CONTINUED FRACTIONS

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§1 Anthyphairesis: how the Greeks compared lengths.

The method we learn at school to compare two lengths is to measure them in centimeters or some other unit, and compute the ratio of the two resulting numbers. The Greek Mathematicians 2.500 years ago used a completely different method, which requires no unit: given two straight segments, they measured how many times the smaller one goes into the bigger one, thus obtaining an integer u_0 and possibly a rest, smaller than the smaller length. Then they measured how many times this rest or remainder goes into the smaller length, thus getting a second integer u_1 and a second remainder smaller than the first, computed how many times this second remainder went into the first, and they continued like that, so that for them a ratio was a sequence, finite or not, of integers. This procedure, which you can physically begin to perform on two rods, was called *Anthyphairesis* ($\alpha\nu\theta\upsilon\phi\alpha\iota\rho\epsilon\sigma\iota s$)

It looks very much like the division algorithm of Euclid, but I insist that it is also a physical manipulation, is not primarily concerned with finding the greatest common divisor of two integers, and as we shall see has a much larger scope.

The mathematical model of this operation is as follows: let $L > l$ be the lengths of the two rods. The integer u_0 is the greatest integer less than $\frac{L}{l}$, denoted by $[\frac{L}{l}]$, so that

$$\frac{L}{l} = u_0 + \frac{L - u_0 l}{l}, \text{ with } u_0 \geq 1 \text{ and } \ 0 \leq \frac{L - u_0 l}{l} < 1.$$

If this remainder is 0, we stop and write $\frac{L}{l} = u_0$. Otherwise we repeat the operation with $\frac{l}{L - u_0 l} > 1$; set

$$u_1 = [\frac{l}{L - u_0 l}],$$

and so on. Finally we write

$$\frac{L}{l} = u_0 + \cfrac{1}{u_1 + \cfrac{1}{u_2 + \cfrac{1}{\ldots u_k + \cfrac{1}{\ldots}}}}$$

The right hand side is symbolized by $[u_0, u_1, \ldots, u_k, \ldots]$ and called the continued fraction expansion of the ratio $\frac{L}{l}$.

§2 Computations on continued fractions.

Let $[u_0, \ldots, u_k, \ldots]$ be a continued fraction, finite or infinite. For any integer k such that u_k is defined, the continued fraction $[u_0, \ldots, u_k]$ represents a rational number $\frac{p_k}{q_k}$, called the $k-th$ convergent, or approximant. The first problem we face is to compute p_k and q_k. A little experimentation gives:

$$p_0 = u_0, q_0 = 1;\ p_1 = u_0 u_1 + 1, q_1 = u_1,\ p_2 = u_0 + u_2(u_0 u_1 + 1), q_2 = 1 + u_2 u_1$$

so that we are led to *conjecture* the general rule of formation

$$(1) \qquad \begin{aligned} p_{k+1} =& p_{k-1} + u_{k+1} p_k, \\ q_{k+1} =& q_{k-1} + u_{k+1} q_k \end{aligned}$$

with the initial conditions $p_0 = u_0,\ q_0 = 1$.
Let us prove these formulas *by induction* on the integer k. Assume that they hold until the computation of p_k; for q_k it is the same proof.
The main idea is to think of p_k or q_k as a function of $u_0, \dots, u_k$. Then we get p_{k+1} or q_{k+1} by substituting $u_k + \frac{1}{u_{k+1}}$ for u_k in this expression. In fact p_k and q_k are *polynomials* in $u_0, \dots, u_k$. If we assume that we know that $p_k = p_{k-2} + u_k p_{k_1}$ this gives

$$p_{k+1} = p_{k-2} + (u_k + \frac{1}{u_{k+1}})p_{k-1} = \frac{1}{u_{k+1}}(p_{k-1} + u_{k+1}(p_{k-2} + u_k p_{k-1})) = \frac{1}{u_{k+1}}(p_{k-1} + u_{k+1} p_k)$$

and so as we have the same computation for q_{k+1}, we may multiply numerator and denominator by u_{k+1} and get

$$\begin{aligned} p_{k+1} &= p_{k-1} + u_{k+1} p_k \\ q_{k+1} &= q_{k-1} + u_{k+1} q_k \end{aligned}.$$

This proves that if our formula is true for an integer k it is true for $k+1$. Since it is true for 0 it is true for all integers.

This will allow us to compute all the p_k, q_k easily if we know the integers u_k, but of course in practice the difficulty is to compute these integers from a given ratio problem. However we shall see that the expression we just proved has many consequences.
The first one is

Theorem 1: For all integers k, we have

$$(2) \qquad p_{k+1} q_k - p_k q_{k+1} = (-1)^k$$

This implies in particular that p_k, q_k have no common divisor.
Proof: look at the two formulas proved by induction; multiply the first by q_k, the second by p_k and compute the difference of the results: it gives $p_{k+1}q_k - p_k q_{k+1} = -(p_k q_{k-1} - p_{k-1} q_k)$, and we have $p_0 = u_0, q_0 = 1, p_1 = u_0 u_1 + 1, q_1 = u_1$, so we get the result by induction.
Let us now see how to compute the u_k.

Proposition 2: Let us start from a rational number $\frac{L}{l}$ with $L,\ l\ > 0$. Then the Euclidean division algorithm coincides with the Anthyphairesis of the ratio $\frac{L}{l}$.

We divide L by l:
$L = u_0 l + r_0$ with $r_0 < l$
$l = u_1 r_0 + r_1$ with $r_1 < r_0$
etc.....
Since at each step the remainder decreases, after a finite number of steps (at most $l - 1$) we get 0 and this means that the last step is
$r_{k-2} = u_k r_{k-1}$
Comparison with the Anthyphairesis shows that we have exactly computed the continued fraction expansion of $\frac{L}{l}$:

$$\frac{L}{l} = [u_0, \ldots, u_k]$$

Note that the greatest common divisor of the integers L and l is r_{k-1}. Nowadays the Euclidean algorithm is often presented as only a means of computing this g.c.d.
Conversely, any finite continued fraction $[u_0, \ldots, u_k]$ represents a rational number, namely $\frac{p_k}{q_k}$.
On the other hand given any positive real number a we can associate to it a sequence of integers by modelling the Anthyphairesis construction (we repeat the construction of §1 in a slightly more abstract setting):
Let $u_0 = [a]$, the integral part of a, i.e the greatest integer less than a. (note that the use of [] here is different from above where we use brackets to denote a continued fraction, but in the continued fraction writing there are only integers between the [], and in the integral part there is only one number. When a is a positive integer the two notations five the same result!). Note also that here u_0 may be 0, if $a < 1$.
Anyway we have $a = u_0 + s_0$ with $0 \leq s_0 < 1$. If $s_0 = 0$, we have $a = u_0$ and we are finished. If not, set $a_1 = \frac{1}{s_0} > 1$ and let u_1 be the integral part of a_1; we repeat the procedure and thus obtain a possibly infinite but uniquely determined sequence of integers, $[u_0, \ldots u_k, \ldots]$ corresponding to the number a.

Note that if this sequence end after finitely many steps, it cannot end with 1, and no integer except possibly the first (u_0) may be zero.
Recall that $|x|$ denotes the absolute value of the number x, and therefore $|a - b|$ is the distance from a to b. Then we have:

Theorem 3: To any such sequence of integers $u_0, \ldots u_k, \ldots$ is associated a unique positive real number a denoted by $[u_0, \ldots u_k, \ldots]$ and characterized by the property that for all integers k we have with the notations introduced above:

$$|a - \frac{p_k}{q_k}| < \frac{1}{q_k q_{k+1}}.$$

In particular a is the *limit* of the sequence of rational numbers $\frac{p_k}{q_k}$.
Proof: Using Theorem 2, we have

$$\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k q_{k+1}}$$

and this implies that the sequence or rational numbers $\frac{p_k}{q_k}$ is convergent, that is there is a real number a such that for an arbitrarily small e the distance from $\frac{p_k}{q_k}$ to a is less than e as soon as k is large enough (depending on e of course). For each integer k we have by the same computation we did in the beginning (think about this!)

$$(3) \qquad a = \frac{p_k + a_{k+2}p_{k+1}}{q_k + a_{k+2}q_{k+1}} \quad \text{where } a_{k+2} = [u_{k+2}, u_{k+3}, \ldots]$$

so using theorem 2 we get

$$(4) \qquad a - \frac{p_k}{q_k} = \frac{(-1)^k a_{k+2}}{q_k(q_k + a_{k+2}q_{k+1})}$$

and since $q_{k+1} > q_k$ we get the result.
Note that in fact this shows with a little work (given as an exercise) a better approximation:

$$(5) \qquad \frac{1}{(u_{k+1}+2)q_k^2} < |a - \frac{p_k}{q_k}| < \frac{1}{u_{k+1}q_k^2}.$$

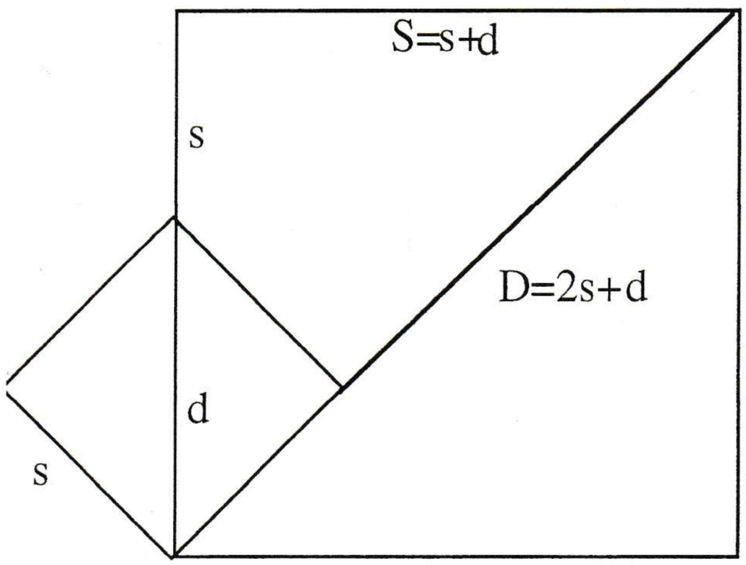
Since the q_k tend to infinity with k, this says that the sequence of rational numbers $\frac{p_k}{q_k}$ *converges* to the real number a, and also that the greater u_{k+1} is the better the approximation. And the reader should check that the continued fraction expansion of the number a is $[u_0, \ldots, u_k, \ldots]$. Note that this correspondance has only one possible ambiguity: if the sequence is finite, then $[u_0, \ldots, u_k, 1] = [u_0, \ldots, u_k + 1]$. This is different from what happens for the decimal expansion where the ambiguity is greater. For example $0,999\ldots = 1,000\ldots$.

Theorem 4: The continued fraction expansion $a = [u_0, \ldots, u_k, \ldots]$ establishes a bijection between positive real numbers and sequences of integers such that the last one is $\neq 1$. In this bijection rational numbers correspond to finite sequences.
Remark that while the continued fraction expansion of $\frac{p_k}{q_k}$ is $[u_0, \ldots, u_k]$, the continued fraction expansion of $\frac{p_{k+1}}{p_k}$ is $[u_{k+1}, \ldots, u_0]$ and the continued fraction expansion of $\frac{q_{k+1}}{q_k}$ is $[u_{k+1}, \ldots, u_1]$. This follows immediately from (1).

§3 Computations of continued fractions. The irrationality of $\sqrt{2}$.

In the time of the Greeks (Archimedes) it was known that a good approximation of the number π is $\frac{22}{7}$; the error is less than $\frac{1}{100}$; this can be seen by measuring with strings the perimeter of a circle and its radius and performing the Anthyphairesis of these two lengths: you get $3+\frac{1}{7}$. If you do it very carefully with a large circle you might even get the next approximants $\frac{333}{106}, \frac{355}{113}$ which were apparently known to ancient mathematicians, and certainly to the Europeans and the Japanese in the 17^{th} century. The continued fraction expansion of π begins by $[3, 7, 15, 1, 292, 1, 1, \ldots]$. You should compute how much more information you store by remembering the first five numbers than by remembering $3,1416$. In general you can compute the first few terms of a continued fraction expansions either "physically" as in this case, or "mathematically" by getting a decimal expansion from theorems in analysis or by "seeing" what the continued fraction expansion has to be, as the famous Indian mathematician Ramanujan did, or by some geometric method. In one series of cases at least one can compute geometrically a whole infinite continued fraction expansion:

It an old story that the Greeks discovered that $\sqrt{2}$ is not a rational number. Today we say that they proved it in the following way: assume that $\sqrt{2} = \frac{p}{q}$ where we may assume that p, q have no common divisor. Then $p^2 = 2q^2$ so p is even, say $p = 2p'$. But then $q^2 = 2p'^2$ so q is also even which is a contradiction!. This proof is in fact (relatively) quite "recent " and it is a reasonable conjecture that the proof of the ancient Greek mathematicians (before 5 centuries B.C.) was much more clever and in fact computed $\sqrt{2}$, as follows: we know by Pythagoras that $\sqrt{2}$ is the length of the diagonal of a square of side 1. So we want to do the Anthyphairesis of the diagonal and the side. It will be more convenient to do the Anthyphairesis of (diagonal+side) and side, which only adds 1 to u_0. Notice that this is independant of the size of the square! We consider the following picture, where we put our square so that its diagonal is vertical, and construct a larger square with side $S = s + d$.



It is easy to see on the picture that the diagonal of the big square is $D = 2s + d$. Notice also that $d < 2s$ because the diagonal is the shortest path between two vertices of the square. So let us do the Anthyphairesis of $S + D = 3s + 2d$ and $S = s + d$: we have $S + D = 2S + s$ so $u_0 = 2$ and now $\frac{S+D}{S} = 2 + \frac{s}{d+s} = 2 + \frac{1}{\frac{d+s}{s}}$ so the next step in the Anthyphairesis of $\frac{S+D}{S}$ is the Anthyphairesis of $\frac{s+d}{s}$ which is the same!
This proves that the sequence of integers is $[2, 2, \ldots, 2, \ldots]$ so we have proved

$$1 + \sqrt{2} = [2, 2, \ldots, 2, \ldots]$$

and this is an infinite continued fraction: this shows that $\sqrt{2}$ is irrational, and in addition we can compute it with arbitrary approximation.

From an algebraic viewpoint, this “ Greek” computation is a proof of the identity

$$1+\sqrt{2}=2+\frac{1}{1+\sqrt{2}}.$$

Now it is true in general that if a positive real number a is a root of a second degree equation with integral coefficients, the continued fraction expansion of a will become periodic after a finite initial part, that is, if

$$Aa^2+Ba+C=0,\quad A,B,C\ \text{ integers},$$

the continued fraction expansion of a is of the form

$$a=[u_0,\ldots u_j,\overline{u_{j+1},\ldots,u_{j+l}}]$$

where the bar means that the same sequence of integers is repeated indefinitely. This is called an *eventually periodic* continued fraction expansion.

This result is due to Lagrange, and is a bit delicate (but you should try to prove it!). The converse is much simpler:
If a positive real number has an eventually periodic continued fraction expansion, it is a root of a second degree equation with integral coefficients. You can do this as an exercise, with the hint that if

$$\frac{qa+p}{ra+s},p,q,r,s\ \text{ integers},\ qs-pr=\pm 1,$$

is a solution of such a quadratic equation, so is a.

Now these numbers with eventually periodic continued fraction expansion are determined by finite data: the numbers u_k before the period starts, and then the period. I believe in Fowler's idea in [F] that these numbers were considered as ”tame” by Greek mathematicians, and that their first serious problems came with numbers like the cubic root of 2, because they could say nothing about its continued fraction expansion not even decide whether the u_k are bounded or not (most people think they are not). And today we have made essentially no progress on this problem. I like to think that it is the oldest unsolved problem in Mathematics:
say something about the continued fraction expansion of the cubic root of an integer which is not the cube of another integer.
Of course we cannot deduce much from a beginning like this:

$$2^{1/3}=[1,3,1,5,1,1,4,1,1,8,1,14,1,10,2,1,4,12,2,3,2,\ldots]$$

but a million terms have now been computed and still no regularity appears, and we have no theorem. It seems the regularity due to the fact that we have a cubic root manifests itself in a very subtle manner.

§4 Continued fractions and approximation.

Now we come to the fact that the convergents of the continued fraction expansion of a real number give "best possible approximation" of this number by rational numbers, in the sense that the difference $a - \frac{p}{q}$ is as small as possible among all rational number with denominator less than q.

The inequalities (5) show that the approximation of a by $\frac{p_k}{q_k}$ depends on the size of u_{k+1} and, through the size of q_k, also on all the preceding u_j, so that in a way it is a mixture between the average size of the u_j for $j \leq k+1$ and the size of the individual u_{k+1}. If all the u_j are small, we expect relatively poor approximation, and we shall make this precise below, but if from time to time there is a very large u_k, then we will have good approximation, which depends both on how large these u_k are, and how often large u_k appear.

Let a be a positive real number and let us consider in the plane with coordinates (p, q) the line $p = aq$ in the quadrant $p \geq 0,\ q \geq 0$. Let us remark first that because of the computations we have made earlier, the convergents of even order of the continued fraction expansion of a form an increasing sequence converging to a and the convergents of odd order form a decreasing sequence converging to a. Consider the line in the plane passing through the points with coordinates (q_k, p_k) and (q_{k+2}, p_{k+2}). Again because of our calculations it is the line with equation

$$q_{k+1}p - p_{k+1}q = (-1)^{k+1}$$

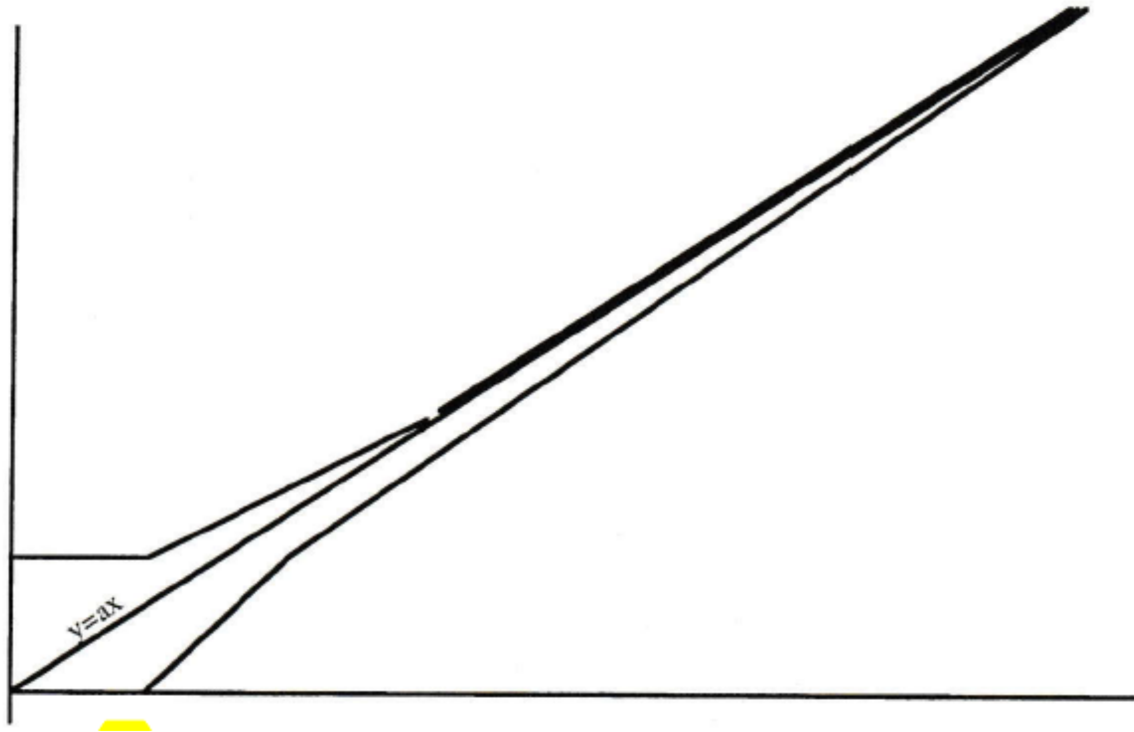
I claim that if k is even any point $(\overline{q}, \overline{p})$ such that $\frac{\overline{p}}{\overline{q}} < a$ is "below " that line, which means that $q_{k+1}\overline{p} - p_{k+1}\overline{q} < (-1)^{k+1}$. Now always by the same computation we have

$$q_{k+1}a - p_{k+1} = \frac{(-1)^{k+1}a_{k+3}}{q_{k+1}(q_{k+1} + a_{k+3}q_{k+2})} > q_{k+1}\frac{\overline{p}}{\overline{q}} - p_{k+1}.$$

Since k is even this implies $q_{k+1}\overline{p} - p_{k+1}\overline{q} < 0$, but this number is an integer, so this means $q_{k+1}\overline{p} - p_{k+1}\overline{q} \leq -1 = (-1)^{k+1}$ which is what we had to prove.

This shows that the set of points with positive integral coordinates below the line $aq - p = 0$ is the intersection of the half planes made of the points lying below the lines $q_{k+1}p - p_{k+1}q = (-1)^{k+1}$ joining the points (q_k, p_k) and (q_{k+2}, p_{k+2}). In other words, in view of the fact that the slopes of these lines are all differerent, the points (q_k, p_k) with k even (odd) are the *extreme points* of the *convex hull* of the set of points in the first quadrant with integral coordinates lying below (above) the line $aq - p = 0$. Recall that a subset of Euclidean space (here the plane) is *convex* if whenever it contains two points, it contains the line segment joining them. The intersection of two convex sets is convex, and the *convex hull* of a set is the smallest convex set containing it; it is also the intersection of all convex sets containing it, and the intersection of all *half spaces* containing it, where a half space (here a half plane) is the set of points lying on one side of a hyperplane (here a line). Extreme points are points on the boundary on a convex set which actually determine this convex set.

The idea of this graphic description of the continued fraction expansion is due to H.J.S. Smith. It was apparently rediscovered by Felix Klein some years later.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It now becomes apparent on the picture, and can be checked by computation, that among all rational numbers with denominator bounded by some number the convergents provide the best approximation to a.
Let us be more precise: first, I leave as an exercise to prove the following:
Given an integer k, with the usual notations, if $a = [u_0, \ldots, u_k, \ldots]$, at least one of the following inequalities is true:

$$|a - \frac{p_k}{q_k}| < \frac{1}{2q_k^2}, \quad |a - \frac{p_{k+1}}{q_{k+1}}| < \frac{1}{2q_{k+1}^2}.$$

Hint: use (2).
It is more difficult to prove the following (see [S]):
Let a positive real number a be given, at let $[u_0, \ldots, u_k, \ldots]$ be its continued fraction expansion. If a rational number $\frac{p}{q}$ satisfies the inequality

$$|a - \frac{p}{q}| < \frac{1}{2q^2},$$

then $\frac{p}{q} = \frac{p_k}{q_k}$ where k is the integer such that $q_k \leq q < q_{k+1}$. So any rational number which approximates well the number a, for a given size of denominator, has to be a convergent. For example, if $\frac{p}{q}$ is a rational number different from $\frac{22}{7}$ and $0 < q \leq 50$, then

$$|\pi - \frac{22}{7}| < |\pi - \frac{p}{q}|.$$

The geometric result above is very striking and synthetic, but it does not suffice by far to prove all results on approximation. Let us first give a quantitative meaning to the intuitive fact that if all the u_k are small, the number a cannot be well approximated.
So we are going to study the numbers having a continued fraction expansion with $1 \leq u_k \leq M$ for some integer M and all $k \geq 0$. I claim that :

Theorem 5: If a is a number such that $u_k \leq M$ for $k \geq 0$, an inequality

$$|a - \frac{p}{q}| < \frac{1}{cq^2}$$

cannot hold for infinitely many values of q if $c > \frac{\sqrt{4+M^2}+\sqrt{4M+M^2}}{2}$. It means that in approximation results $|a - \frac{p}{q}| < \frac{1}{cq^2}$, not only the exponent 2 cannot be improved, but even the constant c cannot be too large.

The proof begins by remarking that we may restrict our consideration to convergents since the critical value of c is > 2, and relies on the formula (4), which we transform to:

$$|a - \frac{p_k}{q_k}| = \frac{1}{q_k^2(\frac{q_{k+1}}{q_k} + \frac{1}{a_{k+2}})}$$

so that the constant c has to be larger that $\frac{q_{k+1}}{q_k} + \frac{1}{a_{k+2}}$ for sufficiently large k. Now we have by formula (1) the equality

$$\frac{q_{k+1}}{q_k} = u_{k+2} + \frac{1}{\frac{q_k}{q_{k-1}}} \; < \; M + 1$$

So that that the sequence $\frac{q_{k+1}}{q_k}$ has an upper limit (smallest upper bound), which we may denote by b. From the equality above it follows that

$$b - \frac{1}{b} \leq M.$$

Therefore b is smaller that the largest root of the second degree equation

$$X^2 - MX - 1 = 0,$$

which is equal to

$$\frac{M + \sqrt{4 + M^2}}{2}$$

It remains to bound the other term of the sum, that is to find a lower bound for the terms

$$a_{k+2} = [u_{k+2}, \ldots,]$$

that is, in effect, a lower bound for all continued fractions having terms bounded by M, which means $1 \leq u_k \leq M$. Here we use formula (2) which has the following consequence: For some integer k, let

$$f(X) = \frac{p_k + Xp_{k+1}}{q_k + Xq_{k+1}}$$

Then the derivative is

$$f'(X) = \frac{(-1)^{k+1}}{(q_k + Xq_{k+1})^2}$$

In particular for k even one must decrease X to increase $f(X)$, and the opposite for k odd. From this it easily follows by induction that:
All infinite continued fractions with $u_0 \geq 1$ and $u_k \leq M$ for all k are greater than the continued fraction

$$[\overline{1, M}].$$

Now we know how to compute the value of this periodic continued fraction: call it d; we compute $p_0 = 1, p_1 = 1 + M, q_0 = 1, q_1 = M$ and we have by the periodicity and formula (3):

$$d = \frac{1 + d(1 + M)}{1 + dM},$$

so that d is the positive root of the equation

$$MX^2 - MX - 1 = 0$$

This root is equal to

$$d = \frac{M + \sqrt{M^2 + 4M}}{2M},$$

and since the product of the roots of this equation is equal to $\frac{-1}{M}$, we find that we have the inequality:

$$\frac{1}{a_{k+2}} \leq \frac{\sqrt{4M + M^2} - M}{2}$$

so that finally, for any $\epsilon > 0$, for large enough k we have

$$\frac{q_{k+1}}{q_k} + \frac{1}{a_{k+2}} < \frac{\sqrt{4 + M^2} + \sqrt{4M + M^2}}{2} + \epsilon$$

which implies that for any $c' > \frac{\sqrt{4+M^2}+\sqrt{4M+M^2}}{2}$, we have

$$|a - \frac{p_k}{q_k}| > \frac{1}{c' q_k^2}$$

for large k, and the result.

As exercises you may show that:
1) If we write

$$a - \frac{p_k}{q_k} = \frac{(-1)^k \delta_k}{q_k^2},$$

as definition of the real number δ_k, then $0 \leq \delta_k < 1$, and the asymptotic lower bound $\underline{\lim}_k \delta_k$ is positive if and only if the u_k are bounded by some M as above.
2) There is a variant of the proof given above, using the equality (which is also a consequence of formula (4))

$$a - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k(q_{k-1} + a_{k+1} q_k)} \quad ;$$

the aim is now to find a lower bound for $a_{k+1} + \frac{q_{k-1}}{q_k}$ and show that it is attained when all $u_k = 1$.

Remarks: of course, the essentially periodic continued fraction expansions of the roots of equations of degree two with integral coefficients, which we saw above, satisfy the condition $u_k \leq M$ for some M, and therefore they satisfy the conclusion of the Theorem.
There is a generalization, which is a deep theorem of number theory, called the Thue-Siegel-Roth Theorem, which implies that if a is an irrational number which is a root of a polynomial equation

$$a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0 = 0$$

with integer coefficients (in short, a is an irrational algebraic number) then for any $\theta > 2$ there exists a positive number δ depending on a and θ such that for all rational numbers $\frac{p}{q}$ we have

$$|a - \frac{p}{q}| > \frac{\delta}{q^\theta}.$$

This implies that for an algebraic number the u_k's cannot grow too fast and is the best result we know about the continued fraction expansions of irrational non quadratic algebraic numbers such as $2^{1/3}$.

§5 Golden ratio, etc..

If we consider the result on approximation of bounded continued fractions, we see that a number which should have the worst approximation by rationals is the number corresponding to $M = 1$, that is, the number

$$\tau = [1, 1, \ldots, 1, \ldots]$$

This number is called the *golden ratio* and it was well known to the Greeks; if you try to find the ratio of the sides of a rectangle such that if you remove from this rectangle the largest possible square contained in it the remaining rectangle has the same ratio of sides, you find that this ratio is the golden section. By now this should be obvious to you by Anthyphairesis of the sides of the rectangle. It should also be easy for you to compute

$$\tau = \frac{1+\sqrt{5}}{2}.$$

In any case this number indeed has the worst possible approximation by rational numbers: the critical constant is $\sqrt{5}$ as we see by taking $M = 1$ in the formula proved above. One can prove that for any positive number the inequality

$$|a - \frac{p}{q}| \leq \frac{1}{\sqrt{5}q^2}$$

has infinitely many solutions; in fact given any three successive convergents of a, at least one of them satisfies

$$|a - \frac{p_k}{q_k}| < \frac{1}{\sqrt{5}q_k^2}$$

but if a is the golden ratio you cannot replace $\sqrt{5}$ by any larger number and keep an infinite number of solutions for this inequality (this result is due to Hurwitz) . For the golden ratio the sequence of p_k is given by $p_{k+1} = p_{k-1} + p_k$, and the same holds for the q_k, with the initial conditions $p_0 = 1, p_1 = 2, q_0 = q_1 = 1$. Thus p_k and q_k are consecutive *Fibonacci numbers* , that is, elements of the sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$ where each number is the sum of the two which precede.
If you observe pine cones, or sunflowers, pineapples, and the leaves of many plants, etc.. you will see that the stems are arranged along spirals, and if you count the spirals turning in one direction and those turning in the other, you will most of the time find two consecutive Fibonacci numbers. This phenomenon can be related to the property of bad rational approximation of the golden ratio. So you can observe a mathematical phenomenon in the growth of plants.

Until recently, there was no way to add or multiply two numbers given by their continued fraction expansion. Perhaps this led to abandon this perfect representation of real numbers for imperfect ones, such that decimal expansion, which admit very simple algorithms for sum, difference, product, etc, which can be easily taught. 20 years ago or so the computer scientist Gosper found an algorithm for adding, multiplying, dividing, etc.. continued fractions. I refer you to the excellent book [F] which describes this algorithm. I was largely inspired by that book, which I highly recommend. Below I give other recommended reading on continued fractions. The theory goes much further. In particular, there are more general definitions of continued fractions (see [P]).

§5 Examples

A bar above a sequence of numbers means that it is repeated indefinitely.

$$\sqrt{2} = [1, \overline{2}]$$

$$\sqrt{3} = [1, \overline{1, 2}]$$

$$\frac{\sqrt{3}}{\sqrt{2}} = [1, \overline{4, 2}]$$

$$\frac{\sqrt{4}}{\sqrt{3}} = [1, \overline{6, 2}]$$

$$\sqrt{(n^2 - 1)} = [n - 1, \overline{1, 2(n - 1)}]$$

$$\sqrt{(n^2 + 1)} = [n, \overline{2n}]$$

$$\sqrt{(n^2 + 2)} = [n, \overline{n, 2n}]$$

$$\frac{e - 1}{2} = [1, 6, 10, 14, 18, 22, \ldots, 4k + 2, \ldots]$$

$$e - 1 = [1, 1, 2, 1, 1, 4, 1, \ldots, 1, 2k, 1, \ldots],$$

where $e = 2,7182818...$ is the basis of logarithms, a transcendental number.

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[P] O. Perron, *Die Lehre von den Kettenbrüchen* Teubner 1934 (?).

[S] Harold M. Stark, *An introduction to number theory* , Paperback edition MIT Press, 1978.

[W] Wall, *Continued fractions* Chelsea reprint Company, New York.