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THE HUNTING OF INVARIANTS

IN THE GEOMETRY OF DISCRIMINANTS

by

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at the Nordic Summer School, Oslo , August 1976

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Introduction

From the table of contents, the reader might have gathered the impression that what is presented here is a survey, a medley, or even a motley of results on singularities.

However, I hope that those who read the text will agree that on the contrary, it is entirely devoted to the illustration of a single idea:

'Primitive invariants of the discriminant D of a map yield rather subtle invariants of the fibre of the map.'

For the purposes at hand, let us agree to a primitive invariant of a germ of a complex hypersurface $(D, 0) \subset (\mathbb{C}^{m+1}, 0)$, being the Newton polyhedron in some coordinates of the restriction of an equation $\delta(v, t_1 \cdots t_m) = 0$ of D to an i-plane in $(\mathbb{C}^{m+1}, 0)$ through the origin. The simplest primitive invariant in this sense is the multiplicity of D, corresponding to $i = 1$ and a line transversal to D. The next simplest is the *Newton polygon* of a plane section of D (see §3) and this is what we study here in the case where D is the discriminant of a miniversal deformation of a hypersurface with isolated singularity $(X_0, 0) \subset (\mathbb{C}^N, 0)$, (i.e. our map is in particular stable).

In fact, it has been an open problem for some time to understand to what extent the geometry of the discriminant determines the geometry of $(X_0, 0)$.

The real-analytic version of this problem is important in the theory of catastrophes of Thom, and problems of a related nature appear in the Jung-Zariski-Abhyankar approach to resolution of singularities (by resolving the discriminant of a projection) (see [1]) and in Zariski's theory of equisingularity. ([5]).

This problem is at least partly solved here in the special case where $(X_0, 0)$ is a plane branch since it is shown, using a theorem of Merle, that the Newton polygon of a general vertical plane section of D (see 5.5.7) is a *complete invariant* of the equisingularity class of $(X_0, 0)$, i.e. in this case, of its topological type. In particular, one can compute the Puiseux characteristic exponents of $(X_0, 0)$ from the inclinations of the edges of this polygon. Of course this case is only the basic test for any theory of invariants of equisingularity, and in the general case the conclusion is not so clear-cut, but we show that one can read from the Newton polygon of a general vertical plane section of D such interesting invariants of $(X_0, 0)$ as the smallest integer a such that if $f(z_1, \cdots, z_N) = 0$ defines X_0, any $g \in \mathbb{C}\{z_1, \cdots, z_N\}$ such that $g - f \in (z_1, \cdots, z_N)^{a+1}$ defines, by $g = 0$, a hypersurface equisingular with $X_0, 0)$,—or the diminution of class which the presence of a singularity isomorphic to $(X_0, 0)$ would impose on a projective hypersurface. Furthermore, this Newton polygon is an invariant of equisingularity, as we define it (see below).

It might seem that all this is not of great practical utility since the equations of discriminants are very hard to compute, but our method also yields a way of computing these Newton polygons which I have found quite usable in practice.

From a geometric viewpoint, what is done here is to take a dynamic view of Morse theory: the Newton polygon mentioned above can also be deemed to describe the various 'speeds' with which the (coordinates of the) quadratic critical points in a generic morsification $v = f(z_1, \cdots, z_N) + u(\sum_1^N \alpha_i z_i)(\alpha_i \in \mathbb{C})$ of an equation of $(X_0, 0)$ vanish to 0 with the parameter of morsification u. It is perhaps a pleasant surprise that in the case of a plane branch these speeds suffice to completely determine the topology of the function $f(z_1, z_2)$, and conversely.

From a formal viewpoint, what we do is this: we introduce on the set of germs of hypersurfaces an equivalence relation: (c)-cosécance (see 2.19), which is our working definition of equisingularity. Two (c)-cosécant germs of hypersurfaces with isolated singularity are topologically equivalent, as well as all their general plane sections. Then, on the set $\mathscr{I}_{\mathscr{I}}$ of (c)-cosécance classes of isolated singularities of hypersurfaces we introduce an operation $[X_0], [X_1] \rightsquigarrow [X_0] \perp [X_1]$, which is induced by the Thom-Sebastiani operation: if X_0 (resp. X_1) is defined by $f_0(z_1, \cdots, z_N) = 0$ (resp. $f_1(w_1, \cdots, w_M) = 0$) then $[X_0] \perp [X_1]$ is the (c)-cosécance class of $f(z_1, \cdots, z_N) - g(w_1, \cdots, w_M) = 0$. The Milnor number gives us a map $\mu : \mathscr{I}_{\mathscr{I}} \to \mathbb{Z}_0$ (where the subscript 0 indicates the non-negative part) satisfying $\mu([X_0] \perp [X_1]) = \mu([X_0]) \cdot \mu([X_1])$.

What we do here is to construct a new ring $\underline{\mathrm{N}}_S$, the *ring of special Newton polygons* and to factor this map μ by a 'Jacobian Newton polygon map' $\boldsymbol{\nu}_j$:

$$\begin{tikzcd} & (\underline{\mathrm{N}}_S)_0 \arrow[dr, "l"] & \\ \mathscr{I}_{\mathscr{I}} \arrow[ur, "\boldsymbol{\nu}_j"] \arrow[rr, "\mu"] & & \mathbb{Z}_0 \end{tikzcd}$$

where l is the length of horizontal projection of a Newton polygon (see 3.6) and $\boldsymbol{\nu}_j$ satisfies

$$\boldsymbol{\nu}_j([X_0] \perp [X_1]) = \boldsymbol{\nu}_j([X_0]) * \boldsymbol{\nu}_j([X_1])$$

where $*$ is the product of Newton polygons defined in 3.6, and the datum of $\boldsymbol{\nu}_i$ is equivalent to the datum of the Newton polygon of a general vertical plane section of the discriminant D of a miniversal deformation of $(X_0, 0)$ (see 5.17).

We remark that μ and $\boldsymbol{\nu}_i$ are given by the two simplest primitive invariants of D, since μ is the multiplicity of D (see 5.5.2). What happens for the higher-dimensional primitive invariants of D is, as yet, unknown to me.

These notes therefore contain material which I believe to be essentially new, but since their aim is partly pedagogical, I decided to include not only the notions and results which I had found necessary to understand what I was doing, but also some illustration of them, to help their assimilation.

For example, to compute invariants from sections of the discriminant, it is indispensable to have a definition of the discriminant compatible with base change. Since the discriminant is – by definition – the image of the critical subspace, we are led towards a definition of the image of a finite map which is compatible with base change. In other words, finding a procedure of *elimination* stable enough to be computable. This is the subject of §1. The discriminant D has the structure of an envelope, in the sense of [4]. Since this fact, although not of direct use to us, is of importance in the study of the geometry of D, we go a little into this in §2, and this gives the idea of our method of exposition. This method has the disadvantage – apart from the lengthening of the text – of provoking brutal changes in the level of exposition, of which I must warn the reader.

Acknowledgements

The proposition on developments in the first part of §2 is a version of a result first proved by Mr. A. Nobile (Some properties of the Nash blowing up, Pacific Journal of Math. Vol. 60 (1975) pp. 297–305). Also, when the definition of the image of a finite complex map given here in §1 was breezily introduced in ([7] Chap. III), I thought it 'well known' since it is so natural, I was mostly happy to have a definition of the discriminant compatible with base change, and an easy and natural purity theorem. Since then, it has appeared that the definition was not so well known or employed by geometers at all, except, as far as discriminants are concerned by D. S. Rim, especially in his papers 'Formal deformation theory' (S.G.A. 7, I, Springer Lecture Notes No. 288 and 'Torsion differentials and deformations' (Transactions A.M.S. Vol. 169 (1972)) where he had also defined the discriminant by a Fitting ideal and proved the purity theorem. Anyway, after some experimentation (partly reproduced in §3) with the Fitting ideal definition of the image of a mapping $f: X \to Y$ such that $R^i f_* \mathcal{O}_X = 0$ (e.g. f finite), I became convinced that it is both a very informative and a very computable elimination process. A part of 3.3 is a maturation of an old result in common with Monique Lejeune-Jalabert and Lê Dũng Tráng (Sur un critère d'équisingularité, Note aux C. R. Acad. Sc. Paris t. 271 (1970) p. 1065). I wish to thank them for their comments on this part, and also Monique Lejeune-Jalabert for discussions on the formal study of an additive monoid of Newton polygons. I also thank P. Barril, M. Merle, and D. Trotman for suggestions which influenced the form and content of this text, and Geir Ellingsrud for writing up notes of the lectures which I used in §§1 and 2. My thanks to to the Matematisk Institutt, Oslo for typing §§1 and 2 first part, and also to Marie-Jo Lécuyer at the Centre de Mathématiques for doing a beautiful and fast typing of the rest of the text. Finally I wish to thank the organizers and participants of the Summer School for creating a very pleasant working atmosphere, and above all Per Holm for his patience with a lecturer who talked extremely fast and wrote the notes extremely slowly!

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§1. Fitting ideals

In this section we will give the definition and elementary properties of Fitting ideals, which we will use later to give a definition of the image, as a complex analytic *space*, of a finite map between complex analytic spaces. We then give a definition of the resultant ideal of two polynomials as a Fitting ideal, and as an application give a proof of Bezout's theorem.

Let A be a ring, and let M be an A-module of finite presentation, that is, M is the cokernel of an A-linear map between two free A-modules of finite rank, or if you prefer, there is an exact sequence, called a presentation of M:

$$A^q \xrightarrow{\psi} A^p \to M \to 0$$

where $p, q \in \mathbb{N}$. For each integer j we associate to M the ideal $F_j(M)$ of A generated by the $(p-j)\times(p-j)$ minors of the matrix (with entries in A) representing ψ. Here we need the convention that if there are no $(p-j)\times(p-j)$ minors because j is too large, i.e. $j \geqq p$, then $F_j(M) = A$ (the empty determinant is equal to 1) and if, at the other extreme, $p-j>q$, set $F_j(M)=0$ (the ideal generated by the empty set is 0).

A theorem of Fitting (for a proof see [1] p. 5 where the $F_j(M)$ are called $\sigma_j'(\psi)$) asserts that $F_j(M)$ depends only on the A-module M and not on the choice of a presentation. We call it the jth Fitting ideal of M.

More generally, if $(X, \mathcal{O}_X)$ is a ringed space, and $\mathcal{M}$ a coherent sheaf of $\mathcal{O}_X$-modules, we can define a sheaf of ideals $\mathcal{F}_j(\mathcal{M})$ of $\mathcal{O}_X$, by defining $\mathcal{F}_j(\mathcal{M})$ locally as above, and then by uniqueness the ideals found locally patch up into a sheaf of ideals. Remark also that since $\mathcal{F}_j(\mathcal{M})$ is locally finitely generated, $\mathcal{F}_j(\mathcal{M})$ will be a coherent sheaf of ideals as soon as $\mathcal{O}_X$ is coherent, e.g. for a complex analytic space by Oka's theorem.

One important fact about Fitting ideals is that 'their formation commutes with base change' as one says. The idea is that if you have $(X, \mathcal{O}_X)$ and a coherent $\mathcal{O}_X$-module $\mathcal{M}$, then for any map $f:(Y, \mathcal{O}_Y)\to(X, \mathcal{O}_X)$ we have that $F_j(f^*\mathcal{M}) = \mathcal{F}_j(\mathcal{M})\mathcal{O}_Y$, where this last expression means, here and in the sequel, the ideal in $\mathcal{O}_Y$ generated by the image of the canonical map, from $f^*(\mathcal{F}_j(M))$ to $\mathcal{O}_Y = f^*\mathcal{O}_X$ coming from the inclusion $\mathcal{F}_j(M)\subset\mathcal{O}_X$. Algebraically, it means that for any A-module M of finite presentation, and any ring-homomorphism $g: A\to B$, we have $F_j(M\otimes_A B)=$ ideal generated by $g(F_j(M))$ in B, which is denoted by $F_j(M)B$. (this is immediate from the fact that tensoring a presentation of M by B gives a presentation of $M\otimes_A B$). In particular, for any maximal ideal m in A we have that $F_j(M)\subset m$ if and only if $\dim_{A/m} M\otimes_A A/m > j$ since the jth Fitting ideal of $M\otimes_A A/m$ is either 0 on A/m depending upon whether the map $\bar{\psi}$ of A/m-vector spaces in the exact sequence: $(A/m)^q \xrightarrow{\bar{\psi}} (A/m)^p \to M\otimes_A A/m \to 0$ (obtained by tensoring a presentation of M by A/m) is of rank $<p-j$ or not, i.e. $M\otimes_A A/m$ is of dimension $>j$ or not.

Geometrically, we think of this as follows: a coherent sheaf of modules $\mathcal{M}$ on a complex-analytic space $(X, \mathcal{O}_X)$ is the sheaf of sections of a mapping of complex analytic spaces $p: L(\mathcal{M})\to X$ such that for any $x\in X$, $p^{-1}(x) = \mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/m_{X,x}$ is a finite dimensional vector space over $\mathcal{O}_{X,x}/m_{X,x}=\mathbb{C}$, and a section is a holomorphic map $\sigma: X\to L(\mathcal{M})$ such that $p\circ\sigma = id_X$.

Note that if $\mathcal{M}$ is not locally free, the dimension of $p^{-1}(x)$ as a vector space can vary with x, and since $F_j(\mathcal{M})_x = F_j(\mathcal{M}_x)$ (as an $\mathcal{O}_{X,x}$-module) we have that the set underlying the analytic subspace $V(\mathcal{F}_j(\mathcal{M}))$ of X defined by the coherent sheaf of ideals $\mathcal{F}_j(\mathcal{M})$ is:

$$|V(F_j(\mathcal{M}))| = \{x\in X / \dim_{\mathbb{C}} p^{-1}(x) > j\}.$$

Let now $f:(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ be a map of complex analytic spaces. We would like to define the image of f as a complex analytic *subspace* of $(Y,\mathcal{O}_Y)$. This is not always possible, and in particular if one hopes to get a closed complex subspace of Y it is better to assume f is proper, and here we will consider only the case where f is *finite* (= proper and with finite fibres).

By theorems of Grauert, the direct image sheaf $f_*\mathcal{O}_X$ is then a coherent sheaf of $\mathcal{O}_Y$-modules, and its formation commutes with base change, i.e. for any complex analytic map $h: Y'\to Y$

$$\begin{tikzcd} X' = X \times_Y Y' \arrow[r, "k"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "h"] & Y \end{tikzcd}$$

we have, in the above cartesian diagram of base change, that $f'_*\mathcal{O}_{X'} = h^*f_*\mathcal{O}_X$ (see [Cartan Seminar] 60–61, p. 15, cor. 1.6).

Now a basic requirement for the definition of the image is that again its formation should commute with base change, i.e.

$$\operatorname{im} f' = h^{-1}(\operatorname{im}(f)) \qquad \text{as complex } \textit{spaces}$$

The first sheaf of ideals that comes to mind as a candidate to define $f(X)$ is the sheaf of functions g on Y such that $g\circ f=0$ on X, i.e. the annihilator sheaf

$$\boldsymbol{Ann}_{\mathcal{O}_Y}(f_*\mathcal{O}_X) = \text{sheaf }\{\text{functions } g \text{ on } Y \text{ such that } g\cdot f_*\mathcal{O}_X = 0\}.$$

This is *not* a good choice because its formation does not commute with base extension, as we will show by an example below.

The second try is the 0th Fitting ideal of $f_*\mathcal{O}_X$, which set theoretically describes also the image of f, since the subspace of Y defined by it, as a set is $\{y\in Y \mid \dim_{\mathbb{C}}(f_*\mathcal{O}_X)(y)>0\} = \{y\in Y \mid (f_*\mathcal{O}_X)_y \neq 0\}$.

Now since, as we have seen, both the formation of direct images and the formation of Fitting ideals commute with base change, in any case we know that the formation of the image, with this definition will also. So we set:

DEFINITION. Let $f: X \to Y$ be a finite morphism of complex analytic spaces. The image of f is the subspace of Y defined by the coherent sheaf of ideals $\mathcal{F}_0(f_*\mathcal{O}_X)$.

LEMMA. *The formation of* $\operatorname{im}(f)$ *commutes with base change. Proof:*

$$\mathcal{F}_0(f'_*\mathcal{O}_{X'}) = \mathcal{F}_0(h^* f_* \mathcal{O}_X) = \mathcal{F}_0(f_*\mathcal{O}_X)\mathcal{O}_{Y'}.$$

REMARK. Cramer's rule tells us that always $\boldsymbol{Ann}\, f_*\mathcal{O}_X \supset \mathcal{F}_0(f_*\mathcal{O}_X)$,

EXAMPLE. Let $f: (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ be given by $x = t^{2k}$, $y = t^{3k}$ for some integer k. Clearly the set-theoretic image of f is the curve $y^2 - x^3 = 0$. However, we wish to obtain an ideal defining a space supported on that curve, but possibly with nilpotent functions. To compute $\mathcal{F}_0(f_*(\mathcal{O}_{\mathbb{C}})$ we will look locally near 0, and therefore compute $\mathcal{F}_0(f_*O_{\mathbb{C}})_0$ as the 0th Fitting ideal of $\mathbb{C}\{t\}$ considered as $\mathbb{C}\{x, y\}$-module via the map of rings $\mathbb{C}\{x, y\} \to \mathbb{C}\{t\}$ sending x to t^{2k} and y to t^{3k}. We must therefore write a presentation of $\mathbb{C}\{t\}$ as $\mathbb{C}\{x, y\}$-module. Let $e_0 = 1$, $e_1 = t, \cdots, e_{2k-1} = t^{2k-1}$. It is easily seen that they form a system of generators of $\mathbb{C}\{t\}$ as $\mathbb{C}\{x, y\}$-module, and that between them we have the following $2k$ relations:

$$
\begin{array}{ll}
xe_k - ye_0 = 0 & x^2e_0 - ye_k = 0 \\
xe_{k+1} - ye_1 = 0 & x^2e_1 - ye_{k+1} = 0 \\
\vdots & \vdots \\
xe_{2k-1} - ye_{k-1} = 0 & x^2e_{k-1} - ye_{2k-1} = 0
\end{array}
$$

which are obviously independent.

Hence we have a sequence of $\mathbb{C}\{x, y\}$-modules.

$$0 \longrightarrow \bigoplus_{i=0}^{2k-1} \mathbb{C}\{x, y\}e_i \xrightarrow{\psi} \bigoplus_{i=0}^{2k-1} \mathbb{C}\{x, y\}e_i \xrightarrow{\varphi} \mathbb{C}\{t\} \longrightarrow 0$$

with $\varphi(e_i) = t^i$, and where ψ is given by the matrix

$$
\underline{\psi} = \begin{array}{cc}
\begin{array}{cc}\overbrace{\qquad\qquad\qquad\qquad}^{k} & \overbrace{\qquad\qquad\qquad\qquad}^{k}\end{array} & \\
\left[\begin{array}{ccccc|ccccc}
-y & 0 & 0 & \cdots & 0 & x & 0 & \cdots & \cdots & 0 \\
0 & -y & 0 & & & 0 & x & & & \vdots \\
\vdots & & \ddots & & & \vdots & & \ddots & & \vdots \\
0 & & & & -y & & & & & x \\
\hline
x^2 & 0 & \cdots & & & -y & & & & 0 \\
0 & x^2 & & & & & \ddots & & & \vdots \\
\vdots & & \ddots & & & & & \ddots & & 0 \\
0 & & & & x^2 & & & & & -y
\end{array}\right]
&
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\}k \\
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\}k
\end{array}
\end{array}
$$

We give it as an exercise to check that the sequence is exact, for there is a general reason why $\mathbb{C}\{t\}$ must have such a resolution of length 1 as $\mathbb{C}\{x, y\}$-module (see §3, 3.5).

By permuting rows and columns of ψ one checks that $\det \underline{\psi} = (y^2 - x^3)^k$ i.e. we have shown that

$$F_0(f_*\mathcal{O}_{\mathbb{C}})_0 = (y^2 - x^3)^k \mathbb{C}\{x, y\}.$$

Let us now calculate $\boldsymbol{Ann}_{\mathbb{C}\{x,y\}}\, \mathbb{C}\{t\}$. Since $1 \in \mathbb{C}\{t\}$, the annihilator is just the kernel of the map $\mathbb{C}\{x, y\} \to \mathbb{C}\{t\}$, which is the ideal generated by $(y^2 - x^3)$, certainly different from our Fitting ideal if $k > 1$.

Let us now make a base change by restricting our map over the x-axis, i.e. by the inclusion $\{y = 0\} \subset (\mathbb{C}^2, 0)$ or algebraically by $\mathbb{C}\{x, y\} \to \mathbb{C}\{x\}$ sending y to 0. Then $\mathbb{C}\{t\} \otimes_{\mathbb{C}\{x,y\}} \mathbb{C}\{x\} = \mathbb{C}\{t\}/(t^{3k})$ viewed as $\mathbb{C}\{x\}$-module via the map sending x to t^{2k}. Then, the annihilator of this $\mathbb{C}\{x\}$-module is $(x^2)\mathbb{C}\{x\}$ while the image in $\mathbb{C}\{x\}$ of $(y^2 - x^3)\mathbb{C}\{x, y\}$ is $(x^3)\mathbb{C}\{x\}$. This shows that the formation of the annihilator does not commute with base change.

We will now construct the resultant of two polynomials as a Fitting ideal. As an application we will prove Bezout's theorem.

Let A be a ring and let $P, Q \in A[X]$ be two polynomials:

$$P = p_0 + p_1X + \cdots + p_nX^n$$
$$Q = q_0 + q_1X + \cdots + q_mX^m$$

We will assume that p_n and q_m are units in A. The resultant $R(P, Q)$ will be an element in A, satisfying the following property:

For any field K and any homomorphism

$$\varphi: A \to K, \quad \text{the polynomials} \quad \varphi(P), \varphi(Q) \in K[X]$$

have a common root in an extension of K if and only if $\varphi(R(P, Q)) = 0$.

Here $\varphi(P)$ denotes the polynomial $\varphi(p_0) + \varphi(p_1)X + \cdots \varphi(p_n)X^n \in K[X]$. Remark first that the A-module $M = A[X, Y]/(P, Q)$ satisfies $M \otimes_A K = 0$ if and only if $\varphi(P)$ and $\varphi(Q)$ have no common root in any extension of K. In fact, they do not have a common root if and only if we can find an extension K' of K and polynomials $S, T \in K'[X]$ with

$$1 = S\varphi(P) + T\varphi(Q)$$

which clearly is equivalent to $A[X]/(P, Q) \otimes_A K = 0$.

NOTE. The existence of such an element $R(P, Q) \in A$ is not at all clear a priori, and anyway if it exists the properties we ask of it will also be satisfied by $UR(P, Q)$, where U is any invertible element of A. Therefore what we can see a priori, is that we can hope to define a resultant *ideal* in A, and then prove it is a principal ideal, which is what we do below.

The geometric meaning of the resultant is the following. Let A correspond to a space S, i.e. let A be the ring of germs of holomorphic functions at a point $s \in S$. Then $A[X]$ corresponds to the space $S \times \mathbb{C}$. Denote by $\pi: S \times \mathbb{C} \to S$ the projection.

The polynomials P, Q define hypersurfaces $V(P)$ and $V(Q)$ in $S \times \mathbb{C}$. Let $m \subset A$ be the maximal ideal, and let $\varphi: A \to A/m$ be the canonical map. Then $\varphi(P)$ and $\varphi(Q)$ have a common root if and only if $V(P) \cap V(Q)$ intersects the fibre $\pi^{-1}(s)$, that is $s \in \pi(V(P) \cap V(Q))$. Thus by the property above the resultant should be a defining equation for the *image* of $V(P) \cap V(Q)$ by π. This motivates the following definition.

Definition. Let M be the A-module $M = A[X]/(P, Q)$. We define the resultant ideal of P and Q as $F_0(M) \subset A$.

Lemma. *$F_0(M)$ is a principal ideal.*
(In fact we will by abuse of language call any generator of $F_0(M)$ the resultant of P and Q and write it $R(P, Q)$.)

Proof. We will write down a presentation of M as an A-module, namely

$$A[X]/(P) \xrightarrow{\psi_1} A[X]/(P) \longrightarrow A[X]/(P, Q) \longrightarrow 0$$

Where ψ_1 is multiplication by Q, that is $\psi_1(\bar{a}) = \overline{Qa}$, where the bar means reduction modulo (P). Now as the highest-coefficient of P was invertible, $A[X]/(P)$ is a free A-module of rank n. Thus we have a finite presentation and moreover ψ_1 is represented by a square matrix so $F_0(M)$ is generated by its determinant.

We could use other presentations too, for example:

$$A[X]/(X^n) \oplus A[X]/(X^m) \to A[X]/(P \cdot Q) \to A[X]/(P \cdot Q) \to 0$$
$$(\bar{a}, \bar{b}) \to \overline{aQ + bP}$$

or the presentation coming from the 'chinese exact sequence' (i.e. Chinese remainder theorem)

$$A[X]/(P \cdot Q) \to A[X]/(P) \oplus A[X]/(Q) \to A[X]/(P, Q) \to 0$$
$$\bar{a} \to (\bar{a}, \bar{a}) \qquad (\bar{a}, \bar{b}) \to \bar{a} - \bar{b}$$

where the bar indicates reduction modulo the ideal in question.

The natural ring in which to treat the resultant is the following

$$\mathcal{A} = \mathbb{Z}\left[p_0, \cdots, p_n, q_0, \cdots, q_m, \frac{1}{p_n}, \frac{1}{q_m}\right]$$

where the p_i and q_j are indeterminates.

In the ring $\mathcal{A}[X]$ we have the two polynomials $\mathcal{P}$ and $\mathcal{Q}$ of degree n and m respectively given by

$$\mathcal{P} = p_0 + p_1 X + \cdots + p_n X^n$$
$$\mathcal{Q} = q_0 + q_1 X + \cdots + q_m X^m$$

They have the following universal property: Take any ring A and any two polynomials $P, Q \in A[X]$ of degree n and m, and suppose that their highest-degree coefficients are invertible in A. Then there exists a unique homomorphism $\varphi: \mathcal{A} \to A$ with $\varphi(\mathcal{P}) = P$ and $\varphi(\mathcal{Q}) = Q$. It is given by sending p_i to the coefficient of X^i in P and q_j to the coefficient of X^j in Q.

The ring $\mathcal{A}$ has a grading given by degree $p_i = n - i$ and degree $q_j = m - j$. If we give X the degree 1, the polynomials $\mathcal{P}$ and $\mathcal{Q}$ are homogeneous of degree m and n respectively for the corresponding grading of $\mathcal{A}[X]$. Moreover if the ring A above is graded and P and Q are homogeneous, the homomorphism φ will be homogeneous of degree 0.

Lemma 0. *In the situation described above, the resultant $R(\mathcal{P}, \mathcal{Q}) \in \mathcal{A}$ is homogeneous of degree mn (i.e. the resultant ideal is homogeneous and generated by an element of degree mn).*

To prove this we need a lemma on graded modules. Let A be any graded ring and let ν be an integer. We define a graded A-module $A(\nu)$. As an A-module it is just A itself, we change only the grading by giving 1 the degree ν. That is, for any homogeneous element $x \in A$, its degree in $A(\nu)$ is $\nu +$(its degree in A).

Lemma 1. *For any homogeneous homomorphism of degree* 0:

$$\psi: \bigoplus_{i=1}^{q} A(e_i) \to \bigoplus_{j=1}^{q} A(f_j).$$

[This represents just a graded homomorphism between two free A-modules, graded as indicated] the Fitting ideals F_k (coker ψ) are homogeneous.

Moreover,

$$\deg(\det \psi) = \sum_{i=1}^{q} e_i - \sum_{j=1}^{q} f_j.$$

Proof. Let ψ be represented by the matrix (ψ_{ij}). By writing what is happening to a basis element in $\bigoplus_{i=1}^{q} A(e_i)$ it is easily verified that ψ_{ij} is homogeneous of degree $e_i - f_j$. In the expansion of $\det(\psi)$ each term is a product $\psi_{i_1 j_1} \cdot \psi_{i_2 j_2} \cdot \psi_{i_3 j_3} \cdot \cdots \cdot \psi_{i_q j_k}$ where each $i \in \{0, \cdots, g\}$ and each $j \in \{0, \cdots, q\}$ appears exactly once. Hence it is homogeneous of degree $\sum_{\nu=1}^{q} e_{i_\nu} - \sum_{\nu=1}^{q} f_{j_\nu} = \sum_{i=1}^{q} e_i - \sum_{j=1}^{q} f_j$. Thus we have proved the part of the lemma concerning $\det \psi$.

As F_k (coker ψ) is generated by minors of ψ, the same argument shows that it is homogeneous.

To calculate deg $R(\mathcal{P}, \mathcal{Q})$ we go back to our presentation:

$$\mathcal{A}[X]/(\mathcal{P}) \xrightarrow{\mathcal{Q}} \mathcal{A}[X]/(\mathcal{P}) \longrightarrow \mathcal{A}[X]/(\mathcal{P}, \mathcal{Q}) \longrightarrow 0$$

The free $\mathcal{A}$-module $A[X]/(\mathcal{P})$ has a basis $1, X, X^2, \cdots, X^n$. We will assign degrees to this such that the multiplication by $\mathcal{Q}$ becomes homogeneous of degree 0, that is, as $X^i\mathcal{Q}$ has degree $m+i$, we must give X^i the degree $m+i$. Hence $e_i = m+i$ and $f_j = j$, which gives $\deg(R(\mathcal{P}, \mathcal{Q})) = \sum_{i=1}^n (m+i) - \sum_{i=1}^n i = nm$.

Exercise. Check this computation using the other presentations of $\mathcal{A}[X]/(\mathcal{P}, \mathcal{Q})$.

Now we will, as an application, give a proof of Bezout's theorem.

Theorem. *Let $C_1, C_2 \subseteq \mathbb{P}^2(\mathbb{C})$ be two curves defined by homogeneous polynomials P and Q of degree n and m. Suppose they have no common component. Then*

$$mn = \sum_{y \in \mathbb{P}^2} \dim_{\mathbb{C}} (\mathcal{O}_{\mathbb{P}^2,y}/(P, Q)\mathcal{O}_{\mathbb{P}^2,y}).$$

In the proof we will need another lemma, very similar in nature to Lemma 1 above:

Lemma 2. *Let $A = \mathbb{C}\{t\}$ and let v be its valuation. Suppose that $\psi: A^p \to A^p$ is an homomorphism whose cokernel is of finite length, i.e. a finite dimensional vector space over $\mathbb{C}$. Then*

$$v(\det \psi) = \dim_{\mathbb{C}} (\operatorname{coker} \psi).$$

Proof. By the main theorem on principal ideal domains we can find bases for A^p such that the matrix representing ψ is a diagonal matrix:

$$\psi = \begin{bmatrix} a_1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & a_p \end{bmatrix}$$

Clearly $v(\det \psi) = \sum v(a_i)$ and

$$\dim (\operatorname{coker} \psi) = \sum \dim_{\mathbb{C}} (A/a_iA)$$

Hence we may assume $p = 1$. Then ψ is just multiplication by an element a. If $v(a) = n$, we can write $a = ut^n$ with u a unit. One easily checks that $1, t, t^2, \cdots, t^{n-1}$ is a $\mathbb{C}$-basis for $A/aA = \mathbb{C}\{t\}/(ut^n)\mathbb{C}\{t\}$. Hence $v(a) = \dim_{\mathbb{C}} A/aA$.

We will now prove Bezout's theorem. Let $A = \mathbb{C}[x_1, x_2]$ and let $P, Q \in A[x_0]$ be the two homogeneous polynomials defining the curves C_1 and C_2. We can write

$$P = \sum_{i=1}^{n} p_i(x_1, x_2)x_0^i \qquad \deg p_i(x_1, x_2) = n - i$$

$$Q = \sum_{j=1}^{m} q_j(x_1, x_2)x_0^j \qquad \deg q_j(x_1, x_2) = m - j$$

After a change of coordinates, we may assume that $p_n(x_1, x_2)$ and $q_m(x_1, x_2)$ are non-zero, that is invertible in A. Geometrically this means that the point $(1, 0, 0)$ is not on any of the curves C_1 and C_2. There exists then a homogeneous homomorphism of degree 0, $\varphi: \mathcal{A} \to A$ with $\varphi(\mathcal{P}) = P$ and $\varphi(\mathcal{Q}) = Q$. Clearly $\varphi(R(\mathcal{P}, \mathcal{Q}) = R(P, Q)$, and we obtain: If $R(P, Q) \not\equiv 0$, then $\deg(R(P, Q)) = mn$. (as a polynomial in x_1, x_2).

I leave it as an exercise to check, using the factoriality of polynomial rings over $\mathbb{C}$, that if $R(P, Q) \equiv 0$ then $P = 0$ and $Q = 0$ have an irreducible component in common.

But the fundamental theorem of algebra can be subsumed by the following formula:

Let $R \in \mathbb{C}[x_1, x_2]$ be a homogeneous polynomial of degree d, and for every point $x = (\bar{x}_1, \bar{x}_2) \in \mathbb{P}^1(\mathbb{C})$, let us denote by $\mathcal{O}_{\mathbb{P}^1,x}$ the local ring of $\mathbb{P}^1(\mathbb{C})$ at x, i.e. the ring of all fractions $S(x_1, x_2)/T(x_1, x_2)$ with T and S homogeneous polynomials and $T(\bar{x}_1, \bar{x}_2) \neq 0$. This is a discrete valuation ring, and we denote by $v_x(R)$ the valuation of the image of R in this ring, i.e. $R/1$. Then:

$$d = \sum_{x \in \mathbb{P}^1(\mathbb{C})} v_x(R)$$

where of course the sum on the right is finite since $v_x(R) \neq 0$ only if $R(\bar{x}_1, \bar{x}_2) = 0$.

Let us now consider the projection $\pi: P^2(\mathbb{C}) \setminus \{(1, 0, 0)\} \to \mathbb{P}^1(\mathbb{C})$ given by $(x_0, x_1, x_2) \to (x_1, x_2)$. For each point $x \in \mathbb{P}^1$ there are finitely many points $y \in C_1 \cap C_2$ such that $\pi(y) = x$, and since $R(P, Q)$ is the ideal $\mathcal{F}_0(\mathcal{O}_{\mathbb{P}^2}/(P, Q))$

it follows by localization that

$$R(P,Q)\mathcal{O}_{\mathbb{P}^1,x}=F_0\Big(\bigoplus_{\pi(y)=x}\mathcal{O}_{\mathbb{P}^2,y}/(P,Q)\mathcal{O}_{\mathbb{P}^2,y}\Big)$$

so that by our Lemma 2 above

$$v_x(R(P,Q))=\sum_{\pi(y)=x}\dim_{\mathbb{C}}\mathcal{O}_{\mathbb{P}^2,y}/(P,Q)\mathcal{O}_{\mathbb{P}^2,y}$$

and by summing up and using the fundamental theorem of algebra, and Lemma 1 above, we obtain Bezout's theorem. Remark that in Bezout's theorem the right-hand side is a sum of *local* terms, called the intersection multiplicity of C_1 and C_2 at $y\in\mathbb{P}^2$, and usually noted $(C_1,C_2)_y$ or $i(C_1,C_2,\mathbb{P}^2,y)$. Bezout's theorem provided the first and most basic examples of computation of local and global intersection multiplicities. Remark also that we deduced Bezout's formula from its analogue in $\mathbb{P}^1$, which is: $\deg R=\sum_{x\in\mathbb{P}^1}v_x(R)$, only by using a good definition of the image. Anyway, the formula can easily be generalized by these methods to the case of n hypersurfaces in $\mathbb{P}^n$.

Exercises on images

Exercise 1. Check that if $Y\xrightarrow{i}X$ is a closed immersion, defined by a sheaf of ideals I, then the image of i is Y.

Exercise 2. Let Y_1 and Y_2 be two closed subspaces of X defined respectively by I_1 and I_2. Then the image of the natural map $p:Y_1\amalg Y_2\to X$ is the subspace defined by I_1I_2, and therefore in general different from $Y_1\cup Y_2$, which is defined by $I_1\cap I_2$ (which is the annihilator ideal of $p_*\mathcal{O}_{Y_1\cup Y_2}$).

Remark. If you are surprised by this, think of mapping the two points in $\mathbb{C}^2$ with coordinates $(x=0,y=1)$ $(x=0,y=2)$ to the x-axis. The image should be defined by $(x^2)\mathbb{C}\{x\}$ since as soon as you move the points a little, you get two distinct images.

Addendum

In what follows, I shall use the definition of the image of a finite map $f:X\to Y$ of complex analytic spaces only in the case where f is not surjective. In general, my definition of the image does not have the property that if $g:Y\to Z$ is another finite map, then $im(g\circ f)$ is the image by g of $im(f)$. To see this, let us consider a finite map $h:X\to Z$ such that $F_0(h_*\mathcal{O}_X)\neq\operatorname{Ann}_{\mathcal{O}_Z}(h_*\mathcal{O}_X)$. We have already seen that such maps exist. Then, let us define Y to be the subspace of Z defined by $\mathbf{Ann}_{\mathcal{O}_Z}(h_*\mathcal{O}_X)$: we can obviously factor h as $g\circ f$ where $g:Y\to Z$ is the inclusion, and $f:X\to Y$. Since f is surjective, $F_0(f_*\mathcal{O}_X)=0$, hence $im(f)=Y$ and its image by g is Y in Z (see exercise 1 on images, §1), but by our assumption, Y is different from the image of $h=g\circ f$, which is the subspace of Z defined by $F_0(h_*\mathcal{O}_X)$. Apparently, if we want a definition of the image which not only is compatible with base changes, but also behaves well under composition, the only possibility is to decide that $f_*\mathcal{O}_X$ 'is' the image, and then to remark that when f is not surjective, then the Fitting ideal construction associates to $f_*\mathcal{O}_X$ a subspace of Y, in a natural way. In fact we have:

Proposition. *Let there be given for every A-module of finite presentation M an ideal $\mathfrak{A}_A(M)$ of A, this correspondence satisfying the following conditions:*

(1) *for any ring homomorphism $\varphi:A\to B$, we have*

$$\mathfrak{A}_B(M\otimes_A B)=\mathfrak{A}_A(M)\cdot B\quad(\textit{compatibility with base change})$$

(2) $$\sqrt{\mathfrak{A}_A(M)}=\sqrt{\operatorname{Ann}_A(M)}\quad(\textit{set-theoretically the right one})$$

(3) *If A is a discrete valuation ring*

$$v(\mathfrak{A}_A(M))=\lg_A M\quad(\lg=\text{length})$$

i.e. Lemma 2 of the proof of Bezout's theorem in §1 holds. Then $\mathfrak{A}_A(M)=F_0(M)$ for all A and M.

Sketch of proof: given a presentation $A^q\xrightarrow{\psi}A^p\to M\to 0$ with matrix $\psi=(\psi_{ij})$, we work in the 'universal ring' $A=\mathbb{Z}[T_{ij}]$ (or $\mathbb{C}\{T_{ij}\}$ if we really want to stay within complex analytic geometry) and consider the 'universal module' $\mathcal{M}=$ cokernel of $A^q\xrightarrow{\psi}A^p$ where the matrix of ψ is (T_{ij}).

Then, thanks to theorems of Macaulay (algebraic theory of modular systems, Cambridge University Press 1919) and Buchsbaum-Rim (see §5) we know that the subspace of $\operatorname{Spec}\mathcal{A}=A^{pq}$ (or $\mathbb{C}^{pq}$ in analytic geometry) defined by the $p\times p$ minors of ψ is *reduced* and all its components are of codimension $q-p+1$. In this case, then $F_0(\mathcal{M})=\operatorname{Ann}_{\mathcal{A}}(\mathcal{M})$. Suppose that $\mathfrak{A}_{\mathcal{A}}(\mathcal{M})\neq F_0(\mathcal{M})$, the subspaces they define must be the same set-theoretically, and if they differ as spaces, we can already see it by restricting to arcs in $\mathbb{C}^{pq}$ (or mapping $\mathbb{Z}[T_{ij}]$ into discrete valuation rings). But condition (3)+(1) implies that $\mathfrak{A}_{\mathcal{A}}(\mathcal{M})$ and $F_0(\mathcal{M})$ cannot become different when we restrict them to arcs in $\mathbb{C}^{pq}$. Therefore they must be equal, i.e. $\mathfrak{A}_{\mathcal{A}}(\mathcal{M})=F_0(\mathcal{M})$. Since our original M is of course $\mathcal{M}\otimes_{\mathcal{A}}A$ where $\varphi:\mathcal{A}\to A$ is the morphism sending T_{ij} to ψ_{ij}, we are done.

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§2. First part: The module of differentials

In this section, we talk about differentials and non-singularity. We will define the *critical subspace* (and not just the critical locus) and the *discriminant subspace* (and not just the branch locus) of some flat complex analytic mappings, using Fitting ideals, and we will prove a small theorem on discriminants.

2.1. There is a complex analytic space, usually denoted by $\mathbb{T}$, which is the subspace of the complex line $\mathbb{C}$ (with coordinate v) defined by the ideal (v^2), i.e. $(v^2)\mathbb{C}\{v\}$. The underlying topological space of $\mathbb{T}$ is therefore just a point, and its 'structure sheaf' is $\mathbb{C}\{v\}/(v^2)$. It is just an infinitesimal direction, and for that reason, Mumford calls it the 'disembodied tangent vector'.

Define the Zariski tangent space $E_{X,x}$ of an analytic space $(X, \mathcal{O}_X)$ at a point $x \in X$ as the set of complex-analytic mappings $\mathbb{T} \to X$ having x as (set-theoretic) image. Algebraically, it is described as the set of morphisms of $\mathbb{C}$-algebras $\mathcal{O}_{X,x} \to \mathbb{C}\{v\}/(v^2)$, which is the same as the set of $\mathbb{C}$-linear mappings $m/m^2 \to \mathbb{C}$ where m is the maximal ideal of $\mathcal{O}_{X,x}$. This gives $E_{X,x}$ its natural structure of $\mathbb{C}$-vector space: $E_{X,x} = (m/m^2)^*$.

Remark that $\dim_{\mathbb{C}} E_{X,x} = \dim_{\mathbb{C}} m/m^2$ and that by Nakayama $\dim_{\mathbb{C}} m/m^2$ is the minimal number of generators of the ideal m, which is therefore also the smallest integer N such that there exists a surjection of $\mathbb{C}$-algebras $\mathbb{C}\{x_1, \cdots, x_N\} \to \mathcal{O}_{X,x}$. Geometrically this integer is the smallest N such that there exists a germ of closed imbedding $(X, x) \subset (\mathbb{C}^N, 0)$: it is called the *imbedding dimension* of X at x.

EXERCISE. Build singular points of curves with arbitrarily large imbedding dimension.

EXERCISE. Compute the image of a morphism $p: \mathbb{T} \to X$ as a *subspace* of X, i.e. given a morphism of $\mathbb{C}$-algebras $\mathcal{O}_{X,x} \xrightarrow{p^*} \mathbb{C}\{v\}/(v^2)$, compute the 0th Fitting ideal of $\mathbb{C}\{v\}/(v^2)$ as $\mathcal{O}_{X,x}$-module. Hint: Use a presentation of $\mathcal{O}_{X,x}$ as quotient of a convergent power series ring as above, and distinguish two cases: p^* is surjective, or not. You'll be surprised!

2.2. It turns out that there is a coherent sheaf of modules on X which has as fibre at $x \in X$ exactly m/m^2: it is the sheaf of differentials on X. I will now

give the construction of this sheaf not only for a space, but also for a morphism, as we will need this later:

Let $f: X \to S$ be a morphism. For simplicity, and because we will examine only local problems, we assume f separated, i.e. that the diagonal $D_X \subset X \times_S X$ is closed. Let I be the sheaf of ideals describing D_X as subspace of $X \times_S X$. Then, by definition, the module of differentials of X/S is I/I^2, which can be viewed as $\mathcal{O}_{X \times X}/I$-module since it is annihilated by I. I/I^2 becomes an $\mathcal{O}_X$-module via the isomorphism $\mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_{X \times X/I}$ given by p_1^* where $p_1: X \times X \to X$ is the first projection.

This sheaf is denoted by $\Omega^1_{X/S}$, and simply Ω^1_X in the case where S is a point. Sometimes it is also written Ω^1_f.

Remark that, for any $s \in S$, $\Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_s} = \Omega^1_{X_s}$ where $X_s = f^{-1}(s)$ is the fibre of $X \to S$ over s, defined by $m_{S,s}\mathcal{O}_X$. More generally, one can check that 'The formation of $\Omega^1_{X/S}$ commutes with base change' i.e. given a cartesian diagram (or fibre-product)

$$\begin{tikzcd} X' \arrow[r, "h'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \\ S' \arrow[r, "h"] & S \arrow[u, "f"'] \end{tikzcd} \qquad (X' \simeq X \times_S S')$$

we have:

$$h'^* \Omega^1_{X/S} = \Omega^1_{X'/S'}.$$

Also, $\Omega^1_{X/S}$ comes equipped with a natural $\underline{\mathbb{C}}$-linear map $d_{X/S}: \mathcal{O}_X \to \Omega^1_{X/S}$ (we write d in the case S is a point) defined as $d = pr_1^* - pr_2^* \bmod I^2$, i.e. given a function $g: U \to \mathbb{C}$ on an open subset of X (i.e. $g \in \Gamma(U, \mathcal{O}_X)$) we associate to g the function on $U \times_S U$ which takes at $(x, x') \in U \times_S U$ the value $g(x) - g(x')$, and we remark that this function belongs to I. dg is its class in I/I^2. Remark that $d(g \cdot g') = g \cdot dg' + g' \cdot dg$.

Now let us consider the following construction: Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. Then for any $x \in X$, there is an open neighborhood U of x in X such that we have an exact sequence

$$\mathcal{O}_U^q \to \mathcal{O}_U^p \to \mathcal{M}|_U \to 0$$

So the symmetric algebra $\mathrm{Sym}_{\mathcal{O}_U} \mathcal{M}|_U$ is a quotient of $\mathcal{O}_U[T_1, \cdots, T_p]$ by an ideal generated by elements which are linear in the T_i, and finite in number. These elements describe a 'linear' subspace of $U \times \mathbb{C}^p$ (linear in the $\mathbb{C}^p$ coordinates) which is therefore a *relative vector space* over U, which we call $\mathrm{Specan}_U \mathrm{Sym}_{\mathcal{O}_U} \mathcal{M}|_U$. This local construction glues up naturally, and therefore we can define a relative vector space $\mathrm{Specan}_X \mathrm{Sym}_{\mathcal{O}_X} \mathcal{M} \to X$ over X. (It is the $L(\mathcal{M})$ of §1.) To give ourselves a section of this is, by the universal property of the symmetric algebra, to give ourselves a map $\mathcal{M} \to \mathcal{O}_X$ of $\mathcal{O}_X$-modules.

DEFINITION. The Zariski tangent space of $(X, \mathcal{O}_X)$ is the relative vector space $T_X = \operatorname{Specan}_X \operatorname{Sym}_{\mathcal{O}_X} \Omega^1_X \to X$.

REMARK. We like to think of a holomorphic vector field on X as a section of the Zariski tangent space $T_X \to X$. Algebraically, it means an $\mathcal{O}_X$-linear map $\Omega^1_X \xrightarrow{\nabla} \mathcal{O}_X$. By composing ∇ with $d: \mathcal{O}_X \to \Omega^1_X$, we get a $\mathbb{C}$-linear map $D: \mathcal{O}_X \to \mathcal{O}_X$ satisfying $D(g \cdot g') = g'Dg + g \cdot Dg'$. i.e. a *derivation* of $\mathcal{O}_X$ into itself. More generally, one can easily check that any $\mathbb{C}$-linear map $D: \mathcal{O}_X \to \mathcal{M}$ satisfying $D(g \cdot g') = g \cdot Dg' + g' \cdot Dg$ ($\mathcal{M}$ an $\mathcal{O}_X$-module) is obtained as $\nabla \cdot d$ where ∇ is an $\mathcal{O}_X$-linear map $\nabla: \Omega^1_X \to \mathcal{M}$. [This is the universal property of the module of differentials Ω^1_X, or more precisely of the differential $d: \mathcal{O}_X \to \Omega^1_X$, it is the 'universal derivation']. In particular, Ω^1_X is generated as $\mathcal{O}_X$-module by the dg, $g \in \mathcal{O}_X$. All this enables us to identify the holomorphic vector fields on X (i.e. sections of $T_X \to X$) with derivations of $\mathcal{O}_X$ into itself. Also, we remark that the fibre $T_X(x)$ of $T_X \to X$ over $x \in X$ is in fact $\operatorname{Hom}_{\mathbb{C}}(m/m^2, \mathbb{C})$ i.e. the Zariski tangent space $E_{X,x}$.

EXERCISE. Check that the datum of an holomorphic vector field on X is the same thing as the datum of a complex-mapping $X \times \mathbb{T} \to X$ which induces the identity of X on $X \times \{0\} \subset X \times \mathbb{T}$.

2.3. Exact sequences of modules of differentials. When we have a map $f: X \to S$, we expect to be able to define a tangent map $Tf: T_X \to T_S$, or more precisely: $Tf: T_X \to T_S \times_S X$. With our definition of T_X, and the fact that Spec is contravariant this means we want to describe the tangent map by: $\partial f: f^*\Omega^1_S \to \Omega^1_X$ and indeed, there is an exact sequence of sheaves of $\mathcal{O}_X$-modules

$$f^*\Omega^1_S \xrightarrow{\partial f} \Omega^1_X \longrightarrow \Omega^1_{X/S} \longrightarrow 0$$

where ∂f is defined as follows: by linearity, it is sufficient to define ∂f on elements of the form $\xi = dg \otimes 1|_V \in \Gamma(V, f^*\Omega^1_S)$, where $g \in \Gamma(U, \mathcal{O}_S)$ for some open U of S. (This is because the dg's generate Ω^1_S as $\mathcal{O}_S$-module.) Then we define $\partial f(\xi)$ as $d(g \cdot f|_V) \in \Gamma(V, \Omega^1_X)$.

We note that ∂f is not in general injective. The fact that the sequence above is exact is often used as a definition of $\Omega^1_{X/S}$, and it is not very hard to check that it coincides with the definition given above.

Another important exact sequence is the following:

Suppose we have been able to imbed our mapping $f: X \to S$ locally around a point $x \in X$. That is, we have, after restricting $(X, \mathcal{O}_X)$ to some

open neighborhood of x in X which we still write X for simplicity, a diagram

$$\begin{array}{ccc} (X, x) & \subset & (S \times C^N, s \times 0) \\ \downarrow & \swarrow & \\ (S, s) & & \end{array}$$

where X is defined in $S \times C^N$ (locally near x) by an ideal $J = (f_1, \cdots, f_k)\mathcal{O}_{S\times\mathbb{C}^N}$, $f_i \in \mathcal{O}_S\{z_1, \cdots, z_N\}$.

Then, we have an exact sequence as follows: (of $\mathcal{O}_X$-modules)

$$J/_{J^2} \xrightarrow{(d)} \Omega^1_{S\times\mathbb{C}^N/S}|_X \longrightarrow \Omega^1_{X/S} \longrightarrow 0$$

where the map (d) is defined as follows: Let $g = \sum_1^k a_i f_i \in J$ then, $dg = \sum a_i \cdot df_i + \sum da_i \cdot f_i$ in $\Omega^1_{S\times\mathbb{C}^N/S}$ (d is actually $d_{S\times\mathbb{C}^N/S}$ here, i.e. we differentiate only with respect to coordinates on $\mathbb{C}^N$) is equal to $\sum a_i\, df_i$ modulo $J\Omega^1_{S\times\mathbb{C}^N/S}$ and in particular is in $J\Omega^1_{S\times\mathbb{C}^N/S}$ if $a_i \in J$, i.e. $g \in J^2$. Now $\Omega^1_{S\times\mathbb{C}^N/S|X} = \Omega^1_{S\times\mathbb{C}^N/S}/J\Omega^1_{S\times\mathbb{C}^N/S}$ so our map (d) is well defined.

If we choose coordinates $z_1, \cdots, z_N$ on $\mathbb{C}^N$, then $\Omega^1_{S\times\mathbb{C}^N/S}$ is the free $\mathcal{O}_{S\times\mathbb{C}^N}$-module generated by the dz_j $(1 \leq j \leq N)$, i.e.

$$\Omega^1_{S\times\mathbb{C}^N/S} = \sum_{j=1}^{N} \mathcal{O}_{S\times\mathbb{C}^N} \cdot dz_j$$

and the exactness of the sequence means that $\Omega^1_{X/S}$ is just the quotient of $\Omega^1_{S\times\mathbb{C}^N/S} \mid_X$ by the submodule generated by the images of the df_k, which is easy to check using the universal property of the module of relative differentials $\Omega^1_{X/S}$, as above (for derivations of $\mathcal{O}_X$ into $\mathcal{M}$ which are 0 on $\mathcal{O}_S$). The sheaf J/J^2 is often called the conormal sheaf of X in $S \times \mathbb{C}^N$ and the above exact sequence is in fact a special case of a general exact sequence, where we have Z instead of $S \times \mathbb{C}^N$

$$\mathcal{N} \to \Omega^1_{Z/S} \mid X \to \Omega^1_{X/S} \to 0$$

which is commonly called the exact sequence of the normal bundle: after dualization, in the case $S = \text{point}$, and X and Z smooth, it gives the usual sequence of sheaves coming from the sequence of vector bundles:

$$0 \to T_X \to T_Z \mid_X \to N^*_{Z,X} \to 0$$

where $N^*_{Z,X}$ is the normal bundle of X in Z.

This aspect has been generalized to normal *cones*, which replace the normal bundle in the singular case, in [Lejeune-Teissier, Normal cones and sheaves of relative jets. Compositio Mathematica *28*, 1974].

2.4. Implicit function theorem, and simplicity theorem

Intuitively, we expect that if the Zariski tangent space $T_X \to X$ is actually a bundle, i.e. the dimension of its fibres is constant, then X will be non-singular. There are actually several versions of this result, according to whether we assume X reduced or not to start with, and whether we like to have a relative theorem or not. Let me start with the relative theorem:

Simplicity Theorem. *Let $f: X \to S$ be a* flat *morphism of complex spaces, and $x \in X$. The following are equivalent*
(i) *there is an S-isomorphism of germs $(X, x) \simeq (S \times \mathbb{C}^d, s \times 0)$ where $s = f(x)$*
(ii) *$f^{-1}(f(x))$ is non-singular of dimension d at x*
(iii) *$\Omega^1_{X/S}$ is locally free of rank d at x.*

(i.e. $T_{X_s} \to X_s$ is a vector bundle of rank d for all $s \in S$). In particular, taking $S =$ a point, we have that: X is non-singular at $x \Leftrightarrow \mathcal{O}_{X,x} \simeq \mathbb{C}\{z_1, \cdots, z_d\} \Leftrightarrow \Omega^1_X$ is locally free, generated by $dz_1, \cdots, dz_d$ at x.

We will see below that (i)$\Leftrightarrow$(ii) can be best interpreted by saying that a non-singular germ of complex space is *rigid* in the sense that any flat deformation of it (such as our f, as deformation of $f^{-1}(s)$) is actually locally a product.

Anyway, let me now describe an avatar of the implicit function simplicity theorem; first, why implicit function: suppose we have a map $f: (\mathbb{C}^N, 0) \to (\mathbb{C}^P, 0)$. The implicit function theorem says: if the tangent map at 0 is surjective, then f is simple, in the sense of the above result. This breaks down into two parts

① f is flat.
② $\Omega^1_{\mathbb{C}^N/\mathbb{C}^P}$ is locally free.

① is a very general fact: For $f: X \to S$ and $x \in X$, $s = f(x)$ if the tangent map $C_{X,x} \to C_{S,s}$ (of tangent cones) is flat, then f is flat at x. If X and S are smooth, then $C_{X,x}$ and $C_{S,s}$ are just the usual tangent maps, and for linear maps, flat is equivalent to surjective.

② comes from the sequence

$$f^*\Omega^1_{\mathbb{C}^P} \xrightarrow{\partial f} \Omega^1_{\mathbb{C}^N} \longrightarrow \Omega^1_{\mathbb{C}^N/\mathbb{C}^P} \longrightarrow 0$$

and the surjectivity of Tf is equivalent (via the usual Jacobian condition on the minors of the matrix describing ∂f as a map between locally free modules, which is the Jacobian matrix) to the fact that ∂f is everywhere of rank P, so that $\Omega^1_{\mathbb{C}^N/\mathbb{C}^P}$ has to be locally free of rank $N-P$.

Now here is the avatar of the simplicity theorem:

Proposition. *Let X be a reduced complex space of pure dimension d. Suppose that Ω^1_X has a locally free quotient of rank d. Then X is non-singular (and hence Ω^1_X is in fact locally free).*

Before going into the proof, let us give the geometric interpretation of this:

Even though Ω^1_X is not the sheaf of sections of a vector bundle, we can define, after Grothendieck, an associated Grassmannian space over X, $G = \mathrm{Grass}_d\, \Omega^1_X \to X$ which has the property that for any $x \in X$, the fibre is the Grassmannian of d-dimensional subspaces of $E_{X,x}$. The characteristic property of $g: G \to X$ is that for any map $h: T \to X$ it is equivalent to give oneself a locally free quotient of rank d of $h^*\Omega^1_X$ or to give oneself a factorization of h through G.

$$\begin{tikzcd} & G = \mathrm{Grass}_d\, \Omega^1_X \arrow[d, "g"] \\ T \arrow[ur, dashed, "h'"] \arrow[r, "h"'] & X \end{tikzcd} \qquad \begin{array}{l} \Leftrightarrow h^*\Omega^1_X \to L \to 0 \\ L \text{ locally free rank } d. \end{array}$$

[This is exactly right if you think of a locally free rank d quotient of Ω^1_X as a sub-relative vector space of $T_X \to X$ which is actually a vector bundle of rank d, i.e. picks a d-dimensional vector space in each fibre, in an 'analytic way'.] In particular, $g^*\Omega^1_X$ has a universal locally free quotient of rank d, corresponding to the tautological bundle on the grassmannian.

Now let $X^0 \subset X$ be the non-singular part of X. Of course, $\Omega^1_X|_{X^0} = \Omega^1_{X^0}$ is locally free of rank d since X is purely of dimension d. Hence we get a section $\sigma^0: X^0 \to G$ of $g: G \to X$. By Cartan's theorem, we can check that the closure in G of the image of σ^0 is a complex analytic reduced subspace of G, which we call X_1

$$X_1 = \overline{\sigma^0(X^0)}.$$

g induces a map $d: X_1 \to X$.

Proposition + Definition. *The induced map $d: X_1 \to X$ is a proper modification of X, which is surjective. We will call it the 'development' of X.*

Remark. For us 'proper modification' means a proper map, which is an isomorphism over an open dense subset (here X^0, of course). d is proper since X_1 is closed in G and g is proper of course. This implies that d is surjective, since its image is closed and dense in X.

Remark. A local version of d, presented in another manner, is often called 'Nash blowing up'. The reason why I chose another terminology is that d, as globally defined, is *not* the blowing up of a coherent sheaf of ideals in general, and also that d is the opposite operation of taking an envelope, which seems significant to me. This construction applied to the module of *relative* differentials, has been used by Hironaka in his lectures, and he called it '∂f-modification'.

REMARK. The fibre $d^{-1}(x)$ is best thought of as the set of limit directions of tangent spaces to X at non-singular points near x. (This is the meaning of our closure operation.)

CLAIM. The assertion of the proposition is equivalent to.

$$d \text{ is an isomorphism} \Leftrightarrow X \text{ is non-singular.}$$

In effect, to say that d is an isomorphism, is equivalent to saying that $\sigma^0 : X^0 \to G$ extends to a section of $g : \sigma : X \to G$ having then necessarily X_1 as its image. But by the universal property of the Grassmannian G, this is equivalent to saying that Ω^1_X has a locally free quotient of rank d.

Remark now that when X is a small representative of a germ of an analytically irreducible curve, d is in fact always an homeomorphism, (the limit of tangents at non-singular points is well defined and unique) so there is in effect something to prove. The proof is by induction on d, and uses the following:

Integration of vector fields (as taught by Zariski in: Studies in equisingularity I, II Amer. Journal of Maths. *87*, 1965). Let $\mathcal{O}$ be a complete local ring containing a field k of characteristic zero. Let $D : \mathcal{O} \to \mathcal{O}$ be a k-derivation of $\mathcal{O}$ such that $D\mathcal{O} \not\subset m$ where m is the maximal ideal of $\mathcal{O}$. Then, there exists $x \in \mathcal{O}$ and a subring $\mathcal{O}_1$ of $\mathcal{O}$ containing k, such that $\mathcal{O} = \mathcal{O}_1[[x]]$. [Translation: If you have an holomorphic vector field which is not 0 at the origin, you can integrate it to get an isomorphism $(X, x) \simeq (X_1 \times \mathbb{C}, x)$, at least formally.] In fact, we could do it in the convergent case, by extending to a non-singular ambient space and using the existence theorems of differential equations, but I think it is more informative to do it in the following way, as Zariski does:

If $D\mathcal{O} \not\subset m$, we can suppose that there exists $x \in \mathcal{O}$ such that $Dx = 1$, after multiplying D by an invertible element of $\mathcal{O}$. Let us consider the operator $e^{-xD} : \mathcal{O} \to \mathcal{O}$ defined by

$$e^{-xD}(h) = h - xDh + \frac{x^2}{2!} D^2 h + \cdots + (-1)^i \frac{x^i}{i!} D^i h + \cdots$$

which is in $\mathcal{O}$ since $\mathcal{O}$ is complete for the m-adic topology. One checks that the image of e^{-xD} is a subring $\mathcal{O}_1$ of $\mathcal{O}$ containing k and that since $Dx = 1$, we have $D_{|\mathcal{O}_1} = 0$. Furthermore, the kernel of e^{-xD} is $x \cdot \mathcal{O}$, so we have an isomorphism $\mathcal{O}/_{x \cdot \mathcal{O}} \simeq \mathcal{O}_1$; now, to check that the natural injection $\mathcal{O}_1[[x]] \subset \mathcal{O}$ is surjective is easy: take $h \in \mathcal{O}$, and define inductively h_i by: $h_0 = h$ and $h_{i-1} - e^{-xD}(h_{i-1}) = x \cdot h_i$. Then, setting $\bar{h}_i = e^{-xD}(h_i) \in \mathcal{O}_1$ we can see formally that

$$h = \bar{h}_0 + x\bar{h}_1 + x^2\bar{h}_2 + \cdots \in \mathcal{O}_1[[x]].$$

Now back to our proposition: The assertion is local, and to prove that an analytic algebra is a convergent power series ring, it is enough to prove that its completion is a formal power series ring. Thus we can reduce to the case of a complete local $\mathbb{C}$-algebra, with a module of differentials having a free quotient

$$\mathcal{O} \quad \text{and} \quad \Omega^1_{\mathcal{O}/\mathbb{C}} \to \mathcal{O}^d \to 0$$

Now let us choose a map $\mathcal{O}^d \to \mathcal{O}$ mapping the basis vector $(1, 0 \cdots 0)$ to 1 and all others to 0. There is an element $x \in \mathcal{O}$ such that the image of dx in $\mathcal{O}$ is invertible, so this gives us a derivation $D : \mathcal{O} \to \mathcal{O}$ and $x \in \mathcal{O}$ with $Dx = 1$. By the lemma, $\mathcal{O} = \mathcal{O}_1[[x]]$ but now we remark, using for example the geometric interpretation, that $\mathcal{O}_1$ satisfies exactly the same hypothesis as $\mathcal{O}$, and $\dim \mathcal{O}_1 = \dim \mathcal{O} - 1$. (e.g. use the fact that $d \times id_{\mathbb{C}} : X_1 \times \mathbb{C} \to X \times \mathbb{C}$ is the development of $X \times \mathbb{C}$ if $d : X_1 \to X$ is the development of X). So by induction on the dimension, we can assume that $\mathcal{O}_1$ is a formal power series ring $C[[x_1, \cdots, x_{d-1}]]$, and get the proposition, provided we can prove it for $d = 0$, but here we use the fact that $\mathcal{O}$ (hence $\mathcal{O}_1$) is reduced. A reduced analytic algebra of dimension 0 is $\mathbb{C}$.

QUESTION. If we iterate the development

$$X = X_0 \xleftarrow{d} X_1 \xleftarrow{d} X_2 \longleftarrow X_3 \longleftarrow \cdots$$

does $d : X_{i+1} \to X_i$ become an isomorphism for some i?

2.5. I hope I have now given enough motivation to define the singular locus of X as the set of those points of X where the dimension of the fibre $\Omega^1_X(x)$ is greater than it should be, i.e. where Ω^1_X has no chance of being locally free. More precisely:

DEFINITION. Let X be a reduced equidimensional complex space of dimension d. We define the *singular subspace* of X by the coherent sheaf of ideals $F_d(\Omega^1_X)$.

And similarly, the simplicity theorem for a map suggests:

DEFINITION. Let $f : X \to S$ be a *flat* map of complex analytic spaces, the fibres of which are of pure dimension d. We define the *critical subspace* C of f by the coherent sheaf of ideals $F_d(\Omega^1_{X/S})$. Let me immediately give two examples:

EXAMPLE 1. Let $f\in\mathbb{C}\{z_0,\cdots,z_n\}$ be such that $f=0$ defines a reduced hypersurface, $(X,0)\subset(\mathbb{C}^{n+1},0)$. Using the second exact sequence of the module of differentials, we get

$$(f)/(f^2)\xrightarrow{(d)}\Omega^1_{\mathbb{C}^{n+1}}|_X\longrightarrow\Omega^1_X\longrightarrow 0$$

and since $\Omega^1_{\mathbb{C}^{n+1}}$ is (locally) freely generated by the dz_i, and $(f)/(f^2)$ is free of rank 1; this is a presentation of Ω^1_X exactly as we need to compute a Fitting ideal. Here the matrix of (d) is obviously $(\partial f/\partial z_0,\cdots,\partial f/\partial z_n)$ and therefore our $F_n(\Omega^1_X)_0$ is $(\partial f/\partial z_0,\cdots,\partial f/\partial z_n)\mathcal{O}_{X,0}$, as one would have wished!

EXAMPLE 2. Consider now f as a map $\mathbb{C}^{n+1}\to\mathbb{C}$, necessarily flat. Then, the 1st exact sequence of the module of differentials gives us, if we take a coordinate v on $\mathbb{C}$:

$$\mathcal{O}_{n+1}\cdot dv\xrightarrow{\psi}\sum\mathcal{O}_{n+1}\cdot dz_i\longrightarrow\Omega^1_{\mathbb{C}^{n+1}/\mathbb{C}}\longrightarrow 0$$

since clearly $\Omega^1_{\mathbb{C}}=\mathcal{O}_1\,dv$ and $f^*\Omega^1_{\mathbb{C}}=\mathcal{O}_{n+1}\,dv$ (here $\mathcal{O}_1=\mathbb{C}\{v\}$, $\mathcal{O}_{n+1}=\mathbb{C}\{z_0,\cdots,z_n\}$).

The map ψ sends dv to $d(v\circ f)=\sum_{i=0}^n(\partial f/\partial z_i)\,dz_i$ hence again the matrix of ψ is $(\partial f/\partial z_0,\cdots,\partial f/\partial z_n)$ and we find that the *critical subspace* of f is defined by $(\partial f/\partial z_0,\cdots,\partial f/\partial z_n)\mathbb{C}\{z_0,\cdots,z_n\}$, again as one would have wished.

EXERCISE. Generalize both examples to the fibre of a *flat* map $F:\mathbb{C}^{n+1}\to\mathbb{C}^k$, and then to the map itself (i.e. compute the singular subspace of $F^{-1}(0)$ and the critical subspace of F) you will find the ideal generated by the minors of the Jacobian matrix, of course. [To say that F is flat is the same as to say that $F^{-1}(0)$ is a complete intersection, in this case, i.e. that it is defined by a regular sequence in $\mathcal{O}_{n+1}=\mathbb{C}\{z_0,\cdots,z_n\}$.]

2.6. Given a flat map $f:X\to S$, and since we know how to define its critical subspace C, we can, in the case where $f\,|\,C:C\to S$ is a finite map, define its image, which we will call the *discriminant subspace* of f, after §1.

DEFINITION. Let $f:X\to S$ be a flat map such that the restriction of f to the critical subspace C of f is finite. The discriminant subspace of f is the subspace of S defined by $F_0(f_*\mathcal{O}_C)$ (0th-Fitting ideal of $f_*\mathcal{O}_C$).

As an example, let us compute the discriminant of $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ in the case where it is defined, i.e. where the critical subspace defined by $(\partial f/\partial z_0,\cdots,\partial f/\partial z_n)$ is finite over $\mathbb{C}$, which means, looking locally around 0, that 0 is an isolated critical point of f. Now we have to compute the 0th Fitting ideal of $M=\mathbb{C}\{z_0,\cdots,z_n\}/(\partial f/\partial z_0,\cdots,\partial f/\partial z_n)$ as $\mathbb{C}\{v\}$-module. Here,

we have no God-given resolution, but we can use Hilbert's Syzygy theorem, which tells us that M has a resolution of length 1 because $\mathbb{C}\{v\}$ is regular of dimension 1: we get an exact sequence of $\mathbb{C}\{v\}$-modules

$$0\longrightarrow\mathbb{C}\{v\}^q\xrightarrow{\psi}\mathbb{C}\{v\}^p\longrightarrow M\longrightarrow 0$$

hence $q\leqq p$. But the support of M is the origin, so necessarily $q\geqq p$ otherwise the support of M would be $\mathbb{C}$. Hence $q=p$ and $F_0(M)=(\det\psi)\subset\mathbb{C}\{v\}$. But remember Lemma 2 in the proof of Bezout's theorem in §1: we have $(\det\psi)=(v^\mu)$ where $\mu=\dim_{\mathbb{C}}M$. Hence we have:

PROPOSITION. *The discriminant of $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ is the subspace of $(\mathbb{C},0)$ defined by $(v^\mu)\mathbb{C}\{v\}$ where*

$$\mu=\dim_{\mathbb{C}}\mathbb{C}\{z_0,\cdots,z_n\}/(\partial f/\partial z_0,\cdots,\partial f/\partial z_n)$$

is the Milnor number of f. (See Orlik's lectures).

EXERCISE. Let now $(X,0)\subset(\mathbb{C}^{n+1},0)$ be a complete intersection with isolated singularity, defined by $f_1,\cdots,f_{k-1}$. Let $f_k:(X,0)\to(\mathbb{C},0)$ be such that the fibre $f_k^{-1}(0)$ still has an isolated singularity. Use the same argument as above to show that the discriminant of f_k is the subspace of $\mathbb{C}$ defined by

$$(v^\Delta)\mathbb{C}\{v\}\quad\text{where}$$

$$\Delta=\dim_{\mathbb{C}}\mathcal{O}_X\Big/\left(\left\{\frac{\partial(f_1,\cdots,f_k)}{\partial(z_{i_1},\cdots,z_{i_k})}\right\}(i_1,\cdots,i_k)\subset\{0,\cdots,n\}\right)$$

§2. Second part: The idealistic Bertini theorem

2.7. In this part, we give the definition and characterization of integral dependence of ideals, which is a powerful tool to translate certain transcendental conditions on complex analytic spaces into geometric, or purely algebraic, conditions. We use this notion, and our space-theoretic viewpoint on singularities, to prove a result which is algebraically much stronger than the Bertini-Sard theorem, and then to show some connections between this result and some equisingularity and incidence conditions, such as Thom's A_F conditions.

One statement of the (second) Bertini theorem, which is a complex analytic version of Sard's theorem on the critical values of a C^∞ mapping (see K. Ueno: Classification theory of algebraic varieties, Springer Lect. Notes No. 439, chap. 1 §4) is as follows:

Let $f:X\to Y$ be a morphism of complex analytic spaces with X reduced and Y non-singular (this is not essential). Then, there exists a nowhere dense closed complex subspace B of X such that:

(i) $f(B)$ has measure 0 in Y

(ii) $|\mathrm{Sing}_x X_{f(x)}| = |\mathrm{Sing}_x X| \cap X_{f(x)}$ for all $x \in X - B$, where $X_{f(x)} = f^{-1}(f(x))$ is the fibre of f through x, and $|\mathrm{Sing}_x X|$ is the germ at x of the singular *locus* $|\mathrm{Sing}\ X|$.

Now that we have defined singular *subspaces*, we can ask whether this statement continues to hold true with singular subspaces instead of just loci. It turns out that it does not, as will be shown by an example below, but still, considerably more than the above statement is true, which can be formulated thanks to the notion of integral dependence on ideals, which I now summarize:

Integral dependence on ideals (for proofs see [1], [2]). Let X be a reduced complex analytic space, and let I be a coherent sheaf of ideals on X. Then, there exists a coherent sheaf of ideals $\bar{I}$ on X, which has the following property (and is characterized by it):

For any point $x \in X$, let $(g_1, \cdots, g_m)$ be generators for the ideal $I_x \subset \mathcal{O}_{X,x}$. Then, let us call $\bar{I}_x$ the ideal of elements $h \in \mathcal{O}_{X,x}$ satisfying an integral dependence relation:

$$h^k + a_1 \cdot h^{k-1} + \cdots + a_k = 0 \quad \text{with} \quad a_i \in I_x^i.$$

Then we have:

① $(\bar{I})_x = \bar{I}_x$

② $h \in \bar{I}_x$ if and only if there exists an open neighborhood U of x in X and a constant $C \in \mathbb{R}_+$ such that h and the g_i converge in U (i.e. come from elements in $\Gamma(U, \mathcal{O}_X)$, denoted by the same letters) and that: $|h(x')| \leqq C.\ \mathrm{Sup}\,|g_i(x')|$ for all $x' \in U$.

③ [*Arcwise condition of integral dependence*]
$h \in \bar{I}_x$ if and only if for *any* mapping of $\mathbb{C}$-algebras $\varphi^*: \mathcal{O}_{X,x} \to \mathbb{C}\{t\}$ we have $v(\varphi^*(h)) \geqq \min_j v(\varphi^*(g_j))$ where $v(\ )$ denotes the order in t of an element of $\mathbb{C}\{t\}$, also known as its valuation.

④ Given $U \overset{\circ}{\subset} X$ and $h \in \Gamma(U, \mathcal{O}_X)$, we have $h \in \Gamma(U, \bar{I})$ if and only if there exists a proper modification $p: Z \to U$ of complex spaces such that $(h \circ p)_z \in (I \cdot \mathcal{O}_Z)_z$ for all $z \in Z$, where $I \cdot \mathcal{O}_Z$ is the image in $\mathcal{O}_Z$ of the natural map $p^*(I|_U) \to p^*\mathcal{O}_U = \mathcal{O}_Z$ coming from the inclusion $I \subset \mathcal{O}_X$.

⑤ $h \in \bar{I}_x$ if and only if h 'asymptotically belongs to I_x' in the sense that there exists a ν_0 such that $h \cdot I_x \subset \bar{I}_x^{\nu+1}$ for $\nu \geqq \nu_0$.

Conditions ②–⑤ can be seen to be different ways of giving a meaning to the idea that h 'almost' belongs to I, which turn out to coincide. $\bar{I}$ (resp. $\bar{I}_x$)

is called the integral closure of the sheaf of ideals I in $\mathcal{O}_X$ (resp. of I_x in $\mathcal{O}_{X,x}$). The elements $h \in \bar{I}_x$ are said to be integrally dependent on I_x. We see that $I_x \subset \bar{I}_x \subset \sqrt{I_x}$ but $\bar{I}_x$ retains a lot of information from I_x which is lost when we look at $\sqrt{I_x}$.

EXERCISE. (1) The integral closure of the ideal $(z_0^a, \cdots, z_n^a)$ in $\mathbb{C}\{z_0, \cdots, z_n\}$ is $(z_0, \cdots, z_n)^a$.

(2) If a ring A is reduced and integrally closed in its total ring of fractions, and $g \in A$ is not a zero-divisor, then $\overline{(g)} = (g)$, (for example: if $A = \mathcal{O}_{X,x}$ and X is normal at x, and g is a local equation of a divisor).

(3) Check that for *any* $f \in \mathbb{C}\{z_0, \cdots, z_n\}$ belonging to the maximal ideal, we have: $f \in \overline{(z_0 \cdot (\partial f/\partial z_0), \cdots, z_n \cdot (\partial f/\partial z_n))}$ in $\mathbb{C}\{z_0, \cdots, z_n\}$, and hence, in particular (since $I \subset J \Rightarrow \bar{I} \subset \bar{J}$) we have $f \in \overline{((\partial f/\partial z_0), \cdots, (\partial f/\partial z_n))}$ always (Hint: use ③). Recall that if $f = 0$ has isolated singularity, $f \in ((\partial f/\partial z_0), \cdots, (\partial f/\partial z_n))$ is equivalent to the fact that $U \cdot f$ is quasi-homogeneous in some coordinate system, with some invertible U (K. Saito: Inventiones Math. *14*, p. 123 (1971)).

REMARK. $I_1 \subset I_2 \Rightarrow \bar{I}_1 \subset \bar{I}_2$; $\bar{\bar{I}} = \bar{I}$.

2.8. Idealistic Bertini theorem

Let $f: X \to Y$ be a flat morphism of complex analytic spaces, X being reduced and Y non-singular. Assume the fibres of f are all of pure dimension d. Set $n = \dim X = d + \dim Y$. Then, we have: There exists a nowhere dense closed complex subspace B of X such that:

(i) $f(B)$ has measure 0 in Y.

(ii) $\overline{F_n(\Omega_X^1)_x} = \overline{F_d(\Omega_{X/Y}^1)_x}$ for all $x \in X - B$.

In words, the germ at x of the singular subspace of X and of the critical subspace of f are defined by ideals having the same integral closure, outside B.

Now let us remark that if we induce $F_d(\Omega_{X/Y}^1)$ to $\mathcal{O}_{X_{f(x)},x}$ we obtain the ideal defining the singular subspace of $X_{f(x)}$ i.e. $F_d(\Omega_{X_{f(x)}}^1)$. Since anyway, as one can check by using the exact sequences, $F_d(\Omega_{X/Y}^1) \subset F_n(\Omega_X^1)$, and since an integral dependence relation can be restricted to a closed subspace, we see that we have

COROLLARY. $\overline{F_n(\Omega_X^1) \cdot \mathcal{O}_{X_{f(x)},x}} = \overline{F_d(\Omega_{X_{f(x)}}^1)_x}$ *for all* $x \in X - B$, *whereas Bertini's theorem quoted above can be translated by:*

$$\sqrt{F_n(\Omega_X^1) \cdot \mathcal{O}_{X_{f(x)},x}} = \sqrt{F_d(\Omega_{X_{f(x)}}^1)_x}$$

which is a much weaker statement, algebraically speaking.

Proof. First, remark that the assertion is local on X, so we can replace X by an open subset, still written X, so small that we have a commutative diagram:

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"'] & Y\times\mathbb{C}^N \arrow[dl, "\mathrm{pr}_1"] \end{tikzcd} \quad \text{(closed immersion)}$$
$$Y$$

where $Y\times\mathbb{C}^N$ also stands for an open subset in $Y_y\in\mathbb{C}^N$, and i is a closed immersion (Y also stands for an open set in Y) [recall, or admit, that a flat morphism is open]. Now let us choose coordinates $y_1,\cdots,y_k$ on Y, $z_1,\cdots,z_N$ on $\mathbb{C}^N$, and let $F_j(y,z)$ $1\leqq j\leqq m$ be generators for the ideal defining X in $Y\times\mathbb{C}^N$. Here $m\geqq N-d=$ codimension of X in $Y\times\mathbb{C}^N$. As we know from the exercises in §2, $F_n(\Omega^1_X)$ is the sheaf of ideals in $\mathcal{O}_X$ generated by the Jacobian determinants $(\partial(F_{j_1},\cdots,F_{j_{N-d}}))/(\partial(y_{i_1},\cdots,y_{i_l},z_{i_{l+1}},\cdots,z_{i_{N-d}}))$, $(j_1,\cdots,j_{N-d})\subset\{1,\cdots,m\}$, $(i_1,\cdots,i_l)\subset\{1,\cdots,k\}$, $(i_{l+1},\cdots,i_{N-d})\subset\{1,\cdots,N\}$ with $0\leqq l\leqq N-d$, and $F_d(\Omega^1_{X/Y})$ is generated in $\mathcal{O}_X$ by those among the above determinants where no $\partial/\partial y_i$ takes place, i.e. $l=0$.

Let us now consider the normalized blowing up of $F_d(\Omega^1_{X/Y})$ in X, i.e. the composed map:

$$\pi: X' \xrightarrow{n} X_1 \xrightarrow{\pi_0} X$$

where π_0 is the blowing up of $F_d(\Omega^1_{X/Y})$ and n is the normalization of X_1, which is reduced since X is so. Since $F_d(\Omega^1_{X/Y})\neq 0$ and X is reduced, π is a proper modification of X, and let us consider the exceptional divisor, i.e. the subspace D of X' defined by the sheaf of ideals $F_d(\Omega^1_{X/Y})\cdot\mathcal{O}_{X'}$, which is now invertible, (i.e. locally on X' generated by only one element, non-zero divisor) by the fundamental property of blowing up. By further restricting X and Y, we may assume that D has only a finite number of irreducible components and write $D=\bigcup_{\alpha\in A}D_\alpha$. Let us write $A=A_0\cup A_1$ where

$$\alpha\in A_0 \Leftrightarrow p(D_\alpha)\neq Y$$
$$\alpha\in A_1 \Leftrightarrow p(D_\alpha)= Y$$

where $p: X'\to Y$ is $f\circ\pi$.

Now set $B=\bigcup_{\alpha\in A_0}\pi(D_\alpha)$. It is a closed complex subspace of X since π is proper, and contained in $|\mathrm{Sing}\, f|$. Furthermore, its image in Y is of measure 0.

Let us now take a point $x\in X-B$, and change our coordinate system so that x is the origin of $\mathbb{C}^k\times\mathbb{C}^N$, (i.e. y_l and z_i all vanish at x). Since X' is normal, hence non-singular in codimension 1, we have that for each open neighborhood U of x in X, we can find a dense open analytic subspace V_α of $D_\alpha\cap\pi^{-1}(U)(\alpha\in A_1)$ such that if $x'\in V_\alpha$, we have:

(i) the germ of D at x' is equal to the germ of D_α at x' and $p\,|\,D_{\alpha,\mathrm{red}}: D_{\alpha,\mathrm{red}}\to Y$ is a submersion of non-singular spaces, at x'.

(ii) X' is non-singular at x', and hence by (i), p is a submersion of non-singular spaces at x'.

By the implicit function theorem we can choose local coordinates on X' centered at x', $(y'_1,\cdots,y'_k,v,u_2,\cdots,u_N)$ such that: $\mathcal{O}_{X',x'}=\mathbb{C}\{y'_1,\cdots,y'_k,v,u_2,\cdots,u_N\}$ and:

$$F_d(\Omega^1_{X/Y})\cdot\mathcal{O}_{X',x'}=v^\nu\cdot\mathcal{O}_{X',x'}. \tag{1}$$

$$(y_l\circ\pi)_{x'}=y'_l. \tag{2}$$

Since $(F_j\circ\pi)_{x'}\equiv 0\ (1\leqq j\leqq m)$ we have:

$$\frac{\partial}{\partial y'_l}(F_j\circ\pi)_{x'}\equiv 0\equiv\left(\sum_{i=1}^N\frac{\partial F_j}{\partial z_i}\circ\pi\cdot\frac{\partial(z_i\circ\pi)}{\partial y'_l}+\frac{\partial F_j}{\partial y_l}\circ\pi\right)_{x'}$$

and hence

$$\left(\frac{\partial F_j}{\partial y_l}\circ\pi\right)_{x'}=-\left(\sum_{i=1}^N\frac{\partial F_j}{\partial z_i}\circ\pi\cdot\frac{\partial(z_i\circ\pi)}{\partial y'_l}\right)_{x'}\quad\text{for all } j \text{ and } l.$$

The multilinearity of determinants now implies immediately that we have:

$$F_n(\Omega^1_X)\cdot\mathcal{O}_{X',x'}=F_d(\Omega^1_{X/Y})\cdot\mathcal{O}_{X',x'}\qquad(x'\in V_\alpha).$$

I claim that this implies in fact that we have $F_n(\Omega^1_X)\cdot\mathcal{O}_{\pi^{-1}(U)}=F_d(\Omega^1_{X/Y})\cdot\mathcal{O}_{\pi^{-1}(U)}$ (U open neighborhood of x). This because anyway $F_d(\Omega^1_{X/Y})\subset F_n(\Omega^1_X)$ on the one hand, and $F_d(\Omega^1_{X/Y})\cdot\mathcal{O}_{X'}$ is invertible by construction on the other hand. Therefore to check equality is to check that certain meromorphic functions generating $(F_d(\Omega^1_{X/Y})\cdot\mathcal{O}_{\pi^{-1}(U)})^{-1}\cdot(F_n(\Omega^1_X)\cdot\mathcal{O}_{\pi^{-1}(U)})$ are in fact holomorphic on $\pi^{-1}(U)$:

this will then imply that $F_n(\Omega^1_X)\cdot\mathcal{O}_{\pi^{-1}(U)}\subseteq F_d(\Omega^1_{X/Y})\cdot\mathcal{O}_{\pi^{-1}(U)}$, and we already know the other inclusion. The meromorphic functions in question are locally the h_i/g where g is a generator of $F_d(\Omega^1_{X/Y})\cdot\mathcal{O}_{X',x'}$ and the h_i are generators of $F_n(\Omega^1_X)\cdot\mathcal{O}_{X',x'}$.

But now we can argue as follows: the polar subspace of these meromorphic functions, if it is not empty, is of codimension 1 in X', by a classical result on normal spaces (see R. Narasimhan: Introduction to the theory of analytic spaces, Springer Lecture Notes No. 25, 1966, p. 89). On the other hand, this polar subspace is certainly contained in the subspace defined by $F_d(\Omega^1_{X/Y})\cdot\mathcal{O}_{X'}$, which is $D\cap\pi^{-1}(U)$, and we have just seen that each component of D contains a dense open analytic subset at each point of which the meromorphic functions in question are holomorphic. Hence the polar subspace in question is empty, and therefore we have $F_n(\Omega^1_X)\cdot\mathcal{O}_{\pi^{-1}(U)}=F_d(\Omega^1_{X/Y})\cdot\mathcal{O}_{\pi^{-1}(U)}$, which implies that $\overline{F_n(\Omega^1_X)_x}=\overline{F_d(\Omega^1_{X/Y})_x}$ at every $x\in X-B$.

REMARK. If we assume f proper, then by a theorem of Frisch (Inventiones Math. *4* (1967), pp. 118–138) there exists a complex-analytic nowhere dense closed subspace $F\subset X$ such that $f(F)$ is a nowhere dense closed complex subspace of Y and $f\,|\,X-F$ is flat, so we may drop the assumption of flatness on f in the theorem, and in this case the conclusion of the theorem is that there exists a nowhere dense closed complex subspace $Z\subset Y$ such that for any $y\in Y-Z$ we have:

$$\overline{F_n(\Omega^1_X)_x}=\overline{F_d(\Omega^1_{X/Y})_x}\quad\text{for all}\quad x\in f^{-1}(y)$$

and hence for $y\in Y-Z$, the equality of sheaves of ideals of $\mathcal{O}_{X_y}$:

$$\overline{F_n(\Omega^1_X)\cdot\mathcal{O}_{X_y}}=\overline{F_d(\Omega^1_{X/Y})\cdot\mathcal{O}_{X_y}}=\overline{F_d(\Omega^1_{X_y})}.$$

REMARK. I have assumed known that the image of a *nowhere surjective* analytic mapping is of measure 0. (Cover the source by countably many semi-analytic compact subsets and use the description of subanalytic sets.)

2.9. EXAMPLE. Let $f:X\to\mathbb{P}^2$ be the restriction of the natural projection $\mathbb{P}^2\times\mathbb{C}^3\to\mathbb{P}^2$ to the subspace X defined by the ideal $(az_1+bz_2+cz_3,\ z_1^6+z_2^6+z_3^6)$ where $(a:b:c)$ are homogeneous coordinates on $\mathbb{P}^2$ and z_1, z_2, z_3 are coordinates on $\mathbb{C}^3$. f is the family of plane sections of the surface in $\mathbb{C}^3$ defined by $z_1^6+z_2^6+z_3^6=0$. The Jacobian matrix of X is

$$\begin{bmatrix}6z_1^5 & 6z_2^5 & 6z_3^5 & 0 & 0 & 0\\ a & b & c & z_1 & z_2 & z_3\end{bmatrix}$$

and therefore $F_1(\Omega^1_{X/\mathbb{P}^2})$ is generated by:

$$(bz_1^5-az_2^5,\ cz_2^5-bz_3^5,\ az_3^5-cz_1^5)$$

while $F_3(\Omega^1_X)=F_1(\Omega^1_{X/\mathbb{P}^2})+(z_1^6,\ z_1z_2^5,\ z_1z_3^5,\ z_2z_1^5,\ z_2^6,\ z_2z_3^5,\ z_3z_1^5,\ z_3z_2^5,\ z_3^6)$. Now I claim that $F_1(\Omega^1_{X/\mathbb{P}^2})\cdot\mathcal{O}_{X_p}\not\subseteq F_3(\Omega^1_X)\cdot\mathcal{O}_{X_p}$ for a general point $p\in\mathbb{P}^2$. We work in the affine open subset of $\mathbb{P}^2$ where $abc\neq 0$ and can therefore assume $c=-1$ and replace z_3 by az_1+bz_2 on X. Using the homogeneity, we see that the remaining equation of X in $\mathbb{C}^2\times\mathbb{C}^2$ (with coordinates a, b, z_1, z_2) already belongs to the ideal generated by $(bz_1^5-az_2^5,\ z_2^5+b(az_1+bz_2)^5)$, which is an ideal on $\mathbb{C}^2\times\mathbb{C}^2$ inducing $F_1(\Omega^1_{X/\mathbb{P}^2})$ on X. Using the homogeneity again, we conclude therefore that the equality $F_3(\Omega^1_X)\cdot\mathcal{O}_{X_p}=F_1(\Omega^1_{X/\mathbb{P}^2})\cdot\mathcal{O}_{X_p}$

if it were true, would imply that there exist complex numbers λ_1, λ_2, μ_1, μ_2 such that we have, for example:

$$z_1\cdot z_2^5=(\lambda_1z_1+\lambda_2z_2)(bz_1^5-az_2^5)+(\mu_1z_1+\mu_2z_2)(z_2^5+b(az_1+bz_2)^5)$$

in the ring $\mathbb{C}[z_1,z_2]$. Looking at the coefficient of each monomial, we obtain 7 linear equations for λ_1, λ_2, μ_1, μ_2, and it can be checked that there are no solutions, provided p is outside a curve in $\mathbb{P}^2$. This shows that when p is outside this curve, $F_1(\Omega^1_{X/\mathbb{P}^2})\cdot\mathcal{O}_{X_p}\not\subseteq F_3(\Omega^1_X)\cdot\mathcal{O}_{X_p}$.

However, using the arcwise condition of integral dependence I will now show that when the coordinates of p satisfy, in addition to $abc\neq 0$, the condition $a^{\frac{6}{5}}+b^{\frac{6}{5}}+c^{\frac{6}{5}}\neq 0$, we have equality of the integral closure of the two ideals. To see this, set again $c=-1$, and check that if $1+a^{\frac{6}{5}}+b^{\frac{6}{5}}\neq 0$, whenever we take an analytic arc $z_1(t)$, $z_2(t)$ *in the fibre* X_p, the lowest order terms in t cannot cancel simultaneously in the two expressions $bz_1(t)^5-az_2(t)^5$, $z_2(t)^5+b(az_1(t)+bz_2(t))^5$, therefore the ideal they generate in $\mathbb{C}\{t\}$ is (t^ν), where $\nu=5\cdot\min\left(v(z_1(t)),v(z_2(t))\right)$, and we see immediately that all the monomials appearing in the expression of $F_3(\Omega^1_X)$ give a greater valuation in t, which shows that $F_3(\Omega^1_X)\cdot\mathcal{O}_{X_p}$ is integrally dependent on $F_1(\Omega^1_{X/\mathbb{P}^2})\cdot\mathcal{O}_{X_p}$.

REMARK. If we denote by $\mathcal{J}_X$ and $\mathcal{J}_{X/Y}$ respectively the ideals *in* $\mathcal{O}_{Y\times\mathbb{C}^N}$ generated by the Jacobian determinants which when restricted to X generate $F_n(\Omega^1_X)$ and $F_d(\Omega^1_{X/Y})$ respectively, it is *not* true in general that for many points $y\in Y$ we have for any $x\in f^{-1}(y)$ that $\overline{J_{X,x}}=\overline{J_{X/Y,x}}$. In the above example, if we did not require that the arc $z_1(t)$, $z_2(t)$ lies on X_p, (or merely in $az_1+bz_2+cz_3=0$) we could always arrange to have cancellation of the lowest order terms in the two expressions in question, the ideal they generate in $\mathbb{C}\{t\}$ then being (t^ν) with $\nu\gg 5\cdot\min\left(v(z_1(t)),v(z_2(t))\right)$, and it is no longer true that, for example, $v(z_1(t)^5\cdot z_2(t))\geqq\nu$, so that $z_1^5\cdot z_2$ is not integrally dependent over $\mathcal{J}_{X/Y}$ in $\mathcal{O}_{Y\times\mathbb{C}^N,x}$. However, when X is a *hypersurface* in $Y\times\mathbb{C}^N$, this phenomenon does not occur, and since this fact is of importance in the theory of equisingularity. I will now prove two results in this direction, which are results of integral dependence of the $\partial F/\partial y_i$ over the ideal $((\partial F/\partial z_1,\cdots,(\partial F/\partial z_N))$ in the ambient space $Y\times\mathbb{C}^N$, and not just on the hypersurface X defined by $F=0$.

2.10. PROPOSITION 1. Let $F(y_1,\cdots,y_k,z_1,\cdots,z_N)$ be convergent in (an open subset of) $Y\times\mathbb{C}^N$, and such that all $\partial F/\partial z_i$ do not vanish identically. Then, there exists a nowhere dense closed complex subspace B' of $Y\times\mathbb{C}^N$ such that:

(i) the image of B' in Y by the projection has measure 0,

(ii) for all $x \in Y\times\mathbb{C}^N - B'$ we have:

$$\left(\frac{\partial F}{\partial y_l}\right)_x \in \overline{\left(\frac{\partial F}{\partial z_1}, \cdots, \frac{\partial F}{\partial z_N}\right)_x}, \quad \text{in } \mathcal{O}_{Y\times\mathbb{C}^N,x} \qquad (1\leqq l\leqq k).$$

PROOF. The idea of the proof is the same as for the idealistic Bertini theorem, and I shall therefore give only the main features: let $\pi: Z\to Y\times\mathbb{C}^N$ be the normalized blowing up of the ideal $\mathcal{J}=((\partial F/\partial z_1), \cdots, (\partial F/\partial z_N))\cdot\mathcal{O}_{Y\times\mathbb{C}^N}$, let $D=\bigcup_{\alpha\in A} D_\alpha$ be the exceptional divisor, defined by $\mathcal{J}\cdot\mathcal{O}_Z$, and D_α its irreducible components. Note that π is a proper modification of $Y\times\mathbb{C}^N$ since $J\neq 0$. Now set $A=A_0 \amalg A_1$

$$\alpha\in A_0 \Leftrightarrow p(D_\alpha)\neq Y$$
$$\alpha\in A_1 \Leftrightarrow p(D_\alpha)=Y$$

where $p: Z\to Y$ is $pr_1\circ\pi$, and let $B'=\bigcup_{\alpha\in A_0}\pi(D_\alpha)$, nowhere dense closed complex subspace of $Y\times\mathbb{C}^N$. I claim that if $x\in Y\times\mathbb{C}^N-B'$, $(\partial F/\partial y_l)_x\in\bar{J}_x$: let U be an open neighborhood of x in $Y\times\mathbb{C}^N$. Each $D_\alpha\cap\pi^{-1}(U)$ contains a dense open analytic subspace V_α at each point z of which the following hold:

(i) the germ of D at z is equal to the germ of D_α at z, and $p\mid D_{\alpha,\mathrm{red}}: D_{\alpha,\mathrm{red}}\to Y$ is a submersion of non-singular spaces at z,

(ii) Z is non-singular at z and hence by (i), $p: Z\to Y$ is a submersion of non-singular spaces at z,

(iii) the strict transform by π of the hypersurface X defined by $F=0$ in $Y\times\mathbb{C}^N$ is empty near z.

[Recall that by definition, the strict transform X' is $\overline{\pi^{-1}(X-F)}$ where F is the subspace defined by $\mathcal{J}$, so that $X'\cap D_\alpha$ is nowhere dense in each D_α.]

Now by the implicit function theorem, we can choose local coordinates on Z centered at z, $(y_1', \cdots, y_k', v, u_2, \cdots, u_N)$ such that:

$$\mathcal{O}_{Z,z}=\mathbb{C}\{y_1', \cdots, y_k', v, u_2, \cdots, u_N\} \quad \text{and:}$$

(1) $\mathcal{J}\cdot\mathcal{O}_{Z,z}=(v^\mu)\cdot\mathcal{O}_{Z,z}$ $\quad\mu\in\ ,$

(2) $(y_l\circ\pi)_z=y_l'$ $(1\leqq l\leqq k)$,

(3) $(F\circ\pi)_z=A\cdot v^\nu$ where A is invertible in $\mathcal{O}_{Z,z}$ (this is the translation of condition (iii)).

Then we have:

$$\frac{\partial(F\circ\pi)}{\partial y_l'}=\sum_{i=1}^N \frac{\partial F}{\partial z_i}\circ\pi\cdot\frac{\partial(z_i\circ\pi)}{\partial y_l'}+\frac{\partial F}{\partial y_l} \quad \text{in} \quad \mathcal{O}_{Z,z} \qquad (*)$$

and

$$\frac{\partial(F\circ\pi)}{\partial y_l'}=\frac{\partial A}{\partial y_l'}\cdot v^\nu.$$

Therefore, to check that $((\partial F/\partial y_l)\circ\pi)_z\in\mathcal{J}\cdot\mathcal{O}_{Z,z}$, it is sufficient to show that we have $\nu\geqq\mu$. To show this, let us restrict the equality (*) to the non-singular subspace W of Z near z defined by $y_1'=\cdots=y_k'=0$. By the exercises on integral dependence, we know that $F(0, z_1, \cdots, z_N)$ is integrally dependent over $((\partial F/\partial z_1)(0, z), \cdots, (\partial F/\partial z_N)(0, z))$ in $\mathbb{C}\{z_1, \cdots, z_N\}$. Since $W=p^{-1}(0)_z$, and since an integral dependence relation can be lifted to $W=\pi^{-1}(0\times\mathbb{C}^N)$ and then localized at z, we see that $A(0, v, u_2, \cdots, u_N)\cdot v^\nu$ is integrally dependent over (v^μ) in $\mathbb{C}\{v, u_2, \cdots, u_N\}=\mathcal{O}_{W,z}$, and since $\mathcal{O}_{W,z}$ is integrally closed in its field of fractions, by the exercises on integral dependence it means $Av^\nu\in(v^\mu)$ but now we use the fact that A is invertible in $\mathcal{O}_{Z,z}$, hence its image in $\mathcal{O}_{W,z}$ is also invertible, and finally $\nu\leqq\mu$.

The end of the proof is as before: $\mathcal{J}\cdot\mathcal{O}_{\pi^{-1}(U)}$ is invertible in a normal space by construction, and the argument above shows that the meromorphic functions $(J\cdot\mathcal{O}_{\pi^{-1}(U)})^{-1}\cdot((\partial F/\partial y_l)\circ\pi)$ are holomorphic on an open dense subset of each component of $D\cap\pi^{-1}(U)$, hence they are holomorphic everywhere on $\pi^{-1}(U)$, hence $(\partial F/\partial y_l)\cdot\mathcal{O}_{\pi^{-1}(U)}\subset\mathcal{J}\cdot\mathcal{O}_{\pi^{-1}(U)}$, hence $((\partial F/\partial y_l))_x\in\bar{J}_x$ $(1\leqq l\leqq k)$.

REMARK. Proposition 1 obviously implies the idealistic Bertini theorem in the case where X is a hypersurface in $Y\times\mathbb{C}^N$, by taking $B=B'\cap X$, since in this case, $F_n(\Omega_X^1)$ is generated by the $(\partial F/\partial z_i)$ and $(\partial F/\partial y_l)$, and $F_d(\Omega_{X/Y}^1)$ by the $(\partial F/\partial z_i)$.

The theory of (c)-equisingularity uses the following result:

2.11. PROPOSITION 2. *Let $F(y_1, \cdots, y_k, z_1, \cdots, z_N)$ be convergent in (an open of) $Y\times\mathbb{C}^N$, such that all $(\partial F/\partial z_i)$ do not vanish identically, and that $F(y_1, \cdots, y_k, 0)\equiv 0$ [i.e. we are now given a section σ $Y\times\mathbb{C}^N \overset{\sigma}{\leftrightarrows} Y$ such that $\sigma(Y)\subset X$]. Then there exists a nowhere dense closed complex subspace $B_0\subset\sigma(Y)$ $(=Y\times\{0\})$ such that at any point $x\in\sigma(Y)-B_0$ we have:*

$$\left(\frac{\partial F}{\partial y_l}\right)_x \in \overline{\left(z_1\cdot\frac{\partial F}{\partial z_1}, \cdots, z_N\cdot\frac{\partial F}{\partial z_N}\right)_x} \quad \text{in} \quad \mathcal{O}_{Y\times\mathbb{C}^N,x} \qquad (1\leqq l\leqq k).$$

PROOF. It is essentially the same as that of proposition 1 above, except that we take the normalized blowing up π of the ideal generated by $(z_1\cdot(\partial F/\partial z_1), \cdots, z_N\cdot(\partial F/\partial z_N))$ in $\mathcal{O}_{Y\times\mathbb{C}^N}$. Then B_0 is the union of the $\pi(D_\alpha)\cap\sigma(Y)$ for those α such that $\pi(D_\alpha)$ does not contain $\sigma(Y)$, where $D=UD_\alpha$ is the exceptional divisor of π. Furthermore, the open analytic V_α which we take in each D_α has to satisfy, in addition to the conditions (i), (ii), (iii) appearing in the proof of proposition 1, the condition (which is also satisfied on an open dense analytic subset):

(iv) the strict transform by π of each of the hyperplanes $z_i=0$ $(1\leqq i\leqq N)$ is empty near $z\in V_\alpha$.

Then, the argument is the same as in the proof of proposition 1, using the equality (*), remarking that condition iv) implies that $(z_i \circ \pi)_z = \zeta_i \cdot v^{\mu_i}$ with ζ_i invertible in $\mathcal{O}_{Z,z}$, and hence $(\partial/\partial y_l')(z_i \circ \pi) = (\partial \zeta_i/\partial y_l') \cdot v^{\mu_i}$ is a multiple of $(z_i \circ \pi)_z$ in $\mathcal{O}_{Z,z}$ and this time using the full strength of the inclusion $f(z_0, \cdots, z_n) \in \overline{(z_0 \cdot (\partial f/\partial z_0), \cdots, z_n \cdot (\partial f/\partial z_n))}$ given in the exercises on integral dependence, to prove that $\nu \geqq \mu$, where μ is the order in v of the ideal $(z_1 \cdot (\partial F/\partial z_1), \cdots, z_N \cdot (\partial F/\partial z_N)) \cdot \mathcal{O}_{Z,z}$.

The details of the adaptation of the proof of proposition 1 are given as an exercise to the reader.

REMARK. My original statement of proposition 2 was that $(\partial F/\partial y_l)_x \in \overline{(z_1, \cdots, z_N) \cdot ((\partial F/\partial z_1), \cdots, (\partial/\partial z_N))_x}$ and I must thank J. P. G. Henry for remarking that my proof actually gave the statement above, which is slightly stronger, and also easier to use. Remark that the ideal $\overline{(z_1 \cdot (\partial F/\partial z_1), \cdots, z_N \cdot (\partial F/\partial z_N))}$ depends on the choice of coordinates $z_1, \cdots, z_N$, and that proposition 2 holds for any choice. The ideal $\overline{(z_1, \cdots, z_N) \cdot ((\partial F/\partial z_1), \cdots, (\partial F/\partial z_N))_x}$ does not depend upon the choice of coordinates, but presumably we have the equality $\overline{(z_1 \cdot (\partial F/\partial z_1), \cdots, z_N \cdot (\partial F/\partial z_N))_x} = \overline{(z_1, \cdots, z_N) \cdot ((\partial F/\partial z_1), \cdots, (\partial F/\partial z_N))_x}$ at a general point x of $\sigma(Y)$ and for a 'sufficiently general' choice of coordinates $z_1, \cdots, z_N$. We are now ready to move into some equisingularity conditions:

2.12. DEFINITION. Let $f: X \overset{\sigma}{\leftrightarrows} Y$ be a family of germs of hypersurfaces, which means that X can be locally around $0 \in X$ imbedded as a hypersurface in $Y \times \mathbb{C}^N$, σ is a section of f, and we may even assume $\sigma(Y) = Y \times \{0\}$. We still assume that Y is non-singular. We say that X is (c)-equisingular along $\sigma(Y)$ at $x \in \sigma(Y)$ if there exists such an imbedding $X \subset Y \times \mathbb{C}^N$ described by $F(y_1, \cdots, y_k, z_1, \cdots, z_N) = 0$ and such that:

$$\left(\frac{\partial F}{\partial y_l}\right)_x \in \overline{(z_1, \cdots, z_N) \cdot \left(\frac{\partial F}{\partial z_1}, \cdots, \frac{\partial F}{\partial z_N}\right)_x} \quad \text{in} \quad \mathbb{C}\{y, z\} = \mathcal{O}_{Y \times \mathbb{C}^N, x} \cdot (1 \leqq l \leqq k)$$

EXERCISE. (1) Check that this condition depends only on f and σ, i.e. is independent of the choice of the equation F and the coordinates. Hint: use the fact that

$$F \in \overline{(y_1 \cdot (\partial F/\partial y_1), \cdots, y_k \cdot (\partial F/\partial y_k), z_1 \cdot (\partial F/\partial z_1), \cdots, z_N \cdot (\partial F/\partial z_N))_x}.$$

(2) Check that in fact this condition depends only upon $\sigma(Y) \subset X$ and not upon the choice of a retraction $Y \times \mathbb{C}^N \to \sigma(Y)$.

2.13. COROLLARY 1. *Given $f: X \overset{\sigma}{\leftrightarrows} Y$, σ a section of f, and f a family of hypersurfaces, then the set of points x of $\sigma(Y)$ such that X is (c)-equisingular along $\sigma(Y)$ at x is the complement of a nowhere dense closed complex subspace of $\sigma(Y)$. This is an immediate consequence of proposition 2 above, and the fact that the set of points where we have an integral dependence relation is open and analytic in any case (possibly empty) because of the coherence of $\bar{I}$.*

2.14. COROLLARY 2. *If $f: X \overset{\sigma}{\leftrightarrows} Y$ is (c)-equisingular at $x \in \sigma(Y)$, then for any local imbedding $X \subset Y \times \mathbb{C}^N$ near x, and for each i, $1 \leqq i \leqq N$, there exists a Zariski open dense $U^{(i)}$ of the Grassmannian of i-planes in $(\mathbb{C}^N, 0)$ such that for any $H_0 \in U^{(i)}$ we have, setting $H = Y \times H_0$,*

(a) $\overline{j_{Y \times \mathbb{C}^N/Y}(F) \cdot \mathcal{O}_{H,x}} = \overline{j_{Y \times H_0/Y}(F \cdot \mathcal{O}_{H,x})}$ *in* $\mathcal{O}_{H,x}$ $\quad [F \cdot \mathcal{O}_{H,x} = (F \mid H)_x]$

(b) *$X \cap H$ is (c)-equisingular along $\sigma(Y)$ at x, [where $j_{Y \times \mathbb{C}^N/Y}(F)$ is the ideal in $\mathcal{O}_{Y \times \mathbb{C}^N, x}$ generated by $((\partial F/\partial z_1), \cdots, (\partial F/\partial z_N))$].*

PROOF. The idea is to apply proposition 2 above to the family of sections of X by such $Y \times H_0$, as follows: We can restrict ourselves to an open subset of the Grassmannian of i-planes, of course, and therefore describe the family of sections $X \cap H$, where $H = Y \times H_0$, H_0 an i-plane in $\mathbb{C}^N$, as follows:

$$F_a = F\Big(y_1, \cdots, y_k, z_1, \cdots, z_i, \sum_{1 \leqq j \leqq i} a_{i+1,j} z_j, \cdots, \sum_{1 \leqq j \leqq i} a_{N,j} z_j\Big) = 0$$

$$F_a \in \mathbb{C}\{y, z_1, \cdots, z_1, (a_{p,j})\} \quad (\text{where } i+1 \leqq p \leqq N, 1 \leqq j \leqq i).$$

Given any function $g(y_1, \cdots, y_k, z_1, \cdots, z_N)$, we will write g_a or $(g)_a$ for the function

$$g\Big(y_1, \cdots, y_k, z_1, \cdots, z_i, \sum_{1 \leqq j \leqq i} a_{i+1,j} z_j, \cdots, \sum_{1 \leqq j \leqq i} a_{N,j} z_j\Big)$$

in $\mathbb{C}\{y, z_1, \cdots, z_i, (a_{p,j})\}$.

Now, by proposition 2, we have that there exists a dense open analytic subset $V^{(i)}$ in the space $\mathbb{C}^{i(N-i)}$ of the coefficients $a = (a_{p,j})$ such that if $a \in V^{(i)}$, then we have:

$$\frac{\partial}{\partial a_{p,j}} F_a \in \overline{\left(y_i \cdot \frac{\partial F_a}{\partial y_1}, \cdots, y_k \cdot \frac{\partial F_a}{\partial y_k}, z_1 \cdot \frac{\partial}{\partial z_1} F_a, \cdots, z_i \cdot \frac{\partial}{\partial z_i} F_a\right)}$$

in $\mathcal{O}_{\mathbb{C}^{i(N-i)} \times (Y \times \mathbb{C}^i), a \times \{0\}}$

which can be explicited by:

$$z_j \left(\frac{\partial F}{\partial z_p}\right)_a \in \overline{\left(y_1 \left(\frac{\partial F}{\partial y_1}\right)_a, \cdots, y_k \left(\frac{\partial F}{\partial y_k}\right)_a, z_1 \left\{\left(\frac{\partial F}{\partial y_1}\right)_a + \sum_p a_{p,1} \left(\frac{\partial F}{\partial z_p}\right)_a\right\}, \cdots \cdots\right.} \qquad (*)$$

$$\overline{\left. z_i \left\{\left(\frac{\partial F}{\partial z_i}\right)_a + \sum_p a_{p,i} \left(\frac{\partial F}{\partial z_p}\right)_a\right\}\right)}$$

for all $1 \leqq j \leqq i$, $i+1 \leqq p \leqq N$. Our assumption that the original family is (c)-equisingular implies:

$$\left(\frac{\partial F}{\partial y_l}\right)_a \in \overline{(z_1, \cdots, z_i)\left(\left(\frac{\partial F}{\partial z_1}\right)_a, \cdots, \left(\frac{\partial F}{\partial z_N}\right)_a\right)} \quad \text{in} \quad \mathcal{O}_{Y \times \mathbb{C}^i \times \mathbb{C}^{i(N-i)}, 0 \times \{a\}}. \qquad (**)$$

We can assume, after a change of the coordinates z_p, that our point $a \in V^{(i)}$ is the origin of $\mathbb{C}^{i(N-i)}$. I now give as an exercise to check, using (*) and (**) that for any mapping

$$\varphi^*: \mathbb{C}\{y_1, \cdots, y_k, z_1, \cdots, z_i, (a_{p,j})\} \to \mathbb{C}\{t\}$$

we have that for any $i+1 \leqq p \leqq N$,

$$v\left(\frac{\partial F}{\partial z_p}\right)_a \geqq \min_{1 \leqq j \leqq i} \left\{ v\left(\frac{\partial F}{\partial z_j}\right)_a \right\}$$

where $v(h) = $ order in t of $\varphi^*(h)$, using the arcwise condition of integral dependence, and the fact that $v(y_l) > 0$, $v(a_{p,j}) > 0$. (Hint: use reductio ab absurdum.) From this, we deduce that

$$\left(\frac{\partial F}{\partial z_p}\right)_{H,x} \in \overline{j_{Y \times H_0/Y}(F \cdot O_{H,x})}, \ (i+1 \leqq p \leqq N), \qquad (***)$$

where H_0 is the plane corresponding to $a \in V^{(i)}$, and from this inclusion, the equality (a) of the proposition follows immediately.

Assertion (b) of the proposition also follows immediately from (**) and (***), in view of assertion (a).

Remarks. (i) We have not even used the fact that the hypersurface defined by $F = 0$ is reduced. The only assumption needed is $(\partial F/\partial z_j) \not\equiv 0$.

(ii) Even when Y is a point, the statement above is not a triviality. Then, condition (c) is automatically satisfied, and we have:

2.15. Corollary 3. *Given $F \in \mathbb{C}\{z_1, \cdots, z_N\}$ such that $F(0) = 0$, $F \not\equiv 0$, for each i, $1 \leqq i \leqq N$, there exists a Zariski open dense subset $U^{(i)}$ in the Grassmannian of i-planes through 0 in $\mathbb{C}^N$ such that for any $H \in U^{(i)}$ we have, denoting as above $F \cdot \mathcal{O}_{H,0}$ the germ at 0 of the restriction of F to H:*

$$\overline{j(F) \cdot \mathcal{O}_{H,0}} = \overline{j(F \cdot \mathcal{O}_{H,0})} \quad \text{in} \quad \mathcal{O}_{H,0}$$

where $j(F)$ is the ideal generated in the ring of functions of the ambient space by the partial derivatives of F. [Equivalently, since $F \in \overline{j(F)}$, we can say $\overline{(F, j(F)) \cdot O_{H,0}} = \overline{(F \cdot O_{H,0}, j(F \cdot O_{H,0})}$.]

Exercise. (1) Compare corollary 3 with the idealistic Bertini theorem.

(2) Find an example where one cannot remove the bars above the ideals in corollary 3. Hint: $F = z_1^3 + z_2^3 + z_3^3$ will do. If you have done exercise 1, you know why such a simple example will not work for the idealistic Bertini theorem.

We now turn to some geometric consequences of condition (c).

2.16. Definition. Let T_1 and T_2 be two vector subspaces of $\mathbb{C}^N$ considered as hermitian space with the form $((z_i), (z_i')) \to \sum_i z_i \bar{z}_i'$ and assume $\dim T_1 \geqq \dim T_2 \geqq 1$. The distance from T_1 to T_2 (in that order) is defined as

$$d(T_1, T_2) = \sup_{\substack{t_1 \in T_1^\perp - \{0\} \\ t_2 \in T_2 - \{0\}}} \left\{ \frac{|(t_1, t_2)|}{\|t_1\| \cdot \|t_2\|} \right\}$$

where $T_1^\perp$ is the vector space of vectors orthogonal to T_1 with respect to the hermitian form.

Remark. By translating to the origin, this gives a distance between *directions* of linear subspaces.

Let now $X \subset Y \times \mathbb{C}^N$ be a hypersurface, defined locally near x by $F(y_1, \cdots, y_k, z_1, \cdots, z_N) = 0$ where we have identified Y with $\mathbb{C}^k$ locally as usual, and x with $0 \in \mathbb{C}^k \times \mathbb{C}^N$, and $F(y, 0) \equiv 0$. Given a point $p \in Y \times \mathbb{C}^N$, let us denote by L_p the level hypersurface through p of the mapping $Y \times \mathbb{C}^N \xrightarrow{F} \mathbb{C}$, i.e. $F^{-1}(F(p)) = L_p$. Whenever p is not a singular point of L_p, we can define the distance between the directions of the tangent hyperplane to L_p at p and $T_{Y,0}$, which we have identified with $\mathbb{C}^k$.

$T_{L_p,p}^\perp$ is generated by the vector: $(\overline{(\partial F/\partial y_1)(p)}, \cdots, \overline{(\partial F/\partial y_k)(p)}, \overline{(\partial F/\partial z_1)(p)}, \cdots, \overline{(\partial F/\partial z_N)(p)})$ and hence

$$d(T_{L_p,p}, T_{Y,0}) = \sup_{\xi \in \mathbb{C}^k - \{0\}} \left\{ \frac{\left| \sum_{l=1}^k \overline{\frac{\partial F}{\partial y_l}}(p) \cdot \xi_l \right|}{\left(\sum_{l=1}^k \left| \frac{\partial F}{\partial y_l}(p) \right|^2 + \sum_{i=1}^N \left| \frac{\partial F}{\partial z_i}(p) \right|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{l=1}^k |\xi_l|^2 \right)^{\frac{1}{2}}} \right\}.$$

2.17. Proposition 3. *If $X \subset Y \times \mathbb{C}^N$ is a hypersurface satisfying condition (c) along $Y \times \{0\}$ at 0, then there exists a neighborhood V of $0 \in Y \times \mathbb{C}^N$, and $C \in \mathbb{R}_+$ such that for any point $p \in V$ such that the level hypersurface L_p is non-singular at p, we have*

$$d(T_{L_p,p}, T_{Y,0}) \leqq C \cdot \operatorname{dist}(p, Y),$$

[where $\operatorname{dist}(p, Y)$ is the distance from the point p to $Y = \mathbb{C}^k \times \{0\}$ in $Y \times \mathbb{C}^N$], and conversely, this inequality implies condition (c).

Proof. It follows easily from the above expression of the distance from

$T_{L_p,p}$ to $T_{Y,0}$, and the criterion ② of integral dependence, if we remark that ② allows us to express condition (c) by:

$$\left|\frac{\partial F}{\partial y_l}(p)\right| \leqq K_l \cdot \operatorname{Sup}_i |z_i(p)| \cdot \operatorname{Sup}_j \left|\frac{\partial F}{\partial z_j}(p)\right| \qquad (1 \leqq l \leqq k,\ K_l \in \mathbb{R}_+)$$

i.e.

$$\left|\frac{\partial F}{\partial y_l}(p)\right| \leqq K_l' \cdot \operatorname{dist}(p, Y) \cdot \operatorname{Sup}_j \left|\frac{\partial F}{\partial z_j}(p)\right|.$$

Corollary 1. *Let X^0 denote the open analytic set of non-singular points of X, and assume X reduced, so that $\overline{X^0} = X$. Set $Y_1 = Y \times \{0\} \subset X$. If X is (c)-equisingular along Y_1 at $x \in Y_1$, we have:*

(1) *the pairs of strata $(Y \times \mathbb{C}^N - Y_1, Y_1)$ and $(X^0 - Y_1, Y_1)$ both satisfy Thom's A_F condition at x, where $F: Y \times \mathbb{C}^N \to \mathbb{C}$ is the map given by F. (See Hironaka's lectures* §3, def. 3).

(2) *The pair strata $(X^0 - Y_1, Y_1)$ satisfies the Whitney condition at x (see Hironaka's lectures* §3, *def.* 2).

[Remark. In practice, most often $Y_1 \subset \operatorname{Sing} X$, so that $X^0 - Y_1 = X^0$.]

(3) *Assuming furthermore that $X^0 = X - Y_1$, we have:*
Any holomorphic vector field on Y_1 can be extended to a vector field on $Y \times \mathbb{C}^N$ (in a neighborhood of x) which is:

(i) *real-analytic outside Y_1,*

(ii) *tangent to the level hypersurfaces L_p,*

(iii) *'rugose' in the sense of Verdier [Stratifications de Whitney et théorème de Bertini-Sard, Inventiones Math. 36 (1976)] with respect to the Whitney stratification: $(Y \times \mathbb{C}^N - Y_1, Y_1)$ of $Y \times \mathbb{C}^N$, or, what amounts to the same, with respect to $(Y \times \mathbb{C}^N - X, X - Y_1, Y_1)$. [rugosity is a Lipschitz-like condition, but relative to a stratification].*

Proof. (1) and (2) follow from proposition 3 and the definitions. As for (3), we think of vector fields as derivations: a (germ of) vector field on Y is given by a derivation ∂ of $\mathcal{O}_{Y,0} = \mathbb{C}\{y_1, \cdots, y_k\}$, and the pull-back of this vector field to $Y \times \mathbb{C}^N$ by the projection $Y \times \mathbb{C}^N \to Y$ is given by the derivations $\tilde{\partial}$ of $\mathcal{O}_{Y \times \mathbb{C}^N,0} = \mathbb{C}\{y_1, \cdots, y_k, z_1, \cdots, z_N\}$ described by $\tilde{\partial} y_l = \partial y_l$, $\tilde{\partial} z_i = 0$.

Now the projection of this vector field, at each point $p \in Y \times \mathbb{C}^N$, on the tangent space $T_{L_p,p}$ is described by the derivation:

$$D = \tilde{\partial} - \sum_{i=1}^{N} \frac{\tilde{\partial} F \cdot \overline{\dfrac{\partial F}{\partial z_i}}}{\sum_1^N \left|\dfrac{\partial F}{\partial z_i}\right|^2} \frac{\partial}{\partial z_i} \quad \text{of} \quad \mathcal{O}_{Y \times \mathbb{C}^N}$$

and now of course $DF = 0$, which means it is tangent to L_p at each $p \in Y \times \mathbb{C}^N - Y_1$, and to check its rugosity is just a matter of applying proposition 3 to the definition.

Since a rugose vector field is always integrable (Verdier, loc. cit.), we obtain by extending the constant vector fields on Y_1 a topological trivialization of X viewed as a family of hypersurfaces with isolated singularity parametrized by Y, and in fact better than topological triviality, a 'rugose' triviality, namely a homeomorphism of pairs

$$(Y \times \mathbb{C}^N, X) \cong (Y \times \mathbb{C}^N, Y \times X_0)$$

compatible with projection to Y, and 'rugose' with respect to the stratifications described above.

Now that we are dealing with isolated singularities, we can ask what happens to the Jacobian ideal $j(F_y)$ associated to the singularities of hypersurfaces $(X_y, y \times \{0\}$ in our family. First we need a definition:

Definition. Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be a germ of hypersurface with isolated singularity. For each $0 \leqq i \leqq N$ we define $\mu^{(i)}(X_0, 0) = \min_H \mu(X_0 \cap H, 0)$ H running through the Grassmannian of i-planes in $\mathbb{C}^N$. It is not difficult to check that in fact the set of those H such that $\mu(X_0 \cap H, 0) = \mu^{(i)}(X_0, 0)$ is a dense Zariski open subset of the Grassmannian. Remark that $\mu^{(N)}(X_0, 0)$ is the usual Milnor number, that $\mu^{(1)}(X_0, 0)$ is the multiplicity (= order of equation) minus none, and $\mu^{(0)}(X_0, 0) = 1$.

2.18. Theorem. *Let $X \subset Y \times \mathbb{C}^N$ (Y non-singular, as always) be a hypersurface such that $|\operatorname{Sing} X| = Y_1 = Y \times \{0\}$. For each $y \in Y$ set $(X_y, 0) = (X \cap (\{y\} \times \mathbb{C}^N), 0)$. Then the following are equivalent*

(1) *$M(X_y, 0) = \sum_{i=0}^N \binom{N}{i} \mu^{(i)}(X_y, 0)$ is independent of $y \in Y$ (in a neighborhood of $0 \in Y$),*

(2) *for each i, $\mu^{(i)}(X_y, 0)$ is independent of $y \in Y$,*

(3) *X satisfies the condition* (c) *along Y_1 at 0×0.*

I will not give the proof, referring to my 'Introduction to equisingularity problems' (Proceedings A.M.S., Conference on Algebraic Geometry, Arcata 1974, A.M.S. Pub., Providence Rhode-Island), and to the forthcoming notes of a course at the Collège de France, Spring 1976, but I will say this: the 'easy part' is to check that condition (c) implies that the Milnor number of the fibres is constant. Then by cor. 2 to prop. 2, all $\mu^{(i)}$ are constant. The converse is more delicate, and uses a connection between integral dependence and the multiplicity of ideals in the sense of algebraic geometers, which implies that equi-multiplicity conditions such as (1) or (2) have the consequence that a condition like (3), when true generically (which we know by corollary 1 to prop. 2 above) remains true at the special point, [and the fact that $M(X_y, 0)$ is the multiplicity of $m \cdot j(F_y)$]. This is the 'principle of specialization of integral dependence' [Appendix 1 to 'Sur diverses conditions numériques d'équisingularité des familles de courbes . . .', preprint No M208.0675, Centre de Mathématiques, Ecole Polytechnique, 91128 Palaiseau Cedex, France, see also 'Cycles évanescents, sections planes et conditions de Whitney', ch. II, §3, No 3.3, Astérisque No 7–8, Société Math. France, 1973].

Remark. One can generalize condition (c) to the incidence of any pair of strata (M, N) with $\bar{M} \supset N$ in a complex-analytic space (in our case, $M = X^0$, $N = Y_1$) and prove that any complex space has a locally finite stratification such that any two distinct strata (M, N) with $N \cap \bar{M} \neq \varnothing$ satisfy: $N \subset \bar{M}$ and (M, N) satisfies condition (c) at every point of N.

Remark. It is known [see 'Cycles évanescents $\cdots$' quoted above] that the Milnor number $\mu^{(N)}(X_y)$ of a hypersurface with isolated singularity depends only on the topological type of the *imbedded* germ $(X_y, 0) \subset (\mathbb{C}^N, 0)$. This is not true for the other $\mu^{(i)}$ (it is conjectured by Zariski for $\mu^{(1)}$ but the answer is not yet known), as is shown by the examples found by Briançon et Speder: X is the family of surfaces in $\mathbb{C}^3$ with isolated singularity defined by $z_2^3 + yz_2z_1^3 + z_1^4z_3 + z_3^9 = 0$ the topological type of the fibres is independent of y, hence also $\mu^{(3)} = 56$, but $\mu^{(2)}(X_0) = 8$, $\mu^{(2)}(X_y) = 7$ for $y \neq 0$. This shows that the topological type of a germ of hypersurface does not determine the topological type of its generic hyperplane section (through the singular point), and that $M(X_y, 0)$ is not an invariant of the topological type of $(X_y, 0)$.

2.19. Definition. Two germs of hypersurfaces $(X_0, 0)$ and $(X_1, 0)$ are (c)-cosécant if there exists a 1 parameter family $f: X \overset{\sigma}{\underset{\longleftarrow}{\longrightarrow}} \mathbb{D}$ of germs of hypersurfaces everywhere (c)-equisingular along $\sigma(\mathbb{D})$, and having one fibre isomorphic to $(X_0, 0)$ and another to $(X_1, 0)$.

We have just seen that the $\mu^{(i)}$ are invariants of (c)-cosécance, and in fact (c)-cosécance implies not only that our hypersurfaces have the same topological type, but all their generic plane sections too. (c)-cosécance will be our working definition of equisingularity, and we will see what it means for plane curves in the next paragraph.

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§3. Families of curves

In this section, we introduce invariants attached to singularities of curves, and study families of curves where some of these invariants remain constant.

3.1. We will mostly study *germs of families of curves*, i.e. germs of *flat* mappings $f: (X, 0) \to (Y, 0)$ such that $(f^{-1}(0), 0)$ is a germ of curve, i.e. purely 1-dimensional analytic space. We will study also (germs of) *families of germs of curves*, which means that we have also given ourselves a section σ of f, so that for any small representative of f, we can speak of the germs $(f^{-1}(y), \sigma(y))$ as the members of our family of germs. Occasionally, we will consider mappings $f: X \to Y$, such that $X_0 = F^{-1}(0)$ is a projective or affine curve. In all cases, we will say that f is a family, or a deformation of its special fibre $X_0 = f^{-1}(0)$ (or the germ $(X_0, 0) = (f^{-1}(0), 0)$).

Let us start by fixing what we mean by the datum of a germ of curve $(X_0, 0)$. Abstractly, as we have said, it is a germ of a purely 1-dimensional analytic space, hence it is described by an analytic algebra $\mathcal{O}_0$ purely of dimension 1. Geometrically, $(X_0, 0)$ can be effectively given in two ways:

① *By equations.* By giving an ideal $I = (F_1, \cdots, F_m)$ in $\mathbb{C}\{z_1, \cdots, z_N\}$ such that $\mathcal{O}_0 = \mathbb{C}\{z_1, \cdots, z_N\}/I$. Saying that $\mathcal{O}_0$ is purely one-dimensional means that the ideal (0) has a primary decomposition $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ where $\sqrt{\mathfrak{q}_j} = \mathfrak{p}_j$ is a minimal prime ideal in $\mathcal{O}_0$, and $\dim \mathcal{O}_0/\mathfrak{p}_j = 1$. All this is easily translated in terms of the primary decomposition of I in $\mathbb{C}\{z_1, \cdots, z_N\}$. The rings $\mathcal{O}_0/\mathfrak{q}_j$ (resp. $\mathcal{O}_0/\mathfrak{p}_j$) correspond to germs of irreducible (resp. irreducible and reduced) analytic curves, called the irreducible components (resp. branches) of the curve. We will mostly study *reduced* curves, which means that $\mathcal{O}_0$ is a reduced ring, or equivalently that $\mathfrak{q}_j = \mathfrak{p}_j$ for all j. In this case, it is well known that the integral closure $\bar{\mathcal{O}}_0$ of $\mathcal{O}_0$ in its total ring of quotients is isomorphic to $\prod_{j=1}^r \mathbb{C}\{t_j\}$, and if we choose generators $z_1, \cdots, z_N$ of the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_0$, the injection $\mathcal{O}_0 \subset \prod_{j=1}^r \mathbb{C}\{t_j\}$ is described by the datum of the N elements $z_i = (z_i(t_j)) \in \prod_{j=1}^r \mathbb{C}\{t_j\}$, and such a datum is usually called a *parametrization* of the germ of curve $(X, 0)$, which brings us to the second way of giving a curve.

② *By a parametrization.* By giving ourselves a germ of a finite map $p: \coprod_{j=1}^r (\mathbb{C}, 0) \to (\mathbb{C}^N, 0)$.

Here one has to be very careful: except when $N = 2$, it is not true, even if $r = 1$, that the image of this mapping in the sense of §1 is a curve: it will have 'imbedded components' concentrated at the singular points, as will be shown in the examples at the end of this section. More precisely, if we give ourselves a reduced germ of curve $(X_0, 0) \hookrightarrow (\mathbb{C}^N, 0)$, then normalize it, as explained above, the normalization $n:(\bar{X}_0, n^{-1}(0)) \to (X_0, 0)$ is isomorphic with $\coprod_{j=1}^r (\mathbb{C}, 0) \to (X_0, 0)$ where r is the number of irreducible components of X_0. Then it is not true, if $N>2$, and X_0 is singular, that the image in the sense of §1 of $p = i \circ n : \coprod_{j=1}^r (\mathbb{C}, 0) \to (\mathbb{C}^N, 0)$ is $X_0 : im(p)$ has some extra 0-dimensional components (imbedded components). However, it is true that the *root* of $F_0(p_* \mathcal{O}_{\bar{X}_0, n^{-1}(0)})$ gives an ideal defining $(X_0, 0) \subset (\mathbb{C}^N, 0)$, (which is also Ann $(p_* \mathcal{O}_{\bar{X}_0, n^{-1}(0)})$. This is another instance of the phenomenon seen in the addendum to §1, and is the price we have to pay for the stability of our images. Anyway, $F_0(p_* \mathcal{O}_{\bar{X}_0, n^{-1}(0)})$ contains more information than is needed to recover $(X_0, 0)$.

3.2. Since a reduced curve is normal outside its singular point we see that the quotient sheaf $\bar{\mathcal{O}}_{X_0}/\mathcal{O}_{X_0}$ is concentrated at the singular points, and hence $\bar{\mathcal{O}}_0/\mathcal{O}_0$ is a finite dimensional vector space, and this will give our first invariant:

Definition. Let $(X_0, 0)$ be a germ of reduced curve (resp. X_0 be a reduced affine algebraic curve). Then the 'invariant δ' of $(X_0, 0)$ is defined by $\delta(X_0, 0) = \delta(\mathcal{O}_0) = \dim_{\mathbb{C}} \bar{\mathcal{O}}_0/\mathcal{O}_0$ (resp. $\delta(X_0) = \dim_{\mathbb{C}} \bar{A}/A$ where $A = \Gamma(X_0, \mathcal{O}_{X_0})$).

Remark. We have: $\delta(X_0) = \sum_{x \in X_0} \delta(X_0, x)$, remarking that the sum on the right is finite, since δ is nonzero only at singular points.

This invariant has to do with the following problem: each of the above descriptions of a germ of a curve suggests a description of what a germ of a family of curves is: abstractly, anyway, we have defined it as a germ of a flat map $f:(X, 0) \to (Y, 0)$. Assume that $Y = \mathbb{C}$ with parameter v, i.e. that we have a 1-parameter family.

We can try to describe our family:

① By a family of equations: Let $f^{-1}(0) = X_0$ be described in $\mathbb{C}^N$ by $(F_1, \cdots, F_m)$ as above. Can we describe $(X, 0)$ as a subspace of $(\mathbb{C} \times \mathbb{C}^N, 0)$ with coordinates $(v, z_1, \cdots, z_N)$, defined by an ideal generated by $(F_1 + v \cdot G_1, \cdots, F_m + v \cdot G_m)$ in $\mathcal{O}_{\mathbb{C} \times \mathbb{C}^N, 0}$, where $G_i \in \mathbb{C}\{v, z_1, \cdots, z_N\} \in \mathcal{O}_{\mathbb{C} \times \mathbb{C}^N, 0}$? [there is a v in front to mark that the perturbation of the equations F_i must vanish for $v = 0$].

② By a family of parametrizations: Can we describe $(X, 0)$ as the *reduced* image of a complex-analytic mapping: $(\mathbb{C}, 0) \times \coprod_{j=1}^r (\mathbb{C}, 0) \to (\mathbb{C} \times \mathbb{C}^N, 0)$ i.e. by giving N elements $z_i = (z_i(v, t_j)) \in \prod_{j=1}^r \mathbb{C}\{v, t_j\}$ where $z_i(v, t_j) = z_i(t_j) + v \cdot \zeta_i(v, t_j)$, with $\zeta_i(v, t_j) \in \mathbb{C}\{v, t_j\}$ and the $z_i(t_j)$ describe a parametrization of $(X_0, 0)$ as above and where the induced map $(\mathbb{C}, 0) \times \coprod_{j=1}^r (\mathbb{C}, 0) \to (X, 0)$ is the normalization and for each fixed value of t induces the normalization of the corresponding fibre?

In fact, any germ of mapping $f:(X, 0) \to (Y, 0)$ can be described as in ①, i.e. $X \subset Y \times \mathbb{C}^N$ defined by $\bar{F}_i \in \mathcal{O}_{Y,0}\{z_1, \cdots, z_N\}$: it is just a matter of remarking that $\mathfrak{m}_{Y,0} \cdot \mathcal{O}_{X,0} + (z_1, \cdots, z_N) = \mathfrak{m}_{X,0}$ ($\mathfrak{m}$ = maximal ideal) and hence we have a surjection of $\mathbb{C}$-algebras $\mathcal{O}_{Y,0}\{z_1, \cdots, z_N\} \to 0_{X,0}$. We are going to see that, on the contrary, a family of reduced curves can be described by a family of parametrizations if and only if it satisfies a numerical condition, essentially that the δ-invariant of its fibres is constant.

3.3. Let $f:(X, 0) \to (\mathbb{C}, 0)$ be a germ of a flat morphism of complex analytic spaces, such that its fibre is a reduced 1-dimensional analytic space (i.e. f is a germ of a family of reduced curves).

Let $n:\bar{X} \to X$ be the normalization of the surface X and let $p = f \circ n : (\bar{X}, n^{-1}(0)) \to (\mathbb{C}, 0)$.

Let us agree to denote $p^{-1}(0)$ by $(\bar{X})_0$ and to set $\delta((\bar{X})_0) = \sum_{x \in n^{-1}(0)} \delta(p^{-1}(0), x)$. Let us also agree to set $\delta(X_0) = \delta(f^{-1}(0))$ and $\delta(X_y) = \delta(f^{-1}(y))$ for $y \in \mathbb{C} - \{0\}$ in a *small enough representative of* f so that all the singular points of $X_y = f^{-1}(y)$ tend to 0 as $y \to 0$, and 0 is the only singular point of $X_0 = f^{-1}(0)$. Then we have:

Proposition. (1) $p = f \circ n$ *is a (multi-germ of a) flat mapping;* (2) $\delta((\bar{X})_0) = \delta(X_0) - \delta(X_y) (y \in \mathbb{C} - \{0\}$, *but near* 0). *In words, the invariant δ of the fibre over $0 \in \mathbb{C}$ of the normalization of the surface which is the total space of our family of curves, is equal to the difference of the invariants δ of the 'special' and 'generic' fibres of our family.*

Proof. The proof is entirely algebraic, and has little to do with complex-analytic geometry, or even characteristic zero for that matter: set $R = \mathcal{O}_{Y,0} = \mathbb{C}\{v\}$, $A = \mathcal{O}_{X,0}$, a reduced R-algebra (by the map $f^*: R \to A$ corresponding to f) and let $\bar{A}$ be the integral closure of A in its total ring of quotients. Thus, $\bar{A} = \mathcal{O}_{\bar{X}, n^{-1}(0)}$. Then, since $\bar{A}$ is a Krull domain, we can apply the results of [Bourbaki, Algèbre commutative, VII, §1.6 Prop. 10 and V, §2.1, Cor. 2]: since by our assumptions $\mathcal{O}_{X_0,0} = A/v \cdot A$ is a reduced 1-dimensional ring, we have that $v \cdot A = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ with $\mathfrak{p}_i$ prime ideals such that $\dim A/\mathfrak{p}_i = 1$. Furthermore, for each $\mathfrak{p}_i$, there exists a prime ideal $\mathfrak{p}'_i$ in $\bar{A}$ such that $A/\mathfrak{p}'_i$ is of dimension 1 and $\mathfrak{p}'_i \cap A = \mathfrak{p}_i$, and furthermore that $v \cdot \bar{A} \subset \mathfrak{p}'_1 \cap \cdots \cap \mathfrak{p}'_r$. Hence we see that $v \cdot \bar{A} \cap A \subseteq v \cdot A$ and since the other inclusion is obvious, we have $v \cdot \bar{A} \cap A = v \cdot A$. Now we need:

Lemma (Universal property of the normalization). *Let A be a reduced ring and let $\bar{A}$ be its integral closure in its total ring of fractions. Assume that the* conductor *of $\bar{A}$ in A, $c = \{d \in \bar{A} \mid d \cdot \bar{A} \subseteq A\}$ is not zero. Then, for a mapping*

$\varphi: A \to B$ where B is a reduced ring integrally closed in its total quotient ring, and φ such that $\varphi(\mathfrak{c})$ contains a nonzero divisor of B, there exists a unique extension $\bar{\varphi}$ of φ to $\bar{A}$.

$$\begin{tikzcd} & \bar{A} \arrow[dr, "\bar{\varphi}"] \arrow[d, phantom, "0"] & \\ A \arrow[ur, hook] \arrow[rr, "\varphi"'] & & B \end{tikzcd}$$

PROOF. Choose $d \in \mathfrak{c}$ such that $\varphi(d)$ is not a zero-divisor in $\operatorname{Tot}(B)$, and set $\bar{\varphi}(a) = (\varphi(d \cdot a)/(\varphi(d)) \in \operatorname{Tot}(B)$, which has a meaning, since $1 \in \bar{A} \Rightarrow \mathfrak{c} \subset A$, and $d \cdot a \in A$.

$\bar{\varphi}(a)$ is integral over $\varphi(A)$ since a is integral over A, hence $\bar{\varphi}(a)$ is integral over B, so it is in B by our assumption. Uniqueness is obvious.

Let us apply this to the composed map

$$A \longrightarrow A/v\cdot A \longrightarrow \overline{A/v\cdot A}.$$

We have that $\bar{A}/A$, which is supported by the singular locus of our surface X, is an R-module of finite type, since by our assumptions, the singular locus of X is finite over $(\mathbb{C}, 0) = Y$ in view of the simplicity theorem of §2, because f is flat, and with non-singular fibre at every point of $X_0 - \{0\}$, for a small enough representative. It then follows from the Weierstrass preparation theorem (see Hironaka's lectures) that any module supported by $|\operatorname{Sing} X|$ is of finite type as an R-module. So $\bar{A}/A$ and also $A/\mathfrak{c}$ are R-modules of finite type.

From this we deduce that it is impossible that $\mathfrak{c} \subset \mathfrak{p}_i$ for some i, otherwise we would have a surjection $A/\mathfrak{c} \to A/\mathfrak{p}_i$ and $\bar{A}/\mathfrak{p}_i$ is definitely not an R-module of finite type since its fibre over 0 is itself because $v \in \mathfrak{p}_i$, and it is of dimension 1, hence certainly not a finite dimensional vector space, in view of Hilbert's nullstellensatz. By a useful lemma [cf. J. P. Serre, Algèbre locale et multiplicités, chap. I, Prop. 2 (old edition: Springer Lect. Notes No. 11)] we have that $\mathfrak{c} \not\subset \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_r$, so the image of $\mathfrak{c}$ in $A/v\cdot A$ or $\bar{A}/v\cdot\bar{A}$ contains a nonzero divisor, and we conclude that the map $A \to \overline{A/v\cdot A}$ factors through $\bar{A}$, and since v goes to zero, it factors in fact through $\bar{A}/v\cdot\bar{A}$ so we have a commutative diagram:

$$\begin{tikzcd} & \bar{A}/v\cdot\bar{A} \arrow[dr, "\bar{\varphi}"] & \\ A \arrow[r] & A/v\cdot A \arrow[u, "i"] \arrow[r, hook] & \overline{A/v\cdot A} \end{tikzcd}$$

We know that i is injective, since its injectivity amounts to: $v\cdot\bar{A} \cap A = v\cdot A$. Now let us show that $\bar{\varphi}$ is injective: to construct our factorization according to the above lemma, we have chosen an element $d \in \mathfrak{c} - \bigcup_{i=1}^{r} \mathfrak{p}_i$.

An element $a \in \bar{A}$ such that $\bar{\varphi}(a) = 0$ must be such that $d \cdot a \in v\cdot A$, but since $d \notin \cup \mathfrak{p}_i$, it means $a \in \mathfrak{p}'_1 \cap \cdots \cap \mathfrak{p}'_r$ hence $a \in v\cdot\bar{A}$, which shows that $\bar{\varphi}$ is injective. Hence we obtain the equality:

$$\dim_{\mathbb{C}} \overline{A/v\cdot A}/A/v\cdot A = \dim_{\mathbb{C}} \overline{A/v\cdot A}/\bar{A}/v\cdot\bar{A} + \dim_{\mathbb{C}} \bar{A}/v\cdot\bar{A}/A/v\cdot A$$

since $A/v\cdot A = 0_{X_0,0}$, the left-hand side is $\delta(X_0, 0)$ and now we remark that $v\cdot\bar{A} \cap A = v\cdot A$ implies that $\bar{A}/A$ is an R-module without torsion. Since R is a discrete valuation ring, this implies that $\bar{A}/A$ is a (locally) free R-module of finite type, and its rank is necessarily $\delta(X_y)$ since for $y \neq 0$ small enough, $(\bar{X})_y = p^{-1}(y)$ is non singular [by Bertini's theorem because $\bar{X}$, being a normal surface, has only isolated singularities], and $(\bar{X})_y \to X_y$ is the normalization.

Since $\bar{A}/A$ is a locally free R-module of rank $\delta(X_y)$, we have

$$\dim_{\mathbb{C}} \bar{A}/v\cdot\bar{A}/A/v\cdot A = \dim_{\mathbb{C}} \bar{A}/A \bigotimes_R R/v\cdot R = \delta(X_y)$$

where of course $R/v\cdot R = \mathbb{C}$, and finally, $\overline{A/v\cdot A}$ is also the integral closure of $\bar{A}/v\cdot\bar{A}$ in its total ring of fractions, so that

$$\dim_{\mathbb{C}} \overline{A/v\cdot A}/\bar{A}/v\cdot\bar{A} = \delta((\bar{X})_0).$$

This shows the equality $\delta((\bar{X})_0) = \delta(X_0) - \delta(X_y)$ and on the way we have seen that $\bar{A}/A$ is flat over R, and since A is by assumption flat over R, this is in fact equivalent to the flatness of $\bar{A}$ as an R-module, as follows: We have the following exact sequence:

$$\operatorname{Tor}_1^R(\bar{A}, \mathbb{C}) \xrightarrow{\ i\ } \operatorname{Tor}_1^R(\bar{A}/A, \mathbb{C}) \longrightarrow A \bigotimes_R \mathbb{C} \xrightarrow{\ j\ } \bar{A} \bigotimes_R \mathbb{C} \longrightarrow \bar{A}/A \bigotimes_R \mathbb{C} \longrightarrow 0$$

coming from the exact sequence $0 \to A \to \bar{A} \to \bar{A}/A \to 0$ and since $v\cdot\bar{A} \cap A = v\cdot A$, j is injective so i is surjective and (see Hironaka's lectures: appendix prop. 2) we see that $\bar{A}/A$ is a flat R-module $\Leftrightarrow \bar{A}$ is a flat R-module.

COROLLARY 1. *Let $f: (X, 0) \to (\mathbb{C}, 0)$ be a germ of* flat *mapping, the fibre of which is reduced of dimension 1. The following are equivalent:*

(i) *f can be described by a deformation of a parametrization of its fibre $(X_0, 0) = (f^{-1}(0), 0)$, in the sense explained at the beginning of this paragraph;*

(ii) *the normalization $\bar{X}$ of X is non singular for a small enough representative of f, and the composite map $\bar{X} \to X \to \mathbb{C}$ is a submersion of non-singular spaces. Furthermore, for any $y \in \mathbb{C}$ (i.e. in a small disk around 0) the induced map $(\bar{X})_y \to X_y$ is the normalization of the curve X_y;*

(iii) *for any small enough representative of f we have $\delta(X_y) = \delta(X_0)$ for any $y \in \mathbb{C}$, i.e. the family of curves has 'δ constant'.*

Proof. (i) $\Leftrightarrow$ (ii) has just been seen. (ii) $\Leftrightarrow$ (iii) follows from the proposition in view of the fact that $\bar{X} \to \mathbb{C}$ is flat, and its fibre is non-singular if and only if $\delta((\bar{X})_0) = 0$, (use the simplicity theorem of §2).

Corollary 2. *Let $X \subset \mathbb{C} \times \mathbb{P}^N$ be a closed complex subspace such that the projection $pr_1 : \mathbb{C} \times \mathbb{P}^N \to \mathbb{C}$ induces on X a flat mapping $f : X \to \mathbb{C}$, the fibres of which are reduced projective connected curves in $\mathbb{P}^N$ (and $\mathbb{C}$ stands for a representative of $(\mathbb{C}, 0)$, as usual). Then the following conditions are equivalent:*

(i) *the family has 'simultaneous normalization', i.e. again the normalization $\bar{X}$ of X is a non-singular surface, and the composed mapping $\bar{X} \to X \to \mathbb{C}$ is a differentiable fibration (since it is proper and submersive) the fibre $(\bar{X})_y$ of which over $y \in \mathbb{C}$ is the normalization of X_y;*

(ii) *$g(X_y) - (r_y - 1)$ is constant ($y \in C$), where $g(X_y)$ is the geometric genus of X_y and r_y is the number of its irreducible components – (equivalently: the topological Euler characteristic $\chi_{\mathrm{top}}(\bar{X}_y)$ of $\bar{X}_y$ is constant).*

The geometric (or effective) genus of a reduced projective curve is the genus of its normalization, which is a non-singular projective curve. It is a birational invariant, and also a topological invariant of the normalization, since $2g(\bar{X}_y) = b_1(\bar{X}_y)$. Since $r_y = b_0(\bar{X}_y)$ is also a topological invariant, it is clear that (i) $\Rightarrow$ (ii).

To prove that (ii) $\Rightarrow$ (i), we use the general genus formula (see H. Hironaka: On the arithmetic genera and the effective genera of algebraic curves, Memoirs College of Science Univ. of Kyoto, series A vol. XXX No 2 (1957) and J. P. Serre: Groupes algébriques et corps de classe, chap. IV §1, 2 Hermann éditeur, Paris (1959). The genus formula states that

$$p_a(X_y) = g(X_y) - (r_y - 1) + \sum_{x \in X_y} \delta(X_y, x)$$

where $p_a(X_y)$ is the arithmetic genus of X_y, which is defined by $1 - p_a(X_y) = \chi(X_y)$ where $\chi(X_y)$ is the constant term of the Hilbert polynomial of $X_y \subset \mathbb{P}^N$, which is also the Euler characteristic $\sum_{i=0}^{2} (-1)^i \dim H^i(X_y, 0_{X_y}) = \chi(0_{X_y})$(here $H^i(X_y, 0_{X_y}) = 0$, $i > 2$). This last definition has the consequence that $p_a(X_y)$ is constant in a flat family of curves (it is perhaps easier to see that the Hilbert polynomial of the fibres of a flat family of projective varieties is constant). We obtain in this way that the assumption of (ii) implies that $\sum_{x \in X_y} \delta(X_y, x)$ is constant. We can now localize near each one of the singular points of the special fibre X_0 say, and then using the semi-continuity of the invariant δ which is implied by the proposition, we obtain that the assumption of (ii) implies that each singular point of X_0 has a neighborhood in X which satisfies the equivalent conditions of corollary 1, whence we get simultaneous normalization, since it is a local property of X.

Remark. A proof in abstract algebraic geometry (characteristic $\neq 0$ for example) would require that we find a hyperplane H in $\mathbb{P}^N$ such that all the singular points of the fibers X_y lie outside H, hence in an affine open subset of X_y, and then repeating the argument of the proof of the proposition in this relatively affine situation (of course such a hyperplane always exists).

3.4. We are now ready to give our geometric interpretation of the invariant $\delta(X_0)$ of a *plane* curve as the maximum number of singular points which one can pile up in the same fibre of a deformation of X_0, meaning of course an arbitrarily small deformation. We consider X_0 either as a sufficiently small representative of a germ of plane curve, or as an affine plane curve, but since all our constructions and invariants are of a local nature on X_0, it is sufficient to treat the first case.

Let $n : \bar{X}_0 \to X_0$ be the normalization of the reduced curve X_0, and consider the closed complex subspace $\bar{X}_0 \times_{X_0} \bar{X}_0 \subset \bar{X}_0 \times \bar{X}_0$, defined by a sheaf of ideals $\mathscr{I}_0$ on the non-singular surface $\bar{X}_0 \times \bar{X}_0$. We will denote by I_0 the multi-germ of $\mathscr{I}_0$ along the finite set $|n^{-1}(0) \times n^{-1}(0)|$.

Let $\delta'(X_0, 0)$ be the maximum number of singular points which can occur in a fibre of an arbitrary small representative of a deformation $f : (X, 0) \to (Y, 0)$ of X_0 [f is flat and $(f^{-1}(0), 0) = (X_0, 0)$]. Remark that if X_y is such a fibre having $\delta'(X_0, 0)$ singular points, since the invariant δ is an integer, $\delta'(X_0, 0) \leqq \delta(X_y)$, and it follows from the proposition above that $\delta(X_y) \leqq \delta(X_0, 0)$, so that we know that $\delta'(X_0, 0) \leqq \delta(X_0, 0)$. We are now going to prove two statements at once:

Corollary 3. *For a germ of a reduced plane curve $(X_0, 0)$:*

(1) *$\delta'(X_0, 0) = \delta(X_0, 0)$;*

(2) *$I_0 = K_0 \cdot \mathfrak{N}_0$ where K_0 is invertible and $\mathfrak{N}_0$ defines a germ of a subspace of $\bar{X}_0 \times \bar{X}_0$ contained in $|n^{-1}(0) \times n^{-1}(0)|$, and:*

(3) *$e(\mathfrak{N}_0) = \dim_{\mathbb{C}} (\mathcal{O}_{\bar{X}_0 \times \bar{X}_0, n^{-1}(0) \times n^{-1}(0)} / \mathfrak{N}_0) = 2 \cdot \delta(X_0, 0)$.*

Here $e(\mathfrak{N}_0)$ is the multiplicity in the sense of Samuel of the ideal $\mathfrak{N}_0$ in the semi-local algebra in which it lives and the first equality in (3) is due to the fact that $\mathfrak{N}_0$ is generated by a regular sequence [see J. P. Serre, Algèbre locale, Multiplicités, Springer Lect. Notes, No. 11].

Comment. We will see that K_0 is in fact the ideal defining the diagonal $\bar{X}_0 \subset \bar{X}_0 \times \bar{X}_0$, and $e(\mathfrak{N}_0)$ in a way measures the difference between $\bar{X}_0 \times_{X_0} \bar{X}_0$ and this diagonal.

Proof. We call z_1 and z_2 generators of the maximal ideal of $\mathcal{O}_0 = \mathcal{O}_{X_0, 0}$, and the normalization is described algebraically by $\mathcal{O}_0 \subset \prod_{i=1}^{r} \mathbb{C}\{t_i\}$ given by

$(z_1(t_j))$ and $(z_2(t_j))$ as explained above. We have $\mathcal{O}_{\bar{X}_0\times\bar{X}_0, n^{-1}(0)\times n^{-1}(0)} = \prod_{i,j}\mathbb{C}\{t_i, t_j'\}$ and to describe an ideal in such a product it is sufficient to describe the ideal it induces in the (i, j)th factor $\mathbb{C}\{t_i, t_j'\}$. For example, we see that our ideal I_0, in view of the definition of the fibre products, is described by the:

$$I_{0,i,j} = (z_1(t_i) - z_1(t_j'), z_2(t_i) - z_2(t_j'))\mathbb{C}\{t_i, t_j'\}.$$

We can define K_0 and $\mathfrak{N}_0$ by:

$$K_{0,i,i} = (t_i - t_i')\mathbb{C}\{t_i, t_i'\} \quad \text{and if} \quad i \neq j,\ K_{0,i,j} = \mathbb{C}\{t_i, t_j'\},$$

$$\mathfrak{N}_{0,i,i} = \left(\frac{z_1(t_i) - z_1(t_i')}{t_i - t_i'}, \frac{z_2(t_i) - z_2(t_i')}{t_i - t_i'}\right)\mathbb{C}\{t_i, t_i'\} \quad \text{and if} \quad i \neq j$$

$$\mathfrak{N}_{0,i,j} = (z_1(t_i) - z_1(t_j'), z_2(t_i) - z_2(t_j'))\mathbb{C}\{t_i, t_j'\}.$$

Clearly K_0 is invertible, and $I_0 = K_0 \cdot \mathfrak{N}_0$. Now we have the:

LEMMA. *Let $(X_0, 0) \subset (\mathbb{C}^2, 0)$ be a germ of reduced plane curve described parametrically by $(z_1(t_j))$ and $(z_2(t_j)) \in \prod_{j=1}^r \mathbb{C}\{t_j\}$. There exists a nowhere dense closed complex surface $B \subset (\mathbb{C}^2)^r$ such that if $((\alpha_j, \beta_j)\ 1 \leq j \leq r) \in (\mathbb{C}^2)^r - B$, the plane curve X described parametrically by $(z_1(t_j) - \alpha_j v t_j, z_2(t_j) - \beta_j v t_j)$ has only ordinary double points as singularities (in a neighborhood of $0 \in \mathbb{C}^2$) for $v \neq 0$ sufficiently small.*

PROOF. Consider the graph $\Gamma \subset (\coprod_{j=1}^r (\mathbb{C}, 0)) \times (\mathbb{C}^2, 0)$ of the map $\coprod_{j=1}^r (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ of which X_0 is the image. Γ is a non-singular curve, and the parametric description above is nothing but that of the projection of Γ to $\mathbb{C}^2$ parallel to the multi-direction with vectors $(1, \alpha_j v, \beta_j v)$. It is a well-known result that a general plane projection of a non-singular curve has only ordinary double points as singularities, 'general' meaning: for all directions of projection except those in a nowhere dense closed subspace of the space of directions of projection.

This proves the lemma.

Remark that the lemma in fact provides as with a 1-parameter deformation of $(X_0, 0)$ (see 3.5.4 below) such that all the fibres except $(X_0, 0)$ have only ordinary double points. Furthermore, this deformation is – by construction – obtained by a deformation of the parametrization, hence by corollary 1, we have $\delta(X_v) = \delta(X_0, 0)$. Since, as is readily checked, the invariant δ of an ordinary double point is 1, we see that X_v must have exactly $\delta(X_0, 0)$ ordinary double points. Hence $\delta'(X_0, 0) \geq \delta(X_0, 0)$ and since we know the other inequality, we get the equality.

Let us now see how $\mathfrak{N}_0$ varies in this deformation: we proceed as follows: consider the ideal I in $\bar{X} \times_{\mathbb{C}} \bar{X}$ defining the closed subspace $\bar{X} \times_X \bar{X} \subset \bar{X} \underset{\mathbb{C}}{\times} \bar{X}$,

where $(X, 0) \to (\mathbb{C}, 0)$ is the family constructed in the lemma and $n : \bar{X} \to X$ is the normalization. By corollary 1, we know that for any v, $(\bar{X})_v = \bar{X}_v$, so I is the family of the ideals I_v corresponding to the fibre X_v. We look at the multi-germs along $|n^{-1}(0) \times n^{-1}(0)|$ and get:

$$\mathcal{O}_{\bar{X}\times_{\mathbb{C}}\bar{X}, n^{-1}(0)\times n^{-1}(0)} = \prod_{i,j}\mathbb{C}\{v, t_i, t_j'\}$$

and again $I = K \cdot \mathfrak{N}$ where

$$\mathfrak{N}_{i,i} = \left(\frac{z_1(t_i) - z_1(t_i')}{t_i - t_i'} - \alpha_i v, \frac{z_2(t_i) - z_2(t_i')}{t_i - t_i'} - \beta_i v\right)\mathbb{C}\{v, t_i, t_i'\}$$

and if $j \neq i$

$$\mathfrak{N}_{i,j} = (z_1(t_i) - z_1(t_j') - (\alpha_i t_i - \alpha_j t_j')v, z_2(t_i) - z_2(t_j') - (\beta_i t_i - \beta_j t_j')v)\mathbb{C}\{v, t_i, t_j'\}.$$

It is easily checked that the natural injection

$$\mathbb{C}\{v\} \to \prod_{i,j}\mathbb{C}\{v, t_i, t_j'\}/\mathfrak{N}$$

makes the quotient a $\mathbb{C}\{v\}$-module without torsion, i.e. flat (essentially because $\mathfrak{N}$ is generated by a regular sequence) and hence the dimension of the fibres is independent of v (Bourbaki, Algèbre Commutative, chap. II) and thus, taking small representatives of X, $\bar{X}$ etc. we find that $e(\mathfrak{N}_v) = e(\mathfrak{N}_0)$ for any sufficiently small value of v. Now, we remark that $e(\mathfrak{N}_v)$ is the sum of the $e(\mathfrak{N}_v, x_i)$ corresponding to the various singular points of X_v, and we give the:

EXERCISE. Describe parametrically an ordinary double point of plane curve, and check that in this case, $e(\mathfrak{N}_0) = 2$.

Since X_v has, for $v \neq 0$, $\delta(X_0, 0)$ ordinary points, we obtain $e(\mathfrak{N}_0) = e(\mathfrak{N}_v) = 2 \cdot \delta(X_0, 0)$.

REMARKS. (1) The lemma above is a parametric version of the Morse lemma which states that if $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ has an isolated critical point at 0, by an arbitrary small generic linear perturbation $f + \sum_0^n \alpha_i z_i$ of f, we obtain a new function which has only μ quadratic non-degenerate critical points with distinct critical values, where

$$\mu = \dim_{\mathbb{C}} \mathbb{C}\{z_0, \cdots, z_n\} \Big/ \left(\frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}\right).$$

For $n = 1$, i.e. plane curves, we see that, while μ is the maximum number of critical points which one can *spread out* by a small perturbation of f, δ is the maximum number of critical points which one can *pile up* (in the same fibre) by a small perturbation of f. We shall see more about this below.

(2) The consideration of the ideal $\mathfrak{N}_0$ defined above, together with its normalized blowing up, and its deformations given by the lemma, provides a wealth of information about the numerical invariants of X_0. [See: F. Pham and B. Teissier, Fractions lipschitziennes d'une algèbre analytique complexe, Centre de Maths. de l'Ecole Polytechnique, Juin 1969, also: F. Pham in Congrès International des Mathématiciens, Nice 1970]. In particular, one can use it to prove algebraically a well-known relation between μ and δ

$$\underline{2\delta = \mu + r - 1}$$

(r = number of irreducible components, or branches), given by Milnor in his book 'Singular points of complex hypersurfaces'. I will not give the proof here, referring to Milnor's book or J. J. Risler: Sur l'idéal jacobien d'une courbe plane, Bull. Soc. Math. Fr. (Risler gives an algebraic proof) but this equality will be used below.

3.5. On the images of parametrizations: the Fitting ideal as a prophet

Let A be a regular local ring (for example $\mathbb{C}\{z_1, \cdots, z_N\}$) and let M be an A-module of finite presentation. By the Hilbert Syzygy theorem, M has a resolution of finite length (in fact of length $\leqq \dim A$), i.e. there exists an exact sequence of A-modules:

$$0 \longrightarrow L_p \longrightarrow \cdots \longrightarrow L_2 \longrightarrow L_1 \xrightarrow{\Psi_1} L_0 \longrightarrow M \longrightarrow 0$$

$p \leqq \dim A$, where each L_i is a *free* A-module of finite type. The smallest integer p such that there exists such a sequence is called the homological dimension of M over A, and denoted $dh_A(M)$. Let us now define an M-sequence of elements of A as a sequence $(a_1, \cdots, a_k)$ of elements of the maximal ideal of A such that for any j, $1 \leqq j \leqq k$, a_j is not a zero-divisor in the module $M/(a_1, \cdots, a_{j-1}) \cdot M$ where $(a_1, \cdots, a_{j-1}) \cdot M$ is the submodule $\sum_{k=1}^{j-1} a_k \cdot M$ (and 0 if $j = 1$). It turns out (see Serre's book on Local algebra) that all the maximal (in the obvious sense) M-sequences have the same length, called the *depth* of M, and that:

$$\underline{dh_A(M) + \operatorname{depth}_A(M) = \dim A.}$$

Application. Suppose we can check that $\operatorname{depth}_A M \geqq \dim A - 1$ and that $\operatorname{Ann}_A M \neq 0$. Then $F_0(M)$ is generated by 1 element, which is not zero.

Proof. By the above equality, $dh_A M \leqq 1$, hence we can find a resolution

$$0 \longrightarrow A^q \xrightarrow{\Psi} A^p \longrightarrow M \longrightarrow 0.$$

We see that $q \leqq p$, but since $\operatorname{Ann}_A M \neq 0$ we have $M \bigotimes_A K = 0$ where $A \to K$ is the field of fractions of A. This implies that $\Psi \bigotimes 1_K : K^q \to K^p$ is surjective, hence $q \geqq p$. So we must have $p = q$ and $F_0(M) = (\det \underline{\Psi}) \cdot A$, where $\underline{\Psi}$ is a matrix representing Ψ.

3.5.1. Suppose now that we parametrize a curve in $(\mathbb{C}^2, 0)$ by $x(t), y(t)$, both $\neq 0$, thus making $\mathbb{C}\{t\}$ a $\mathbb{C}\{x, y\}$-module of finite representation. Clearly, $x(t)$ (for example) is not a zero-divisor in $\mathbb{C}\{t\}$, so the depth of $\mathbb{C}\{t\}$ as $\mathbb{C}\{x, y\}$-module is $\geqq 1$. On the other hand, we know our curve has an equation, so that $\operatorname{Ann} \mathbb{C}\{t\} \neq 0$, hence we know that $F_0(\mathbb{C}\{t\})$ is a principal ideal in $\mathbb{C}\{x, y\}$, and a generator is what we call an equation of the image curve. If we have arranged that for a given 'general' point (x_0, y_0) on the image curve, the equations $x(t) = x_0$, $y(t) = y_0$ have only one solution then we know even our equation will be reduced, that is, will be a prime element in $\mathbb{C}\{x, y\}$. (Compare all this with the example given in §1.)

3.5.2. Suppose now that we consider the same plane curve but *as lying in* $\mathbb{C}^3$, that is, the curve in $\mathbb{C}^3$ given by $x = x(t)$, $y = y(t)$, $z = 0$. Then, let us compute the Fitting ideal of $\mathbb{C}\{t\}$ as $\mathbb{C}\{x, y, z\}$-module. Take for example $x(t) = t^3$, $y(t) = t^4$. Then $\mathbb{C}\{t\}$ is generated as $\mathbb{C}\{x, y, z\}$-module by $e_0 = 1$, $e_1 = t$, $e_2 = t^2$ and it is not difficult to see that the relations are described by the matrix

$$\underline{\Psi} = \begin{bmatrix} y & -x & 0 \\ 0 & y & -x \\ x^2 & 0 & -y \\ z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix}$$

i.e. $\underline{\Psi}$ is the matrix of a presentation

$$\mathbb{C}\{x, y, z\}^6 \xrightarrow{\Psi} \mathbb{C}\{x, y, z\}^3 \longrightarrow \mathbb{C}\{t\} \longrightarrow 0$$

here $F_0(\mathbb{C}\{t\}) = (y^3 - x^4, z^3, zx^2, zy^2, z^2y, z^2x)$ so that the image of our curve computed by $F_0(\mathbb{C}\{t\})$ consists of the curve $y^3 - x^4 = 0$, $z = 0$, plus an extra 0-dimensional component sticking out of the (x, y)-plane. Of course, $\sqrt{F_0(\mathbb{C}\{t\})} = (y^3 - x^4, z)\mathbb{C}\{x, y, z\}$. If we had computed $F_0(\mathbb{C}\{t\})$ for a curve which really lies in $\mathbb{C}^3$, such as $x = t^4$, $y = t^6$, $z = t^7$ which is a complete intersection with ideal: $(y^2 - x^3, z^2 - x^2y)\mathbb{C}\{x, y, z\}$, we would similarly have found $F_0(\mathbb{C}\{t\}) = ((y^2 - x^3)\varphi_1, (z^2 - x^2y)\varphi_2)\mathbb{C}\{x, y, z\}$ where $\sqrt{\varphi_1} = \sqrt{\varphi_2} = (x, y, z)\mathbb{C}\{x, y, z\}$. In general, given a morphism $\mathbb{C}\{z_1, \cdots, z_N\} \to \mathbb{C}\{t\}$ corresponding to a parametric representation of a curve, $F_0(\mathbb{C}\{t\})$ will define a curve, in the sense that $\mathbb{C}\{z_1, \cdots, z_N\}/F_0(\mathbb{C}\{t\})$ is purely 1-dimensional, only if $N=2$ or if $\min v(z_i(t))=1$, which is the case where our germ of curve is non-singular.

3.5.3. Let us now turn to the following problem: given again a parametrization of a curve in $\mathbb{C}^N$, let us consider the generalization of the construction made above for plane curves, namely, for general values of $(\alpha_i)\in\mathbb{C}^N$, the algebra $A=\mathbb{C}\{v, z_1(t)+\alpha_1 vt, \cdots, z_N(t)+\alpha_N vt\}\subset\mathbb{C}\{v, t\}$. It is not difficult to see that if $N\geqq 3$, for general values of α_i, the curve thus described for each value of the parameter v is *non-singular* for $v\neq 0$. This might seem to contradict the proposition proved above: we apparently have a family of curves given by a deformation of the parametrization, such that $\delta(X_0)>0$ (we have of course chosen our curve to be singular) but $\delta(X_v)=0$ for $v\neq 0$. What happens here is that $A/v\cdot A$ is not a purely 1-dimensional ring if $N\geqq 3$, again it has an imbedded component. Setting $\bar{A}=\mathbb{C}\{v, t\}$, it amounts to the fact that we do *not* have $v\cdot\bar{A}\cap A=v\cdot A$. Therefore A does *not* describe a deformation of our original curve, if $N\geqq 3$, and the proposition above is not contradicted. [Remark that to say that $v\cdot\bar{A}\cap A=v\cdot A$ amounts to saying that $A/v\cdot A$ can be computed by setting $v=0$ in A.]

In order to see this, take for example $A=\mathbb{C}\{v, t^4+\alpha vt, t^6+\beta vt, t^7+\gamma vt\}$ in $\bar{A}=\mathbb{C}\{v, t\}$, and look at the element $(t^6+\beta vt)^2-(t^4+\alpha vt)^3$: it is in $v\cdot\bar{A}\cap A$ but *not* in $v\cdot A$. The choice of this element was dictated by the fact that y^2-x^3 lies in the annihilator of $\mathbb{C}\{t\}$ (as module over $\mathbb{C}\{x, y, z\}$ via $x=t^4$, $y=t^6$, $z=t^7$) but does not lie in the Fitting ideal $F_0(\mathbb{C}\{t\})$. In fact, the Fitting ideal *predicts*, by its imbedded components, before any deformation is made, that such 'accidents' will happen when deforming the parametrization.

3.5.4. When $N=2$, however, we can apply the results of the beginning of 3.5, and check that $F_0(\mathbb{C}\{v, t\})$ is a principal ideal. By the compatibility of the Fitting ideal with base change, the fibre over $v=0$ of the hypersurface in $\mathbb{C}\times\mathbb{C}^2$ thus described is precisely our original plane curve by 3.5.1. Hence in this case $\mathbb{C}\{v, x(t)+\alpha vt, y(t)+\beta vt\}$ does describe a flat family of curves and the argument used above in the proof of corollary 3 is indeed valid.

3.6. The Newton polygon of a plane curve

We do not leave the subject of parametrization, or prophecy for that matter, since the Newton polygon construction gives information in reverse of the Fitting ideal: given an equation $f(x, y)=0$ for a plane curve, it allows us to predict at least the *ratio* of the smallest exponents appearing in a parametric representation of our curve, that is, which ratios $v(x(t_j))/v(y(t_j))$ we get in the parametric representation $(x(t_j), y(t_j))$ (v is the order in t_j).

Take $f\in\mathbb{C}\{x, y\}$, assume for simplicity $f(0, y)\not\equiv$ and $f(x, 0)\not\equiv 0$. Write $f=\Sigma c_{ij}x^iy^j$ and set $\Delta(f)=\{(i, j)\subset\mathbb{N}^2/c_{ij}\neq 0\}$. The Newton polygon $\mathfrak{N}(f)$ (in the coordinates x, y) is the convex hull of $\Delta(f)$ in the following sense: $\mathfrak{N}(f)$ is a convex polygon with two sides of infinite length and a finite number of edges of finite length, and a line l in $\mathbb{R}^2$ contains an edge of finite length of $\Delta(f)$ if and only if contains at least 2 points of $\Delta(f)$, and there are no points of $\Delta(f)$ on the same side of l as $0\in\mathbb{R}^2$.

Since we have assumed $f(0, y)\not\equiv 0$ we can, by the Weierstrass preparation theorem, assume, up to multiplication by an invertible element $U\in\mathbb{C}\{x, y\}$, which does not change the Newton polygon, that $f=y^m+a_{m-1}(x)y^{m-1}+\cdots+\cdots+a_0(x)$. It is clear that $\mathfrak{N}(f)$ is also the convex hull of the finite set $\{(v(a_k(x)), k)\}\subset\mathbb{N}^2$. Let us consider f as an element of $\mathbb{C}\{\{x\}\}[y]$, where $\mathbb{C}\{\{x\}\}$ is the field of 'meromorphic functions', valued by the order in x (whether $\geqq 0$ or <0). I think it is clear what a valued extension of a valued field is: $(K, v)\subset(L, w)$ means $w\mid K=v$. (Note that w can take values in a subgroup of $\mathbb{Q}$ *isomorphic* to $\mathbb{Z}$, such as $1/k\cdot\mathbb{Z}$.)

3.6.1. We now describe a formalism on Newton polygons which will simplify matters later. We think of a Newton polygon as the following picture

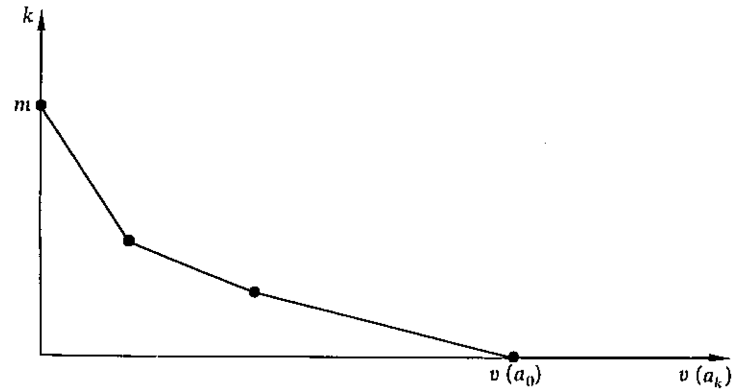

Figure 1

the coordinates of the vertices being integers. We define an *elementary* polygon as one having only one edge of finite length. An elementary polygon P has a height $h(P)$, length $l(P)$ and inclination $i(P)$ as shown in Fig. 2.

We count the polygon consisting of the two coordinate axis as the 0-polygon. Its inclination is not defined. For any Newton polygon, we can still define $h(P)$ and $l(P)$. We now define the *sum* of two elementary polygons

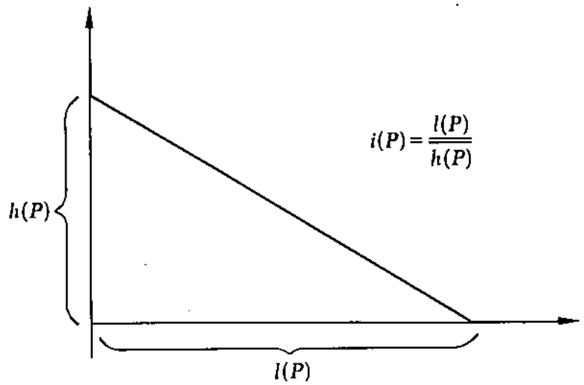

Figure 2

P, Q as the only *convex* polygon having an edge of inclination $i(P)$ and length (of horizontal projection) $l(P)$, and an edge of inclination $i(Q)$ and length (of horizontal projection) $l(Q)$. If $i(P)=i(Q)$ then $P+Q$ is again elementary, of length $l(P)+l(Q)$ and height $h(P)+h(Q)$. We need a notation for an elementary polygon, and I propose $P=\{l(P)//h(P)\}$. We see that any Newton polygon has a decomposition as a (convex) sum of elementary polygons, and this decomposition is unique if we require that they all have different inclinations. This enables us to define the sum of any two Newton polygons P and Q by first decomposing them into elementary polygons, $P=\Sigma P_i$, $Q=\Sigma Q_j$ and then making the only convex polygon sum of all the P_i, Q_j. This operation is associative and commutative, and the 0-polygon is a neutral element, so at this stage we have made the set of polygons into a *semi-group* $\mathfrak{N}_0$. We now formally symmetrize it to imbed it into a group $\mathfrak{N}$. Remark that $l(P+Q)=l(P)+l(Q)$ and $h(P+Q)=h(P)+h(Q)$.

Exercise. Prove that $\mathfrak{N}(f)+\mathfrak{N}(g)=\mathfrak{N}(f\cdot g)$ in $\mathfrak{N}_0$.

We now proceed to define a *product* on $\mathfrak{N}$: first we define it for two elementary polygons by

$$P*Q=\begin{cases}\left\{\dfrac{l(P)\cdot l(Q)}{l(P)\cdot h(Q)}\right\} & \text{if} \quad i(P)\leqq i(Q)\\[2ex] \left\{\dfrac{l(Q)\cdot l(P)}{l(Q)\cdot h(P)}\right\} & \text{if} \quad i(P)\geqq i(Q).\end{cases}$$

We remark that for elementary polygons, $i(P*Q)=\max(i(P), i(Q))$ and we now extend $*$ to all polygons by requiring distributivity, i.e. decomposing $P=\Sigma P_i$, $Q=\Sigma Q_j$ into elementary polygons, define:

$$P*Q=\sum_{i,j} P_i * Q_j$$

it is easy to see that $*$ is commutative and associative. Furthermore, define the set of *special Newton polygons* $\mathfrak{N}_S\subset\mathfrak{N}$ as those having only edges of inclination $\geqq 1$. $\mathfrak{N}_S$ is stable under sum and $*$, and defining the unit Newton polygon to be $\mathbb{1}=\{1//1\}$, we have:

$$P\in\mathfrak{N}_S \Leftrightarrow P*\mathbb{1}=P.$$

So that we have now endowed our symmetrized semi-group of Newton polygons $\mathfrak{N}$ with a *commutative ring structure* and $\mathfrak{N}_S\subset\mathfrak{N}$ is even a commutative ring with unit, which we call the *ring of special Newton polygons.*

There are two other useful operations on $\mathfrak{N}$:

(1) the symmetry σ defined by

$$\sigma\left(\sum_i\left\{\frac{l(P_i)}{h(P_i)}\right\}\right)=\sum_i\left\{\frac{h(P_i)}{l(P_i)}\right\}.$$

(2) The horizontal expansion ε defined by

$$\varepsilon\left(\sum_i\left\{\frac{l(P_i)}{h(P_i)}\right\}\right)=\sum_i\left\{\frac{l(P_i)+h(P_i)}{h(P_i)}\right\}.$$

Exercise. (1) Show that for any Newton polygons:

$$l(P*Q)=l(P)\cdot l(Q), \qquad h(P*P)=2\cdot S(P), \qquad l(P*P)=l(P)^2,$$

where $S(P)$ is the *area* between the polygon and the two axis.

(2) The application $\mathbb{Z}\to\mathfrak{N}_S$ defined by $n\to\{n//n\}$ identifies $\mathbb{Z}$ with a subring of $\mathfrak{N}_S$.

We can now state:

Theorem (Newton-Puiseux) of decomposition of a polynomial along the sides of its Newton polygon [see R. J. Walker, Algebraic curves, Dover books, and J. Dieudonné, Calcul infinitésimal, Hermann, Paris].

(1) *Let* (L, w) *be a finite valued extension of the valued field* $(\mathbb{C}\{\{x\}\}, v)$ [*where* v *is the order in* x, *whether* $\geqq 0$ *or* <0 *of an element of the field of fractions* $\mathbb{C}\{\{x\}\}$ *of* $\mathbb{C}\{x\}$, *and* w *is a valuation of* L *with values in a subgroup of* $\mathbb{Q}$ *isomorphic to* $\mathbb{Z}$ (i.e. $(1/d)\,\mathbb{Z}$ *for some* d) *and* $w\,|\,\mathbb{C}\{\{x\}\}=v$] *assume that* $f\in\mathbb{C}\{\{x\}\}[y]$ *has all its roots in* L, *call them* $y_1,\cdots,y_m$. *Then*

$$\underline{\mathfrak{N}(f)=\sum_{\rho\in\mathbb{Q}}\left\{\frac{m_\rho\cdot\rho}{m_\rho}\right\}}$$

where m_ρ *is the number of roots* y_k *of* f *having valuation* $w(y_k)=\rho$. *Of course except for a finite number of values of* ρ, *the number so found is zero, so the sum on the right is finite.*

(2) *Any finite extension of* $\mathbb{C}\{\{x\}\}$ *is isomorphic with* $\mathbb{C}\{\{x^{1/d}\}\}$ *where d is the degree of the extension, and in particular, the algebraic closure of* $\mathbb{C}\{\{x\}\}$ *is* $\bigcup_{d\geq 1}\mathbb{C}\{\{x^{1/d}\}\}$.

(3) *Let* $\rho_1, \cdots, \rho_k$ *be the rational numbers such that* $m_{\rho_i}\neq 0$. *Then there is in* $\mathbb{C}\{\{x\}\}[y]$ *a decomposition* $f=f_{\rho_1}\cdots f_{\rho_k}$ *where each* f_{ρ_i} *is of degree* m_{ρ_i} *in y, all its roots have the same valuation* ρ_i, *and* $m_{\rho_i}\cdot\rho_i$ *is the order in x (or the valuation) of the constant term (in y) of* f_{ρ_i}.

TRANSLATION. Assume f to be irreducible. Then all its roots in y lie in $\mathbb{C}\{\{x^{1/m}\}\}$ where m is the degree of f in y which means that the roots of $f(x, y)=0$ are convergent series in $x^{1/m}$ or equivalently that we can describe the curve $f(x, y)$ parametrically by

$$\begin{cases} x=t^m \\ y=\varphi(t) \quad \varphi \text{ a convergent power series.}\end{cases}$$

Finally we have that the valuation of y, which is $(1/m)\times$(order in t of $\varphi(t)$) is uniquely determined by the equation

$$\left\{\frac{m\cdot\rho}{m}\right\}=\left\{\frac{e}{m}\right\}$$

where $\{e//m\}$ is the Newton polygon of f, which is elementary in view of 3 above. So finally, we see that if f is irreducible, its Newton polygon has only one edge, hence is elementary, say $\{e/m\}$ and from this we can predict that the parametric representation of our curve will be

$$\begin{cases} x=t^m \\ y=u(t)(t^e+c_{e+1}t^{e+1}+\cdots) \quad \text{with} \quad u(0)\neq 0.\end{cases}$$

Of course if $\mathfrak{N}(f)$ is elementary, we *cannot* conclude that f is irreducible, but only that the ratios e_k/m_k occurring in the parametrizations of the various branches of the curve $f(x, y)=0$ are equal.

EXERCISE. Convince yourself that at least the valuation $w(y_k)$ of a root of $f(x, y)=0$ in a valued extension is the inclination of an edge of $\mathfrak{N}(f)$.

3.6.2. One can also use the Newton polygon construction *in reverse;* namely, knowing a parametric representation of a plane curve, say $(x(t_j), y(t_j))$ in $\prod_{j=1}^r \mathbb{C}\{t_j\}$ we can immediately say something non-trivial about the equation $f=0$ of our plane curve, without having to compute the Fitting ideal;

$$\mathfrak{N}(f)=\sum_{j=1}^{r}\left\{\frac{v(y(t_j))}{v(x(t_j))}\right\} \quad (\text{where } v=\text{order in } t_j).$$

REMARKS. (1) The Newton Polygon depends very much upon the choice of the coordinates (x, y) in which we expand f and if we take a 'generic choice' of coordinates (x, y), $\mathfrak{N}(f)=\{m//m\}$ where m is order of f at 0, so that $\mathfrak{N}(f)$ contains very little information about f. There are two ways in which the Newton polygon can be made to yield information: by taking coordinates 'as special as possible', basically asking that the inclination of $\mathfrak{N}(f)$ in these coordinates should be as large as possible: this is the theory of maximal contact of Hironaka, or alternately by looking only at those functions which are *well represented* by their Newton polygons in given coordinates: this approach is that of A. Kushnirenko (Inv. Math. *32*, 1 (1976)). Both theories were actually developed in arbitrary dimensions, and in Hironaka's theory of infinitely near points, the behaviour of Newton Polyhedra and polygons under some special modifications (permissible blowing ups) plays an important role.

Anyway, given a family of equations $f_v=\sum c_{ij}(v)x^iy^j$ with $c_{ij}(v)\in\mathbb{C}\{v\}$ we can observe that $\mathfrak{N}(f_0)$ lies *above* $\mathfrak{N}(f_v)$ simply because the coefficients $c_{ij}(v)$ can only vanish when $v=0$, and not suddenly become $\neq 0$, hence $\Delta(f_0)\subseteq\Delta(f_v)$, and the condition that $\mathfrak{N}(f_v)$ should be independent of v, say for some coordinate system, is a rather weak condition on the family of curves described by $f=0$, unless we assume that they are 'well represented' by their Newton polygon. However, this condition, as explained above, does contain some interesting information about the parametrizations of the curves in our family. We now turn to much stronger conditions.

3.7. A short summary of equisingularity for plane curves (after [6], [1])

One of the things we try to do here is to describe pertinent numerical invariants of the geometry 'up to equisingularity' of isolated singularities of hypersurfaces. The model is what happens for plane curves, where the situation is very agreeable. Let us first consider the case of a germ of an irreducible plane curve $(X_0, 0)$, defined by $f(x, y)=0$. Then, as we have seen we can obtain a parametric representation of our curve by $x=t^n$, $y=\varphi(t)$ and if we have chosen our coordinates so that of f (which is the multiplicity of X_0), and that the x-axis is in the tangent cone of X_0, then $v(\varphi(t))>n$ and the topology of $(X_0, 0)$ *as an imbedded germ* in $(\mathbb{C}^2, 0)$ (i.e. up to a germ of a homeomorphism extending to $\mathbb{C}^2$) is completely determined by the integer n and the values $v(\varphi(t)-\varphi(\omega t))$, ω running through the nth roots of unity. Call these values $\beta_1<\cdots<\beta_g$; then our curve is *topologically* equivalent to the curve described parametrically by

$$\begin{cases} x=t^n \\ y=t^{\beta_1}+t^{\beta_2}+\cdots+t^{\beta_g}.\end{cases}$$

The set of integers $(n, \beta_1, \cdots, \beta_g)$ is called the *characteristic* of the branch $(X_0, 0)$.

Define integers l_i by $l_0 = n$, $l_1 = (n, \beta_1), \cdots, l_i = (l_{i-1}, \beta_i)$ and remark that since our branch is not multiple, $l_g = 1$. Now define n_i by: $l_{i-1} = n_i \cdot l_i$: we have $n = n_1 \cdots n_g$, and finally define m_i by $\beta_i = m_i \cdot l_i$. Then $(m_i, n_i) = 1$ $(1 \leqq i \leqq g)$ and we can check that any branch with characteristic $(n, \beta_1, \cdots, \beta_g)$ can be described by a *Puiseux expansion*

$$y = \sum_{i=0}^{k_0} a_{0,i} x^i + \sum_{i=0}^{k_1} a_{1,i} x^{(m_1 + i/n_1)} + \cdots + \sum_{i=0}^{k_2} a_{p,i} x^{(m_p + i/n_1 \cdots n_q)} + \\ + \cdots + \sum_{i=0}^{\infty} a_{g,i} x^{(m_g + i/n_1 \cdots n_g)}.$$

Another way to obtain the characteristics is this: consider $\mathcal{O}_0 = \mathbb{C}\{x(t), y(t)\} \subset \mathbb{C}\{t\}$. Then, the valuations of the elements of $\mathcal{O}_0$ form a semi-group $\Gamma \subset \mathbb{N}$, and $\Gamma = \mathbb{N} \Leftrightarrow 1 \in \Gamma \Leftrightarrow \mathcal{O}_0 = \mathbb{C}\{t\}$ i.e. X_0 is non-singular. One constructs a minimal system of generators $\bar{\beta}_0, \cdots, \bar{\beta}_g$, (i.e. $\bar{\beta}_i \in \Gamma$ does not belong to the semi-group $\langle \bar{\beta}_0, \cdots, \bar{\beta}_{i-1} \rangle$ generated by the previous ones, and is the smallest non-zero element of Γ with this property). Then, by a theorem (see [7]) one has that $g' = g$, the number of characteristic exponents, and furthermore, the β_q's and $\bar{\beta}_q$'s are linked by the following relation: $\bar{\beta}_0 = \beta_0$, $\bar{\beta}_1 = \beta_1$

$$\bar{\beta}_q = n_{q-1} \cdot \bar{\beta}_{q-1} - \beta_{q-1} + \beta_q \qquad (1 < q \leqq g).$$

And so we see that the datum of $(n, \beta_1, \cdots, \beta_g)$ is equivalent to the datum of $\Gamma = \langle \bar{\beta}_0, \cdots, \bar{\beta}_g \rangle$.

Let us now consider a reducible, but reduced, (i.e. without nilpotent functions) plane curve, $(X_0, 0)$ with equation $f = 0$. Let us decompose f in $\mathbb{C}\{x, y\}$, $f = f_1, \cdots, f_r$, where each f_i is a prime element in $\mathbb{C}\{x, y\}$. Then each of the branches $X_{0,i}$ (defined by $f_i = 0$) has its own characteristic, and in addition we consider the intersection numbers $(X_{0,i}, X_{0,j}) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(f_i, f_j)$ (compare §1).

Let us now consider a *germ of family* of reduced plane curves, $(X, 0) \subset (Y \times \mathbb{C}^2, 0)$ defined by $(F)\mathcal{O}_{Y \times \mathbb{C}^2, 0}$ (where $(Y, 0) \simeq (\mathbb{C}^k, 0)$). By the Weierstrass preparation theorem, at the price of a linear change of the coordinates (x, y), we can assume

$$F = y^n + a_{n-1}(y, x) y^{n-1} + \cdots + a_0(y, x) \quad \text{where} \quad a_i \in \mathbb{C}\{y_1, \cdots, y_k, x\},$$

and we can define the *discriminant* of the projection π of X to $Y \times \mathbb{C}$ $(\mathcal{O}_{Y \times \mathbb{C}, 0} = \mathbb{C}\{y, x\})$ it is, as we know $F_0(\mathcal{O}_{Y \times \mathbb{C}^2}/(F, (\partial F/\partial y)))$, F_0 being the Fitting ideal as $\mathcal{O}_{Y \times \mathbb{C}, 0}$-module, of course. The critical locus of π is finite over $Y \times \mathbb{C}$ because $\pi^{-1}(0)$, being defined by $(y^n)\mathbb{C}\{y\}$, has an isolated singularity. This discriminant is a hypersurface Δ_π in $Y \times \mathbb{C}_1$ defined by the resultant of F and $(\partial F/\partial y)$. We say Δ_π is *trivial* at 0 if $\Delta_{\pi,\mathrm{red}}$ is a non-singular hypersurface at 0. (This implies that Δ_π is itself analytically trivial along $\Delta_{\pi,\mathrm{red}}$, i.e.: $(\Delta_\pi, 0) = (Z_0 \times \Delta_{\pi,\mathrm{red}}, 0)$ where Z_0 is isomorphic to the subspace of $\mathbb{C}$ defined by $(x)^\Delta$ for some integer Δ.) Then we have:

Theorem of equisingularity for plane curves (see [6], [1], [2], [5]):

I. *For a family* $(X, 0) \subset (Y \times \mathbb{C}^2, 0)$ *as above, assuming that* $X \supset Y_1 = Y \times \{0\}$, *we have equivalence of the following conditions:*

(1) *All the fibres* $(X_y, 0)$ *have the same topological type.*

(2) *All the fibres* $(X_y, 0)$ *have the same Milnor number*, i.e. $\mu^{(2)}(X_y, 0)$ *is independent of* $y \in Y$.

(3) *The invariant* $\delta(X_y, 0)$ and *the number of branches* r_y *of each fibre are independent of* $y \in Y$.

(4) *The composed map* $p: \bar{X} \xrightarrow{n} X \longrightarrow Y$ *is a submersion of non-singular spaces in a neighborhood of* $n^{-1}(0)$, n *induces the normalization* $(\bar{X})_y = \bar{X}_y \to X_y$ *in each fibre, and the map induced by* $p: (n^{-1}(Y_1))_{\mathrm{red}} \to Y$ *is a (trivial) covering of degree* r (= *the number of branches of the fibres*).

(5) *There exists a projection* $\pi: X \to Y \times \mathbb{C}$ *such that the discriminant is trivial (hence* $(\Delta_{\pi,\mathrm{red}}, 0) \simeq (Y \times \{0\}, 0)$.

(6) *There exists a projection* π *such that the multiplicity of the discriminant* Δ_π *is constant along* $Y \times \{0\} \subset \Delta_\pi$.

(7) *The sum* $\mu^{(2)}(X_y) + m(X_y) - 1 = \mu^{(2)}(X_y) + \mu^{(1)}(X_y)$ *is independent of* $y \in Y$.

(8) *For any projection* $\pi: X \to Y \times \mathbb{C}$ *such that the multiplicity of* $\pi^{-1}(0)$ *at* 0 *is equal to the multiplicity of* X *at* 0, *the discriminant* Δ_π *is trivial.*

(9) *Any holomorphic vector field on* $Y_1 = Y \times \{0\}$ *can be locally extended to a vector field which is* Lipschitz *and* meromorphic *on* X *(hence extends to a holomorphic vector field on the normalization* $\bar{X}$, *see condition* 4).

(10) *For any two values* y_1, y_2 *of* $y \in Y$ *(near* 0) *there is a bijection* b *from the set of branches of* $(X_{y_1}, 0)$ *to the set of branches of* $(X_{y_2}, 0)$ *such that* $b(X_{y_1,i})$ *has the same characteristic as* $X_{y_1,i}$ $(1 \leqq i \leqq r)$ *and we have equality of intersection numbers:*

$$(b(X_{y_1,i}), b(X_{y_1,j}))_0 = (X_{y_1,i}, X_{y_1,j})_0.$$

(11) $(X, 0)$ *satisfies the condition* (c) *of* §2 (2*nd part*) *along* $Y_1 = Y \times \{0\}$ *at* 0.

(12) $\bar{X} \xrightarrow{n} X \longrightarrow Y$ *is a submersion of non-singular spaces and* $j_{Y \times \mathbb{C}^2/Y}(F) \cdot \mathcal{O}_{\bar{X}, n^{-1}(0)}$ *is an invertible ideal.*

II. *Given two germs of plane curves* $(X_1, 0), (X_2, 0)$ *having the same topological type, or satisfying the numerical condition of* (10), *there exists a*

1-parameter family of germs of reduced plane curves

$$\begin{tikzcd} X \subset Y \times \mathbb{C}^2 \arrow[dr] & \\ & Y \arrow[ul, shift left, "\sigma"] \end{tikzcd}$$

and $y_1, y_2 \in Y(\dim Y = 1)$ *such that:*

(i) *the family satisfies the* 12 *equivalent conditions at every point of* $\sigma(Y)$.

(ii) $(X_{y_1}, \sigma(y_1)) \cong (X_1, 0), (X_{y_2}, \sigma(y_2)) \cong (X_2, 0)$, *where* $\cong$ *means analytic isomorphism.*

This theorem summarizes results, mostly due to Zariski [6], also to Lê Dũng Tráng and C. P. Ramanujam [2], Pham and Teissier [3] and Teissier [4]. See [1]. Complete detailed proofs will appear in [5].

We will see examples of deformations of curves in the next section.

EXERCISE. (1) Check in 12 different ways that the family of curves defined by $y^2 - x^3 + v \cdot x^2 = 0$ has constant invariant δ but is not equisingular. Check that $y^2 - x^3 = 0$ has no equisingular deformation which is not analytically a product. Check that the same holds for $y^2 - x^2 = 0$, $y^3 - x^3 = 0$.

(2) Check that the family $y^4 - x^4 + v \cdot x^2 y^2 = 0$ is equisingular at 0, but not analytically trivial.

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§4. Unfoldings and deformations

In this section, we give the basic definitions of the theory of unfoldings and deformations. Our purpose is not to prove the existence of versal unfoldings, but rather to illustrate the definitions by examples and above all to emphasize the close connection between the complex-analytic avatar of

the differential geometers' theory of unfoldings (Thom-Mather) and the algebraic geometers' theory of deformations (Schlessinger-Tjurina-Grauert). In fact, the local theory of unfoldings becomes simpler in the complex-analytic frame, and we have thought that the best way to show the connection mentioned above was to give a proof of the existence theorem of versal deformations for those singularities which can be presented as the fibres of a morphism $f : (\mathbb{C}^N, 0) \to (\mathbb{C}^p, 0)$ of finite singularity type in the sense of Mather, a class which includes complete intersections with isolated singularities and finite analytic spaces. This proof uses the existence of versal unfoldings and the existence of a local flattener for a map of complex-analytic spaces.

We use a transcription in complex-analytic geometry of the theory of unfoldings as found in the notes of Mather in these proceedings, and in the notes of Martinet and of Mather in 'Singularités d'applications différentiables', Springer Lecture Notes No. 535.

4.1. DEFINITION 1. Let $f : X \to Y$ be a morphism of complex analytic spaces. An unfolding of f, with base a germ of complex space $(S, 0)$, is a morphism:

$$F : X \times S \to Y \times S$$

commuting with the natural projections to S and such that $F \mid X \times \{0\} = f$.

We immediately remark that the datum of such an F is equivalent to the datum of $\mathbf{F} = pr_1 \circ F : X \times S \to Y$ (and $\mathbf{F} \mid X \times \{0\} = f$).

The first example is that of an *infinitesimal* unfolding of f: it is the case where $S = \mathbb{T}$(see §2). Then, we can view $\mathbf{F} : X \times \mathbb{T} \to Y$ as a vector field on Y parametrized by X (vector field in the sense of §2, i.e. of Zariski tangent vectors). Namely, for every $x \in X$, $\mathbf{F}$ gives a vector tangent to Y at $f(x)$, varying analytically with x. In this sense, as we saw in an exercise in §2, a vector field on X is nothing but an infinitesimal unfolding of the identity of X.

DEFINITION 2. Two unfoldings F and G of $f : X \to Y$ with the same base are said to be $\mathscr{A}$-isomorphic if there exists a commutative diagram

$$\begin{tikzcd} X \times S \arrow[r, "F"] \arrow[d, "\xi"'] & Y \times S \arrow[d, "\eta"] \\ X \times S \arrow[r, "G"] & Y \times S \end{tikzcd}$$

where ξ and η are isomorphisms, unfoldings of the identity of X and Y respectively.

An unfolding of f is said to be $\mathscr{A}$-*trivial* if it is $\mathscr{A}$-isomorphic to the morphism $f \times id_S : X \times S \to Y \times S$.

Let us see what it means for our infinitesimal unfolding to be $\mathscr{A}$-trivial; in order to simplify notations, we will write our unfoldings as $\mathbf{F}: X \times S \to Y$.

We can make many infinitesimal unfoldings of f as shown in the following diagram:

$$\begin{tikzcd} X \times \mathbb{T} \arrow[r, "f \times id_{\mathbb{T}}"] \arrow[d, "\xi"'] \arrow[dr, dashed, "\mathbf{F}"] & Y \times \mathbb{T} \arrow[d, dashed, "\eta"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Take a vector field ξ on X (resp. η on Y) and take $\mathbf{F} = f \circ \xi$ (resp. $\mathbf{F} = \eta \circ (f \times id_{\mathbb{T}})$) and you get an infinitesimal unfolding of f. However, by our definition, all such unfoldings are trivial.

REMARK. In the case where X and Y are non-singular, $\xi \mapsto f \circ \xi$ is Mather's tf, and $\eta \to \eta \circ (f \times id_{\mathbb{T}})$ is his ωf.

We now remark (see §2) that to give such an F is the same as to give a derivation of $f^{-1}\mathcal{O}_Y$ in $\mathcal{O}_X$ and that therefore, the set of such F is canonically endowed with a structure of $\mathcal{O}_X$-module. [Essentially, $\mathbf{F}$ corresponds to a map $\mathbf{F}^*: \mathcal{O}_Y \to \mathcal{O}_X[\varepsilon]$ such that the composed map $\mathcal{O}_Y \xrightarrow{\mathbf{F}^*} \mathcal{O}_X[\varepsilon] \longrightarrow \mathcal{O}_X[\varepsilon]/(\varepsilon) = \mathcal{O}_X$ is the map 'composition with f', and the derivation is defined by: $\Delta(h) =$ coefficient of ε in $F^*(h)$].

DEFINITION 3. f is infinitesimally stable if every infinitesimal unfolding of f is A-trivial that is, if any $\mathbf{F}: X \times \mathbb{T} \to Y$ such that $\mathbf{F} \mid X \times \{0\} = f$ can be written as

$$\mathbf{F} = f \circ \xi + \eta \circ (f \times id_{\mathbb{T}})$$

where ξ (resp. η) is a vector field on X (resp. Y).

Let us now consider *germs* of mappings $f: (X, 0) \to (Y, 0)$. Then the set of infinitesimal unfoldings of f can be identified with the set of $\mathbb{C}$-derivations of $\mathcal{O}_{Y,0}$ in $\mathcal{O}_{X,0}$ where $\mathcal{O}_{X,0}$ is considered as $\mathcal{O}_{Y,0}$-module via the map $f^*: \mathcal{O}_{Y,0} \to \mathcal{O}_{X,0}$ 'composition with f', which is a map of $\mathbb{C}$-algebras. The vector fields on $(X, 0)$ (resp. $(Y, 0)$) then correspond to the $\mathbb{C}$-derivations of $\mathcal{O}_{X,0}$ (resp. $\mathcal{O}_{Y,0}$) in itself. Let then $D \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{Y,0}, \mathcal{O}_{Y,0})$ be a derivation of $\mathcal{O}_{Y,0}$ into itself corresponding to η, and let $D' \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,0}, \mathcal{O}_{X,0})$ correspond similarly to ξ. Then, $f \circ \xi$ corresponds to the derivation $D' \circ f^*$ of $\mathcal{O}_{Y,0}$ in $\mathcal{O}_{X,0}$ and

$\eta \circ (f \times id_{\mathbb{T}})$ corresponds to $f^* \circ D$. Assume that $(X, 0)$ and $(Y, 0)$ are non-singular, so that we can set $\mathcal{O}_{X,0} = \mathcal{O}_{\mathbb{C}^N,0} = \mathbb{C}\{z_1, \cdots, z_N\}$ and $\mathcal{O}_{Y,0} = \mathcal{O}_{\mathbb{C}^p,0} = \mathbb{C}\{y_1, \cdots, y_p\}$, with f^* described by

$$f^*(y_j) = f_j(z_1, \cdots, z_N) \in \mathbb{C}\{z_1, \cdots, z_N\} \qquad (1 \leq j \leq p).$$

We can see that a derivation $\Delta \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^p,0}, \mathcal{O}_{\mathbb{C}^N,0})$ is described by the datum of the $\Delta y_j \in \mathbb{C}\{z_1, \cdots, z_N\}$, so that, having made a choice of coordinates, we can identify the set of $\mathbb{C}$-derivations of $\mathcal{O}_{\mathbb{C}^N,0}$ as $\mathcal{O}_{X,0}$-module with $\mathcal{O}^p_{\mathbb{C}^N,0}$, while the 'trivial' ones coming from derivation of $\mathcal{O}_{Y,0}$ into itself are identified with the sub $\mathcal{O}_{Y,0}$-module $(f^*(\mathcal{O}_{\mathbb{C}^p,0}))^p$ of $\mathcal{O}^p_{\mathbb{C}^N,0}$ and the 'trivial' ones coming from derivations of $\mathcal{O}_{\mathbb{C}^N,0}$ into itself are identified with the sub $\mathcal{O}_{\mathbb{C}^N,0}$-module of $\mathcal{O}^p_{\mathbb{C}^N,0}$ generated by the N elements $(\partial \vec{f}/\partial z_1), \cdots, (\partial \vec{f}/\partial z_N)$ where $\vec{f} = (f_1, \cdots, f_p)$. [N.B. This is clear because it means that we can write $\Delta y_k = D' f_k = \sum_{i=1}^N (\partial f_k/\partial z_i) D' z_i$ if and only if the element $(\Delta y_1, \cdots, \Delta y_p)$ of $\mathcal{O}_{\mathbb{C}^N,0}$ belongs to that submodule.]

We will write the submodule of $\mathcal{O}^p_{\mathbb{C}^N,0}$ generated by $((\partial \vec{f}/\partial z_1), \cdots, (\partial \vec{f}/\partial z_N))$ by: $((\partial \vec{f}/\partial z_1), \cdots, (\partial \vec{f}/\partial z_N))$ when no confusion is to be feared.

Finally we see that the infinitesimal unfoldings of f modulo the A-trivial ones, which we will call the non(-$\mathscr{A}$-)trivial infinitesimal unfoldings of f can be identified with the quotient:

$$A^1_f = \mathcal{O}^p_{\mathbb{C}^N,0}/(f^*(\mathcal{O}_{\mathbb{C}^p,0}))^p + ((\partial \vec{f}/\partial z_1), \cdots, (\partial \vec{f}/\partial z_N)).$$

It is very easy to check, using Hilbert's Nullstellensatz that if 0 is an isolated critical point of $f: (\mathbb{C}^N, 0) \to (\mathbb{C}^p, 0)$ then A^1_f is a finite dimensional vector space.

We also remark that $A^1_f = 0 \Leftrightarrow f$ is infinitesimally stable.

In the case where $(X, 0)$ and $(Y, 0)$ are allowed to have singularities, the situation is more delicate and I refer the reader to the work of Mount and Villamayor (Publ. Math. IHES No 43, P.U.F. 1974).

4.2. Now that we have studied infinitesimal unfoldings of a map $f: (\mathbb{C}^N, 0) \to (\mathbb{C}^p, 0)$, let us turn to arbitrary unfoldings of such an f: Remark that we can always describe an unfolding $F: (\mathbb{C}^N \times S, 0) \to (\mathbb{C}^p \times S, 0)$ of f as follows: if we describe f by $f^*(y_j) = f_j(z_1, \cdots, z_N) (1 \leq j \leq p)$, then F can be described by:

$$F^*(y_j) = f_j(z) + \sum_{A \in \mathbb{N}^N} h_{j,A} z^A,$$

where $h_{j,A} \in m_{S,0}$, maximal ideal of $\mathcal{O}_{S,0}$.

In this way, we can clearly see what the unfolding $F': (\mathbb{C}^N \times S', 0) \to (\mathbb{C}^p \times S', 0)$, obtained from F by a base change $\varphi: (S', 0) \to (S, 0)$, is: letting

$\varphi^*: \mathcal{O}_{S,0} \to \mathcal{O}_{S',0}$ be the map of algebras corresponding to φ, F' is described by

$$F'^*(y_j) = f_j(z) + \sum_{A \in \mathbb{N}^N} \varphi^*(h_{j,A}) z^A.$$

Definition 4. An unfolding $F: (\mathbb{C}^N \times S, 0) \to (\mathbb{C}^p \times S, 0)$ of $f: (\mathbb{C}^N, 0) \to (\mathbb{C}^p, 0)$ is said to be $\mathcal{A}$-versal (resp. $\mathcal{A}$-miniversal) if any other unfolding $H: (\mathbb{C}^N \times S', 0) \to (\mathbb{C}^p \times S', 0)$ of f can be obtained, up to $\mathcal{A}$-isomorphism, from F by a base change $\varphi: (S', 0) \to (S, 0)$ (resp. and S has the smallest possible dimension for this property to hold).

Theorem (Mather). *An unfolding $F: (\mathbb{C}^N \times \mathbb{C}^s, 0) \to (\mathbb{C}^p \times \mathbb{C}^s, 0)$ of f is $\mathcal{A}$-versal (resp. $\mathcal{A}$-miniversal) if and only if, when we describe it by*

$$F^*(y_j) = f_j(z) + \sum_{A \in \mathbb{N}^N} h_{j,A}(u) z^A$$

$[u = (u_1, \cdots, u_s)$, coordinates on $(\mathbb{C}^s, 0)]$; we have that setting $\vec{F} = (F^*(y_1), \cdots, F^*(y_p))$, the elements

$$\left.\frac{\partial \vec{F}}{\partial u_1}\right|_{u=0}, \cdots, \left.\frac{\partial \vec{F}}{\partial u_s}\right|_{u=0} \quad \text{in} \quad \mathbb{C}\{z_1, \cdots, z_N\}^p$$

have images in A_f^1 which generate it as a $\mathbb{C}$-vector space (resp. form a basis of it). All A-miniversal unfolding of f are obtained from one another, up to A-ismorphism, by base change by (non-canonical) isomorphisms $(\mathbb{C}^s, 0) \simeq (\mathbb{C}^s, 0)$.

In particular, we see that f has an $\mathcal{A}$-miniversal unfolding $F: (\mathbb{C}^N \times \mathbb{C}^s, 0) \to (\mathbb{C}^p \times \mathbb{C}^s, 0)$ if and only if A_f^1 is a finite-dimensional vector space, and then $s = \dim_{\mathbb{C}} A_f^1$ and F is the unfolding of f constructed as follows: one takes in $\mathbb{C}\{z_1, \cdots, z_N\}^p$ s elements $\vec{h}_1, \cdots, \vec{h}_s$ such that their images in A_f^1 form a basis of it over $\mathbb{C}$. Then F is described by:

$$F^*(y_j) = f_j(z) + \sum_{i=1}^{s} u_i \cdot h_{i,j}(z) \qquad (1 \leqq j \leqq p)$$

where $h_{ij}(z)$ is the jth component of $\vec{h}_i$.

4.2.1. Remark 1. Strictly speaking, the proof given in differential geometry, once transcribed in analytic geometry, proves only the versality of such an F with respect to unfoldings having a non-singular base, while we have in our definitions allowed arbitrary bases, e.g. a *finite* local analytic space: the analytic spectrum of an Artinian analytic algebra (a.k.a* 'thick point'). This difficulty, however, is inessential because if we begin a theory of

* a.k.a. = also known as.

unfoldings over such bases in the spirit of Schlessinger ('Functors of Artin rings', Trans. A.M.S. 130 (1968) 208–22) we see immediately that the functor of unfoldings is *unobstructed*, having the property that given $(S_1, 0)$ and $(S_2, 0)$ closed in $(S, 0)$ and unfoldings of base $(S_1, 0)$ and $(S_2, 0)$ which coincide on $(S_1 \cap S_2, 0)$, they can be extended to an unfolding over $(S, 0)$. This follows from the explicit description given at the beginning of 4.2, and implies that the base of the A-miniversal unfolding is non-singular, as implied by Mather's theorem above.

4.3. Until now, we have been considering the set of mappings $\mathbb{C}^N \to \mathbb{C}^p$ modulo the group action of $A = \operatorname{Aut}\mathbb{C}^N \times \operatorname{Aut}\mathbb{C}^p : f \mapsto \eta \circ f \circ \xi^{-1}$, and the sub $\mathcal{O}_{\mathbb{C}^p,0}$-module of $\mathcal{O}^p_{\mathbb{C}^N,0}$, $(f^*(\mathcal{O}_{\mathbb{C}^p,0}))^p + ((\partial \vec{f}/\partial z_1), \cdots, (\partial \vec{f}/\partial z_N)) \subset \mathcal{O}^p_{\mathbb{C}^N,0}$ which we saw above is deemed to be the 'tangent space at f of the orbit $A \circ f$ of f', the vector space A_f^1 then being its 'supplementary' in the tangent space at f to the space of functions such as f.

We now look at another group action, corresponding to an interest in the geometry of the *fibres* of mappings more than in the mappings themselves. It is the group of *contact transformations* in the terminology of Mather, or V-isomorphisms in that of Martinet. The idea is that we allow more than the usual automorphisms in the target space $\mathbb{C}^p$, namely we consider f and f' as equivalent if $(f^{-1}(0), 0)$ and $(f'^{-1}(0), 0)$ are isomorphic, or, what amounts to the same, if there exists a complex map $\mathbb{C}^N \xrightarrow{M} GL(p, \mathbb{C})$ such that $f'(x) = M(x) \cdot f(x)$ [which simply means that the ideals generated by $(f_1, \cdots, f_p)$ and $(f'_1, \cdots, f'_p)$ in $\mathbb{C}\{z_1, \cdots, z_N\}$ are equal]. In this spirit, we have:

Definition 5. Two unfoldings $\mathbf{F}, \mathbf{F}': (X \times S, 0) \to (Y, 0)$ of $f: (X, 0) \to (Y, 0)$ are $\mathcal{K}$-isomorphic if there exists a germ of an analytic isomorphism $\Phi: X \times S \xrightarrow{\sim} X \times S$, which is an unfolding of the identity of X, and such that $((F' \circ \Phi)^{-1}(0), 0)$ and $(F^{-1}(0), 0)$ are isomorphic as germs of complex-analytic spaces.

Let us now see what it means for an infinitesimal unfolding to be $\mathcal{K}$-trivial, i.e. $\mathcal{K}$-isomorphic to the trivial unfolding $X \times S \to Y$ which is $f \circ pr_1$.

We are given an unfolding $\mathbf{F}: (X \times \mathbb{T}, 0) \to (Y, 0)$ viewed as a derivation D of $\mathcal{O}_{Y,0}$ into $\mathcal{O}_{X,0}$. We assume $(X, 0)$ and $(Y, 0)$ non-singular again, and set $\mathcal{O}_{X,0} = \mathbb{C}\{z_1, \cdots, z_N\}$, $\mathcal{O}_{Y,0} = \mathbb{C}\{y_1, \cdots, y_p\}$. To say that F is $\mathcal{K}$-trivial is to say that there exists an infinitesimal unfolding of id_X, i.e. a vector field on X, i.e. a derivation $D': \mathcal{O}_{\mathbb{C}^N,0} \to \mathcal{O}_{\mathbb{C}^N,0}$, such that

$$Dy_j - D'f_j \in (f_1, \cdots, f_p) \cdot \mathcal{O}_{\mathbb{C}^N,0} \qquad (1 \leqq j \leqq p)$$

where $f_j = f_j(z) \in \mathcal{O}_{\mathbb{C}^N,0} = \mathbb{C}\{z_1, \cdots, z_N\}$ describe f (i.e. $f^*(y_j) = f_j$). We remark that $(f_1, \cdots, f_p) \cdot \mathcal{O}_{\mathbb{C}^N,0}$ is the ideal generated in $\mathcal{O}_{\mathbb{C}^N,0}$ by the image by

$f^*:\mathcal{O}_{\mathbb{C}^p,0}\to\mathcal{O}_{\mathbb{C}^N,0}$ of the maximal ideal $\mathfrak{m}_y$ of $\mathcal{O}_{\mathbb{C}^p,0}$, i.e. what we usually write simply by $\mathfrak{m}_y\cdot\mathcal{O}_{\mathbb{C}^N,0}$.

Therefore, we see that the 'tangent space to the $\mathcal{K}$ orbit' of f is exactly the sub-$\mathcal{O}_{\mathbb{C}^N,0}$-module:

$$\mathfrak{m}_y\cdot\mathcal{O}^p_{X,0}+\left(\frac{\partial\vec{f}}{\partial z_1},\cdots,\frac{\partial\vec{f}}{\partial z_N}\right)\subset\mathcal{O}^p_{\mathbb{C}^N,0}$$

because to say that D describes a $\mathcal{K}$-trivial infinitesimal unfolding is exactly to say that $(Dy_1,\cdots,Dy_p)\in\mathcal{O}^p_{\mathbb{C}^N,0}$ belongs to this submodule. We shall write:

$$K^1_f=\mathcal{O}^p_{\mathbb{C}^N,0}/\mathfrak{m}_y\cdot\mathcal{O}^p_{\mathbb{C}^N,0}+\left(\frac{\partial\vec{f}}{\partial z_1},\cdots,\frac{\partial\vec{f}}{\partial z_N}\right)$$

and remark that, since the fibre $(X_0,0)$ of $f:(\mathbb{C}^N,0)\to(\mathbb{C}^p,0)$ is the subspace defined by $\mathfrak{m}_y\cdot\mathcal{O}_{\mathbb{C}^N,0}$, we have also:

$$K^1_f=\mathcal{O}^p_{X_0,0}\Big/\sum_{i=1}^{N}\mathcal{O}_{X_0,0}\cdot\frac{\partial\vec{f}}{\partial z_i}.$$

Definition. Those maps $f:(\mathbb{C}^N,0)\to(\mathbb{C}^p,0)$ such that $\dim_{\mathbb{C}}K^1_f<\infty$ are called 'of finite singularity type' or 'T.S.F.' by Mather.

Exercise 1. Check that when f is finite, f is T.S.F.

Exercise 2. Check that when f is *flat* and its fiber $(X_0,0)$ has an isolated singularity, f is T.S.F. In general, f is T.S.F. if and only if the critical locus of f is finite over $\mathbb{C}^p$.

Exercise 3. Consider the map $f:(\mathbb{C}^4,0)\to(\mathbb{C}^4,0)$ given by $(z_1,z_2,z_3,z_4)\mapsto(z_1\cdot z_3,z_2\cdot z_3,z_1\cdot z_4,z_2\cdot z_4)$ and check that it is *not* T.S.F. although its fibre $(X_0,0)$, the union of two 2-dimensional planes in $\mathbb{C}^4$ meeting in one point, has an isolated singularity. Deduce that no map $f:(\mathbb{C}^N,0)\to(\mathbb{C}^p,0)$ having $(X_0,0)$ as fibre is T.S.F.

Definition 6. An unfolding $F:(\mathbb{C}^N\times S,0)\to(\mathbb{C}^p\times S,0)$ of $f:(\mathbb{C}^N,0)\to(\mathbb{C}^p,0)$ is $\mathcal{K}$-versal (resp. $\mathcal{K}$-miniversal) if any unfolding of f, say $H:(\mathbb{C}^N\times S',0)\to(\mathbb{C}^p\times S',0)$, is $\mathcal{K}$-isomorphic to an unfolding obtained from F by a base change $\varphi:(S',0)\to(S,0)$ (resp. and S has the smallest possible dimension for this property to hold).

Theorem (Mather). *An unfolding $F:(\mathbb{C}^N\times\mathbb{C}^t,0)\to(\mathbb{C}^p\times\mathbb{C}^t,0)$ is $\mathcal{K}$-versal (resp. $\mathcal{K}$-miniversal) if and only if, when we describe it by:*

$$y_j\circ F=F^*(y_j)=f_j(z)+\sum_{A\in\mathbb{N}^N}k_{j,A}(v)z^A$$

$[v=(v_1,\cdots,v_t)$, *coordinates on* $(\mathbb{C}^t,0)]$ *we have that, setting* $\vec{F}=(F^*(y_1),\cdots,F^*(y_p))\in\mathcal{O}^p_{\mathbb{C}^N\times\mathbb{C}^t,0}$, *the elements* $(\partial\vec{F}/\partial v_1)|_{v=0},\cdots,(\partial\vec{F}/\partial v_t)|_{v=0}$ *in*

$\mathcal{O}^p_{\mathbb{C}^N,0}$ *have images in* K^1_f *which generate it as* $\mathbb{C}$-*vector space (resp. form a basis of it).*

All $\mathcal{K}$-miniversal unfoldings of f can be obtained from one another, up to $\mathcal{K}$-isomorphism, by base change by (non-canonical) isomorphisms $(\mathbb{C}^t,0)\simeq(\mathbb{C}^t,0)$.

In particular, f has a $\mathcal{K}$-miniversal unfolding if and only if K^1_f is a finite-dimensional vector space, and it is built in the manner analogous to that explained for $\mathcal{A}$-miniversal unfoldings. In particular, it has the form

$$F^*(y_j)=f_j(z)+\sum_{i=1}^{t}v_ih_{i,j}(z)\qquad(1\leqq j\leqq p)$$

where the elements $\vec{h}_i=(h_{i,j})$ have images in K^1_f which form a basis of it.

4.4. We now turn to the theory of deformations; here we (temporarily) forget mappings, and think only of spaces. Let $(X_0,0)$ be a germ of a complex analytic space:

Definition 7. A deformation of $(X_0,0)$ is a Cartesian diagram of germs:

$$\begin{tikzcd}(X_0,0) \arrow[r, hook, "i"] \arrow[d] \arrow[dr, phantom, "\square"] & (X,0) \arrow[d, "G"] \\ \{0\} \arrow[r, hook] & (S,0)\end{tikzcd}$$

where G is a *flat* map. [Cartesian diagram means in this case that we are given an isomorphism of $(G^{-1}(0),0)$ with $(X_0,0)$.] A morphism of deformations is a morphism of squares inducing the identity on $(X_0,0)$. A deformation is said to be *trivial* if it is isomorphic with the product deformation $pr_2:(X_0\times S,0)\to(S,0)$, or equivalently if there exists a commutative diagram

$$\begin{tikzcd}(X,0) \arrow[rr, "\Psi"] \arrow[dr, "G"'] & 0 & (X_0\times S,0) \arrow[dl, "pr_2"] \\ & (S,0) & \end{tikzcd}$$

with Ψ an isomorphism inducing the identity on $(X_0,0)$.

Definition 8. A deformation $G:(X,0)\to(S,0)$ of $(X_0,0)$ is said to be versal (resp. miniversal) if any deformation $H:(X',0)\to(S',0)$ of $(X_0,0)$ is isomorphic to a deformation obtained from G by a base change $\varphi:(S',0)\to(S,0)$ i.e. H is isomorphic to G^φ:

$$\begin{tikzcd}(X\times S',0) \arrow[r, "pr_1"] \arrow[d, "G^\varphi"'] \arrow[dr, phantom, "\square"] & (X,0) \arrow[d, "G"] \\ (S',0) \arrow[r, "\varphi"] & (S,0)\end{tikzcd}$$

(resp. and S has the smallest possible dimension for this property to hold).

4.5. Now let us recall that by definition, any germ of complex analytic space $(X_0, 0)$ can be presented as the fibre of a germ of complex analytic map $f:(\mathbb{C}^N, 0)\to(\mathbb{C}^p, 0)$. Namely, the map is described by generators of the ideal defining $(X_0, 0)\subset(\mathbb{C}^N, 0)$. Recall also (see Hironaka's lectures), that $(X_0, 0)$ is a *complete intersection* if and only if it can be presented with a *flat* map f.

Finally recall that, given any complex-analytic map-germ $h:(X, 0)\to (Y, 0)$, there exists a germ of a complex subspace $(Z_h, 0)\subset(Y, 0)$, the *flattener* of h, which satisfies the following universal property:

For any complex-analytic map $\varphi:(Y', 0)\to(Y, 0)$, the map obtained by base change by φ:

$$\begin{tikzcd}
(X \underset{Y}{\times} Y', 0) \arrow[r, "pr_1"] \arrow[d, "h^{\varphi}"] \arrow[dr, phantom, "\square"] & (X, 0) \arrow[d, "h"] \\
(Y', 0) \arrow[r, "\varphi"] & (Y, 0)
\end{tikzcd}$$

is flat (at 0, of course) if and only if φ factors through the subspace $(Z_\varphi, 0)\subset(Y, 0)$.

(This was originally proved in [1] and is proved by a different method in Hironaka's lectures, §3).

Now the theory of unfoldings and the theory of deformations meet in the:

4.5.1. Proposition. *Let $(X_0, 0)$ be a germ of complex space with isolated singularity. Suppose there is a presentation of $(X_0, 0)$ as the fibre of an $f:(\mathbb{C}^N, 0)\to(\mathbb{C}^p, 0)$ such that* $\dim_{\mathbb{C}} K_f^1<\infty$, *let $F:(\mathbb{C}^N\times\mathbb{C}^t, 0)\to(\mathbb{C}^p\times\mathbb{C}^t, 0)$ be a $\mathcal{K}$-miniversal unfolding of f. Consider the subspace $(X', 0)=(F^{-1}(0\times\mathbb{C}^t), 0)$ and the induced map $F':(X', 0)\to(\mathbb{C}^t, 0)$. Let now $(S, 0)\hookrightarrow(\mathbb{C}^t, 0)$ be the flattener of F' and $G:(X, 0)\to(S, 0)$ be the map obtained from F' by base change by the inclusion $(S, 0)\hookrightarrow(\mathbb{C}^t, 0)$, i.e. restriction of F' over $(S, 0)$. Then, G is a* miniversal deformation *of $(X_0, 0)$. The construction is summarized in the diagram:*

$$\begin{tikzcd}
(X_0, 0) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & (X, 0) \arrow[r, hook] \arrow[d, "G"] \arrow[dr, phantom, "\square"] & (X', 0) \arrow[r, hook] \arrow[d, "F'"] & (\mathbb{C}^N\times\mathbb{C}^t, 0) \arrow[d, "F"] & (\mathbb{C}^N\times 0, 0) \arrow[l, hook'] \arrow[d, "f"] \\
\{0\} \arrow[r, hook] & (S, 0) \arrow[r, hook, "\text{flattener}"'] & (0\times\mathbb{C}^t, 0) \arrow[r, hook] & (\mathbb{C}^p\times\mathbb{C}^t, 0) & (\mathbb{C}^p\times 0, 0) \arrow[l, hook']
\end{tikzcd}$$

Proof of the proposition. First, remark that any deformation of $(X_0, 0)$ can be represented by an unfolding: let $f_j\in\mathbb{C}\{z_1, \cdots, z_N\}(1\leqq j\leqq p)$ be the equations of $(X_0, 0)\subset(\mathbb{C}^N, 0)$. As we saw in §3, any deformation $H:(X', 0)\to(S', 0)$ of $(X_0, 0)$ can be put in a commutative diagram:

$$\begin{tikzcd}
(X', 0) \arrow[rr, hook] \arrow[dr, "H"'] & 0 & (S'\times\mathbb{C}^N, 0) \arrow[dl, "pr_3"] \\
& (S', 0) &
\end{tikzcd}$$

where $(X', 0)\subset(S'\times\mathbb{C}^N, 0)$ is defined by an ideal generated by $F_j\in\mathcal{O}_{S',0}\{z_1, \cdots, z_N\}(1\leqq j\leqq p)$. These F_j of course also describe an unfolding $\boldsymbol{F}:(\mathbb{C}^N\times S', 0)\to(\mathbb{C}^p, 0)$ of the map $f:(\mathbb{C}^N, 0)\to(\mathbb{C}^p, 0)$ described by the f_j. Now by definition of a $\mathcal{K}$-miniversal unfolding, there exists a base change $\varphi:(S', 0)\to(\mathbb{C}^t, 0)$ such that $\boldsymbol{F}$ is $\mathcal{K}$-isomorphic to the unfolding obtained from F by the base change φ, and by the definition of $\mathcal{K}$-isomorphism, this means exactly that our deformation is obtained from the map F' in the diagram above by the base change φ. Since H is a flat map by definition, φ must factor through the flattener $(S, 0)\subset(\mathbb{C}^t, 0)$, and this shows that G is a *versal* deformation of $(X_0, 0)$. To check that G is in fact miniversal, we first stop to examine the Zariski tangent space to the base of the miniversal deformation of $(X_0, 0)$, which of course coincides with the set of infinitesimal deformations of $(X_0, 0)$ modulo isomorphism of deformations: suppose again $(X_0, 0)\subset(\mathbb{C}^N, 0)$ given by the ideal $I_0=(f_1, \cdots, f_p)\subset\mathbb{C}\{z_1, \cdots, z_N\}$. Then, any deformation of $(X_0, 0)$ with base $\mathbb{T}$ can be described in $\mathbb{C}^N\times\mathbb{T}$ by an ideal $I=(f_1+\varepsilon\cdot g_1, \cdots, f_p+\varepsilon\cdot g_p)\subset\mathbb{C}\{\varepsilon, z_1, \cdots, z_N\}$, with $\varepsilon^2=0$. It is an exercise on flatness (use the appendix to Hironaka's lectures) to check that $X\subset\mathbb{C}^N\times\mathbb{T}$ described by such an I is flat over $\mathbb{T}$ (by the restriction to X of $pr_2:\mathbb{C}^N\times\mathbb{T}\to\mathbb{T}$) if and only if the following condition is satisfied:

(*) For any relation $\sum_1^p a_i(z)\cdot f_i(z)=0$ between the f_i in $\mathbb{C}\{z_1, \cdots, z_N\}$, we have that:

$$\sum_1^p a_i(z)\cdot g_i(z)\in I_0=(f_1, \cdots, f_p).$$

Let us now remark that the datum of a set of (g_i) is exactly the datum of a map $\mathbb{C}\{z_1, \cdots, z_N\}^p\to\mathbb{C}\{z_1, \cdots, z_N\}$ of $\mathbb{C}\{z_1, \cdots, z_N\}$-modules (by sending the ith base element to g_i) and (*) is equivalent to the fact that the composed map:

$$\mathbb{C}\{z_1, \cdots, z_N\}^p\to\mathbb{C}\{z_1, \cdots, z_N\}\to\mathbb{C}\{z_1, \cdots, z_N\}/I_0=\mathcal{O}_{X_0,0}$$

factors through the map

$$\mathbb{C}\{z_1, \cdots, z_N\}^p\longrightarrow I_0\hookrightarrow\mathbb{C}\{z_1, \cdots, z_N\}$$

sending the ith base element to f_i.

Finally, we can identify the set of infinitesimal deformations of $(X_0, 0)$ with $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^N,0}}(I_0, \mathcal{O}_{X_0,0})$. Now those infinitesimal deformations which are trivial are those such that the ideal generated by the $(f_i+\varepsilon g_i)$ in $\mathbb{C}\{\varepsilon, z_1, \cdots, z_N\}$ becomes equal, after a change of variables, to the ideal generated by the (f_i), and since a change of variables adds to each g_i an element of the form $\sum_{j=1}^N (\partial f_i/\partial z_j)h_j$ it is clear that the deformation is trivial if and only if, setting $\vec{g}=(g_1, \cdots, g_p)\in \mathcal{O}^p_{\mathbb{C}^N,0}$, we have:

$$\vec{g}\in I_0\cdot \mathcal{O}^p_{\mathbb{C}^N,0}+\left(\left(\frac{\partial \vec{f}}{\partial z_1}\right), \cdots, \left(\frac{\partial \vec{f}}{\partial z_N}\right)\right)$$

where $((\partial\vec{f}/\partial z_1), \cdots, (\partial\vec{f}/\partial z_N))=\sum_1^N \mathcal{O}_{\mathbb{C}^N,0}\cdot(\partial\vec{f}/\partial z_i)$ (as submodule of $\mathcal{O}^p_{\mathbb{C}^N,0}$). Compare this with 4.3, after remarking that if we view $\mathcal{O}_{\mathbb{C}^N,0}$ as $\mathcal{O}_{\mathbb{C}^p,0}$-module via f^*, we have $\mathfrak{m}_y\cdot \mathcal{O}^p_{\mathbb{C}^N,0}=I_0\cdot \mathcal{O}^p_{\mathbb{C}^N,0}$.

Exercise. The natural map of $\mathcal{O}_{\mathbb{C}^N,0}$-modules: $I_0\to \Omega^1_{\mathbb{C}^N,0}\underset{\mathcal{O}_{\mathbb{C}^N,0}}{\otimes}\mathcal{O}_{X_0,0}$ given by $h\longmapsto dh\otimes 1$ induces a map

$$d^*:\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^N,0}}(\Omega^1_{\mathbb{C}^N,0}\underset{\mathcal{O}_{\mathbb{C}^N,0}}{\otimes}\mathcal{O}_{X_0,0}, \mathcal{O}_{X_0,0})\to \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^N,0}}(I_0, \mathcal{O}_{X_0,0}).$$

Check that the trivial infinitesimal deformations of $(X_0, 0)$ correspond exactly to those elements $\vec{g}$ inducing elements of $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^N,0}}(I_0, \mathcal{O}_{X_0,0})$ which are in the image $\mathrm{Im}\, d^*$ of d^*.

Therefore, the Zariski tangent space to the base of the miniversal deformation of $(X_0, 0)$ can be naturally identified with the $\mathbb{C}$-vector space:

$$T^1_{X_0,0}=\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^N,0}}(I_0, \mathcal{O}_{X_0,0})/\mathrm{Im}\, d^*=\mathrm{Hom}_{\mathcal{O}_{X_0,0}}(I_0/I_0^2, \mathcal{O}_{X_0,0})/\mathrm{Im}\, d^*$$

[which is a finite-dimensional vector space if $(X_0, 0)$ has an isolated singularity].

I claim that $T^1_{X_0,0}$ is precisely the Zariski tangent space to the flattener of the map F' of the proposition. The reason is very simple: an infinitesimal unfolding of f is given by $(f_i+\varepsilon g_i)$ with no condition on the g_i. To say that it is $\mathcal{K}$-trivial is to say, as we saw in 4.3, that

$$\vec{g}\in I_0\cdot \mathcal{O}^p_{\mathbb{C}^N,0}+((\partial\vec{f})/(\partial z_1), \cdots, (\partial\vec{f})/(\partial z_N)) \quad\text{in}\quad \mathcal{O}^p_{\mathbb{C}^N,0},$$

and therefore the Zariski tangent space to the $\mathbb{C}^r$ parametrizing the $\mathcal{K}$-miniversal unfolding is naturally identified with

$$K^1_f=\mathcal{O}^p_{X_0,0}\Big/\left(\frac{\partial\vec{f}}{\partial z_1}, \cdots, \frac{\partial\vec{f}}{\partial z_N}\right)\cdot \mathcal{O}^p_{X_0,0}$$

again as we saw in 4.3. To prove the claim is to check that $T^1_{X_0,0}$ is precisely the subset (in fact vector subspace) of K^1_f corresponding to those infinitesimal unfoldings of f which give infinitesimal deformations of $(X_0, 0)$, i.e. to check that in the natural map

$$\Psi:\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^N,0}}(I_0, O_{X_0,0})\to \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^N,0}}(\mathcal{O}^p_{\mathbb{C}^N,0}, \mathcal{O}_{X_0,0})\cong O^p_{X_0,0}$$

coming from the map $\mathcal{O}^p_{\mathbb{C}^N,0}\to I_0$ sending the ith base element to f_i, we have that $\Psi^{-1}((\partial\vec{f})/(\partial z_1), \cdots, (\partial\vec{f})/(\partial z_N))\cdot \mathcal{O}^p_{X_0,0}=\mathrm{Im}\, d^*=$ those elements in $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^N,0}}(I_0, \mathcal{O}_{X_0,0})$ corresponding to trivial infinitesimal deformations of $(X_0, 0)$.

But this is exactly what was checked above, hence the claim.

This shows that the flattener $(S, 0)$ of F' is a versal deformation of $(X_0, 0)$ having as its Zariski tangent space $T^1_{X_0,0}$, and which therefore has the smallest possible Zariski tangent space: it must be a miniversal deformation of $(X_0, 0)$.

4.5.2. Remark 1. If $(X_0, 0)$ is a complete intersection, then F is flat, hence F' is also flat since flatness is preserved by base change, and in this case F' itself is the miniversal deformation of $(X_0, 0)$: we recover the fact that the miniversal deformation of a complete intersection has a non-singular base – and source –. In fact, from the viewpoint we take here, we see that 'all the obstruction comes from the flatness requirement', and this is quite different from the way algebraic deformation theory constructs the obstruction. (See M. Schlessinger's papers [2].)

4.5.3. Remark 2. It is a theorem of Grauert (Inventiones Math. *15*, 3 (1972)) that any isolated singularity has a miniversal deformation. Since such an isolated singularity cannot always be presented as the fibre of a T.S.F. map f, (see exercise 3 in 4.3) it motivates the construction of a theory of unfoldings with an infinite-dimensional base, and the extension of the construction of the flattener to mappings between such infinite-dimensional spaces, so that hopefully the base of the miniversal deformation of an isolated singularity would appear as the (finite-dimensional) flattener of a map of infinite-dimensional spaces. (See Astérisque No 16, Soc. Math. Fr. 1974). Our viewpoint is always, given a deformation problem, to embed it in a 'bigger' problem which is unobstructed, [instead of directly constructing a prorepresentable hull of our original problem] and then to seek the base of our original problem as a subspace of the base of that bigger problem. Here of course, the 'bigger problem' is the $\mathcal{K}$-miniversal unfolding of the mapping having our space as fibre. We will see other instances of this in §5.

4.6. Examples of miniversal deformations

4.6.1. Exercise. Check that if $(X_0, 0)$ is *non-singular*, then the canonical map $(X_0, 0)\to\{0\}$ is the miniversal deformation of $(X_0, 0)$: in other words any flat map having non-singular fibre is locally a product: this is the simplicity theorem of §2.

4.6.2. EXERCISE. (1) Let $(X_0, 0) \subset (\mathbb{C}, 0)$ be defined by the ideal (z^{n+1}). The miniversal deformation of $(X_0, 0)$ is the map $(X, 0) \to (\mathbb{C}^n, 0)$ induced by the projection $(\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ on the hypersurface $(X, 0) \subset (\mathbb{C} \times \mathbb{C}^n, 0)$ defined by:

$$z^{n+1} + t_1 z^{n-1} + \cdots + t_n = 0,$$

(where $t_1, \cdots, t_n$ are coordinates on $(\mathbb{C}^n, 0)$).

(2) Check that in characteristic zero, where one can remove by a change of variables the term in z^n of a polynomial of degree $n+1$ in z, the statement of the existence of a miniversal deformation of this $(X_0, 0)$ is equivalent to the statement of the classical Weierstrass preparation theorem.

4.6.3. (Taken from the appendix to [6]). Fix an integer s, and consider the curve in $\mathbb{C}^3$ (with coordinates z_1, z_2, z_3) defined by $(z_1^2 - z_0^3, z_2^2 - z_0^{s+2} z_1)$. (It is the curve parametrized by $z_0 = t^4, z_1 = t^6, z_2 = t^{2s+7}$). The miniversal deformation of this curve is the restriction of the natural projection $\mathbb{C}^3 \times \mathbb{C}^{2s+10} \to \mathbb{C}^{2s+10}$ to the subspace X of $\mathbb{C}^3 \times \mathbb{C}^{2s+10}$ defined by the ideal generated by (F_1, F_2) in $\mathbb{C}\{z_1, z_2, z_3, v_1, \cdots, v_{2s+10}\}$ where

$$\begin{aligned}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = {} & \begin{pmatrix} z_1^2 - z_0^3 \\ z_2^2 - z_0^{s+2} \cdot z_1 \end{pmatrix} + v_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + v_3 \begin{pmatrix} z_0 \\ 0 \end{pmatrix} + v_4 \begin{pmatrix} 0 \\ z_0 \end{pmatrix} + v_5 \begin{pmatrix} z_1 \\ 0 \end{pmatrix} \\ & + v_6 \begin{pmatrix} 0 \\ z_1 \end{pmatrix} + v_7 \begin{pmatrix} z_2 \\ 0 \end{pmatrix} \\ & + v_8 \begin{pmatrix} z_0^2 \\ 0 \end{pmatrix} + v_9 \begin{pmatrix} z_0 \cdot z_1 \\ 0 \end{pmatrix} + v_{10} \begin{pmatrix} z_0 \cdot z_2 \\ 0 \end{pmatrix} + v_{11} \begin{pmatrix} 0 \\ z_0^2 \end{pmatrix} \\ & + \sum_{j=3}^{s+1} v_{9+j} \begin{pmatrix} 0 \\ z_0^j \end{pmatrix} + \sum_{j'=1}^{s} v_{10+s+j'} \begin{pmatrix} 0 \\ z_0^{j'} \cdot z_1 \end{pmatrix}.\end{aligned}$$

EXERCISE. Check that in this example all the curves corresponding to points in $\mathbb{C}^{2s+10}$ with $v_k = 0$ for $k \neq 7$, 10 and $v_7 \neq 0$ are isomorphic to the plane curve with equation $(z_1^2 - z_0^3)^2 - z_0^{s+2} \cdot z_1 = 0$, and that all the germs of curves corresponding to points with $v_k = 0$ for $k \neq 10$, and $v_{10} \neq 0$ are isomorphic to one another, but *not* isomorphic to the special curve $(X_0, 0)$.

Check that, given $\bar{v}_7 \in \mathbb{C} - \{0\}$, the mapping $F: (\mathbb{C}^3 \times \mathbb{C}, 0) \to (\mathbb{C}^2 \times \mathbb{C}, 0)$ described by

$$\begin{cases} y_1 \circ F = z_1^2 - z_0^3 + \bar{v}_7 \cdot z_2 + v_{10} \cdot z_1 z_2 \\ y_2 \circ F = z_2^2 - z_0^{s+2} \cdot z_1 \\ v_{10} \circ F = v_{10} \end{cases}$$

and considered as an unfolding of the map:

$$f: \mathbb{C}^3 \to \mathbb{C}^2$$

described by

$$\begin{aligned} y_1 \circ f &= z_1^2 - z_0^3 + \bar{v}_7 z_2 \\ y_2 \circ f &= z_2^2 - z_0^{s+2} z_1 \end{aligned}$$

is $\mathscr{K}$-trivial but not $\mathscr{A}$-trivial.

4.7. In the special case of *functions* $f: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$, there is another useful notion of equivalence, known as $\mathscr{R}$-equivalence (for Right-equivalence). Two unfoldings $F, F': (\mathbb{C}^N \times S) \to (\mathbb{C} \times S, 0)$, with the same base $(S, 0)$ of a function $f: (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$ are Right-equivalent if there exists an S-isomorphism $\Psi: (\mathbb{C}^N \times S, 0) \to (\mathbb{C}^N \times S, 0)$, unfolding of the identity of $\mathbb{C}^N$, such that $F' \circ \Psi = F$.

By the same methods as used above, we can check that an infinitesimal unfolding $f(z_1, \cdots, z_N) + \varepsilon g(z_1, \cdots, z_N)$ is $\mathscr{R}$-trivial if and only if $g \in ((\partial f/\partial z_1), \cdots, (\partial f/\partial z_N)) \cdot \mathbb{C}\{z_1, \cdots, z_N\}$ so that the space 'transversal to the $\mathscr{R}$-orbit' is

$$R_f^1 = \mathbb{C}\{z_1, \cdots, z_N\}/((\partial f/\partial z_1), \cdots, (\partial f/\partial z_N))$$

which is finite-dimensional if and only if $(X_0, 0) = (f^{-1}(0), 0)$ has an isolated singularity, and then we have:

$$\dim_{\mathbb{C}} R_f^1 = \mu^{(N)}(X_0, 0)$$

where $\mu^{(N)}(X_0, 0)$ is the Milnor number of $(X_0, 0)$, which we have already met in §2 as a discriminant and also (2.18) in equisingularity conditions.

REMARK. The reason why one does not consider $\mathscr{R}$-equivalence when $p > 1$ is that the corresponding R_f^1 would then be infinite dimensional, except when it is zero, which is the case where f is a germ of submersion. To see this, please do the:

EXERCISE. Generalize the notion of $\mathscr{R}$-equivalence to unfoldings of an $f: (\mathbb{C}^N, 0) \to (\mathbb{C}^p, 0)$, and check that the corresponding R_f^1 is:

$$R_f^1 = \mathbb{C}\{z_1, \cdots, z_N\}^p / ((\partial \vec{f}/\partial z_1), \cdots, (\partial \vec{f}/\partial z_N)).$$

This shows that R_f^1 is the cokernel of a map $\Psi: \mathcal{O}^N_{\mathbb{C}^N,0} \to \mathcal{O}^p_{\mathbb{C}^N,0}$. Using the fact (easy consequence of what has been recalled in the addendum to §1) that the non-empty components of the subspace of $\mathbb{C}^N$ defined by the Fitting ideal of the cokernel of such a map Ψ are of codimension $\leq N - p + 1$, check that if $p > 1$, $\dim_{\mathbb{C}} R_f^1 < \infty \Leftrightarrow R_f^1 = (0) \Leftrightarrow f$ is a submersion.

4.7.1. There is a notion of $\mathscr{R}$-miniversal unfolding of a germ of function $f:(\mathbb{C}^N,0)\to(\mathbb{C},0)$: it is the function described by

$$y=f+\sum_1^{\mu} t_i\cdot s_i(z_1,\cdots,z_N)\qquad (s_i\in\mathbb{C}\{z_1,\cdots,z_N\})$$

$$\text{i.e.: } \mathbb{C}^N\times\mathbb{C}^{\mu}\to\mathbb{C}\times\mathbb{C}^{\mu}\qquad [(\mathbb{C}^{\mu},0)\text{ with coordinates } t_1,\cdots,t_{\mu})]$$

where the images of the s_i form a basis of R_f^1.

However, with this definition, one can always take one of the s_i to be 1, and the corresponding t_i changes the function only by a translation.

The custom therefore is to enlarge the Right-equivalence by allowing translations in the target space, and if we call $\tilde{\mathscr{R}}$-equivalence the corresponding notion, we have of course that

$$\tilde{R}_f^1=(z_1,\cdots,z_N)\cdot\mathbb{C}\{z_1,\cdots,z_N\}/((\partial f/\partial z_1),\cdots,(\partial f/\partial z_N))$$

so that $\dim \tilde{R}_f^1=\mu^{(N)}(X_0,0)-1$ and an $\tilde{\mathscr{R}}$-miniversal unfolding of f is a map $F;(\mathbb{C}^N\times\mathbb{C}^{\mu-1},0)\to(\mathbb{C}\times\mathbb{C}^{\mu-1},0)$ where $\mu=\mu^{(N)}(X_0,0)$ described by

$$t_0\circ F=f(z_1,\cdots,z_N)+\sum_1^{\mu-1} t_i\cdot s_i(z_1,\cdots,z_N)\qquad (s_i\in\mathbb{C}\{z_1,\cdots,z_N\})$$

$$t_j\circ F=t_j\quad\text{if}\quad 0<j\leqq\mu-1$$

where the images of s_i form a basis of $\tilde{R}_f^1$.

This should be compared with the miniversal *deformation* of the hypersurface $(f^{-1}(0),0)=(X_0,0)\subset(\mathbb{C}^N,0)$: from what we saw above, this miniversal deformation G appears in a diagram

$$\begin{tikzcd}[column sep=small]
(X,0)\subset(\mathbb{C}^N\times\mathbb{C}^t,0) \arrow[d, "G"', shift right=6] \arrow[d, "pr_2", shift left=6] \\
(\mathbb{C}^t,0)
\end{tikzcd}$$

where X is defined by

$$f(z_1,\cdots,z_N)+\sum_1^{t} v_i\cdot g_i(z_1,\cdots,z_N)=0\qquad (g_i\in\mathbb{C}\{z_1,\cdots,z_N\})$$

and the images of the g_i form a basis of

$$K_f^1=\mathbb{C}\{z_1,\cdots,z_N\}\Big/\left(f,\frac{\partial f}{\partial z_1},\cdots,\frac{\partial f}{\partial z_N}\right).$$

Again, one of the g_i must be invertible, so can be chosen to be -1. Also, tradition imposes that in this case we write $\tau(X_0,0)$ instead of t for $\dim_{\mathbb{C}} K_f^1$.

Finally, setting $m=\tau(X_0,0)-1$, the miniversal deformation of $(X_0,0)$ is isomorphic to the map:

$$G:(\mathbb{C}^N\times\mathbb{C}^m,0)\to(\mathbb{C}\times\mathbb{C}^m,0)$$

given by

$$\begin{cases} v_0\circ G=f(z_1,\cdots,z_N)+\sum_1^m v_i\cdot g_i(z_1,\cdots,z_N)\\ v_j\circ G=v_j\qquad 0<j\leqq m\end{cases}$$

where $(\mathbb{C}^m,0)$ has coordinates $v_1,\cdots,v_m$.

Exercise. Write similarly the miniversal deformation of a complete intersection $(X_0,0)\subset(\mathbb{C}^N,0)$ given by p equations as a map $(\mathbb{C}^N\times\mathbb{C}^m,0)\to(\mathbb{C}^p\times\mathbb{C}^m,0)$ where $m=\dim_{\mathbb{C}} K_f^1-p$ (and $f:(\mathbb{C}^N,0)\to(\mathbb{C}^p,0)$ has $(X_0,0)$ as fibre).

4.8. Basic results on the openness and economy of unfoldings and deformations

4.8.1. Let $F:(\mathbb{C}^N\times S,0)\to(\mathbb{C}^p\times S,0)$ be an unfolding of a map-germ $f:(\mathbb{C}^N,0)\to(\mathbb{C}^p,0)$ which is of finite singularity type. Let $\mathfrak{m}_y$ be the ideal defining $0\times S$ in $\mathbb{C}^p\times S$, i.e. the ideal in $\mathcal{O}_{\mathbb{C}^p\times S,0}$ generated by coordinates $(y_1,\cdots,y_p)$ on $(\mathbb{C}^p,0)$. We can consider

$$A_{F/S}^1=\mathcal{O}_{\mathbb{C}^N\times S,0}^p\Big/\mathcal{O}_{\mathbb{C}^p\times S,0}^p+\left(\frac{\partial\vec{F}}{\partial z_1},\cdots,\frac{\partial\vec{F}}{\partial z_N}\right)$$

where $\mathcal{O}_{\mathbb{C}^p\times S,0}^p$ designates the sub-$\mathcal{O}_{\mathbb{C}^p\times S,0}$-module of $\mathcal{O}_{\mathbb{C}^N\times S,0}$ generated by the basis elements and $\vec{F}=(y_1\circ F,\cdots,y_p\circ F)$ and similarly:

$$K_{F/S}^1=\mathcal{O}_{\mathbb{C}^N\times S,0}^p\Big/\mathfrak{m}_y\cdot\mathcal{O}_{\mathbb{C}^N\times S,0}+\left(\frac{\partial\vec{F}}{\partial z_1},\cdots,\frac{\partial F}{\partial z_N}\right)$$
$$=\mathcal{O}_{X,0}^p\Big/\left(\frac{\partial\vec{F}}{\partial z_1},\cdots,\frac{\partial\vec{F}}{\partial z_N}\right)$$

and in the case $p=1$, we consider:

$$\tilde{R}_{F/S}^1=(z_1,\cdots,z_N)\cdot\mathcal{O}_{\mathbb{C}^N\times S,0}\Big/\left(\frac{\partial F}{\partial z_1},\cdots,\frac{\partial F}{\partial z_N}\right).$$

Consider now the map of $\mathcal{O}_{S,0}$-modules

$$\Omega_{S,0}^{1\vee}\xrightarrow{\theta_B}B_{F/S}^1\quad\text{where}\quad B=A, K\text{ or }\tilde{R}$$

defined as follows: take $D\in\Omega_{S,0}^{1\vee}=\mathrm{Hom}_{\mathcal{O}_{S,0}}(\Omega_{S,0}^1,\mathcal{O}_{S,0})$ and extend it to a derivation $\tilde{D}$ of $\mathcal{O}_{\mathbb{C}^N\times S,0}$ by setting $\tilde{D}z_i=0$. Then define $\theta_B(D)$ to be the residue class in $B_{F/S}^1$ of $\tilde{D}\vec{F}=(\tilde{D}(y_1\circ F),\cdots,\tilde{D}(y_p\circ F))$, and the kernel of $\theta_B(0)$ corresponds to elements of $\Omega_{S,0}^{1\vee}(0)=E_{S,0}$ which give $\mathscr{B}$-trivial infinitesimal unfoldings and F is a $\mathscr{B}$-miniversal (resp. versal) unfolding if and only if $\theta_B(0)$ is an isomorphism (resp. is onto), which is just another formulation of what we saw above.

Now as soon as $B^1_{F/S}$ has a support which is finite over S (by $H = pr_2 \circ F$) θ_B can be sheafified in a map of coherent $\mathcal{O}_S$-modules:

$$\Omega_S^{1\vee} \xrightarrow{\theta_B} H_* B^1_{F/S}$$

(this finiteness will occur if f is T.S.F.) and if F is miniversal at 0, $\theta_B(0): E_{S,0} \to B^1_f$ is an isomorphism, and by Nakayama's lemma this implies that $\theta_B(s)$ is *onto* for all $s \in S$ sufficiently near 0. This is the source of results of *openness of versality*, of which I will now quote only what I will use:

4.8.2. Theorem (Product decomposition theorem, see [5] chap. III §1). *Let $(X_0, 0)$ be a germ of complete intersection with isolated singularity. Any sufficiently small representative $G:(X, 0) \to (\mathbb{C}^\tau, 0)$ of a miniversal deformation of $(X_0, 0)$ has the following property: for any $s \in \mathbb{C}^\tau$, if the fibre $X_s = G^{-1}(s)$ has $l(=l(s))$ singular points $x_i(s)(1 \leqq i \leqq l)$ there is a (non-canonical) decomposition of $\mathbb{C}$ in the neighborhood of s: $\mathbb{C}^\tau \simeq S_1 \times \cdots \times S_l \times \mathbb{C}^r$ where $r = \tau(X_0, 0) - \sum_{i=1}^{l} \tau(X_s, x_i)$ such that in a neighborhood of $x_i(s)$, G is isomorphic as a deformation to a map:*

$$id_{S_1} \times \cdots \times id_{S_{i-1}} \times G_i \times id_{S_{i+1}} \times \cdots \times id_{S_l} \times \mathbb{C}^r:$$

$$S_1 \times \cdots \times S_{i-1} \times X_i \times S_{i+1} \times \cdots \times S_l \times \mathbb{C}^r \to S_1 \times \cdots \times S_{i-1} \times S_i \times S_{i+1} \times \cdots \times S_l \times \mathbb{C}^r$$

where $G_i: X_i \to S_i$ is the miniversal deformation of the isolated singularity of complete intersection $(X_s, x_i(s))$ [I have omitted marked points for simplicity of notation and we have $S_i \cong \mathbb{C}^{\tau(X_s, x_i(s))}$.]

Remark. The theorem implies in particular that G remains a *versal* deformation of X_s at every point of X_s near 0, hence the terminology 'openness of versality' but it is stronger than this and has in particular the:

4.8.3. Corollary. *Let $\Sigma \subset X$ be a closed complex subspace of X (for a small enough representative) defined by conditions concentrated at each singular point of a fibre of G. Then, setting $\Delta = G(\Sigma)$ we have in a neighborhood of every points $s \in \mathbb{C}^t$ (and in particular of course if $s \in \Delta$) a decomposition according to the singular points $(x_i(s), 1 \leqq i \leqq l(s))$ of $G^{-1}(s)$:*

$$\Delta = \bigcup_{i=1}^{l} \tilde{\Delta}_i$$

where $\quad \tilde{\Delta}_i = S_1 \times \cdots \times S_{i-1} \times \Delta_i \times S_{i+1} \times \cdots \times S_l \times \mathbb{C}^r$

and $\Delta_i \subset S_i$ is $G_i(\Sigma_i)$, where $\Sigma_i \subset X_i$ is the subspace of X_i defined by the conditions which define $\Sigma \subset X$.

In particular this shows that near every $s \in \Delta$, Δ is a union of subspaces *in general position* of $\mathbb{C}^\tau$, which are in 1–1 correspondence with the singular points of $G^{-1}(s)$.

If we recall the way we built the miniversal deformation $G:(X, 0) \to (S, 0)$ (where $S = \mathbb{C}^{\tau(X_0, 0)}$) of an isolated singularity of complete intersection, we see that we can define a coherent $\mathcal{O}_S$-module $G_* C^1_{F/S}$ with the property that for any $s \in S$

$$\dim_{\mathbb{C}} G_* C^1_{F/S}(s) = \sum_{i=1}^{l} \tau(X_s, x_i(s))$$

where $x_i(s)(1 \leqq i \leqq l)$ are the singular points of X_s. From this, one can deduce the existence of a (locally) finite partition of S, $S = \cup S_t$ into locally closed complex subspaces, such that

$$s \in S_t \Leftrightarrow \sum_{i=1}^{l} \tau(X_s, x_i(s)) = t$$

$S_{\tau(X_0, 0)}$ is the subspace containing 0, and is a closed complex subspace of S. We remark that if $s \in S_{\tau(X_0, 0)}$ and if X_s has only one singular point, $x(s)$, then the restriction of G to a neighborhood of $x(s)$ is a miniversal deformation of $(X_s, x(s))$.

4.8.4. Theorem (of economy of miniversal deformations, see [T_1] exp. 1, §1). *Let $(X_0, 0)$ be a germ of a complete intersection with isolated singularity. Any sufficiently small representative $G:(X, 0) \to (\mathbb{C}^\tau, 0)$ of a miniversal deformation of $(X_0, 0)$ has the following property:*

the set of points $x \in X$ such that the fibre of G through x, (X_s, x) where $s = G(x)$, is analytically isomorphic to $(X_0, 0)$, is reduced to $\{0\}$.

The meaning of this theorem can be seen as follows:

Corollary 1 (proved by Seidenberg when $(X_0, 0)$ is a plane curve). *For any deformation $H:(Z, 0) \to (Y, 0)$ of $(X_0, 0)$ where $(Y, 0)$ is reduced, the following conditions are equivalent:*

(1) there exists a representative of H such that each fibre Z has a point $z(y)$ such that $(Z_y, z(y))$ is analytically isomorphic to $(X_0, 0)$.

(2) There exists a germ of a section $\sigma:(Y, 0) \to (Z, 0)$ of H and H is isomorphic (over $(Y, 0)$) to the trivial deformation $(X_0 \times Y, 0) \xrightarrow{pr_2} (Y, 0)$ in such a way that $\sigma(Y)$ is sent to $0 \times Y$.

In words: if all the fibres are isomorphic, the deformation is (locally) trivial.

Corollary 2 (See my appendix to [6]):

A. *Same situation as above, but we only assume that there exists a nowhere dense closed subspace $(F, 0) \subset (Y, 0)$ such that for any $y \in Y - F$, the fibre Z_y*

has a singular point $z(y)$ such that: given any other $y' \in Y-F$, $(Z_y, z(y))$ is isomorphic to $(Z_{y'}, z(y'))$ (but not necessarily to $(X_0, 0)$ since $F \ni 0$.
Then we have: If $(Z_y, z(y))$ is not isomorphic to $(X_0, 0)$ for $y \in Y-F$

$$\tau(X_0, 0) > \tau(Z_y, z(y)) \qquad (y \in Y-F).$$

B. *Let $S^1 \subset S_{\tau(X_0,0)}$ be a complex subspace of the 'τ constant stratum of 0' constructed above, such that for any $s \in S^1$, X_s has only one singular point $x(s)$. Then the 'analytic type of the fibres varies continuously' on S^1 meaning that any $s \in S^1$ has a neighborhood V_s such that $(X_{s'}, x(s'))$ is not isomorphic to $(X_s, x(s))$, for all $s' \in V_s$.*

Remark 1. We have nowhere been explicit about the uniqueness of the base change through which a given unfolding or deformation comes from the miniversal unfolding (or deformation): it is only the *Zariski tangent map* to this base change which is uniquely determined.

Remark 2. In the proof of all the theorems above, the integration of holomorphic vector fields is often used, a procedure which is very far from being algebraic. Renée Elkik has proved the 'algebraicity' of the construction of miniversal deformations of isolated singularities (Ann. Sc. E. N. S. 4ème série, t. 6 (1973) 553–604) and Bruce Bennett has given a beautiful new proof, without 'integration', of the product decomposition theorem (Normalization theorems for certain modular discriminantal loci, Compositio Math. *32* (1976) 13–32).

Exercise. Let $f(z_1, \cdots, z_N) = 0$, $f \in \mathbb{C}\{z_1, \cdots, z_N\}$ define a germ of complex hypersurface $(X_0, 0) \subset (\mathbb{C}^N, 0)$ with isolated singularity. Write a miniversal *deformation* of $(X_0, 0)$ as

$$G : (\mathbb{C}^N \times \mathbb{C}^m, 0) \to (\mathbb{C} \times \mathbb{C}^m, 0) \quad \text{where} \quad m = \tau(X_0, 0) - 1$$

and an $\mathcal{R}$-miniversal unfolding of f as

$$F : (\mathbb{C}^N \times \mathbb{C}^{\mu-1}, 0) \to (\mathbb{C} \times \mathbb{C}^{\mu-1}, 0) \quad \text{where} \quad \mu = \mu^{(N)}(X_0, 0).$$

Show that there is a germ of submersion

$$\varphi : (\mathbb{C} \times \mathbb{C}^{\mu-1}, 0) \to (\mathbb{C} \times \mathbb{C}^m, 0)$$

such that F is obtained up to *isomorphism of deformations* from G by pull back, and that the restriction of G to $\varphi^{-1}(0)$ is trivial as deformation, but not as unfolding.

Exercise. Define the 'μ constant stratum' as a closed complex subspace of the target space of the $\mathcal{R}$-miniversal unfolding, in analogy with the 'τ constant stratum' defined above, and show that the dimension of the 'τ constant stratum' in the base of the miniversal deformation is *not* upper semi-continuous.

Hint. Use the plane curve $(z_1^2 - z_0^3)^2 + z_0^5 z_1 = 0$. Use what you have seen about it in an exercise above to prove that its τ-constant stratum is $\{0\}$. However, for any $v_0 \neq 0$, the curve $(z_1^2 - z_0^3)^2 + z_0^5 z_1 + v_0 z_0^4) = 0$ has a one-dimensional τ-constant stratum in the base of its miniversal deformation. Check that this phenomenon does not occur for the μ-constant stratum in the base of the $\mathcal{R}$-miniversal *unfolding* of $f = (z_1^2 - z_0^3)^2 + z_0^5 z_1$. For more details on the μ-constant stratum in the base of the $\mathcal{R}$-miniversal unfolding, see Arnold's article in the Proceedings, International Congress of Mathematicians, Vancouver 1974.

Exercise. Let $F : (\mathbb{C}^A, 0) \to (\mathbb{C}^B, 0)$ be a *flat* map. Show that it is infinitesimally stable if and only if it is a versal deformation of $(F^{-1}(0), 0)$.

Hint. Use Nakayama's lemma and the Weierstrass preparation theorem.

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§5. Discriminants

In this section, we define and study the discriminant of versal deformations $G : (X, 0) \to (S, 0)$ (with S non-singular) of complete intersections with isolated singularities, i.e. equivalently, of stable and flat maps between non-singular spaces. In the case of hypersurfaces, we go into more detail, and finally reach the goal of these notes, which is to show connections between naive invariants of the discriminant, on one hand, and invariants of the geometry *up to (c)-cosécance* (§ 2) of the hypersurface, on the other hand.

On the way, we meet some naive invariants of the discriminant which are *not* invariants of (c)-cosécance of the fibre, and we emphasize the structure of the discriminant as an envelope.

5.1. To study versal deformations with a non-singular base of a complete intersection, it is sufficient to study the miniversal ones, since a versal deformation is the product of a miniversal one by the identity of some space.

Let $G:(X,0)\to(S,0)$ be a miniversal deformation of a germ of complete intersection with isolated singularity $(X_0,0)=(G^{-1}(0),0)$. As we saw in §4, G is an infinitesimally stable and flat map between non-singular spaces (and is stable, by a theorem of Mather), and G can be described as the restriction over $0\times\mathbb{C}^\tau$ of a $\mathcal{K}$-miniversal unfolding $F:(\mathbb{C}^N\times\mathbb{C}^\tau,0)\to(\mathbb{C}^p\times\mathbb{C}^\tau,0)$ of a flat map $f:(\mathbb{C}^N,0)\to(\mathbb{C}^p,0)$ having $(X_0,0)$ as fiber.

Tradition imposes that we write $T^1_{X/S}$ for the $\mathcal{O}_X$-module $K^1_{F/S}$ of §4 (here $(S,0)=(\mathbb{C}^\tau,0)$) and $\tau=\tau(X_0,0)$ for $\dim_{\mathbb{C}}K^1_f$, which was denoted by t in §4. We also saw in §4 that G itself can be described as an unfolding of f, namely $G:(\mathbb{C}^N\times\mathbb{C}^m,0)\to(\mathbb{C}^p\times\mathbb{C}^m,0)$ commuting to projections to $\mathbb{C}^m$, where $m=\tau(X_0,0)$-p.

We wish to study the *critical subspace* C of G in the sense of §2: it is the subspace of X defined by $F_{N-p}(\Omega^1_{X/S})$.

EXERCISE. Show that $F_{N-p}(\Omega^1_{X/S})=F_0(T^1_{X/S})$ and that $T^1_{X/S}$ has a presentation:

$$\mathcal{O}_X^N\xrightarrow{\Psi}\mathcal{O}_X^p\longrightarrow T^1_{X/S}\longrightarrow 0.$$

Since G is flat and $(X_0,0)$ has an isolated singularity, we see by using the simplicity theorem of §2 (at the non-singular points of $(X_0,0)$) and the Weierstrass preparation theorem for a sufficiently small representative of G, the critical subspace C will be finite over $S=\mathbb{C}^\tau$, i.e. $G\mid C:C\to\mathbb{C}^\tau$ is a finite map. By §1, $G_*(C)=\operatorname{im}(G\mid C)$ is therefore a subspace of $\mathbb{C}^\tau$, and by the theorem of Bertini, (see §2, second part) since X is non-singular, $G(C)$ is a *strict* closed subspace of $\mathbb{C}^\tau$, so that $\operatorname{Ann}_{\mathcal{O}_{\mathbb{C}^\tau}}(G_*\mathcal{O}_C)\neq 0$.

5.1.1. CLAIM. $\operatorname{depth}_{\mathcal{O}_{\mathbb{C}^\tau,0}}(G_*\mathcal{O}_C)\geqq\tau-1$.

The proof is as follows: by an easy computation of local algebra, (see A. Grothendieck: E.G.A. IV, 0.16.4.8) $\operatorname{depth}_{\mathcal{O}_{\mathbb{C}^\tau,0}}(G_*\mathcal{O}_C)=\operatorname{depth}_{\mathcal{O}_{X,0}}(\mathcal{O}_{C,0})$ since $\mathcal{O}_C$ is a finite $\mathcal{O}_{\mathbb{C}^\tau,0}$-module (see above). Now we use a result of Buchsbaum-Rim ([5] cor. 2.7) (already used in §1) to the effect that since $T^1_{X/S}$ has a presentation as above, the maximum length of a sequence of elements of $F_0(T^1_{X/S})$ which is a regular sequence for $\mathcal{O}_{X,0}$ (called the $F_0(T^1_{X/S})$-depth of $\mathcal{O}_{X,0}$ is $\leqq N-p+1$, and also that if it is equal to $N-p+1$, then $\operatorname{dh}_{\mathcal{O}_{X,0}}(\mathcal{O}_{C,0})$ i.e., $\operatorname{depth}_{\mathcal{O}_{X,0}}(\mathcal{O}_{X,0}/F_0(T^1_{X/S}))$, is also equal to $N-p+1$. Therefore, if we can prove that the $F_0(T^1_{X/S})$-depth of $\mathcal{O}_{X,0}$ is at least $N-p+1$, we will obtain $\operatorname{dh}_{\mathcal{O}_{X,0}}(\mathcal{O}_{C,0})=N-p+1$ and then by the equality quoted in 3.5: $\operatorname{depth}(\mathcal{O}_{C,0})=m+p-1=\tau-1$, and we win. Now, a basic property of a Cohen-Macaulay local ring A is that for any proper ideal I in A, we have I-depth $A=\dim A-\dim A/I$. Since $\mathcal{O}_{X_0,0}$ is regular, it is Cohen-Macaulay, and hence, setting $d=F_0(T^1_{X/S})$-depth of $\mathcal{O}_{X,0}$, we have:

$$\dim\mathcal{O}_{X,0}/F_0(T^1_{X/S})=\dim\mathcal{O}_{X,0}-d$$

and hence:

$$d=\dim\mathcal{O}_{X,0}-\dim C$$

but we have already seen that $\dim C=\dim G_*(C)\leqq m+p-1$ by Bertini's theorem, hence $d\geqq N+m-(m+p-1)=N-p+1$ which proves the claim.

REMARK. If you wish to understand the geometry behind this kind of computation, as well as that in the addendum to §1, and more, I suggest reading the beautiful papers of G. Kempf ([15] and [16]). The above presentation partly follows suggestions of Patrick Barril.

Our goal is to deduce from this:

5.2. THEOREM (of purity of discriminants. See [7]). *Let $G:(X,0)\to(S,0)$ be a miniversal deformation for a germ of a complete intersection with isolated singularity. Then the discriminant $D=G_*(C)$ of G, as defined in* §2, (i.e. by *$F_0(G_*\mathcal{O}_C)$ is a reduced and irreducible hypersurface germ in the non-singular space $(S,0)=(\mathbb{C}^\tau,0)$. Furthermore, C is normal and $G\mid C:C\to D$ is the normalization map.*

PROOF. Since D is defined as the image in the sense of §1 of C by $G\mid C$, the fact that it is a hypersurface follows immediately from the claim above and 3.5. To prove the rest of the assertions, we stop a while to give a typical application of the stability of G (which we know thanks to Mather's theorem) and the product decomposition theorem of §4:

The Thom-Boardman strata of G. (See [4], [23]).

Given a stable G such as above, for every sufficiently small representation there are non-singular complex subspaces $\Sigma^I(G)$ of X, indexed by sequences of integers $I=(i_1,\cdots,i_k)$ and defined inductively as follows: $\Sigma^{\varnothing}=X$, and

$$\Sigma^{I,j}(G)=\{x\in\Sigma^I/\dim\operatorname{Ker}(T_x(G\mid\Sigma^I))=j\}$$

in words: points where the dimension of the kernel of the tangent map of $G\mid\Sigma^I$ is exactly j.

In fact these can be defined for any map G as *subspaces* using Fitting ideals, and they can also be defined as coming from a stratification of the jet-space $J^\infty(X,S)$, and in this way one sees that if G is stable, $\Sigma^I(G)$ is non-singular.

5.2.1. EXERCISE. Show that in the case of hypersurfaces $(p=1)$ one has $C=\Sigma^N(G)$, and that the set of points of C which are of multiplicity $a=m_0(X_0,0)$ (the multiplicity in the usual sense of $(X_0,0)$) in their fibre (of G) is $\Sigma^{N,\cdots,N}(G)$ when N occurs $a-1$ times.

Boardman also gave a formula for the codimension of his strata $\Sigma^I(G)$ as follows: if $\Sigma^I(G)$ is not empty, with $I=(i_1,\cdots,i_k)$, then

(α) $i_1 \geqq i_2 \geqq \cdots \geqq i_k$

(β) $N-p \leqq i_1 \leqq N+m \quad (m=\dim K_f^1 - p)$

(γ) if $i_1 = N-p$, $i_2 = \cdots = i_k = N-p$,

and then the codimension of $\Sigma^I(G)$ in X is

$$\nu_I = (p-N+i_1)\mu_{i_1,\ldots,i_k} - (i_1-i_2)\mu_{i_2,\ldots,i_k} - \cdots - (i_{k-1}-i_k)\mu_{i_k}$$

where $\mu_{i_1,\ldots,i_k}$ is the number of non-increasing sequences $(j_1,\cdots,j_k)$ with $j_1 \neq 0$ and $j_l \leqq i_l$.

We will see these Thom-Boardman strata again later, but for the moment we use them to prove the theorem of purity: first, remark that it is an easy consequence of 4.8.3 that the mapping $G \mid \Sigma^I : \Sigma^I \to G(\Sigma^I)$ is generically one-one, just because $G(\Sigma^I)$ has to be locally irreducible outside of a nowhere dense closed subspace of itself. If $i_1 > N-p$, then $\Sigma^I \subset C$ and hence $G \mid \Sigma^I$ is finite. If we denote by S^I the closure of Σ^I in X, which is a closed complex subspace of X, we have:

5.2.2. PROPOSITION. *If $i_1 > N-p$, $G \mid S^I : S^I \to G_*(S^I)$ is a* proper modification *of $G_*(S^I) = \overline{G_*(\Sigma^I)}$, and in particular $G_*(S^I)$ is reduced. (See §2 for 'proper modification'): this follows from the above remark and the definition of the Σ^I by rank conditions.*

End of the proof of the purity theorem. $\mathcal{O}_C$ is of depth (as $\mathcal{O}_{\mathbb{C}^\tau,0}$-module) equal to its dimension $\tau-1$ (5.1.1) hence all its irreducible components at 0 are of the same dimension $\tau-1$, which is also the dimension of $\Sigma^{N-p+1}(G)$, and of no other Boardman stratum, by the codimension formulas. Hence, $C = S^{N-p+1}(G)$, and hence D is reduced. Moreover, the singular locus of C is contained in the union of $\Sigma^{I'}$ where $I' > (N-p+1, 0\cdots 0)$ in the lexicographic order, and this is of codimension at least two in C. Since we already know that C has depth $\tau-1$ at each of its points, it follows from the criterion of Serre ([30]) that C is *normal* hence it is locally analytically irreducible, and hence its image D is also locally analytically irreducible at 0, and finally since the induced map $G \mid C : C \to D$ is a proper modification of D (5.2.2), it has to be the normalization.

REMARKS. (1) As we saw in 5.2.1, in the case $p=1$, C is even non-singular, so that $G \mid C$ is a resolution of singularities of the hypersurface D. However, there is a deeper fact about this map $C \to D$: it is in fact also the *development* of D in the sense of §2, as we shall see below.

(2) I want to illustrate the proof above in the case $p=1$ as follows: how do we prove in this case that the discriminant D is reduced? as follows: the Boardman stratum $\Sigma^N(G)$ is dense in C, and $x \in \Sigma^N(G)$ if and only if the singularity at x of the fibre of G through x is isomorphic to the ordinary quadratic singularity A_1 with equation $z_1^2 + \cdots + z_N^2 = 0$. Now D is of course locally irreducible outside a nowhere dense closed subspace B, so by 4.8.3 there is an homeomorphism $C - n^{-1}(B) \to D - B$. At every point of $(C - n^{-1}(B)) \cap \Sigma^N(G)$, which is dense in C, we can apply the openness of versality of §4 (4.8.2) and see that our whole situation is locally isomorphic to a cylinder over the situation for the singularity A_1: it remains to check that for this singularity, the map from critical subspace to discriminant subspace is an isomorphism, which is obvious since the versal deformation is given by $t_0 \circ F = z_1^2 + \cdots + z_N^2$: critical subspace and discriminant are both reduced points (here $\mu^{(N)} = 1$).

5.3. Examples of discriminants

The purpose of these examples is to convince the reader that it is much more convenient to give oneself the discriminants *parametrically*, i.e. by $C \xrightarrow{n} D$ $(n = G \mid C)$ (more precisely, as image of $G \mid C : C \to \mathbb{C}^p \times \mathbb{C}^m$) than to actually compute the equation of D in $\mathbb{C}^p \times \mathbb{C}^m$. Here is what happens in the simplest cases: for the versal deformations of the singularities defined in $\mathbb{C}$ by $(z^N) \cdot C\{z\}$: the versal deformation is the restriction of $(\mathbb{C} \times \mathbb{C}^{N-1}, 0) \xrightarrow{pr_2} (\mathbb{C}^{N-1}, 0)$ to the hypersurface X defined in $\mathbb{C} \times \mathbb{C}^{N-1}$ by

$$z^N + t_{N-2} z^{N-2} + \cdots + t_0 = 0$$

and the equation of the discriminant in $\mathbb{C}^{N-1}$ is of course just the z-resultant of this equation and its derivative with respect to z.

Here is what you get for the equation of the discriminant:

$N=1$: non singular, discriminant empty.

$N=2$: $t_0 = 0$.

$N=3$: $27t_0^2 + 4t_1^3 = 0$.

$N=4$: $256t_0^3 - 27t_1^4 - 128t_0^2t_2^2 + 144t_0t_1^2t_2 + 16t_0t_2^4 - 4t_1^2t_2^3 = 0$.

$N=5$: $3.125t_0^4 - 3.750t_0^3t_2t_3 + 2.250t_0^2t_1t_2^2 + 825t_0^2t_2^2t_3^2 + 108t_0^2t_3^5$
$-900t_0^2t_1t_3^3 + 2.000t_0^2t_1^2t_3 - 630t_0t_1t_2^3t_3 + 16t_0t_2^3t_3^3 + 108t_0t_2^5$
$-72t_0t_1t_2t_3^4 + 560t_0t_1^2t_2t_3^2 - 1.600t_0t_1^3t_2 + 256t_1^5 - 128t_1^4t_3^2$
$+16t_1^3t_3^4 + 144t_1^3t_2^2t_3 - 27t_1^2t_2^4 - 4t_1^2t_2^2t_3^3 = 0$.

$N=6$: the equation has 76 monomials (available on request).

(I thank Gérard Lejeune for programming this for me.)

The real part of the discriminant for $N=4$ is well known (not as discriminant, but as bifurcation set, see below) under the name *swallowtail:*

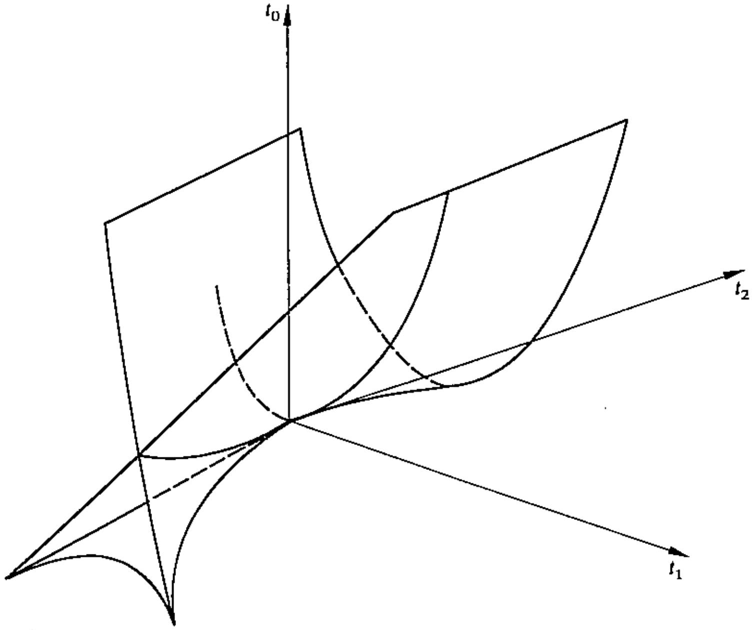

Figure 3

N.B. You can find pictures of sections of the (real part of the) discriminant also for $N=5$ (again considered as bifurcation set) in [38].

Indeed, in the case of the discriminant of the general unitary polynomial in one variable, there is a very pleasant way of building the discriminant geometrically, due to Dominique Thillaud (unpublished):

Consider $P(z)=z^N+t_{N-1}z^{N-1}+\cdots+t_0=0$.

Notice that here we have *not* removed the term in z^{N-1}, so this is in fact a versal deformation of $(z^N)\cdot C\{z\}$. Now the set of points where $P(z)=0$ has an N-uple root is a curve M_N in $\mathbb{C}^N\,(t_0,\cdots,t_{N-1})$ given parametrically by

$$t_i=(-1)^{N-i}\binom{N}{i}u^{N-i}$$

where u is the value of the root.

Proposition (Thillaud). *The set M_p of points in $\mathbb{C}^N$ where P has at least a $p$uple root is the* $(N-p)$th developable variety D_{N-p} *of the curve M_N, where*

the $(N-p)$th developable variety of M_N is the closure in $\mathbb{C}^N$ of the set $\bigcup_{\ell\in M_N^*}T_{N-p}(M_N,\ell)$, where $T_{N-p}(M_N,\ell)$ is the osculating space of dimension $N-p$ to the curve M_N at ℓ, meaning the affine subspace of $\mathbb{C}^N$ made of elements of the form $\vec{M}_N(v)+\sum_{i=1}^{N-p}\lambda_i\vec{M}^{(i)}(u)$, $\underline{\lambda}\in\mathbb{C}^{N-p}$. M_N^* designates the points ℓ of M_N where the dimension of $T_{N-p}(M_N,\ell)$ is actually $N-p$. Remark that $\dim M_p=N-p+1$.

Corollary. *The discriminant of $P(z)$ is the* $(N-2)$th *developable variety of the curve M_N.*

Proof. Given as an exercise.

Question. Take a non-singular curve M in projective N-space, and consider its $(N-2)$th developable, D_{N-2} which is a hypersurface on $\mathbb{P}^N$ containing M. Is it true that there exists a Zariski dense set of points $U\subset M$ such that if $p\in U$, D_{N-2} is locally at p analytically isomorphic to the discriminant of $z^N+t_{N-1}z^{N-1}+\cdots+t_0$?

(For $N=3$ it is well known that a curve is a cuspidal edge for its developable surface.) It would then follow that the section of D_{N-2} by a generic hypersurface through p is locally at p analytically isomorphic to the N-swallowtail, i.e. the discriminant of $z^N+t_{N-2}z^{N-2}+\cdots+t_0$.

5.4. We now come back to the general discriminants, with the following easy consequence of the product decomposition theorem of §4:

Remark. Given a discriminant $(D,0)\subset(\mathbb{C}^p\times\mathbb{C}^m,0)$ as above, for each integer k, $1\leqq k\leqq m+p$, there exists for any sufficiently small representative of $D\subset\mathbb{C}^p\times\mathbb{C}^m$ a closed complex subspace $B_k\subset D$, of codimension at least k in D, such that at every point $p\in D-B_k$, D is locally analytically isomorphic to a finite union $\bigcup_i\tilde{D}_i$ of cylinders $\tilde{D}_i$ in general position over discriminants $D_i\subset\mathbb{C}^{\tau_i}$ of miniversal deformations of isolated singularities of complete intersections with $\Sigma_i\tau_i\leqq k$.

Corollary. *For each k, there exists a Zariski open dense subset U in the Grassmannian $G(m+p,k)$ of directions of k-planes in $\mathbb{C}^{m+p}$, such that given $H_0\in U$, for a sufficiently small representative of $D\subset\mathbb{C}^p\times\mathbb{C}^m$, there exists an open analytic dense subset V of the space of affine k-planes in $\mathbb{C}^p\times\mathbb{C}^m$ having direction H_0 such that: If $H\in V$, $D\cap H$ is locally near each of its points isomorphic to a finite union $\bigcup_i\tilde{D}_i$ of cylinders, in general position in H, over discriminants $D_i\subset\mathbb{C}^{\tau_i}$ with $\Sigma\tau_i\leqq k$.*

In particular:

5.4.1. Exercise (from [7]). Show that for any sufficiently small representative of a discriminant $D\subset\mathbb{C}^p\times\mathbb{C}^m$ as above, for 'almost every' affine

2-plane H (in the sense made precise above), $H\cap D$ is a plane curve having as singularities only nodes and cusps.

HINT. Use the local irreducibility of discriminants at their origin, the general position coming from the product decomposition theorem, and the fact that the only singularities of a complete intersection $(X_0, 0)$ such that $\tau(X_0, 0)=2$ is, up to isomorphism, the singularities called A_2, or 'suspension' of $z^3=0$, namely $\sum_{i=1}^{N-1} z_i^2+z_N^3=0$, where $N=\dim X_0+1$.

By an easy flatness argument, one sees that in fact the number d of nodes and the number k of cusps of $D\cap H$ are the same for 'almost all H' in the sense made precise above. These numbers k and d depend only upon the geometry of D, i.e. of G. They are mentioned here because they are related to the geometry of certain plane sections of the discriminant D *through the origin*, and these plane sections, in the case $p=1$, will be one of our main points of interest.

5.4.2. We therefore have the following picture in mind: a non-singular point p in the discriminant corresponds to a singularity in $G^{-1}(p)$ at which $G^{-1}(p)$ is locally isomorphic to an ordinary quadratic singularity: $z_1^2+\cdots+z_{N-p+1}^2=0$ (of the right dimension). And it is the only singularity of $G^{-1}(p)$.

A point p at which the discriminant is cusp-like (i.e. cylinder over a cusp) corresponds to a point in the fiber $G^{-1}(p)$ at which it is isomorphic to $z_1^2+\cdots+z_{N-p}^2+z_{N-p+1}^3=0$ (i.e. again cusp-like) a point at which the discriminant is node-like corresponds to a fibre $G^{-1}(p)$ which has *two* ordinary quadratic singularities, and there are no points more complicated than that outside a subspace of D of codimension 2.

5.5. We will from now on consider any versal deformations of germs of *hypersurfaces* with isolated singularities, i.e. we restrict to the case $p=1$.

5.5.1. THEOREM. *The map induced by $G:(X, 0)\to(S, 0)$ from the critical subspace C to the discriminant $D=G_*(C)$ is the development of D, in the sense of §2. Since C is non-singular, we see that the singularities of D are resolved by one development (and in this case the development turns out to coincide with the normalization).*

PROOF. We need the following

LEMMA. *Any $f\in\mathbb{C}\{z_1,\cdots, z_N\}$ can be written after a change of variables:*

$$f(z_1,\cdots, z_N)=z_1^2+\cdots+z_p^2+\tilde{f}(z_{p+1},\cdots, z_N)$$

where $\tilde{f}\in(z_{p+1},\cdots, z_N)^3$ is uniquely determined, and has an isolated critical point if such was the case for f.

The proof is immediate by using the classical form of the Weierstrass preparation theorem and the removal of the term in z in a polynomial of degree 2 in z.

Next we make the

REMARK. The bases of miniversal deformations for f and $\tilde{f}$ are isomorphic, and the discriminants are also isomorphic.

PROOF. A miniversal deformation for $f=0$ is described by

$$G:(\mathbb{C}^N\times\mathbb{C}^m, 0)\to(\mathbb{C}\times\mathbb{C}^m, 0)\quad(\text{coordinates } v, t_1,\cdots, t_m)\text{ on }\mathbb{C}\times\mathbb{C}^m)$$

$$v\circ G=f(z_1,\cdots, z_N)+\sum_{i=1}^m t_i g_i(z_1,\cdots, z_N)$$

$$t_i\circ G=t_i\quad(1\leqq i\leqq m)$$

where the images of 1 and the g_i in $C\{z_1,\cdots, z_N\}/(f, (\partial f/\partial z_1,\cdots, (\partial f/\partial z_N))$ form a basis over $\mathbb{C}$. Clearly

$$\mathbb{C}\{z_1,\cdots, z_N\}\Big/\left(f, \frac{\partial f}{\partial z_1},\cdots, \frac{\partial f}{\partial z_N}\right)=\mathbb{C}\{z_{p+1},\cdots, z_N\}\Big/\left(\tilde{f}, \frac{\partial\tilde{f}}{\partial z_{p+1}},\cdots, \frac{\partial\tilde{f}}{\partial z_N}\right)$$

and therefore we can assume that the g_i depend only upon $z_{p+1},\cdots, z_N$. Hence

$$\tilde{G}:(\mathbb{C}^{N-p}\times\mathbb{C}^m, 0)\to(\mathbb{C}\times\mathbb{C}^m, 0)$$

$$v\circ\tilde{G}=\tilde{f}(z_{p+1},\cdots, t_N)+\sum_1^m t_i g_i(z_{p+1},\cdots, z_N)$$

$$t_i\circ\tilde{G}=t_i$$

is a miniversal deformation of $\tilde{f}(z_{p+1},\cdots, z_N)=0$, and has the same discriminant as G.

APPLICATION. For the study of f from the discriminant of its miniversal deformation, we can always assume that $f\in(z_1,\cdots, z_N)^3$, and we will do so. It implies that we can choose $g_1,\cdots, g_N$ equal to $(z_1,\cdots, z_N)$, and from now on we will write our miniversal deformations by

$$v\circ G=F(z, t)\stackrel{\text{def}}{=}f(z_1,\cdots, z_N)$$
$$+t_1z_1+\cdots+t_Nz_N+t_{N+1}g_{N+1}(z)+\cdots+t_mg_m(z),$$

where $g_k\in(z_1,\cdots, z_N)^2$ if $k\geqq N+1$.

Let now $\delta(v, t_1,\cdots, t_m)=0$ be an equation for our discriminant $(D, 0)\subset(\mathbb{C}\times\mathbb{C}^m, 0)$.

PROPOSITION. *We have for all* $1 \leqq j \leqq m$

$$\frac{\partial \delta}{\partial t_j} \circ G \Big|_C = -\frac{\partial \delta}{\partial v} \circ G \Big|_C \cdot \frac{\partial F}{\partial t_j}\Big|_C.$$

PROOF. C is defined in $\mathbb{C}^N \times \mathbb{C}^m$ by the ideal $((\partial F/\partial z_1), \cdots, (\partial F/\partial z_N))$ [which incidentally shows again that C is non-singular $(\partial F/\partial z_i)$ begins with t_i] since $D = G_* C$, we have that $\delta \circ G$ vanishes on C (i.e. $\delta \in \mathrm{Ann} \cdot G_* \mathcal{O}_C$: in fact $(\delta)\mathcal{O}_{\mathbb{C} \times \mathbb{C}^m} = \boldsymbol{Ann}\ G_* \mathcal{O}_C$ since D is reduced, see §1).

Since C is reduced (being non-singular!) it implies that

$$\delta \circ G = \sum_{i=1}^{N} A_i(z, t) \frac{\partial F}{\partial z_i}(z, t)$$

$$\frac{\partial(\delta \circ G)}{\partial t_j} = \sum_{i=1}^{N} A_i \cdot \frac{\partial^2 F}{\partial z_i \partial t_j} \bmod \left(\frac{\partial F}{\partial z_1}, \cdots, \frac{\partial F}{\partial z_N}\right)$$

and since δ comes from $\mathbb{C} \times \mathbb{C}^m$:

$$0 = \frac{\partial(\delta \circ G)}{\partial z_i} = \sum_{i=1}^{N} \sum_{j=1}^{N} A_i(z, t) \frac{\partial^2 F}{\partial z_i \partial z_j}(z, t) \bmod \left(\frac{\partial F}{\partial z_1}, \cdots, \frac{\partial F}{\partial z_N}\right)$$

but as we saw, on an open-analytic dense subset of C, the hessian determinant $\det((\partial^2 F/\partial z_i \partial z_j))$ is different from 0. Hence $A_i(z, t)$ vanishes on C, hence $A_i \in ((\partial F/\partial z_1), \cdots, (\partial F/\partial z_N))$ and therefore on C we have:

$$\begin{aligned} 0 = \frac{\partial(\delta \circ G)}{\partial t_j}\Big|_C &= \frac{\partial \delta}{\partial v} \circ G \Big|_C \cdot \frac{\partial(v \circ G)}{\partial t_j}\Big|_C + \frac{\partial \delta}{\partial t_j} \circ G \Big|_C \\ &= \frac{\partial \delta}{\partial v} \circ G \Big|_C \cdot \frac{\partial F}{\partial t_j}\Big|_C + \frac{\partial \delta}{\partial t_j} \circ G \Big|_C \end{aligned}$$

which proves the proposition.

In particular, we can write

$$\frac{\partial \delta}{\partial t_i} \cdot \mathcal{O}_{C,0} = -\frac{\partial \delta}{\partial v} \cdot \frac{\partial F}{\partial t_i} \cdot \mathcal{O}_{C,0}$$

which shows that the restriction to D of the Jacobian ideal of δ becomes invertible on C:

$$j(\delta) \cdot \mathcal{O}_{C,0} = \frac{\partial \delta}{\partial v} \cdot \mathcal{O}_{C,0}.$$

Since for a hypersurface, the development is isomorphic to the blowing up of the restriction to the hypersurface of the Jacobian ideal, as is easily checked (see §2), we see that C must dominate the development D_1 of D, i.e. we have a factorization for $n = G \mid C$:

$$\begin{tikzcd} C \arrow[d, "n"'] \arrow[dr, dashed, "p"] & \\ D & D_1 \arrow[l, "d"] \end{tikzcd}$$

In particular, $d^{-1}(0)$ has only one point, and to prove that p is a local isomorphism at 0, it is sufficient to prove that $\mathfrak{m}_1 \cdot \mathcal{O}_{C,0} = \mathfrak{m}_{C,0}$ where $\mathfrak{m}_1$ is the maximal ideal of D_1 at $d^{-1}(0)$, and $\mathfrak{m}_{C,0}$ is the maximal ideal of C at 0. Because of the equations described above for C in $\mathbb{C}^N \times \mathbb{C}^m$, we see that $\mathfrak{m}_{C,0}$ is generated by the restrictions to C of $z_1, \cdots, z_N, t_{N+1}, \cdots, t_m$. Since $t_{N+1}, \cdots, t_m$ give elements which are already in $\mathfrak{m}_{D,0}$ hence a fortiori in $\mathfrak{m}_{D_1,0}$, it is sufficient to check that $z_i \in \mathfrak{m}_{D_1,0}$. But by definition of a blowing up, the ratios $(\partial\delta/\partial t_i)/(\partial\delta/\partial v)$ which are meromorphic on D and tend to 0 since they do so when lifted to C by the proposition above, become already holomorphic on D_1, which means that $(\partial F/\partial t_j)|_C$, a priori meromorphic on D_1 is in fact in $\mathfrak{m}_{D_1,0}$ $(1 \leqq j \leqq m)$. But for $1 \leqq j \leqq N$, $(\partial F/\partial t_j) = z_j$, which shows that $z_j \in \mathfrak{m}_{D_1,0}$ hence $D_1 = C$ and this concludes the proof of the theorem.

REMARK 1. The fact that $(C, 0) \to (D, 0)$ is the development seems to me to lie deeper than the fact it is the normalization. Are there similar results for the other $G_*(S^I)$?

REMARK 2. This result enables one to identify C with the 'projectivized conormal bundle to D' in the 'projectivized cotangent bundle to $\mathbb{C} \times \mathbb{C}^m$' and Pham has shown to me that the real-analytic version of this result (which I had proved only for aesthetic reasons) was of great use in the theory of caustics. It is not my purpose to go into this here, and I refer to [29], [1].

REMARK 3. It follows from the theorem and the fact that there is only one point of C lying over 0, that there is only one limit position at 0 of tangent hypersurfaces to D, and since the $(\partial F/\partial t_j)$ tend to 0 with (z, t), this limit position is the 'horizontal hyperplane' $dv = 0$. This can be expressed by saying that the discriminant 'flattens' on the hyperplane $0 \times \mathbb{C}^m$ (a flattening very different from that used in §4!). We shall see more about this below.

EXERCISE. Show that the tangent cone to D at 0 is set-theoretically given by $v = 0$ [strictly speaking one would say $V = 0$ where V is the initial form of v in the associated graded ring $gr_{\mathfrak{m}}\mathcal{O}_{\mathbb{C} \times \mathbb{C}^m, 0}$].

HINT. Use the fact that the tangent cone to D must be contained in the image of the restriction to $T_{C,0}$ of the tangent map to G, which can be written explicitly.

5.5.2. Proposition. *The multiplicity of D at 0 is equal to the Milnor number $\mu^{(N)}(X_0, 0)$, where $(X_0, 0)$ is the hypersurface with isolated singularity, fibre of G.*

Proof. The multiplicity of a hypersurface at 0 can be computed as its intersection multiplicity at 0 (in its ambiant space) with any non-singular curve C such that $T_{C,0} \not\subset C_{D,0}$ ($C_{D,0}$ = tangent cone). It follows from the exercise above that the line $t_1 = \cdots = t_m = 0$ (i.e. the v-axis) satisfies this. Therefore we have to compute the multiplicity of the restriction of our discriminant to this v-axis. But *since our definition of the discriminant is compatible with base change*, this is exactly what was computed in 2.6.

Remark. Consider now the restriction $\pi: D \to \mathbb{C}^m$ to D of the natural projection $\mathbb{C} \times \mathbb{C}^m \to \mathbb{C}^m$: it has a discriminant which we will denote by $(B, 0) \subset (\mathbb{C}^m, 0)$. B_{red} is called the *bifurcation locus* of G: it is exactly the set of values $t \in \mathbb{C}^m$ such that the corresponding function $v = f(z) + \sum_1^m t_i g_i(z)$ is not an excellent Morse function (near the origin) i.e. a function having only ordinary quadratic singularities giving distinct critical values.

5.5.3. We remark that these notions can be defined just as well in the real-analytic case, and then the discriminant and bifurcation locus will be *semi-analytic hypersurfaces* in their respective ambient spaces $\mathbb{R} \times \mathbb{R}^m$ and $\mathbb{R}^m$, and a fundamental object of study in Thom's theory of elementary catastrophes is the bifurcation locus of an $\vec{R}$-miniversal unfolding of a function $f: (\mathbb{R}^N, 0) \to (\mathbb{R}, 0)$ having an (algebraically) isolated critical point. The fact that these sets, defined as images, are semi-analytic, follows from Galbiati's theorem (see Hironaka's lectures) in this case, because they are images of maps having a finite (hence proper) complexification. The same remark applies to all the images occurring in a stratification of a stable map-germ in the real-analytic case (see below). In fact, using finite determinacy, Mather even made them semi-algebraic. Anyway, going back to the complex case, if we take $t \in \mathbb{C}^m - B$ (for a small representative of G), the line $\mathbb{C} \times \{t\}$ will meet D in $\mu^{(N)}(X_0, 0)$ non-singular points of D, and transversally. It means the function $v = f + \sum t_i g_i$ will then have $\mu^{(N)}(X_0, 0)$ non-degenerate critical points.

5.5.4. One should not confuse $B \subset \mathbb{C}^m$ with the discriminant of the composed map $C \xrightarrow{n} D \xrightarrow{\pi} \mathbb{C}^m$ which is easily seen to be flat (it is just that $t_1, \cdots, t_m$ lift to regular sequence on C) and to be a ramified covering of degree $\mu^{(N)}(X_0, 0)$ in view of 5.5.2. The discriminant Δ of this composed map $p: C \to \mathbb{C}^m$, set-theoretically, is the image by π of the locus of cusp-like points of D, but does not take into account the node-like points of D.

We will use very much from now on the following:

5.5.5. Proposition (See [7]). *Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be a germ of hypersurface with isolated singularity, and let $l: (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$ be a linear function (i.e. a coordinate function if we want). Assume that $l^{-1}(0) \cap X_0$ still has an isolated singularity. Then the discriminant of $l \mid X_0: (X_0, 0) \to (\mathbb{C}, 0)$ is the origin counted with a multiplicity equal to $\mu^{(N)}(X_0, 0) + \mu^{(N-1)}(X_0 \cap H, 0)$, where H is the hyperplane $l^{-1}(0)$. Furthermore, if $f(z_1, \cdots, z_N) = 0$ is an equation for $(X_0, 0) \subset (\mathbb{C}^N, 0)$, the ideal generated by the coefficients of the 2-form $df \wedge dl$ defines a curve S_H in $(\mathbb{C}^N, 0)$ (the* polar curve *of f with respect to l) and we have*

$$\underline{(X_0, S_H)_0 = \mu^{(N)}(X_0, 0) + \mu^{(N-1)}(X_0 \cap H, 0)}$$

where $(\ , \ )_0$ is the intersection number at 0 in $\mathbb{C}^N$.

Finally, when l ranges through the $\mathbb{P}^{N-1}$ of linear maps, $\mu^{(N-1)}(X_0 \cap H, 0)$ takes its minimal value if and only if $l^{-1}(0) = H$ is not *a limit direction of tangent hyperplanes to X_0 at non-singular points near 0 (in the sense of the first part of §2). This minimal value is the $\mu^{(N-1)}(X_0, 0)$ defined at the end of* 2.17.

Remark 1. In view of the last exercise in 2.6, we have, taking $l = z_1$, and defining H by $z_1 = 0$:

$$\mu^{(N)}(X_0, 0) + \mu^{(N-1)}(X_0 \cap H, 0) = \dim_{\mathbb{C}} \mathbb{C}\{z_1, \cdots, z_N\} \Big/ \left(f, \frac{\partial f}{\partial z_2}, \cdots, \frac{\partial f}{\partial z_N}\right).$$

We will meet this curve S_H defined by $((\partial f/\partial z_2), \cdots, (\partial f/\partial z_N))$ again. Remark that because $(X_0, 0)$ has an isolated singularity, the $(\partial f/\partial z_i)$ form a regular sequence, so that S_H is a complete intersection.

Remark 2. The numerical part of the above proposition has been shown to be a special case of a nice general formula, proved topologically by Lê ([17]) and more algebraically by Greuel ([8]): Assume we know the Milnor number of an isolated singularity of complete intersection is. Then if $(X_0, 0)$ is such an isolated singularity, and $h: (X_0, 0) \to (\mathbb{C}, 0)$ any function such that $(h^{-1}(0), 0)$ again has an isolated singularity, than taking a coordinate v on $\mathbb{C}$, the discriminant of h is $(v^{\Delta})\mathbb{C}\{v\}$ where $\Delta = \mu(X_0, 0) + \mu(h^{-1}(0), 0)$.

The 'vanishing cycles' aspect of this proposition and the interpretation of $\mu^{(N)}(X_0, 0) + \mu^{(N-1)}(X_0 \cap H, 0)$ as an intersection number have also been generalized to arbitrary singularities of hypersurfaces by Lê ([18]).

Anyway, please do the

Exercise 1. Let $(X_0, 0) \subset (\mathbb{C}^2, 0)$ be a germ of reduced plane curve. Let $\pi: (X_0, 0) \to (\mathbb{C}, 0)$ be a projection parallel to a line H in $(\mathbb{C}^2, 0)$. Show that the discriminant of π is $(v^\Delta)\mathbb{C}\{v\}$ (v:coordinate on the target of π) where

$$\Delta=\mu^{(2)}(X_0,0)+(X_0,H)_0-1.$$

In particular, if H is not tangent to $(X_0,0)$, then $\Delta=\mu^{(2)}(X_0,0)+m_0(X_0,0)-1$ where $m_0(\ )$ is the multiplicity (= order of the equation).

In particular, show that if $(X_0,0)$ is a cusp (resp. a node) then if H is not tangent, $\Delta=3$ (resp. 2).

Exercise 2 (from [7]). Let $D\subset\mathbb{C}\times\mathbb{C}^m$ be the discriminant of a miniversal deformation of an isolated singularity of a hypersurface, and let $(B,0)\subset(\mathbb{C}^m,0)$ be the corresponding bifurcation subspace discriminant of the projection $\pi:(D,0)\to(\mathbb{C}^m,0)$. Show that

5.5.6. $$\underline{m_0(B,0)=2d+3k=\tilde{\mu}+\mu-1.}$$

Where $\mu=\mu^{(N)}(X_0,0)$ and where $\tilde{\mu}$ is the Milnor number of the plane curve $(D\cap(\mathbb{C}\times H),0)$, $(H,0)\subset(\mathbb{C}^m,0)$ being a 'generic' line (in fact, a line not in the tangent cone to B at 0).

(We remark that B is not reduced, so that we have to use again that we have a good definition of the discriminant.)

Hint. Use the fact that if you move H away from the origin, it will meet B 'transversally' in a number of points which (counted with multiplicities) is $m_0(B,0)$.

5.5.7. Definition. I shall call a section of the discriminant D by an i-plane of the form $\mathbb{C}\times H$, where $H\subset\mathbb{C}^m$ is an $(i-1)$-plane, a *vertical* section. We will be very interested in general vertical plane sections of D.

5.5.8. *The invariant δ of curves and the geometry of the discriminant: first movement.*

In fact, there is another relation between d, k and the general vertical plane sections of D, basically that found by Lê and Iversen [20], but which here we shall deduce from the results of §3: take a 2-plane $(H,0)$ in $\mathbb{C}^m$, and write it $(H_0\times H_1,0)$. If H is sufficiently general, for every $t\in H_1\setminus\{0\}$, the line $H_t=H_0\times\{t\}$ in $\mathbb{C}^m$ will be such that $D\cap(\mathbb{C}\times H_t)$ has as singularities only k cusps and d nodes. On the other hand, we can view $D\cap(\mathbb{C}\times H)$ as a flat family of plane curves parametrized by H_1. Furthermore, $n^{-1}(D\cap(\mathbb{C}\times H))$ is a surface in C which is Cohen-Macaulay since the two coordinates on H form a regular sequence in $\mathcal{O}_C$ (see the beginning of this paragraph) and is non-singular in codimension 1 (by Bertini's theorem) hence is normal, and is the normalization of $D\cap(C\times H)$. Let therefore Γ denote the curve $(D\cap(\mathbb{C}\times H_0),0)$ and Γ' the curve $(n^{-1}(\Gamma),0)\subset(C,0)$. It follows from 3.3 and the fact that the invariant δ of a cusp or a node is equal to 1 that:

5.5.9. $$\delta(\Gamma',0)=\delta(\Gamma)-(d+k),\quad\text{i.e.}\quad \underline{d+k=\delta(\Gamma)-\delta(\Gamma').}$$

Remark 3. If we assume known the fact, recently proved by Marc Giusti [9], that the formula $2\delta=\mu+r-1$ of 3.4 remains valid for reduced curves which are complete intersections, we deduce from 5.5.9 the equality:

5.5.10. $$k=\mu'+\mu-1$$

where μ' is the Milnor number of the complete intersection $\Gamma'\subset(C,0)$.

Exercise. Give another proof of the equality 5.5.10 using the formula of Lê and Greuel quoted in remark 2 above.

Hint. Remark that k is precisely the multiplicity at 0 of the discriminant Δ of the μ-fold branched cover $(C,0)\xrightarrow{p}(\mathbb{C}^m,0)$, and use the fact that $(p^{-1}(0),0)$ being a 0-dimensional complete intersection, its Milnor number is equal to its multiplicity minus 1. Remark that $\Gamma'=p^{-1}(H_0)$.

Remark. The computation of k and d for the discriminants of versal deformations of complete intersections with isolated singularity has been done by Lê and Greuel ([19]). The new feature is that the discriminant non longer 'flattens' so that 5.5.6 (for a generic projection of D to a hyperplane) becomes $m_0(B,0)=2d+3k+\tau$ where τ is a number of 'vertical tangents' which has to be evaluated.

Remark. One can seek estimates for k and d in terms of $\mu=\mu^{(N)}(X_0,0)$ only. It follows from 5.5.6 that ([7] chap. III).

(α) $\underline{2d+3k\geqq\mu^2-1}$

and it follows from 5.5.9 and a well-known formula for the behavior of δ by blowing up (see [10]) that

(β) $\underline{d+k\geqq\dfrac{\mu(\mu-1)}{2}}$

of course (α) follows from (β) and 5.5.10.

5.6. The invariant δ and the geometry of the discriminant: second movement

5.6.1. Let $(D,0)\subset(\mathbb{C}^{m+1},0)$ be a germ of hypersurface. For any representative of D, define $\mathrm{Cr}_D\ (k)$ to be the set of points $s\in D$ at which D is locally analytically isomorphic to the union of k non-singular hypersurfaces in general position in $\mathbb{C}^{m+1}$. (Cr is for Cross). $\mathrm{Cr}_D\ (k)$ is a locally closed complex analytic subspace of D, of codimension k in $\mathbb{C}^{m+1}$, if it is not empty.

5.6.2. Definition. Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be a germ of a hypersurface with isolated singularity. Define $\delta(X_0, 0)$ to be the maximum number of singular points which one can pile up in the same fibre of an arbitrarily small deformation of $(X_0, 0)$.

Exercise 1 (from [36]). Show that, if D is the discriminant of a versal deformation of $(X_0, 0)$, one has:

$$\delta(X_0, 0) = \operatorname{Max}\{k/0 \in \overline{\mathrm{Cr}_D(k)}\}.$$

Hint. Use 4.8.2.

Exercise 2. Check that when $(X_0, 0)$ is a plane curve, this definition agrees with that given in §3.

5.6.3. Remark. $\delta(X_0, 0)$, maximum number of critical points of a function nearby the function $f:(\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$ having $(X_0, 0)$ as fibre which one can *pile up* in the same level variety of this function, should be compared with $\mu^{(N)}(X_0, 0)$, which is the maximum number of critical points of such a function which one can *spread out* (in the sense of critical values). In a way, δ and μ correspond to the same preoccupation, but δ is to the geometer, interested in spaces, i.e. fibres, what μ is to the function-theorist. Anyway, Mr. I. N. Iomdin has communicated to me the following results:

5.6.4. Proposition (I. N. Iomdin [12]). For an isolated singularity of hypersurface $(X_0, 0)$, one has the following inequalities:

$$\frac{\mu^{(N)}(X_0, 0)}{\mu^{(N-1)}(X_0, 0)} \leqq 2 \cdot \delta(X_0, 0) \leqq \mu^{(N)}(X_0, 0) + \mu^{(N-1)}(X_0, 0).$$

Proof. The upper bound for δ is obtained as follows: it is a theorem in ([7], chap. II) that the multiplicity in the sense of algebraic geometry of the Jacobian ideal *on the hypersurface* i.e. of $j(f) \cdot \mathcal{O}_{X_0,0}$, is equal to $\mu^{(N)}(X_0, 0) + \mu^{(N-1)}(X_0, 0)$. Suppose we have a deformation of $(X_0, 0)$ where the general fibre has $\delta(X_0, 0)$ singular points. By exercise 1 above, they have to be all ordinary quadratic singularities, and by the upper semi-continuity of multiplicities we must have:

$$\mu^{(N)}(X_0, 0) + \mu^{(N-1)}(X_0, 0) \geqq a \cdot \delta(X_0, 0)$$

where a is the value of $\mu^{(N)} + \mu^{(N-1)}$ for an ordinary quadratic singularity, which is obviously equal to 2.

5.6.5. To prove the other inequality, we must go back to the theory of polar curves according to ([14] §3, [35]). Anyway, this theory will be of essential use below: the polar curve of $f(z_1, \cdots, z_N) = 0$ with respect to a hyperplane H is the curve S_H defined in 5.5.5. It is not difficult to convince oneself [and it is proved in [35] with the help of generic simultaneous normalization as in §3] that there exists a Zariski open dense subset $V \subset \mathbb{P}^{N-1}$ such that if the hyperplane $H \in V$, if we choose coordinates so that H is given by $z_1 = 0$, the polar curve S_H, which is now $(\partial f/\partial z_2) = \cdots = (\partial f/\partial z_N) = 0$, is reduced, has a number of irreducible components 1 in its decomposition $S_H = \bigcup_{q=1}^{1} \Gamma_q$ which is independent of $H \in V$ and furthermore, setting $m = m_0(\Gamma_q)$, we can define integers $e_q \geqq 0$ by $e_q + m_q = (X_0, \Gamma_q)_0$ (intersection multiplicity in $(\mathbb{C}^N, 0)$) and the sequence of integers (e_q, m_q) is independent of $H \in V$. Furthermore, e_q is equal to the intersection multiplicity of Γ_q with the hypersurface defined by $(\partial f/\partial z_1) = 0$, and we have $(H \in V)$

$$5.6.6. \quad \begin{cases} \sum_{q=1}^{1} e_q = \mu^{(N)}(X_0, 0) \\ \sum_{q=1}^{1} m_q = \mu^{(N-1)}(X_0, 0) = m_0(S_H) = (S_H, H)_0. \end{cases}$$

This means in particular that no component of S_H has its reduced tangent cone (a line) contained in H, and therefore we can think of the branches Γ_q as given parametrically by:

$$\Gamma_q \begin{cases} z_1 = t_q^{m_q} \\ z_i = t_q^{k_{q,i}} + \cdots \end{cases} \qquad k_{q,i} \geqq m_q \quad (2 \leqq i \leqq N).$$

While $f_{|\Gamma_q}$ has an expansion

$$f_{|\Gamma_q} = \gamma_q \cdot t_q^{e_q + m_q} + \cdots \qquad (\gamma_q \in \mathbb{C}^*)$$

and

$$\left.\frac{\partial f}{\partial z_1}\right|_{\Gamma_q} = \zeta_q \cdot t^{e_q} + \cdots \qquad (\zeta_q \in \mathbb{C}^*)$$

(by the equivalence of the several definitions of intersection multiplicities).

(Remark that the (e_q, m_q) thus defined are independent of the choice of coordinates and equation as follows from [35] th. 2.)

Now Iomdin argues as follows: take $a = [\operatorname{Sup}_q [(e_q/m_q)]$ and a point p on a component Γ_q on which this sup is attained. Then we can find a polynomial

$P_p(z_1)$ depending on p such that $f_{|\Gamma_q} - P_{p|\Gamma_q}$ vanishes with multiplicity $a+1$ at p, and furthermore, as $a \leqq \text{Sup}\ (e_q/m_q)$, P_p tends to zero as $p \to 0$ on Γ_q.

The polar curve of $f(z_1, \cdots, z_N) - P_p(z_1) = 0$, which is Γ_q near p, is non-singular and intersects it with intersection multiplicity $a+1$. Hence $f(z) - P_p(z_1) = 0$ has at p a singularity of type A_a i.e. locally isomorphic to $z_1^{a+1} + z_2^2 + \cdots + z_N^2 = 0$. (Recall that $\mu^{(N-1)} = 1$ implies that the general hyperplane section has only a quadratic singularity.) Therefore we have proved:

Lemma (Iomdin). *Arbitrarily close to a singularity $(X_0, 0)$ with invariants (e_q, m_q) as constructed above, there are singularities of type A_a where $a = [\text{Sup}_q\ (e_q/m_q)]$.*

Now there only remains to prove that for A_a one can find a nearby fibre with $[(a+1/2)]$ quadratic singularities. But according to exercise 1 above, this depends only upon the geometry of the discriminant and hence by the lemma in 5.5.1 we are reduced to $z_1^{a+1} = 0$, where it is obvious. All that remains now is to observe that

$$\text{Sup}_q \frac{e_q}{m_q} \geqq \frac{\mu^{(N)}(X_0, 0)}{\mu^{(N-1)}(X_0, 0)} \quad \text{by} \quad (5.6.6).$$

5.6.7. Remark. The method used in §3 when $N=2$ to realize δ will not work in higher dimensions: take a resolution of singularities $\tilde{X}_0 \to X_0$ and perturb the composed map $\tilde{X}_0 \to X_0 \to \mathbb{C}^N$: the image will have generalized pinch-points!

5.6.8. Question. How many points of $\text{Cr}_D\ (k)$ are there in a sufficiently general section of D by an affine k-plane near the origin? (for $k=2$, it is the computation of d).

5.7. Remark. The map $G \mid C : (C, 0) \to (\mathbb{C} \times \mathbb{C}^m, 0)$ is *not* sufficiently general for its image $(D, 0) \subset (\mathbb{C} \times \mathbb{C}^m, 0)$ to be *weakly normal* in the sense of [2], which means that every complex map-germ which is *continuous* on D and becomes holomorphic on C (i.e. when composed with n) is already holomorphic on D. (Weak normality would imply no cusps in codimension 1) but it is sufficiently general for D to be *Lipschitz saturated* as was remarked in [7 chap. III, 5.18] which means that every *locally Lipschitz* function on D which becomes holomorphic on C is already holomorphic on D.

The question above is linked to what happens when one makes a generic perturbation of the map $G \mid C: C \to \mathbb{C} \times \mathbb{C}^m$.

5.8. Stratifications (see Hironaka's and Mather's lectures in these proceedings).

5.8.1. Let $G: (\mathbb{C}^N \times \mathbb{C}^m, 0) \to (\mathbb{C} \times \mathbb{C}^m, 0)$ be a versal deformation of a germ of hypersurface with isolated singularity $(X_0, 0) \subset (\mathbb{C}^N, 0)$. We want to decompose the source and target of any small representative of G in a *finite* number of locally closed non-singular subspaces,

$$\mathbb{C}^N \times \mathbb{C}^m = \bigcup_\alpha Z_\alpha, \qquad \mathbb{C} \times \mathbb{C}^m = \bigcup_\beta S_\beta$$

such that

(i) for each α there exists a β such that G induces a subsmersive surjection $Z_\alpha \to S_\beta$

(ii) some interesting feature of G remains constant along each 'stratum' Z_α.

We shall be interested in the following three features of G:

(1) the local topological type at $x \in \mathbb{C}^N \times \mathbb{C}^m$ of the fibre $(G^{-1}(G(x)), x)$ of G through x (again a hypersurface with isolated singularity).

(2) The (c)-cosécance class (2.19) of this same fibre.

(3) The topological type of the germ at x of the map G:

$$G: (\mathbb{C}^N \times \mathbb{C}^m, x) \to (\mathbb{C} \times \mathbb{C}^m, s). \qquad (s = G(x))$$

5.8.2. It is a basic fact of life in these problems that if we require nothing about frontier conditions and such, the constancy of (1) along a Z_α is in general *strictly weaker* than the constancy of (2) if $N \geqq 3$, and the constancy of (2) is in general *strictly weaker* than the constancy of (3), even for $N=2$, (i.e. plane curves). (See below 5.12.1 and 5.12.4).

5.8.3. Anyway, a process for building Thom stratifications of G [i.e. of the source and target of a small enough representative of G] along each stratum of which (3) is constant, has been described in the lectures of Hironaka and Mather. Basically one finds Whitney stratifications of $\mathbb{C}^N \times \mathbb{C}^m$ and $\mathbb{C} \times \mathbb{C}^m$ such that (i) is satisfied and also Thom's condition A_G: then one has an isotopy lemma to prove the topological triviality of G along the strata (by the integration of controlled vector fields), a fact which is even stronger than the constancy of 3). The point is that the finiteness of C over $\mathbb{C} \times \mathbb{C}^m$ implies the 'no blowing up' condition of Thom.

Remark 1. One can prove exactly the same results but with the generalized condition (c) (2.18, Remark) instead of Whitney conditions. One then integrates rugose vector fields, which gives a stronger trivialization.

REMARK 2. Of course, $G^{-1}(\mathbb{C}\times\mathbb{C}^m - D)$ will be one stratum, and $G^{-1}(D)-C$ will be a union of strata. I wish to remark that for any stratum $Z_\alpha\subset C$, $G\mid Z_\alpha$ will in fact have to be a *local isomorphism* onto a stratum $S_\beta\subset D$. Furthermore, one can build at the same time a stratification of the *parameter space* $\mathbb{C}^m$, say $\mathbb{C}^m=\bigcup T_\gamma$ such that if $t\in T_\gamma$, then the topological type of the maps $F_t:\mathbb{C}^N\to\mathbb{C}$ are the same.

Furthermore, denoting by S_0 (resp. T_0) the stratum of the origin in the Thom stratification in $\mathbb{C}\times\mathbb{C}^m$ (resp. $\mathbb{C}^m$) the projection $\mathbb{C}\times\mathbb{C}^m\to\mathbb{C}^m$ induces a local isomorphism $S_0\xrightarrow{\sim} T_0$: This is simply because by the properties of our stratifications, as long as $t\in T_0$, F_t has only one critical value.

Finally, all this remains true in the real analytic case, with semi-analytic (even semi-algebraic) strata.

We will be mostly interested in the constancy of (2), but let us stop a while to sketch an application of the idea of the invariant δ above:

5.9. A real interlude. The catastrophic version of the Gibbs phase rule (after [32], [36]).

In the theory of elementary catastrophes of Thom, a family of physical systems depending on parameters is 'represented' by a family of functions $F_t:\mathbb{R}^N\to\mathbb{R}$, representing the family of potentials which govern the evolution of the systems. Stability conditions imply that we can think of this family of functions as an $\mathfrak{R}$-miniversal unfolding (see §4) of a given function $f=F_0:(\mathbb{R}^N,0)\to(\mathbb{R},0)$ with an algebraically isolated critical point. Our family of functions is then represented by

$$F:(\mathbb{R}^N\times\mathbb{R}^{\mu-1},0)\to(\mathbb{R}\times\mathbb{R}^{\mu-1},0).$$

Now take a Thom stratification of any small representative of this map (what follows is true for any Thom stratification, but Mather has proved the existence of a canonical one in his lectures) and let $T_0\subset\mathbb{R}^{\mu-1}$ be the image of the stratum of the origin $S_0\subset\mathbb{R}\times\mathbb{R}^{\mu-1}$ by the cannonical projection. Recall that $pr_2\mid S_0:S_0\to T_0$ is a local isomorphism, and that F induces a local isomorphism $Z_0\to S_0$ where Z_0 is the stratum of 0 in $\mathbb{R}^N\times\mathbb{R}^{\mu-1}$.

Since the morphology of our family of functions does not vary along T_0, the true number of parameters of our systems is the dimension of the supplementary subspace to T_0 i.e. $\operatorname{codim}_{\mathbb{R}^{\mu-1}}T_0=\operatorname{codim}_{\mathbb{R}\times\mathbb{R}^{\mu-1}}S_0-1$ and on the other hand, the *stable states* of the physical system represented by F_t correspond to the ordinary minima of the function F_t on $\mathbb{R}^N$ (i.e. of course in a ball around 0 in $\mathbb{R}^N$): therefore, the maximum number $\nu(F)$ of minima which a function F_t in (an arbitrary small representative of) our family can present, will represent the maximum number of *phases* which can coexist in a system of our family (if we think of our systems as chemical or thermodynamical), and the well-known Gibbs phase rule to the effect that the number of phases is at most the number of parameters plus one, becomes a purely geometric statement:

$$\nu(F)\leqq\operatorname{codim}_{\mathbb{R}^{\mu-1}}T_0+1$$

that is

$$\underline{\nu(F)\leqq\operatorname{codim}_{\mathbb{R}\times\mathbb{R}^{\mu-1}}S_0.}\qquad 5.9.1$$

One proves this in two steps: first, one defines the real analogue $\delta_{\mathbb{R}}(X_0,0)$ of the $\delta(X_0,0)$ of 5.6 and since one has a real analogue of the product decomposition theorem, one proves that $\delta_{\mathbb{R}}(X_0,0)=\max\{k/0\in\overline{\mathrm{Cr}_D(k)}\}$ where $\mathrm{Cr}_D(k)$ is the k-cross locus of the real discriminant D of F. Since a homeomorphism of ambient spaces respecting D cannot help to respect $\mathrm{Cr}_D(k)$ for all k, we see that $S_0\subset\overline{\mathrm{Cr}_D(\delta_{\mathbb{R}}(X_0,0))}$ and hence $\operatorname{codim}_{\mathbb{R}\times\mathbb{R}^{\mu-1}}(S_0)\geqq\delta_{\mathbb{R}}(X_0,0)$. On the other hand, following ideas of Smale, since all minima of a function have the same index as real critical points, there is no obstruction to putting them all at the same level, i.e. in the same fibre of F: this shows that $\delta_{\mathbb{R}}(X_0,0)\geqq\nu(F)$ and proves 5.9.1.

5.10. The partition of the discriminant D according to multiplicity (a.k.a. its Samuel stratification. See [33])

The fact that the Milnor number $\mu^{(N)}(X_0,0)$ of a germ of hypersurface with isolated singularity is an invariant of its topological type (as embedded germ in $(\mathbb{C}^N,0)$) is easily deduced from the material in Milnor's book. There is a partial converse, harder to prove:

THEOREM (Lê Dũng Tráng and C. P. Ramanujam [21]). *In an analytic family of germs of hypersurfaces with isolated singularity* (*in the sense of* 2.12), *the constancy of the Milnor number of the fibres implies the constancy of their topological type, if* $N\neq 3$.

REMARK. The restriction $N\neq 3$ comes from the fact that the proof uses the h-cobordism theorem. The author has recently given a purely algebraic proof of a stronger result in the case $N=2$ (i.e. for plane curves). See [35].

5.10.1. Anyway, from our point of view this theorem and 5.5.2 show that the most primitive invariant of D at 0, namely its multiplicity $\mathrm{m}_0(D)$, already contains some good information about $(X_0,0)$. This fact led me to introduce in [33] the so-called Samuel stratum D_μ of the origin in D as a candidate for the base of a deformation 'miniversal for deformations where the topological type of the fibres remains constant'.

We must first make precise what we mean by Samuel stratum, since we can define it either as a reduced subspace of D or not. Let us go back to the notations of 5.5.1 and write

$$v \circ G = F(z,t) = f(z) + t_1 z_1 + \cdots + t_N z_N + t_{N+1} g_{N+1}(z) + \cdots + t_m g_m(z)$$

assuming $f \in (z_1, \cdots, z_N)^3$.

I will also write an equation $\delta(v, t_1, \cdots, t_m)$ for D in $\mathbf{C}\times\mathbf{C}^m$ as follows:

$$\delta = v^\mu + a_{\mu-1}(t)v^{\mu-1} + \cdots + a_0(t) = 0 \qquad (t = (t_1, \cdots, t_m)).$$

5.10.2. Definition. The (non-reduced) Samuel stratum D_μ of the origin in D is the subspace of $\mathbf{C}\times\mathbf{C}^m$ defined by the ideal generated by the

$$\left(\frac{\partial^\alpha \delta}{\partial v^i \partial t^\gamma} \in \mathcal{O}_{\mathbf{C}\times\mathbf{C}^m,0}\right) \quad \text{where} \quad (i, \gamma) = \alpha, \quad |\alpha| < \mu = \mu^{(N)}(X_0, 0).$$

Of course $(D_\mu)_{red} = \{(v, t) \in D / m_{(v,t)}(D) = \mu\}$.

5.11. I now want to make a list, with references, of positive and negative facts known about $(D_\mu)_{red}$.

5.11.1. ([33], [7], using results on the monodromy proved by Lê, Lazzeri, Gabrielov.)

Above any point $(v,t) \in D_\mu$, there is only one point of C, which is given by an analytic section described by:

$$z_i = \frac{-1}{\mu}\frac{\partial a_{\mu-1}(t)}{\partial t_i} \qquad (1 \leqq i \leqq N)$$

(recall that $z_1, \cdots, z_N$ and $t_{N+1}, \cdots, t_m$ form a system of coordinates on C). This shows that $n^{-1}((D_\mu)_{red})$ is the image of a section $\sigma : (D_\mu)_{red} \to C$. This is the 'non splitting' principle, (see [14]).

5.11.2. In particular, after 4.8.2 and 5.2, D remains locally analytically irreducible at every point of $(D_\mu)_{red}$ and in particular, its tangent cone remains a μ-fold hyperplane, the hyperplane in question being given by the tangent space at the point (v,t) to the non-singular subvariety W of $\mathbf{C}\times\mathbf{C}^m$ with equation

$$W : v = -\frac{1}{\mu} a_{\mu-1}(t_1, \cdots, t_m).$$

(Remark that the flattening property of D of 5.5.1 implies that $a_{\mu-i} \in (t_1, \cdots, t_m)^{i+1}$.)

Remark. In the language of the theory of contact of Hironaka [11], W has the *maximal contact* at every point of $(D_\mu)_{red}$ (in fact $W \supset D_\mu$) and if $s \in D_\mu$, $T_{W,s}$ in one case coincides with the *strict tangent space* to D at s.

This is important for us because it will give us a *natural frame in which to take Newton polygons* (namely the v axis and $T_{W,s}$) at every point s of D_μ.

5.11.3. The problem of the non-singularity of the μ-constant stratum was as far as I know raised in [33], where it was shown that:

Theorem. *If D is the discriminant of the miniversal deformation of a germ of reduced plane curve, $(D_\mu)_{red}$ is non-singular.*

Remark 1. In that first proof, the hard work had been done by J. Wahl who had shown in his thesis [37] the existence of a (formal) miniversal Zariski equisingular deformation for a germ of plane curve, and had shown it was (formally) non-singular. All I had to do was to identify the completion at 0 of the local ring of $(D_\mu)_{red}$ at 0 with that constructed by J. Wahl, thanks to the fact that μ-constant $\Leftrightarrow$ equisingularity in the sense of Zariski (see 3.7).

Recently however, in the appendix to [39], I succeeded in giving a completely different proof, at least in the case of branches: the idea, which is new I believe, is to make any branch with a given semi-group of values Γ (see 3.7) appear as a deformation of the *monomial* curve C^Γ which has the same semi-group. Of course one has to go out of the paradise of plane curves: if $\Gamma = \langle \bar\beta_0, \cdots, \bar\beta_g \rangle$, the monomial curve C^Γ, which is given parametrically by $v_i = t^{\bar\beta_i}$ $(0 \leqq i \leqq g)$ has imbedding dimension $g+1$. What one wins is that it has a $\mathbf{C}^*$-action, and if it is a complete intersection, which is the case if Γ is the semi-group of a plane curve, then it is easy to see that it has a *miniversal constant-semi-group deformation* which is a *linear* subspace in the base of a $\mathbf{C}^*$-equivariant miniversal deformation of C^Γ. Now for plane branches, equisingularity is equivalent to 'same semi-group' so that by 4.8.2, since our original plane branch $(X_0, 0)$ appears as fibre of a deformation (arbitrary small) of C^Γ, the μ-constant stratum of its discriminant, multiplied by a non-singular subspace, is non-singular: therefore this μ-constant stratum is non-singular (the example 4.6.3 is lifted from that theory). It seems to be easy to pass from this result to an arbitrary reduced plane curve, but I think the generalization of the ideas of this proof to higher dimensions (where the question is still open) is quite an intriguing problem (see below), and is for me the main motivation for the study of complete intersections and/or quasi-homogeneous singularities. Another form of the question, which is stronger and therefore perhaps easier to answer, is the following:

Question. Is the Samuel stratum D_μ a *resolved* space in the sense of Hironaka? (see [11])

(A resolved space is a (non-reduced) space which cannot be improved by permissible blowing up: its underlying reduced space is non-singular and it is normally flat along it.)

REMARK 2. The non-singularity of $(D_\mu)_{\text{red}}$ of course would imply the non-singularity of the μ-constant stratum in a $\bar{\mathbb{R}}$-miniversal unfolding of f, after the first exercise in 4.8.4.

5.12. We now come to the negative facts concerning $(D_\mu)_{\text{red}}$, which are extremely important.

5.12.1. THEOREM (Pham [26], [3]). *Even when $N=2$, the topological type of $G:\mathbb{C}^N\times\mathbb{C}^m\to\mathbb{C}\times\mathbb{C}^m$ can vary along $\sigma((D_\mu)_{\text{red}})$ where σ is the section built in 5.11.1. The example given by Pham is the miniversal deformation of the reduced plane curve $y^3+x^9=0$. The miniversal deformation is described by:*

$$v\circ G=y^3+\mathscr{a}(x)\cdot y+\mathscr{b}(x)$$

where

$$\mathscr{a}(x)=a_0+a_1\cdot x+\cdots+a_7\cdot x^7$$
$$\mathscr{b}(x)=b_1\cdot x+\cdots+b_7\cdot x^7+x^9.$$

The μ-constant stratum is given in this case by $v=0$, $b_j=0$ $(1\leqq j\leqq 7)$, $a_i=0$ $(0\leqq i\leqq 5)$ and the corresponding family is given by

$$y^3+(a_6x^6+a_7x^7)y+x^9=0.$$

EXERCISE. Check that it is an equisingular family, using 3.7. Now Pham shows that near every point of $(D_\mu)_{\text{red}}$ with $a_6=0$, there is a point of the discriminant such that the corresponding fibre has 2 singular points of multiplicity 3 which have in suitable coordinates Newton polygons of smallest inclination $\geqq 4/3$ and $\geqq 5/3$ respectively, and that this is *not* the case if $a_6\neq 0$. Since the maximum over the set of coordinate choices of the inclination of the first side of the Newton polygon is a topological invariant, [11], this shows that the topological type of G cannot be the same if $a_6=0$ or $a_6\neq 0$ (near a point where $a_6=0$, $a_7\neq 0$, say).

5.12.2. Furthermore, the counterexamples of Briançon and Speder (2.18, see [6]) show that even if $(D_\mu)_{\text{red}}$ is non-singular, or if we restrict to a non-singular subspace of it, it is not true that the corresponding fibres will be equisingular in any strong sense, as soon as $N\geqq 3$. (However, they will have the same topological type, at least if $N\neq 3$, by the theorem of Lê-Ramanujam).

My reaction to this was to abandon 'constant topological type' as a definition of equisingularity in high dimension, in favour of (c)-cosécance, since in my opinion the topology of the general sections is an important part of the geometry of a singularity (see for example 5.20.2 below). Furthermore (c)-cosécance can be easily generalized to the non-hypersurface case, coincides with Zariski's definition of equisingularity when $N=2$, and in the case of hypersurfaces implies that the *functions* having our hypersurfaces as fibres are topologically equivalent, a fact often deemed useful.

One could think that in the definition of equisingularity conditions 'the stronger the better' provided the condition is always satisfied for 'almost all' fibres in a family. However, I do not go all the way to the really strong possible definitions of equisingularity, such as 'miniversal deformation topologically constant', which is called also 'universal equisingularity' or 'τ constant', which both satisfy the openness condition mentioned above (see also [14] §1). The reason is that with these definitions, some deformations which I like, such as the specialization of a plane branch to the monomial curve with the same semi-group, mentioned above, would not be equisingular.

5.12.3. By the results in [14] §2, all the $\mu^{(i)}(X_y,0)$ are analytically upper semi-continuous in a family of hypersurfaces. Hence we can define a closed complex subspace of the μ-constant stratum D_μ, the μ^*-constant stratum D_{μ^*}, at least as a reduced subspace. Of course, when $N=2, D_{\mu^*}=(D_\mu)_{\text{red}}$. It seems that the extensions of the methods of proof of the non-singularity of $(D_\mu)_{\text{red}}$ for plane branches are better geared to answer the

QUESTION. Is D_{μ^*} non-singular?

5.12.4. The last negative fact on D_μ (or D_{μ^*}), also due to Pham, is quite important for us:

THEOREM (Pham). *Even when $N=2$, the topology of a general plane section of the discriminant $D\subset\mathbb{C}\times\mathbb{C}^m$ of the miniversal deformation of $(X_0,0)\subset(\mathbb{C}^N,0)$, through the point $s\in D$, can vary when s varies as the μ-constant stratum D_μ. As a consequence* (see 5.5.6) *the number k of cusps in a general plane section of D near s* (see 5.4.1) *can vary as s varies on D_μ.*

5.12.5. IMPORTANT REMARK. I want to emphasize that after the counterexamples of Briançon and Speder (5.12.2) one is aware of the fact that a hypersurface (such as our D) can be topologically trivial along a non-singular subspace S (such as $(D_\mu)_{\text{red}}$) while the topological type of its general i-plane section ($i=2$ for example) through $s\in S$ *varies* with s. There is no example where the topology of the discriminant varies along $(D_\mu)_{\text{red}}$ as far as I know. See [22].

5.12.6. We shall analyze the phenomenon in 5.12.4 in a way different from Pham's. The idea is that there is a very close connection between a general plane section of D through the origin and the *polar curve* we saw in 5.6.5 and 5.6.6. This is seen as follows:

5.13. PROPOSITION. *The topological type (indeed, the (c)-cosécance class 2.19), of a general* vertical *plane section of the discriminant D (5.5.7) is the same as that of a general plane section of D.*

PROOF. Since the set of 2-planes H in $(\mathbb{C}\times\mathbb{C}^m, 0)$ giving the general topological type for $D\cap H$ is Zariski open and dense in the Grassmannian $Gr(m+1, 2)$, it is sufficient to check that the family of plane sections of D through 0 satisfies an equisingularity condition near a plane section of the form:

$$v=\alpha_0 x, \qquad t_i=\beta_i y \qquad (1\leqq i\leqq m)$$

where $\alpha_0\neq 0$.

Let us then consider the family of plane curves X defined in $\mathbb{C}^{2m+2}\times\mathbb{C}^2$ by

$$F(\alpha,\beta,x,y)=\delta(\alpha_0x+\beta_0y, \alpha_1x+\beta_1y, \cdots, \alpha_mx+\beta_my)=0$$

where $\delta(v, t_1, \cdots, t_m)=0$ is the equation of the discriminant D. We see that

$$\frac{\partial F}{\partial\alpha_0}=x\cdot\frac{\partial\delta}{\partial v}, \qquad \frac{\partial F}{\partial\beta_0}=y\cdot\frac{\partial\delta}{\partial v}$$

$$\frac{\partial F}{\partial\alpha_i}=x\cdot\frac{\partial\delta}{\partial t_i}, \qquad \frac{\partial F}{\partial\beta_i}=y\cdot\frac{\partial\delta}{\partial t_i}$$

$$\frac{\partial F}{\partial x}=\alpha_0\frac{\partial\delta}{\partial v}+\sum\alpha_i\frac{\partial\delta}{\partial t_i}$$

$$\frac{\partial F}{\partial y}=\beta_0\frac{\partial\delta}{\partial v}+\sum\beta_i\frac{\partial\delta}{\partial t_i}.$$

CLAIM. Near a point $p:(\alpha,\beta,0)$ where $\alpha_0\neq 0$ we have

$$\frac{\partial F}{\partial\alpha_i}\cdot\mathcal{O}_{X,p}\in\overline{(x,y)\cdot\left(\frac{\partial F}{\partial x},\frac{\partial F}{\partial y}\right)\cdot\mathcal{O}_{X,p}} \qquad (0\leqq i\leqq m)$$

$$\frac{\partial F}{\partial\beta_i}\cdot\mathcal{O}_{X,p}\in\overline{(x,y)\cdot\left(\frac{\partial F}{\partial x},\frac{\partial F}{\partial y}\right)\cdot\mathcal{O}_{X,p}} \qquad (0\leqq i\leqq m).$$

PROOF. Since an integral dependence relation can be pulled back by a map such as the map $X\to D$ induced from the map $\varphi:\mathbb{C}^{2m+2}\times\mathbb{C}^2\to\mathbb{C}\times\mathbb{C}^m$ given by $v=\alpha_0x+\beta_0y$, $t_1=\alpha_ix+\beta_iy$ $(1\leqq i\leqq m)$, on $X=\Phi^{-1}(D)$ we see that as soon as α_0 or β_0 is different from 0, since $(\partial\delta/\partial t_i)\cdot\mathcal{O}_{D,0}$ is integrally dependent on the ideal $(\partial\delta/\partial v)\cdot\mathcal{O}_{D,0}$ for all i, as follows from the proof of 5.5.1, and 4 of 2.7, we see that if $\alpha_0\neq 0$ the ideal $((\partial F/\partial x), (\partial F/\partial y))\,\mathcal{O}_{X,p}$ is generated by $(\partial\delta/\partial v)\cdot\mathcal{O}_{X,p}$ and all $(\partial\delta/\partial t_i)\cdot\mathcal{O}_{X,p}$ are integrally dependent on it. The claim follows from this. Now we remark that the claim is nothing but the restriction to X of the integral dependence required for X to satisfy condition (c) along $\mathbb{C}^{2m+2}\times\{0\}$ near p. Of course, condition (c) does not follow directly from this, but a direct computation shows that the pull-back by φ of the map $G\,|\,C: C\to\mathbb{C}\times\mathbb{C}^m$ is non-singular and is the normalization of X, provided $\alpha_0\neq 0$. Then one can apply the criterion (12) of 3.7 since $j(F)\cdot\mathcal{O}_{\bar{x},n^{-1}(p)}$ is then generated by $(\partial\delta/\partial v)\cdot\mathcal{O}_{\bar{x},n^{-1}(p)}$.

REMARK. The same argument shows that general vertical sections of any dimension give the general (c)-cosécance class.

5.14. PROPOSITION. *The Newton polygon (in the coordinates given by the decomposition $H=\mathbb{C}\times H_1$) of the intersection of D with a general vertical plane is equal to the Newton polygon of the intersection of D with the plane $\mathbb{C}\times H_1$ where H_1 is the line $t_2=\cdots=t_m=0$ in $\mathbb{C}^m$, provided $z_1=0$ is a sufficiently general hyperplane in the sense of 5.6.5. Furthermore, this section also has the same topological type ((c)-cosécance) as a general plane section of D.*

PROOF. A vertical plane in $\mathbb{C}\times\mathbb{C}^m$ can be given parametrically by:

$$\begin{cases} v=v \\ t_i=\alpha_i u \qquad \alpha_i\in\mathbb{C} \qquad (1\leqq i\leqq m)\end{cases}$$

and if we write the corresponding functions, with the conventions of 5.5, we find

$$v=f(z_1,\cdots,z_N)+\sum_{i=1}^{N}\alpha_iuz_i+\sum_{i=N+1}^{m}\alpha_iug_i(z_1,\cdots,z_N)$$

which, after a linear change of coordinates, becomes (if our vertical plane is general in the sense that $\alpha\neq 0$)

$$v=f(z_1,\cdots,z_N)+u\left(z_1+\sum_{i=N+1}^{m}\alpha_ig_i(z_1,\cdots,z_N)\right)$$

with $g_i\in(z_1,\cdots,z_N)^2$.

After a new change of coordinates, we can write it as

$$\underline{v=f(z_1,\cdots,z_N)+uz_1.}$$

Now since the formation of the discriminant commutes with base change, $D\cap H$ is just the discriminant of the map $(\mathbb{C}^N\times\mathbb{C},0)\to(\mathbb{C}\times\mathbb{C},0)$ thus described. The *critical subspace* is defined by the ideal

$$\left(\frac{\partial f}{\partial z_1}+u, \frac{\partial f}{\partial z_2}, \cdots, \frac{\partial f}{\partial z_N}\right)\cdot \mathcal{O}_{\mathbb{C}^N\times\mathbb{C}}$$

which is nothing but the intersection of the surface $S_H\times\mathbb{C}$, where S_H is our *polar curve* with respect to $z_1=0$, with the hypersurface $(\partial f/\partial z_1)+u=0$.

As we have already assumed $z_1=0$ to be general, we are ready to obtain a parametric representation of our general vertical plane section of D, as follows:

Take the parametric representation of the branches Γ_q of S_H as in 5.6.6:

$$\Gamma_q \begin{cases} z_1=t_q^{m_q} \\ z_i=a_{q,i}t^{k_{q,i}}+\cdots & (2\leqq i\leqq N) \qquad k_{q,i}\geqq m_q. \end{cases}$$

We have now a parametric representation of the branches C_q of the critical subspace above (which is a reduced curve) as follows:

$$C_q \begin{cases} u=-\partial f/\partial z_1|_{\Gamma_q}=-\zeta_q t_q^{e_q}+\cdots & (\zeta_q\in\mathbb{C}^*) \\ z_1=t_q^{m_q} \\ z_i=a_{q,i}t^{k_{q,i}}+\cdots \end{cases}$$

and therefore also a parametric representation of the branches of $D\cap H$, say D_q, as follows: computing $v=f+uz_{1\Gamma_q}$ gives: (see 5.6.6 and remark that differentiating gives $\gamma_q=\zeta_q\cdot(m_q)/(e_q+m_q)$)

$$D_q \begin{cases} u=-\zeta_q t_q^{e_q}+\cdots \\ v=\eta_q t_q^{e_q+m}+\cdots & \left(\eta_q=-\dfrac{e_q}{e_q+m_q}\zeta_q\in\mathbb{C}^*\right). \end{cases}$$

Hence we have proved:

THEOREM. *The number of branches of a general vertical plane section $D\cap H$ of D is equal to that of a general polar curve of the function $f(z_1,\cdots,z_N)$, and its Newton polygon in the natural coordinates (u,v) on $H=\mathbb{C}\times H_1$ is*

$$\mathfrak{N}(\delta\mid H)=\sum_{q=1}^{1}\left\{\frac{e_q+m_q}{e_q}\right\} \qquad \text{(notations of 3.6).}$$

This now follows immediately from 3.6.2.

5.15. We remark that since $k_{q,i}\geqq m_q$, it is reasonable to say that the ratios m_q/e_q describe the vanishing rates of the critical points of the Morse function $f+uz_1$ *as functions of* u since for example, on C_q:

$$z_1=(-\zeta_q)^{(1/e_q)}u^{(m_q/e_q)}+\cdots$$

and that the $(e_q+m_q)/(e_q)$ describe the vanishing rates in function of u of the critical *values* of this function.

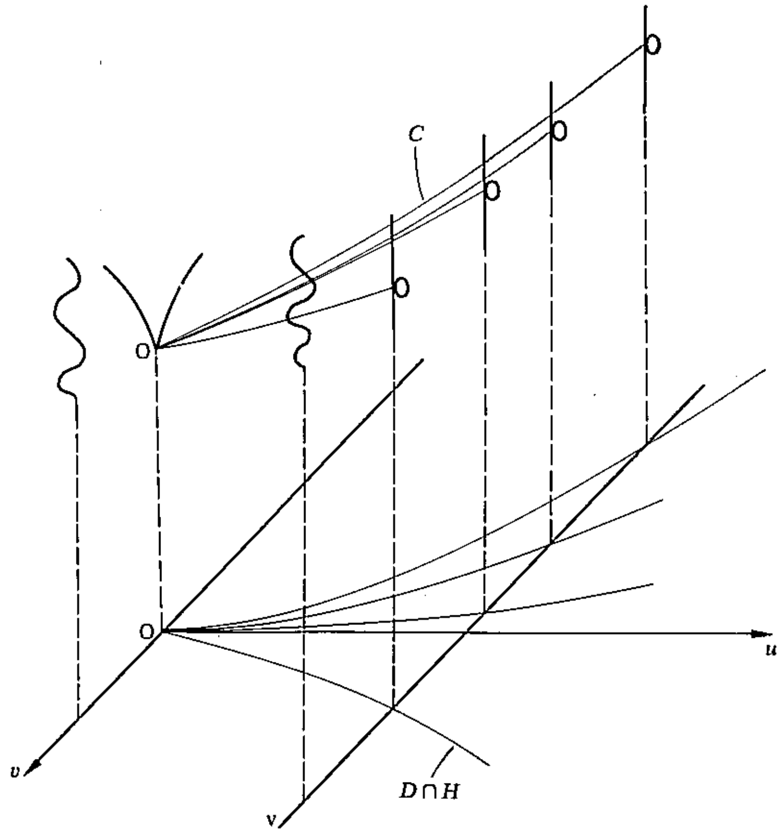

Figure 4

5.16. The point is that while it is quite hard to compute the discriminant, it is relatively easy to compute the polar curve (in practice, of course, we have to decompose it into its irreducible components, which is not so easy, but usually manageable by using the following trick: a general plane projection, and then Newton polygon again!) For example, to prove Pham's theorem in 5.12, all we have to do is to find an equisingular (i.e. μ-constant) family of plane curves such that the number of irreducible components of a general polar curve does not remain constant:

Consider for example the family of plane curves given by

$$z_2^3+z_1^7+yz_1^5z_2=0 \quad \text{in} \quad \mathbb{C}\times\mathbb{C}^2$$

since it is cubic in z_2 it is easy to see by Zariski's discriminant criterion that it is equisingular along $\mathbb{C}\times\{0\}$. However, consider a general polar curve:

$$\lambda(3z_2^2+yz_1^5)+\mu(7z_1^6+5yz_1^4z_2)=0 \qquad (\lambda:\mu)\in\mathbb{P}^1 \text{ 'general'}.$$

It is not difficult to see that for $y=0$, the general polar curve is reducible: it has two non-singular components. However for $y\neq 0$, it is topologically the same as $z_2^2+z_1^5=0$ hence irreducible. Therefore we can assure that the topology of a general 2-plane section of the discriminant D varies along the μ-constant stratum.

Exercise. Parametrizing the components of the polar curve and computing adequate intersection numbers, check that whether y is 0 or not, the Newton polygon of a general vertical plane section of the discriminant is $\{14//12\}$.

Exercise. Consider the family of plane curves

$$z_2^4+z_1^9+yz_2^2z_1^5=0$$

(1) Check that it is equisingular.

(2) This time, the general polar curve is irreducible for $y=0$, but becomes reducible for $y\neq 0$ (two components).

(3) Whatever the value of y, the Newton polygon of a general vertical section of the discriminant of the miniversal deformation is $\{27//24\}$ (but for $y\neq 0$ it appears as $\{9//8\}+\{18//16\}$).

5.17. This Newton polygon $\mathfrak{N}(\delta\mid H)=\sum_{q=1}^{l}\{(e_q+m_q)//(e_q)\}$ of a general vertical plane section of the discriminant of a miniversal deformation of $(X_0,0)$ is the main new invariant which I wish to associate to an isolated singularity of hypersurface $(X_0,0)$, but I am used to working with another polygon, namely $\sum_{q=1}^{l}\{e_q//m_q\}$ which I will call the *Jacobian Newton polygon* of X. Of course, the datum of $\boldsymbol{\nu}_j(X_0,0)=\sum_{q=1}^{l}\{e_q//m_q\}$ is equivalent to that of $\mathfrak{N}(\delta\mid H)$. In fact with the notations of 3.6, we have

$$\mathfrak{N}(\delta\mid H)=\varepsilon(\sigma(\boldsymbol{\nu}_j(X_0,0))).$$

We also remark that since e_q is the intersection number of Γ_q with a hypersurface, $e_q\geqq m_q$ so that $\boldsymbol{\nu}_j(X_0,0)$ is a *special* Newton polygon. Remark also that

$$l(\boldsymbol{\nu}_j(X_0,0))=\mu^{(N)}(X_0,0), \qquad h(\boldsymbol{\nu}_j(X_0,0))=\mu^{(N-1)}(X_0,0).$$

5.18. I now quote results concerning $\boldsymbol{\nu}_j(X_0,0)$. First, it is important to remark that the analytic type of the hypersurface defined by $f(z_1,\cdots,z_N)-g(w_1,\cdots,w_M)=0$ does *not* depend only upon the analytic type of the hypersurfaces $(X_0,0)$ and $(X_1,0)$ defined respectively by $f=0$ and $g=0$ (e.g. multiply f by a unit). However, we have

Proposition. *The (c)-cosécance class of* $f(\underline{z})-g(\underline{w})=0$ *depends only upon the (c)-cosécance classes of* $(X_0,0)$ *and* $(X_1,0)$.

The proof is given as an exercise on (c)-cosécance. Therefore we have defined an operation on the set $\mathcal{I}_\delta$ of (c)-cosécance classes of isolated singularities of hypersurfaces. This operation will be denoted by $\perp$: the (c)-cosécance class of $f(\underline{z})-g(\underline{w})=0$ will be denoted by $[X_0]\perp[X_1]$, where $[X]$ is the (c)-cosécance class of the germ of hypersurface $(X,0)$. Now we have

5.19. Theorem ([35] th. 4). Assuming that $(X_0,0)$ and $(X_1,0)$ (given by $f(\underline{z})=0$, $g(\underline{w})=0$) have isolated singularities, the Jacobian Newton polygon of $f(\underline{z})-g(\underline{w})=0$ is given by

$$\boldsymbol{\nu}_j(f(\underline{z})-g(\underline{w})=0,0)=\boldsymbol{\nu}_j(X_0,0)*\boldsymbol{\nu}_j(X_1,0).$$

This theorem is not stated in this way in [35], but the proof of theorem 4 there actually proves this. ($*$ is the product defined in 3.6).

5.20. Theorem ([35] th. 6′). *The Jacobian Newton polygon is an invariant of (c)-cosécance.*

The idea of the proof is as follows, and relies on Theorem 2.18. First, one checks that if a family $F(y,z_1,\cdots,z_N)=0$ of hypersurfaces is (c)-equisingular, then so is the *double* family $F(y,z_1,\cdots,z_N)+F(y,w_1,\cdots,w_N)=0$. After Theorem 2.18, the Milnor number of the generic hyperplane section of the fibres of this family is constant (independent of y). But by an exercise in 3.6, and 5.19, this number is *twice the area* under the Jacobian Newton polygon $\boldsymbol{\nu}_j(X_y,0)$. Since a Newton polygon can only rise under specialization we have that $\boldsymbol{\nu}_j(X_0,0)$ is above $\boldsymbol{\nu}_j(X_y,0)$ $(y\neq 0)$ and encloses the same area: therefore they are equal.

5.21. Now we have defined our Jacobian Newton polygon map $\boldsymbol{\nu}_j:\mathcal{I}_\delta\to\underline{\mathrm{N}}_S$ and shown that $\boldsymbol{\nu}_j([X_0]\perp[X_1])=\boldsymbol{\nu}_j([X_0])*\boldsymbol{\nu}_j([X_1])$.

Remark. Since for plane curves a deformation is (c)-equisingular if and only if it is equisingular in the sense of Zariski (3.6), the two exercises given above are nothing but verifications of 5.20. However, they show that 5.20 is a rather fine phenomenon since the number of branches of the polar curve varies.

5.21.1. Remark. I want to emphasize that 5.20 has also the following geometric meaning: Along the μ^*-constant stratum D_{μ^*} in the discriminant $D\subset\mathbb{C}\times\mathbb{C}^m$ of a miniversal deformation of an isolated singularity of hypersurface $(X_0,0)$, the Newton polygon of a general vertical plane section (which has a meaning in view of 5.11.2) is constant.

For the reader familiar with the theory of contact of Hironaka ([11]) this implies in particular that the *contact exponent* $\delta_s(W, D)$ of the discriminant D with the non-singular hypersurface W of 5.11.2 (which has maximal contact) is constant (and equal to $1+(1/\eta)$ where $\eta=\max_q(e_q/m_q)$) along D_{μ^*}.

5.21.2. Furthermore, we have also that the intersection multiplicity at $s\in D_{\mu^*}$ of D with a general non-singular curve contained in W is also constant along D_{μ^*} and equal to $\mu^{(N)}(X_s, 0)+\mu^{(N-1)}(X_s, 0)$ where $X_s=G^{-1}(s)\subset\mathbb{C}^N$ is the fibre of the miniversal deformation G over s. It is rather intriguing that this number, which is the *diminution of class* due to the presence of the singularity $(X_s, 0)$ on a projective hypersurface (i.e. what comes in the Plücker formula, see Kleiman's lectures) is also very closely linked with the contact of the discriminant D with the limit direction at $s\in D_{\mu^*}$ of tangent hyperplanes, at nearby non-singular points, namely $T_{W,s}$.

I will end by quoting from [35] theorems showing that $\boldsymbol{\nu}_j(X_0, 0)$, or $\mathfrak{N}(\delta \mid H)$, contains real information on the geometry of the singularity $(X_0, 0)$ up to (c)-cosécance:

THEOREM. *A necessary and sufficient condition for $(X_0, 0)$ to be (c)-cosécant to a hypersurface defined by an equation 'with one variable separated' i.e. of the form $f(z_2, \cdots, z_N)+z_1^{a+1}$ where $z_1=0$ is a general hyperplane section, is that $\boldsymbol{\nu}_j(X_0, 0)$ is of the form $\{a\cdot m/m\}$ where a is an integer.*

THEOREM. *Let $f\in\mathbb{C}\{z_1, \cdots, z_N\}$ define a function with an isolated critical point at 0, and $(X_0, 0)=(f^{-1}(0), 0)$. For an integer, the following conditions are equivalent*

(i) *$a>\operatorname{Sup}(e_q/m_q)$ where (e_q/m_q) are those associated to f as in 5.6.5.*

(ii) *Any function $g\in\mathbb{C}\{z_1, \cdots, z_N\}$ such that $g-f\in(z_1, \cdots, z_N)^{a+1}$ defines by $g=0$ a hypersurface having the same topological type as $(X_0, 0)$.*

(iii) *Any function $g\in\mathbb{C}\{z_1, \cdots, z_N\}$ such that $g-f\in(z_1, \cdots, z_N)^{a+1}$ defines by $g=0$ a hypersurface (c)-cosécant to $(X_0, 0)$ (and hence g has the same topological type as f, as a function).*

Finally, the test of any invariant is that it recovers, for plane branches, the classical complete systems of invariants to the geometry up to (c)-cosécance namely the characteristic $(\beta_0, \cdots, \beta_g)$ or the semi-group $\Gamma=\langle\bar{\beta}_0, \cdots, \bar{\beta}_g\rangle$: the translation in the language presented here of a theorem of M. Merle [24] gives a complete answer:

THEOREM (M. Merle). *Let $(X_0, 0)\subset(\mathbb{C}^2, 0)$ be a plane branch. Then (notations of 3.6 and 3.7) we have:*

$$\boldsymbol{\nu}_j(X_0, 0)=\sum_{q=1}^{g}\left\{\frac{(n_q-1)\bar{\beta}_q-n_1\cdots n_{q-1}(n_q-1)}{n_1\cdots n_{q-1}(n_q-1)}\right\}.$$

COROLLARY. *Given two plane branches $(X_0, 0)$, $(X_1, 0)$, letting $\mathfrak{N}(\delta_0 \mid H)$ (resp. $\mathfrak{N}(\delta_1 \mid H)$) denote the Newton polygon of a general vertical plane section of the discriminant of a miniversal deformation of $(X_0, 0)$, resp. $(X_1, 0)$. The following are equivalent:*

(1) *$(X_0, 0)$ and $(X_1, 0)$ have the same topological type.*

(2) *$(X_0, 0)$ and $(X_1, 0)$ are (c)-cosécant.*

(3) *$\boldsymbol{\nu}_j(X_0, 0)=\boldsymbol{\nu}_j(X_1, 0)$.*

(4) *$\mathfrak{N}(\delta_0 \mid H)=\mathfrak{N}(\delta_1 \mid H)$.*

EXERCISE. Using the construction in Appendix II of [34] show that $\boldsymbol{\nu}_j(X_0, 0)$ plays with respect to the local Plücker formula exactly the role that $\mathfrak{N}(\delta \mid H)$ plays with respect to Milnor's formula: it is a dynamic version. (See also [14] §3.)

EXERCISE. Let $(X_0, 0)\subset(\mathbb{C}^3, 0)$ and $(X_1, 0)\subset(\mathbb{C}^3, 0)$ be defined by $Z_2^3+Z_1^\beta Z_3+Z_3^{3\alpha}=0$ and $(X_1, 0)\subset(\mathbb{C}^3, 0)$ be defined by $Z_2^3+Z_1^\alpha Z_2+Z_1^\beta Z_3+Z_3^{3\alpha}$ where $3\alpha=2\beta+1$ and $\alpha\geqq 3$ (see [6]).

Show that

$$\boldsymbol{\nu}_j(X_0, 0)=\left\{\frac{2\beta}{2}\right\}+\left\{\frac{2\beta(2\beta-2)}{2\beta-2}\right\}$$

$$\boldsymbol{\nu}_j(X_1, 0)=\left\{\frac{2\beta(2\beta-1)}{2\beta-1}\right\}.$$

HINT. Use 5.19 and 5.20.

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NOTES ADDED AFTER PROOFREADING

p.567 line -5 of text : the $\underline{N}_S$ should be an $\underline{\mathfrak{N}}_S$ as on p. 619 line 2.

p.571 in the diagram : $X' = X \underset{Y}{\times} Y'$.

p.592 line 4 of text : $Y \times \mathbb{C}^N$ instead of $X_y \in \mathbb{C}^N$.

p.603 line -18 : $\mu^{(1)}(X_o,0)$ instead of $\mu^{(\ell)}(X_o,0)$.

p.618 in the picture : $P = \left\{\frac{\ell(P)}{h(P)}\right\}$ instead of $i(P) = \frac{\ell(P)}{h(P)}$.

p.644 line -8 : replace $\mathrm{depth}_{\mathcal{O}_{X,0}}(\ )$ by $dh_{\mathcal{O}_{X_o}}(\ )$.

p.660 line 13 : read [(a+1)1/2] instead of [(a+1/2)].

<u>Addendum</u> : The facts stated in Corollary 3 of §3 were apparently known to various people for different purposes. I want to quote two papers giving beautiful pictures and applications to monodromy of the deformation of the Lemma :

N. A'Campo : Le groupe de monodromie du déploiement des singularités isolées de courbes planes, Math. Ann. 213 (1975) pp. 1-22.

S. M. Gusein-Zade : Intersection matrices of some singularities of functions of two variables, Funktional. Anal. i. Prilozen. <u>8</u> (1974) No 1 pp. 11-15.

Also, this Corollary 3 shows geometrically that the conductor gives the right structure for the double-point scheme of a map from a curve to a non-singular surface. (See Kleiman's lectures, chap. 5, and the paper quoted p. 614.)
