume that $\langle a^i, m-n \rangle = 0$ for $1 \leq i \leq t$ and that $\langle a^{t+1}, m-n \rangle, \ldots, \langle a^N, m-n \rangle$ are > 0. But then we can rewrite:

$$u^m - \lambda_{mn} u^n \mapsto y_1^{\langle a^1, n \rangle} \ldots y_N^{\langle a^N, n \rangle} (y_{t+1}^{\langle a^{t+1}, m-n \rangle} \ldots y_N^{\langle a^N, m-n \rangle} - \lambda_{mn}).$$

And if $u^m - \lambda_{mn} u^n$ is one of the generators of a prime binomial ideal, the vector $m - n \in \mathbf{Z}^N$ has to be primitive (its non zero components are coprime) and because the a^i are a basis of the integral lattice, the vector with coordinates $\langle a^i, m - n\rangle$ is also primitive. But since $\lambda_{mn} \neq 0$ this implies that the hypersurface

$$y_{t+1}^{\langle a^{t+1}, m-n\rangle} \ldots y_N^{\langle a^N, m-n\rangle} - \lambda_{mn} = 0,$$

which is the strict transform of our binomial, is non singular, *whatever the characteristic of k is*.

Embedded resolution of singularities of toric varieties (End)

Now you will find it easy to believe that things work in the same way for a prime binomial ideal

$$(u^{m^\ell} - \lambda_\ell u^{n^\ell})_{\ell \in L} \subset k[u_1, \ldots, u_N]$$

by taking a regular fan in $\check{\mathbf{R}}^N_{\geq 0}$ compatible with all the hyperplanes $H_{m^\ell - n^\ell}$.
This is always possible because of the theorem of resolution of normal toric varieties.
The fact that the vector $m - n$ is primitive has to be replaced by the fact that the lattice $\mathcal{L} \subset \mathbf{Z}^N$ generated by the vectors $m^\ell - n^\ell$ is *saturated*, which means that it is a direct factor in $\mathbf{Z}^N$.
The fact that $\mathcal{L}$ is saturated if the ideal is prime uses that k is algebraically closed.
Finally, the jacobian minors of a binomial ideal are related in a very simple manner with the minors of the matrix of the vectors $m^\ell - n^\ell$.

Deforming binomial ideals (1)

For this lecture, a weight will be a morphism of semigroups $\mathbf{N}^N \to \mathbf{R}_{\geq 0}$ which attributes to each variable u_i a weight which is a non negative real number.

• A weight is compatible with a binomial ideal if each binomial is homogeneous.

Deforming binomial ideals (2)

Given a weight which is compatible with it, an *overweight deformation* of a binomial is an expression

$$F = u^m - \lambda_{mn} u^n + \sum_{w(u^p) > w(u^m)} c_p u^p, \; c_p \in k.$$

If we have a finite number of binomials and a compatible weight we do the same and consider the deformations

$$F_\ell = u^{m^\ell} - \lambda_\ell u^{n^\ell} + \sum_{w(u^p) > w(u^{m^\ell})} c_p^{(\ell)} u^p, \; c_p^{(\ell)} \in k.$$

One has to add the condition that the initial binomials of the F_ℓ generate the ideal of initial forms of the elements of the ideal generated by the F_ℓ.

Deforming binomial ideals (3)

Let us consider an overweight deformation (F_ℓ) of a prime binomial ideal in the power series ring $k[[u_1, \ldots, u_N]]$, and the map

$$\pi\colon k[[u_1, \ldots, u_N]] \to R = k[[u_1, \ldots, u_N]]/(F_1, \ldots, F_s).$$

Proposition

1) The map which associates to $x \in R$ the *maximum* of the weights of the elements of $\pi^{-1}(x)$ is well defined and is a valuation ν on R.
2) Given $\tilde{x} \in \pi^{-1}(x)$, we have $w(\tilde{x}) = \nu(x)$ if and only if $\mathrm{in}_w(\tilde{x}) \notin F_0$, where F_0 is the ideal generated by the binomials.

An example

Let $F_1 = u_2^2 - u_1^3 - u_3, F_2 = u_3^2 - u_1^5 u_2$ be an overweight deformation of the binomial ideal $F_0 = (u_2^2 - u_1^3, F_2 = u_3^2 - u_1^5 u_2)$, with u_1 of weight 4, u_2 of weight 6 and u_3 of weight 13.
The element $x \in k[[u_1, u_2, u_3]]/(F_1, F_2)$ which is the image of the element $u_2^2 - u_1^3$, of weight 12, has another counterimage of weight 13, namely u_3, which gives the maximum since it is not in the binomial ideal.

What is a valuation?

A valuation on a domain R is a map $\nu\colon R\setminus\{0\}\to\Phi_{\geq 0}$ where Φ is a totally ordered abelian group, satisfying:

$$\nu(xy)=\nu(x)+\nu(y)$$

and

$$\nu(x+y)\geq \min(\nu(x),\nu(y)) \qquad .$$

If K is the field of fractions of R, the valuation ν extends to a map $\nu\colon K^*\to\Phi$ by $\nu(x/y)=\nu(x)-\nu(y)$.

One should think of a valuation as an "order of vanishing" (or of pole) for rational functions x/y along "something" which may be a divisor, a point, or in restriction to something which may be invisible to algebraic geometry, like a very transcendental curve.
One usually sets $\nu(0)=\infty$, an element larger than any element of Φ.

If you are surprised that the value group is not $\mathbf{Z}$, think of measuring the order of vanishing of a power series $f(x,y)$ along the curve $y = x^{\tau}$, with $\tau \in \mathbf{R}_{+} \setminus \mathbf{Q}$. It is the order in x of $f(x, x^{\tau})$ and is an element of $\mathbf{Z} + \mathbf{Z}\tau$. And there are much more complicated examples.
We need to look at all valuations of K, because they form a quasi-compact space by a theorem of Zariski, and that finiteness is essential.

The valuations we consider in this lecture are special, but in a sense they are those which govern all the others.

Because a valuation is essentially a birational object whenever we have a proper and birational map $b\colon X' \to X$ a valuation centered at $x \in X$, that is a valuation of the local ring $\mathcal{O}_{X,x}$ such that $\nu(m_{X,x}) > 0$ will determine a unique point $x' \in b^{-1}(x)$ such that $\nu(m_{X',x'}) > 0$. It is called the center of the valuation.
Rational valuations are characterized by the fact that all centers in birational models have the same residue field as $x \in X$, or of you prefer are rational points over the base field.

The problem of local uniformization is to show the existence of a proper and birational map $b\colon X' \to X$ such that the center of ν on X' is a regular point. It is a weak form of resolution of singularities. It suffices to prove it for rational valuations.

On the other hand, a valuation defines a filtration of the ring R on which it is defined. Let

$$\mathcal{P}_\phi(R) = \{x \in R | \nu(x) \geq \phi\}, \;\; \mathcal{P}^+_\phi(R) = \{x \in R | \nu(x) > \phi\},$$

and

$$\mathrm{gr}_\nu R = \bigoplus_{\phi \in \Phi_{\geq 0}} \frac{\mathcal{P}_\phi(R)}{\mathcal{P}^+_\phi(R)}.$$

In the example where the valuation is the order of vanishing on the curve $y = x^\tau$, the associated graded ring is $k[X, Y]$ graded by giving X weight one and Y weight τ.

The sum $\bigoplus_{\phi\in\Phi_{\geq 0}} \frac{\mathcal{P}_\phi(R)}{\mathcal{P}^+_\phi(R)}$ is actually indexed by the semigroup $\Gamma = \nu(R\setminus\{0\}) \subset \Phi_{\geq 0}$. Since R is noetherian, this semigroup is well ordered, of ordinal $\leq \omega^{\dim R}$.

As a consequence it has a minimal system of generators

$$\Gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_i, \ldots, \gamma_\omega, \ldots \rangle,$$

indexed by an ordinal $\leq \omega^{\dim R}$.

There are valuations of $k[[x,y]]$ where the value group is any non discrete subgroup of **Q** and the semigroup is minimally generated by an infinite sequence of rational numbers.

Theorem: the relation between valuations and toric geometry

For the class of valuations of integral local domains most important for us, the *rational valuations* we have the following

Let ν be a rational valuation of the local domain (R, m) and let $(\gamma_i)_{i\in I}$ be the minimal set of generators of the semigroup $\nu(R\setminus\{0\})$. The graded algebra is a quotient of a polynomial algebra over $k = R/m$ in variables indexed by the ordinal I by a prime binomial ideal (possibly generated by infinitely many binomials).

$$k[(U_i)_{i\in I}] \to \mathrm{gr}_\nu R,\ \ U_i \mapsto \overline{\xi}_i,$$

where $\overline{\xi}_i$ is the initial form of an element $\xi_i \in R$ of valuation γ_i

There is a general principle of commutative algebra: a filtered ring, such as R with the filtration by the valuation ideals $\mathcal{P}_\phi$, specializes, in a faithfully flat way, to the associated graded ring, in our case $\mathrm{gr}_\nu R$.

If the local domain R is complete and equicharacteristic, it has a field of representatives $k \subset R$, and we can do even better:

Let $(u_i)_{i\in I}$ be variables indexed by the elements of the minimal system of generators $(\gamma_i)_{i\in I}$ of the semigroup Γ of the valuation ν on R. Give each u_i the weight $w(u_i)=\gamma_i$ and let us consider the set of power series $\sum_{e\in E} d_e u^e$ where $(u^e)_{e\in E}$ is any set of monomials in the variables u_i and $d_e\in k$.

By a theorem of Campillo-Galindo, the semigroup Γ being well ordered is combinatorially finite, which means that for any $\phi\in\Gamma$ the number of different ways of writing ϕ as a sum of elements of Γ is finite. This is equivalent to the fact that the set of exponents e such that $w(u^e)=\phi$ is finite: for any given series the map $w\colon E\to\Gamma,\ e\mapsto w(u^e)$ has finite fibers. Each of these fibers is a finite set of monomials in variables indexed by a totally ordered set, and so can be given the lexicographical order and order-embedded into an interval $1\leq i\leq n$ of $\mathbf{N}$.

This defines an injection of the set E into $\Gamma \times \mathbf{N}$ equipped with the lexicographical order and thus induces a total order on E, for which it is well ordered. When E is the set of all monomials, this gives a total monomial order.
The combinatorial finiteness implies that this set of series is a k-algebra, which we denote by

$$k\widehat{[(u_i)_{i\in I}]}$$

Since the weights of the elements of a series form a well ordered set and only a finite number of terms of the series have minimum weight, the associated graded ring of $k\widehat{[(u_i)_{i\in I}]}$ with respect to the filtration by weights is the polynomial ring $k[(U_i)_{i\in I}]$

The valuative Cohen theorem

Assuming that the local noetherian equicharacteristic domain R is complete, with a rational valuation ν, and fixing a field of representatives $k \subset R$, there exist choices of representatives $\xi_i \in R$ of the $\overline{\xi_i}$ such that the surjective map of k-algebras $k[(U_i)_{i\in I}] \to \mathrm{gr}_\nu R,\ U_i \mapsto \overline{\xi_i}$, is the associated graded map of a continuous surjective map

$$k\widehat{[(u_i)_{i\in I}]} \to R,\ u_i \mapsto \xi_i,$$

of topological k-algebras, with respect to the weight and valuation filtrations respectively. The kernel of this map is generated up to closure by overweight deformations of binomials generating the kernel of $k[(U_i)_{i\in I}] \to \mathrm{gr}_\nu R$, $U_i \mapsto \overline{\xi}_i$.

This can be seen as a ”comfortable” embedding for the singularity represented by R, but it is in general not finite dimensional. So we must try to extract from it a finite dimensional comfortable embedding, using the fact that R is noetherian.

The good thing is that we have equations for R of the form

$$F_\ell = u^{m^\ell} - \lambda_\ell u^{n^\ell} + \sum_{w(u^p) > w(u^{m^\ell})} c_p^{(\ell)} u^p,$$

and the bad thing is that there are infinitely many variables and equations in general.

But there is a case where all is well:

As a consequence, whenever R is complete and equicharacteristic, if the valuation ν is rational and the semigroup Γ is finitely generated, the ring R is an overweight deformation of the binomial ideal defining its associated graded ring.

Indeed, in this case, the frightening ring $k\widehat{[(u_i)_{i\in I}]}$ is an ordinary power series ring $k[[u_1, \ldots, u_N]]$, except that the variables have weights $\gamma_i \in \Gamma$.

Let k be an algebraically closed field of characteristic 0 and let $R = k[[u_1, u_2]]/((u_2^2 - u_1^3)^2 - u_1^5 u_2)$ be the ring of a plane branch C. It has only one valuation, induced by the t-adic order of its normalization $k[[t]]$. The semigroup of values is the numerical semigroup $\Gamma = \langle 4, 6, 13 \rangle$. The relations between the generators are $3\gamma_1 = 2\gamma_2$ and $2\gamma_3 = 5\gamma_1 + \gamma_2$.

This tells us that our ring is an overweight deformation of the quotient of $k[[u_1, u_2, u_3]]$ by the binomial ideal $(u_2^2 - u_1^3, u_3^2 - u_1^5 u_2)$. Indeed, R is isomorphic to $k[[u_1, u_2, u_3]]/(u_2^2 - u_1^3 - u_3, u_3^2 - u_1^5 u_2)$.

No toric modification of $\mathbf{A}^2(k)$ will resolve the branch $C \subset \mathbf{A}^2(k)$, but there is a regular fan Σ in $\check{\mathbf{R}}^3$ containing the regular cone σ_1 spanned by $\{(4,6,13),\ (2,3,6),\ (1,1,3)\}$, which corresponds to the monomial map

$$\pi(\sigma_1): \begin{array}{rcl} u_1 & = & {y_1}^4{y_2}^2y_3 \\ u_2 & = & {y_1}^6{y_2}^3y_3 \\ u_3 & = & {y_1}^{13}{y_2}^6{y_3}^3 \end{array}$$

in which the strict transform of the curve $C \subset \mathbf{A}^3(k)$ has equations $1 - y_3 = 0, 1 - y_2 = 0$.

Then, the overweight condition implies, by a purely combinatorial argument, that one can find regular fans in $\check{\mathbf{R}}^N_{\geq 0}$ defining birational toric maps which not only resolve the singularities of the toric variety associated to the graded ring $\mathrm{gr}_\nu R$ but also uniformize the valuation ν on R.

The idea is very simple to explain in the case of a single equation

$$F = u^m - \lambda u^n + \sum_{w(p)>w(m)} c_p u^p.$$

Let

$$E' = \langle \{p - n / c_p \neq 0\}, m - n \rangle \subset \mathbf{R}^N,$$

where $\langle a, b, ... \rangle$ denotes the cone generated by $a, b, ...$. Since there may be infinitely many exponents p, the smallest closed convex cone containing E' may not be rational. However, the power series ring being noetherian, there exist finitely many exponents $(p_f - n)_{f \in F}$ as above, with F finite, such that E' is contained in the rational cone E generated by $m - n$, the vectors $(p_f - n)_{f \in F}$ and the basis vectors of $\mathbf{R}^N$. This cone is strictly convex because all its elements have a strictly positive weight except the positive multiples of $m - n$.
We can define the *weight vector* $\mathbf{w} = (w(u_1), \dots, w(u_N)) \in \check{\mathbf{R}}^N$. The weight of a monomial u^m is then the evaluation, or scalar product, $\langle \mathbf{w}, m \rangle$. We note that, by construction, we have $\mathbf{w} \in \check{E}$.

So if Σ is a regular fan subdividing $\check{\mathbf{R}}^N_{\geq 0}$ which is compatible with H and $\check{E}$, it will contain a regular cone σ of dimension N whose intersection with H is of dimension $N-1$, which contains $\mathbf{w}$ and is contained in $\check{E}$. By the resolution theorem for normal toric varieties, since $\check{E}$ is a rational convex cone and H a rational hyperplane, we know that there exist such regular fans.

As a first step, let us examine the transforms in the charts $Z(\sigma)$ corresponding to cones $\sigma = \langle a^1, \ldots, a^N \rangle$ which contain **w** and are compatible with $\check{E}$ and H.

• *Because our fan is compatible with H, the convex cone σ has to be entirely on one side of H and its intersection with H is a face*. We may assume that $a^1, \ldots, a^t$ are those among the a^j which lie in the hyperplane H and all the other $\langle a^j, m-n \rangle$ are of the same sign, say $\langle a^j, m-n \rangle > 0$. We have then

$$u^m - \lambda u^n \longmapsto y_1^{\langle a^1, n \rangle} \cdots y_N^{\langle a^N, n \rangle} (y_{t+1}^{\langle a^{t+1}, m-n \rangle} \cdots y_N^{\langle a^N, m-n \rangle} - \lambda).$$

• *By compatibility with* $\check{E}$ *and since it contains* $\mathbf{w}$ *which is in* $\check{E}$, the cone σ is contained in $\check{E} \subseteq \check{E}'$ so that *all* $\langle a^i, p-n \rangle$ *are* ≥ 0. After perhaps re-subdividing σ and choosing a smaller regular cone containing $\mathbf{w}$ and whose intersection with H does not meet the boundary of $\check{E}$, we have that the $\langle a^i, p-n \rangle$ are > 0 at least for those i such that $a^i \in H$.

In the corresponding chart $Z(\sigma)$ the transform of our equation F by the monomial map can then be written:

$$y_1^{\langle a^1,n\rangle}\dots y_N^{\langle a^N,n\rangle}\big(y_{t+1}^{\langle a^{t+1},m-n\rangle}\dots y_N^{\langle a^N,m-n\rangle}-\lambda+\sum_p c_p y_1^{\langle a^1,p-n\rangle}\dots y_N^{\langle a^N,p-n\rangle}\big).$$

Since $\sum_{i=1}^N \langle a^i, p-n\rangle w(y_i)=\langle \mathbf{w}, p-n\rangle$, this shows that the strict transform F' of F by the monomial map $Z(\sigma)\to \mathbf{A}^N(k)$, which is the quantity between parenthesis, is an overweight deformation of the strict transform, of weight zero, of the initial part of F.
This implies the result we seek since the hypersurface defined by the initial part of F' is non singular.

The charts where the strict transform intersects the maximal number of components of the toric boundary are obtained by choosing the regular cone $\sigma \in \Sigma$ in such a way that its intersection with the hyperplane H is of maximal dimension $N-1$, which means that $N-1$ of the vectors a^i are in H. The $N-1$ corresponding coordinates y_i will be of positive value and provide a system of local coordinates for the strict transform of our hypersurface at the point picked by the valuation. In fact, if a^N is the vector which is not in H, according to what we saw above we must have $\langle a^N, m-n\rangle = 1$ and our local equation becomes

$$F' = y_N - \lambda + \sum_{w(p)>w(n)} c_p y_1^{\langle a^1, p-n\rangle} \dots y_N^{\langle a^N, p-n\rangle}.$$

Since the weight of y_N is zero and since this is an overweight deformation, we see immediately that F' is a power series in $y_1, \dots, y_{N-1}$ and $w_N = y_N - \lambda$ and the hypersurface $F' = 0$ is non singular and transversal to the toric boundary at the point $y_1 = \dots = y_{N-1} = 0, y_N = \lambda$, with local coordinates $y_1, \dots, y_{N-1}$. This point is the point picked by the valuation because on $F' = 0$ the valuation of $y_N - \lambda$ has to be positive.

Now you will find it easy to believe that things work essentially in the same way whene there is a finite number of equations.
Again the fact that an affine toric variety corresponds to a saturated lattice and the special form of the jacobian minors play an essential role.

All this leads to the following question: Given a singular algebraic subvariety X of an affine space $\mathbf{A}^N(k)$ over an algebraically closed field k, can one re-embed X in another affine space $\mathbf{A}^M(k)$ is such a way that:

• There exists a system of coordinates $z_1, \ldots, z_M$ for $\mathbf{A}^M(k)$ such that there are regular fans Σ subdividing $\check{\mathbf{R}}^M_{\geq 0}$ and such that the corresponding birational toric map $Z(\Sigma) \longrightarrow \mathbf{A}^M(k)$ gives an embedded resolution (or pseudo-resolution) of X.

• The singular locus of X is a union of intersections of X with strata of the canonical stratification of the union of coordinate hyperplanes (the toric boundary, in the toric jargon).

Very optimistic, some will say. Well...

Tevelev's Theorem, Collectanea Math., 2014

Let k be an algebraically closed field of characteristic zero.
Let $X \subset \mathbf{P}^n$ be an irreducible algebraic variety. For a sufficiently high order Veronese re-embedding $X \subset \mathbf{P}^N$ one can choose homogeneous coordinates $z_0, \ldots, z_N$, a smooth toric variety Z' of the algebraic torus $T = \mathbf{P}^N \setminus \bigcup\{z_i = 0\}$ and a toric birational morphism $Z' \to \mathbf{P}^N$ such that the following conditions are satisfied: $X \cap T$ is non-empty, the strict transform of X in Z' is smooth and intersects the toric boundary transversally, and $Z' \to \mathbf{P}^N$ is a composition of blowing-ups with smooth torus-invariant centers.

Very roughly speaking, a suitable embedding of a singular space in some affine space must have the property that there are enough coordinate hyperplanes for the strict transforms of the corresponding hyperplane sections to ”read” all the important information of the exceptional divisor of an embedded resolution in that embedding.

There is both local and global information along the exceptional divisor, and it is really complicated.
But things are easier if we use valuations, one at a time, as probes.

How to prove the existence of suitable embeddings without having an embedded resolution?

Prove the existence of a suitable embedding for each valuation, for which there is a toric modification of the new ambient space which uniformizes the valuation, then use the compactness of the space of valuations to prove that there exist a finite collection of such suitable embeddings having the property that for every valuation, at least one of these embeddings is suitable. Then glue up those finitely many embeddings into a single one.

Well, we are not there yet!

We only know how to prove the existence of suitable embeddings for the simplest valuations, called ”Abhyankar valuations”, but they are very important since they can be used to approximate any valuation.

And I have hidden many difficulties, in particular for the reduction to the case of complete local rings (work in progress with Herrera, Olalla, Spivakovsky).

Muchas gracias por su atención,
y Viva Pepe !!!
(May his singularity never be resolved!)

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