On the paper *Un résultat sur la monodromie*

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1 Présentation

The local study of complex hypersurfaces in $\mathbf{C}^{n+1}$, $n = 0, 1, \cdots$ has a long history.

First let us study the case $n = 0$. Let $U \subset \mathbf{C}$ be an open connected neighborhood of 0 and let $f : U \to \mathbf{C}$ be a non constant holomorphic function that vanishes at 0. Then after shrinking U, there exists on U a complex coordinate function z, that vanishes at 0, such that f equals z^m, $m = 1, 2, \cdots$. So the complex hypersurface $H_f = \{p \in U \mid f(p) = 0\}$ is the singleton $H_f = \{0\}$. The complex hypersurface $H_{f-t} = \{p \in U \mid f(p) = t\}$ consists for $|t|$ small enough of m points provided $t \neq 0$. If $t = t(\tau) = r(\cos(\tau) + i\sin(\tau))$, $\tau \in [0, 2\pi]$ the set H_{f-t} moves in such a way that the graph $\tau \in [0, 2\pi] \mapsto H_{f-t(\tau)}$ is a braid in $[0, 2\pi] \times U$ (Fig. 1).
The monodromy of this braid is a cyclic permutation of the set H_{f-t}.

The cases $n \geq 1$ are more complicated. As before, $0 \in U \subset \mathbf{C}^{n+1}$, and let $f : U \to \mathbf{C}$ be a non-constant holomorphic function that vanishes at 0. Two assumptions will simplify the study : the differential df does not vanish on $U \setminus \{0\}$ and, instead of the open U, one considers a euclidean closed ball $\mathbf{B} = \mathbf{B}(0, R)$ of small radius R. A seminal Theorem of F. Bruhat and H. Cartan, see [B-C], [W-B], valid for real analytic, semianalytic or subanalytic sets, is essential. The Theorem goes today under its name given by Milnor : *Curve Selection Lemma.* Using this Lemma, a transversality result of Milnor follows : the radius R can be chosen such that for all $0 < r \leq R$ the intersections of the spheres $\partial\mathbf{B}(0, r)$ and the complex hypersurface H_f are transverse. It follows that the space $H_f \cap \mathbf{B}(0, R)$ is contractible and that 0 is the only critical value of the restriction of f to the interior of B. Moreover, there exist $\rho > 0$ such that for all $t \in \mathbf{C}$, $0 < |t| \leq \rho$, the intersection $X_{f,t} = B \cap H_{f-t}$ is a manifold with boundary. Putting $t(\tau) = \rho(\tau) = \rho(\cos(\tau) + i\sin(\tau))$, the graph of $\tau \in [0, 2\pi] \mapsto X_{t(\tau)}$ sweeps out a cobordism $W_{f,\rho}$ from $X_{f,\rho} = X_{f,t(0)}$ back to $X_{f,\rho}$. A trivialisation of $W_{f,\rho}$ induces a diffeomorphism $T : X_{f,\rho} \to X_{f,\rho}$. The isotopy class of T is called the *local monodromy* at 0 of the complex hypersurface H_f. The induced map by T on the homology is commonly called the monodromy.

Only few monodromies were known at the time of the work of Sebastiani and Thom. The local monodromy of the quadratic function $f = z_0^2 + z_1^2 + \cdots + z_n^2$:

$\mathbf{C}^{n+1} \to \mathbf{C}$ by Picard and Lefschetz and the cohomological monodromy for functions $f = z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n} : \mathbf{C}^{n+1} \to \mathbf{C}$, $a_0, a_1, \cdots a_n \geq 2$, by Pham. These polynomials f do not seem very complicated, and the discoveries of Brieskorn and Hirzebruch showing that many exotic spheres previously discovered by Milnor appear as boundaries $\partial X_{f,0}$ came as a surprise. Milnor gave a course on this topic at Princeton ; see the well known book [M].

The work of Sebastiani and Thom is an extension of the work of Pham.
First let us explain very briefly the work of Pham. For $f = z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n} : \mathbf{C}^{n+1} \to \mathbf{C}$, $a_0, a_1, \cdots, a_n \geq 2$, Pham constructs inside $X_{f,t} = \mathbf{B}(0,R) \cap H_{f-t}, 0 < |t| << R$, a simplicial complex $\Phi_{f,t}$ on which $X_{f,t}$ retracts by deformation. The complex is a union of n-simplices $\Delta_\mu(t) \subset X_{f,t}$ that are parametrized by the standard n-simplex $\Delta^n = \{s_0, s_1, \cdots, s_n \mid s_i \geq 0, \sum_{i=0}^n s_i = 1\}$ as follows :

$$\phi_{\mu(t)} : (s_0, s_1, \cdots, s_n) \in \Delta^n \mapsto (s_0^{\frac{1}{a_0}} \mu_0(t), s_1^{\frac{1}{a_1}} \mu_1(t), \cdots, s_n^{\frac{1}{a_n}} \mu_n(t)) \in X_{f,t}$$

where $\mu(t) = (\mu_0(t), \mu_1(t), \cdots, \mu_n(t))$ and $\mu_i(t)$ is a chosen a_i-th root of t. The number of possible choices, i.e., the number of n-simplices in $\Phi_{f,t}$, is the product $a_0 a_1 . \cdots . a_n$ of the exponents. Moreover, the realization $Z_{f,t}$ of $\Phi_{f,t}$ is homeomorphic to the join $U_{a_0} * U_{a_1} * \cdots * U_{a_n}$ where $U_{a_i} \subset \mathbf{C}$ is the set of a_i-roots of unity. The dimension of $Z_{f,t}$ is half the real dimension of $X_{f,t}$. The simplicial map induced by the primitive cyclic permutations of U_{a_i} is a homotopic model for the local monodromy of f.

The ramification of the integrals

$$t \mapsto \int_{\Delta_\mu(t)} z^b dz_0 \wedge dz_1 \wedge \cdots \wedge dz_n / df$$

for monomials $z^b = z_0^{b_0} \cdots z_n^{b_n}$ with $0 \leq b_i < a_i$ is studied by Pham and he computes the monodromy action on the De Rham cohomology of $X_{f,t}$.

The work of Sebastiani and Thom concerns hypersurfaces defined by polynomials $f = f_0 + f_1 + \cdots + f_k : \mathbf{C}^{N+1} \to \mathbf{C}, k \geq 2$ where $N + 1 = (n_0 + 1) + (n_1 + 1) + \cdots + (n_k + 1)$. The term f_0 depends only on the first $(n_0 + 1)$ variables, f_1 depends only on the next $(n_1 + 1)$ variables, ... Again assume $f_i(0) = 0$, $i = 1, 2, \cdots, k$, and that df has an isolated zero at $0 \in \mathbf{C}^{N+1}$. Clearly, df_i has an isolated zero at $0 \in \mathbf{C}^{n_i+1}$. Let $T_i : X_{f_i,t} \to X_{f_i,t}$ be a monodromy diffeomorphism for the hypersurface $\{f_i = 0\} \subset \mathbf{C}^{n_i+1}$, well defined up to isotopy. Here $X_{f_i,t}$ is constructed as above for $0 < |t| << R_i$.

A concrete example with $k = 2$, $n_0 = 1$, $n_1 = 0$ would be $f = ((z_0^3 + z_1^2)^2 + z_0^5 z_1) + z_2^3$.

In this language, and with above notations, Sebastiani-Thom's Theorem reads as follows :

Theorem 1.1 (Sebastiani & Thom). *Let f be as above. Choose R, R_i for f, f_i with $R_i << R$. Choose $0 < |t| << \mathrm{Min}\{R_i\}$ for f. Choose monodromy diffeomorphisms $T_i : X_{f_i,t} \to X_{f_i,t}$. For $s \in \Delta^k$ define*

$$Y_{f,t,s} = X_{f_0,s_0t} \times X_{f_1,s_1t} \times \cdots \times X_{f_k,s_kt} \subset X_{f,t} \subset \mathbf{C}^{(n_0+1)+(n_1+1)+\cdots+(n_k+1)}$$

Define $Y_{f,t} = \cup_{s\in\Delta^k} Y_{f,t,s}$. *Then* $Y_{f,t} \subset X_{f,t}$ *is a deformation retract of* $X_{f,t}$. *Moreover, the space* $Y_{f,t}$ *is homotopic to the join* $Z_{f,t} = X_{f_0,t} * X_{f_1,t} * \cdots * X_{f_k,t}$ *and the join map* $T_0 * T_1 * \cdots * T_k : Z_{f,t} \to Z_{f,t}$ *is a homotopic model for the local monodromy of* f *at* $0 \in \mathbf{C}^N$.

Remark. If $k = n+1$, i.e. for polynomials studied by Pham, the spaces $Y_{f,t}$ and $Z_{f,t}$ coincide and are a polyhedron of real dimension n inside the fiber $X_{f,t}$ of real dimension $2n$. If $2 \leq k < n+1$, i.e. for polynomials studied by Sebastiani and Thom, the space $Y_{f,t}$ is a closed semi-algebraic subset of real dimension $2N-k+1$ in $X_{f,t}$. The homotopy equivalence with the join $Z_{f,t}$ is due to the homotopy equivalences $Y_{f_i,0} \sim \{*\}$.

In case of sums $f = f_0 + f_1 + \cdots + f_k$ of functions in disjoint sets of variables having at 0 non-isolated singularities, the Curve Selection Lemma still applies and extends Milnor's transversality result in stratified sense. As a consequence, the homotopy equivalences $Y_{f_i,0} \sim \{*\}$ and, without any change of words, that the above Theorem of Sebastiani and Thom remains valid in case of summands with non-isolated singularities.

2 Commentaires

The addition defined by Sebastiani-Thom has turned out to have many more applications than the construction of many explicit examples of monodromies and boundaries of isolated singularities, such as exotic spheres. First of all, the proof constructs not only the monodromy of the sum, but also its Seifert form and hence the intersection form of vanishing cycles.

Assuming isolated singularities, keep notations, and introduce the notation of $X'_{f,t}$ for the Milnor fiber obtained by pushing the tube fiber $X_{f,t}$ radially, by constant argument of t, into the sphere of ambiant space. This is possible as done by Milnor, again using the Curve Selection Lemma. The spaces $X_{f,t}$, $X'_{f,t}$, X'_{f,t^+} are canonically homotopic. Here t^+ denotes a complex number $e^{2\pi\delta}t$ for small positive δ. The subspaces $X'_{f,t}$, X'_{f,t^+} in the local sphere are disjoint, and X'_{f,t^+} is a homotopy retract of $X_{f,t}$, so by the Alexander linking one obtains a non-degenerate pairing

$$A_f : \tilde{H}_N(X'_{f,t}, \mathbb{Z}) \times \tilde{H}_N(X'_{f,t^+}, \mathbb{Z}) \to \mathbb{Z}$$

of reduced homology groups, which together with above homotopy equivalence yields the Seifert form

$$S_f : \tilde{H}_N(X'_{f,t}, \mathbb{Z}) \to (\tilde{H}_N(X'_{f,t}, \mathbb{Z}))^*$$

Remember the join formula $S^N = S^{n_0} * S^{n_1} * \cdots * S^{n_k}$ for the local ambient spheres. Also remember that the reduced homology of a join product is canonically isomorphic to the graded-shifted tensor product of the reduced homologies of the factors. The shift is by $k-1$. Now it is straightforward that

$$S_f = \pm S_{f_0} \otimes S_{f_1} \otimes \cdots \otimes S_{f_k}$$

holds for the Seifert form. The Seifert S form determines the intersection I form on the reduced homology and also the homological monodromy T by $I = S \pm S^t$ and $T = \pm S \circ (S^t)^{-1}$.

As one sees by considering Sebastiani-Thom sums such as

$$U(z_0, \dots, z_n) f(z_0, \dots, z_n) + g(w_0, \dots, w_m),$$

where U is a unit, the analytic isomorphism type of the sum is not in general well defined by the analytic type of the summands. However, its equisingularity class for any reasonable definition of equisingularity is well defined and this may partly explain its usefulness in the study of geometric invariants. Here are some examples :

— The study of special case of the Thom-Sebastiani operation where one adds a function of a single variable, $f(z_0, \dots, z_n) + w^k = 0$ is already quite rewarding. From a differential topology viewpoint the intersection of this hypersurface with a small $2n + 3$-sphere centered at the origin in $\mathbf{C}^{n+2}$ is a k-fold cyclic covering of the sphere $\mathbf{S}^{2n+1}$ ramified along its intersection with $f(z_0, \dots, z_n) = 0$.

— Thom had asked whether any hypersurface with isolated singularity is a generic hyperplane section of a hypersurface with isolated singularity, that is, a section by a hyperplane which is not a limit of tangent hyperplanes at non singular points. In [Te] it is shown that when the integer k is larger than the largest polar invariant (see the commentary to the manuscript "Un résultat sur la monodromie" at the end of this volume) $\eta(f) \in \mathbf{Q}_{>0}$ of the hypersurface with isolated singularity $f(z_0, \dots, z_n) = 0$, then the hyperplane $w = 0$ is a generic hyperplane section of the hypersurface with isolated singularity $f(z_0, \dots, z_n) + w^k = 0$. The same type of argument, based on the study of the polar invariants of a Sebastiani-Thom sum, is used there to show that $[\eta(f)] + 1$ is the degree of topological sufficiency of the map germ $f \colon (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$.

— Integral dependence criteria for equisingularity imply that if $F(t, z_0, \dots, z_n) = 0$ and $G(t, w_0, \dots, w_m) = 0$ determine two germs of families of isolated singularities of hypersurfaces which are both equisingular in the sense that the t-axis is the singular locus and a Whitney stratum, then the Sebastiani-Thom sum $F(t, z_0, \dots, z_n) + G(t, w_0, \dots, w_m) = 0$ is also equisingular. In [Te], applying this with $G = F$ is the key to a proof of the constancy, in an equisingular family of hypersurfaces with isolated singularities, of the Jacobian Newton polygon, and in particular of the polar invariants.

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There have recently appeared many generalizations of the Thom-Sebastiani addition ; we can mention :

— Goulwen Fichou and Yimu Yin have in [F-Y] proved a Sebastiani-Thom formula for the non-archimedean Milnor fiber introduced by Hrushovski-Loeser which, in their words, is "a richer embodiment of the philosophy of the Milnor fiber".

— In [GLM], Guibert, Loeser and Merle consider the Sebastiani-Thom addition as a special case of the composition of a family of functions $f_1, \dots, f_k$ in disjoint sets of variables with a polynomial $P(y_1, \dots y_k)$ which is non degenerate with respect to its Newton polyhedron. This is of course the case for $P(y_1, y_2) = y_1 + y_2$. This viewpoint had already appeared, in the case of two functions, in the article [N] of A. Némethi.

— In [Ma], David Massey extended the Sebastiani-Thom result to the case of maps $f\colon X \to \mathbf{C}$ and $g\colon Y \to \mathbf{C}$, the Milnor fiber being that of the map $f \circ pr_1 + g \circ pr_2 \colon X \times Y \to \mathbf{C}$, where X and Y may be singular. He shows that the Sebastiani-Thom result is the consequence of a natural isomorphism in the derived category of bounded constructible sheaves on $X \times Y$.

— The concept of Milnor fiber also appears in the theory of constructible sheaves, and a Sebastiani-Thom formula has been proved by Schürmann in [S].

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