Newton Polyhedra of Discriminants: A Computation

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Abstract

We compute the Newton polyhedron in the natural coordinates of the discriminant of a germ of complex analytic mapping $(\mathbf{C}^3 \times \mathbf{C}, 0) \to (\mathbf{C}^3 \times \mathbf{C}, 0)$ associated by the polar hypersurface construction to the degeneration of a plane analytic branch with two characteristic pairs to the monomial curve with the same semigroup. The result shows that the jacobian Newton polyhedron is not in general constant in an equisingular family of complete intersection branches (whereas it is constant in an equisingular family of plane branches). However, in this case the information that it contains, namely the semigroup, is constant and only the encoding changes.

Introduction

To any germ of an isolated complex analytic hypersurface singularity defined by a convergent power series equation $f(u_0, \ldots, u_n) = 0$, one can associate its *jacobian Newton polygon*, which is the Newton polygon *in the coordinates* (t_0, t_1) of the discriminant of the map

$$(\ell, f)\colon (\mathbf{C}^{n+1}, 0) \longrightarrow (\mathbf{C}^2, 0)$$

given by $t_0 = \ell(u_0, \ldots, u_n), t_1 = f(u_0, \ldots, u_n)$, where ℓ is a sufficiently general linear form. We say that a family of hypersurfaces with isolated singularities is equisingular if the singular locus of the total space of the family is a stratum of the minimal Whitney stratification of that total space. For a family of germs of plane complex analytic curves, this is equivalent to the usual definitions of equisingularity, and in particular to the constancy of the local embedded topological type.

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The discriminants associated in the way just described to the members $f_v = 0$ of an equisingular family of equations for germs do not in general form an equisingular family; the numbers of their branches may vary. It is therefore remarkable that their Newton polygons in the coordinates (t_0, t_1), which are the jacobian Newton polygons, are constant (see [6]).

Thanks to a result of Merle ([5]), it is even true that the jacobian Newton polygon of a plane branch is a complete invariant of its equisingularity type; it determines and is determined by the Puiseux characteristic (see §2). In particular the jacobian Newton polygon has g compact edges, where $g+1$ is the number of Puiseux characteristic exponents, and they can be computed from the Puiseux exponents; we call this the *decomposition theorem*.

In the case of a plane curve C defined by $f(u_0, u_1) = 0$, the information contained in the jacobian Newton polygon concerns the possible contacts with C at 0 of the germs of analytically irreducible components (the branches) of the relative polar curve $\frac{\partial f}{\partial u_1} + \tau \frac{\partial f}{\partial u_0} = 0$ for a general value of τ. The invariants extracted from the jacobian Newton polygon appear in many different types of objects related to the singularity. For example in the JSJ decomposition of the complement in the sphere $\mathbf{S}^3_\epsilon$ (of radius ϵ centered at 0) of a small tubular neighborhood of the knot $\mathbf{S}^3_\epsilon \cap C$ for small enough ϵ. In fact, alternative proofs of the topological invariance of the inclinations of the edges of the jacobian Newton polyhedron mentioned above have been given using this fact (see [4]). They also appear in the description of the asymptotic behaviour of the Lipschitz-Killing curvature (as a real surface) of the Milnor fiber $\mathbf{B}^4_\epsilon \cap f^{-1}(t) \subset \mathbf{B}^4_\epsilon$ for $0 < |t| \ll \epsilon \ll 1$ (see [3]), in the Łojasiewicz exponent at 0 of $f(u_0, u_1)$ and so on. The constancy of the jacobian Newton polygon then appears as a tool to understand how the local topology determines geometric structures such as the JSJ decomposition, or even metric information.

It seems therefore interesting to examine whether this phenomenon of constancy of the jacobian Newton polygon in an equisingular family extends to other equisingular families of curves, for example those which are local complete intersections.

There is a particularly interesting such family, which is the specialization of a given plane branch to the monomial curve with the same semigroup (see [7]). The general fiber of this family is the plane branch suitably reimbedded in affine $(g+1)$-dimensional space, where g is the number of its Puiseux exponents.

In this paper, we compute the jacobian Newton polyhedra of the jacobian discriminants of the fibers of such a family of complete intersections in the case of a branch with two characteristic pairs, and we obtain the following information:

- The jacobian Newton polyhedron is *not* constant, although the family is Whitney-equisingular.

- However, the information contained in the jacobian Newton polyhedra of the special and general fibers is the same, and is equivalent to the topology of the branch.

It is interesting to verify on this example that although the equations of the discriminants which we consider are, as usual, rather complicated, the method of computation by Fitting ideals makes it possible to determine at least their Newton polyhedron.

The interested reader will also note that the system of equations which we study is degenerate with respect to its Newton polyhedron in the usual sense for $v \neq 0$, so that the generic methods of computation of Newton polyhedra of discriminants à la Gel'fand-Kapranov-Zelevinski (see [2]) do not apply. This system of equations becomes non degenerate for $v = 0$, of course with respect to a different Newton polyhedron.

This work has a strongly computational flavour, and computer algebra tools did play a role in computing the first examples which led to conjecture the general shape of the result. Although Singular was not used, we are happy to dedicate it to Gert-Martin Greuel, who did so much to develop computer algebra tools for singularists.

1 Plane Branches, Semigroups and Monomial Curves

(A reminder)

For us, a branch is an irreducible germ of a complex analytic curve. A plane branch is given by a convergent power series $f(u_0, u_1) \in \mathbf{C}\{u_0, u_1\}$ which is not a unit and is irreducible in that ring. The branch is the germ at 0 of the set of solutions of $f(u_0, u_1) = 0$. By the theorem of Newton, after possibly a change of coordinates to achieve that $u_0 = 0$ is transversal to it at 0, the branch C can be parametrized near 0 as follows

$$\begin{aligned} u_0(t) &= t^n \\ u_1(t) &= a_m t^m + a_{m+1} t^{m+1} + \cdots + a_j t^j + \cdots \qquad \text{with } m \geq n. \end{aligned}$$

Let us now consider the following grouping of the terms of the series $u_1(t)$: set $\beta_0 = n$ and let β_1 be the smallest exponent appearing in $u_1(t)$ which is not divisible by β_0. If no such exponent exists, it means that u_1 is a power series in u_0, so that our branch is analytically isomorphic to $\mathbf{C}$, hence non singular. Let us suppose that this is not the case, and set $e_1 = (n, \beta_1)$, the greatest common divisor of these two integers. Now define β_2 as the smallest exponent appearing in $u_1(t)$ which is not divisible by e_1. Define $e_2 = (e_1, \beta_2)$; we have $e_2 < e_1$, and we continue in this manner. Having defined $e_i = (e_{i-1}, \beta_i)$, we define β_{i+1} as the smallest exponent appearing in $u_1(t)$ which is not divisible by e_i. Since the sequence of integers

$$n > e_1 > e_2 > \cdots > e_i > \cdots$$

is strictly decreasing, there is an integer g such that $e_g = 1$. At this point, we have structured our parametric representation as follows:

$$\begin{aligned} u_0(t) &= t^n \\ u_1(t) &= a_n t^n + a_{2n} t^{2n} + \ldots + a_{kn} t^{kn} + a_{\beta_1} t^{\beta_1} + a_{\beta_1+e_1} t^{\beta_1+e_1} + \ldots + a_{\beta_1+k_1e_1} t^{\beta_1+k_1e_1} \\ &\quad + a_{\beta_2} t^{\beta_2} + a_{\beta_2+e_2} t^{\beta_2+e_2} + \ldots + a_{\beta_q} t^{\beta_q} + a_{\beta_q+e_q} t^{\beta_q+e_q} + \ldots \\ &\quad + a_{\beta_g} t^{\beta_g} + a_{\beta_g+1} t^{\beta_g+1} + \ldots \end{aligned}$$

where, by construction, the coefficients of the t^{β_i} for $i \geq 1$ are not zero.

The integers $(n = \beta_0, \beta_1, \ldots, \beta_g)$ are called the Puiseux characteristic exponents of the branch.

Let $\mathbf{C}\{u_0, u_1\}/(f(u_0, u_1)) = \mathcal{O}$ be the analytic algebra of a germ of analytically irreducible curve C, and let $\overline{\mathcal{O}}$ be its normalization; we have an injection $\mathcal{O} \hookrightarrow \overline{\mathcal{O}}$, in fact given by $u_0 \mapsto t^n,\ u_1 \mapsto u_1(t)$, which makes $\overline{\mathcal{O}}$ an $\mathcal{O}$-module of finite type, and $\overline{\mathcal{O}}$ is a subalgebra of the fraction field of $\mathcal{O}$. Since $\overline{\mathcal{O}}$ is isomorphic to $\mathbf{C}\{t\}$, the order in t of the series defines a mapping $\nu\colon \mathbf{C}\{t\} \setminus 0 \to \mathbf{N}$ which satisfies

i) $\nu(a(t)b(t)) = \nu(a(t)) + \nu(b(t))$ and

ii) $\nu(a(t) + b(t)) \geq \min(\nu(a(t)), \nu(b(t)))$ with equality if $\nu(a(t)) \neq \nu(b(t))$;

in other words, ν is a *valuation* of the ring $\mathbf{C}\{t\}$.

We consider the valuations of the elements of the subring $\mathcal{O}$, i.e., the image Γ of $\mathcal{O} \setminus \{0\}$ by ν; in view of i), it is a semigroup contained in $\mathbf{N}$. The fact that $\overline{\mathcal{O}}$ is a finite $\mathcal{O}$-module implies that $\mathbf{N} \setminus \Gamma$ is finite.

Now, we seek a minimal set of generators of Γ as a semigroup: Let $\overline{\beta_0}$ be the smallest nonzero element in Γ, let $\overline{\beta_1}$ be the smallest element of Γ which is not a multiple of $\overline{\beta_0}$, let $\overline{\beta_2}$ be the smallest element of Γ which is not a combination with non negative integral coefficients of $\overline{\beta_0}$ and $\overline{\beta_1}$, i.e., is not in the semigroup $\langle \overline{\beta_0}, \overline{\beta_1} \rangle$, and so on. Finally, since $\mathbf{N} \setminus \Gamma$ is finite, we find in this way a minimal set of generators:

$$\Gamma = \langle \overline{\beta_0}, \overline{\beta_1}, \ldots, \overline{\beta_g} \rangle.$$

This set of generators is uniquely determined by the semigroup Γ, and of course determines it.

Let us take the notations introduced for the Puiseux exponents; it is easy to check that if we set $\beta_0 = n$, the multiplicity, then $\overline{\beta_0} = \beta_0 = n, \overline{\beta_1} = \beta_1$.

After that is becomes more complicated. Zariski ([9], Th. 3.9) proved the following recursive formula: $\overline{\beta}_0 = \beta_0 = n$, $\overline{\beta}_1 = \beta_1$ and for $q \geq 2$,

$$\overline{\beta_q} = n_{q-1}\overline{\beta_{q-1}} - \beta_{q-1} + \beta_q,$$

where the integers n_i are defined inductively by $e_0 = n$ and $e_{i-1} = n_i e_i$, where the e_i are the successive greatest common divisors introduced at the beginning of the section, so that we have

$$n = \beta_0 = \overline{\beta}_0 = n_1 \dots n_g.$$

Thus, the datum of these generators, or of the semigroup, is equivalent to the datum of the Puiseux characteristic of $(X, 0)$, or of its topological type. The proof relies on a formula of Max Noether which computes the *contact exponent* $\frac{(C,D)_0}{m_0(D)}$ of two analytic branches at the origin in terms of the coincidence of their Puiseux expansions in fractional powers of x.
The semigroups coming from plane branches are characterized among all semigroups of analytically irreducible germs of curves by the following two properties:

1) $n_i\overline{\beta_i} \in \left\langle \overline{\beta_0}, \dots, \overline{\beta_{i-1}} \right\rangle$
2) $n_i\overline{\beta_i} < \overline{\beta}_{i+1}$.

That the semigroups of plane branches have these properties follows from the induction formula and the inequalities $\beta_i < \beta_{i+1}$. The converse can be proved by the construction outlined below (see [7]).

Conversely, given a semigroup Γ in $\mathbf{N}$ with finite complement, we can associate to it an analytic (in fact algebraic) curve, called the *monomial curve* associated to Γ. If $\Gamma = \left\langle \overline{\beta_0}, \overline{\beta_1}, \dots, \overline{\beta_g} \right\rangle$, the monomial curve C^Γ is described parametrically by

$$u_0 = t^{\overline{\beta_0}}, \quad u_1 = t^{\overline{\beta_1}}, \quad \dots, \quad u_g = t^{\overline{\beta_g}}.$$

On the other hand, the relations 1) above mean that there exist natural numbers $\ell_i^{(j)}$ satisfying

$$1') \qquad \begin{aligned} n_1\overline{\beta_1} &= \ell_0^{(1)}\overline{\beta_0}, \\ n_2\overline{\beta_2} &= \ell_0^{(2)}\overline{\beta_0} + \ell_1^{(2)}\overline{\beta_1}, \\ &\vdots \\ n_j\overline{\beta_j} &= \ell_0^{(j)}\overline{\beta_0} + \dots + \ell_{j-1}^{(j)}\overline{\beta_{j-1}} \\ &\vdots \\ n_g\overline{\beta_g} &= \ell_0^{(g)}\overline{\beta_0} + \dots + \ell_{g-1}^{(g)}\overline{\beta_{g-1}}. \end{aligned}$$

These relations translate into equations for the curve $C^\Gamma \subset \mathbf{C}^{g+1}$; since $u_i = t^{\overline{\beta_i}}$, our curve satisfies the g equations

$$f_j \;=\; u_j^{n_j} - u_0^{\ell_0^{(j)}} u_1^{\ell_1^{(j)}} \dots u_{j-1}^{\ell_{j-1}^{(j)}} = 0, \quad 1 \le j \le g,$$

and it can be shown that they actually define $C^\Gamma \subset \mathbf{C}^{g+1}$, so that if Γ is the semigroup of a plane branch, C^Γ is a complete intersection.

The relations $1'$) are not uniquely determined, but there is a canonical choice: dividing each $\ell_k^{(j)}$ by n_k we can request that for every $k \ge 1$ we have $\ell_k^{(j)} < n_k$; it is the choice we shall make in the sequel.

Remark that if we give to u_i the weight $\overline{\beta_i}$, the i-th equation is homogeneous of degree $n_i\overline{\beta_i}$.

The connection between a plane curve C having semigroup Γ and the monomial curve is much more precise and interesting than the formal relation we have just seen; by small deformations of the monomial curve one obtains all the branches with the same semigroup. In fact, the best way to understand all branches with semigroup Γ is to consider the not necessarily plane curve C^Γ (C^Γ is plane if and only if C has only one characteristic exponent).

By definition of Γ, there are elements $\xi_q \in \mathcal{O}$ with $\nu(\xi_q) = \overline{\beta_q}$. We can write these elements in $\mathbf{C}\{t\}$ as

$$\xi_q = t^{\overline{\beta_q}} + \sum_{j > \overline{\beta_q}} \gamma_{q,j} t^j .$$

Let us consider the one-parameter family of parametrizations

$$u_0 = t^m\,, \quad u_1 = t^{\overline{\beta_1}} + \sum_{j>\overline{\beta_1}} v^{j-\overline{\beta_1}} \gamma_{1,j} t^j\,, \quad \dots\,, \quad u_g = t^{\overline{\beta_g}} + \sum_{j>\overline{\beta_g}} v^{j-\overline{\beta_g}} \gamma_{g,j} t^j\,.$$

The reader can check that for $v \neq 0$, the curve thus described is isomorphic to our original curve C. (hint: make the change of parameter $t = vt'$ in the ξ_q and the change of coordinates $u_j = v^{\overline{\beta_j}} u'_j$, and remember the definition of the ξ_j). For $v = 0$, we have the parametric description of the monomial curve.

So we have, in fact, described a map $\mathbf{C} \times \mathbf{C} \to \mathbf{C}^{g+1} \times \mathbf{C}$ which induces the identity on the second factors (with coordinate v). The image of this map is a surface, which is the total space of a deformation of the monomial curve, all of its fibers except the one for $v = 0$ being isomorphic to our plane curve C. It follows that the monomial curve is a specialization, in this family, of our plane curve. In this specialization the multiplicity and the semigroup remain constant; in a rather precise sense it is an equisingular specialization, or one may say that *the plane curve is an equisingular deformation of the monomial curve with the same semigroup.*

The same phenomenon can be also observed in the language of equations rather than parametrizations. Let us consider a one parameter family of equations for curves in $\mathbf{C}^{g+1}$, of the form

$$\begin{array}{ll} F_1 = & u_1^{n_1} - u_0^{\ell_0^{(1)}} - v u_2 = 0\,, \\ F_2 = & u_2^{n_2} - u_0^{\ell_0^{(2)}} u_1^{\ell_1^{(2)}} - v u_3 = 0\,, \\ \vdots & \qquad \vdots \\ F_{g-1} = & u_{g-1}^{n_{g-1}} - u_0^{\ell_0^{(g-1)}} u_1^{\ell_1^{(g-1)}} \ldots u_{g-2}^{\ell_{g-2}^{(g-1)}} - v u_g = 0\,, \\ F_g = & u_g^{n_g} - u_0^{\ell_0^{(g)}} u_1^{\ell_1^{(g)}} \ldots u_{g-1}^{\ell_{g-1}^{(g)}} = 0\,. \end{array}$$

For $v = 0$ we get the equations of the monomial curve, and for $v \neq 0$ we get a curve which has semigroup Γ; this is a general heuristic principle of equisingularity: we have added to each equation of the monomial curve, homogeneous of degree $n_i\overline{\beta_i}$, a perturbation of degree $\overline{\beta_{i+1}} > n_i\overline{\beta_i}$, and this should not change the equisingularity class (the perturbation is "small" compared to the equation).

Notice that for each fixed $v \neq 0$ the curve described by the above equations is a plane curve: for simplicity take $v = 1$; then use the first equation to compute $u_2 = u_1^{n_1} - u_0^{\ell_0^{(1)}}$, substitute this in the next equation, and use this to compute u_3 as a function of u_0, u_1, and so on. Finally, the last equation gives us the equation of a plane curve of the form

$$\left(\cdots\left(\left(u_1^{n_1} - u_0^{\ell_0^{(1)}}\right)^{n_2} - u_0^{\ell_0^{(2)}} u_1^{\ell_1^{(2)}}\right)^{n_3} - \cdots\right)^{n_g} - u_0^{\ell_0^{(g)}} u_1^{\ell_1^{(g)}} \left(u_1^{n_1} - u_0^{\ell_0^{(1)}}\right)^{\ell_2^{(g)}} \cdots = 0\,.$$

The first consequence (see [7]) is that we can produce explicitely the equation of a plane curve with given characteristic exponents: compute the semigroup and its generators, and then write the equation above.

A more important fact is that one can show (*loc. cit*) that any plane curve with a given semigroup appears up to isomorphism as a fiber in a deformation depending on a finite number of parameters: it is a deformation of the monomial curve obtained by adding to the j-th equation a polynomial in the u_i's of order $> n_j\overline{\beta_j}$, where u_{j+1} appears linearly if $j < g$, and these polynomials can in principle be explicitely computed.

In fact it is shown in [7] that we can in this manner produce equations for *all* branches having the same semigroup (or equisingularity type) up to an analytic isomorphism.

In view of the constancy of the jacobian Newton polygon for equisingular families of plane branches, it is plausible that the special family above represents all degenerations of plane branches to the associated monomial curve, as far as the variation of jacobian Newton polyhedra are concerned. We shall therefore make computations for this family.

2 The Discriminant

Set $v = (v_1, \dots, v_{g-1})$, and consider the map

$$\begin{array}{rcl} \phi : \mathbf{C}^{g+1} \times \mathbf{C}^{g-1} & \longrightarrow & \mathbf{C}^{g+1} \times \mathbf{C}^{g-1} \\ (u_0, \dots, u_g, v) & \longmapsto & (u_0, F_1, \dots, F_g, v) \end{array}$$

in the coordinates $(t_0, \dots, t_g, v)$ on the right-hand copy of $\mathbf{C}^{g+1} \times \mathbf{C}^{g-1}$.

Let us first verify that the morphism ϕ is flat. We shall see below that we have even better. Indeed, it is a map between two non singular spaces, whose fiber over 0 is a complete intersection. The flatness follows. Since the special fiber has an isolated singularity at the origin, the critical subspace $\mathcal{C}$ of ϕ is finite over its image in $\mathbf{C}^{g+1} \times \mathbf{C}^{g-1}$, at least locally, by the Weierstrass preparation theorem. This image (or at least its germ at 0) is then a complex analytic space which is by definition the discriminant of the map ϕ ([6], §1).

Let us now compute the discriminant of the mapping ϕ as the image of critical subspace using the Fitting ideal of the algebra of the critical subspace $\mathcal{C}$ as in [6].

The critical subspace is defined by the ideal generated by the coefficients of the differential form

$$dF_1 \wedge \cdots \wedge dF_g \wedge dt_0 \wedge dt_1 \cdots \wedge dt_g \wedge dv_1 \cdots \wedge dv_{g-1}.$$

Since $F_i = u_i^{n_i} - u_0^{\ell_0^{(i)}} \cdots u_{i-1}^{\ell_{i-1}^{(i)}} - v_i u_{i+1}$ for $j < g$ and $F_g = u_g^{n_g} - u_0^{\ell_0^{(g)}} \cdots u_{g-1}^{\ell_{g-1}^{(g)}}$, we see that a generator for the ideal of the critical subspace can be taken of the form

$$\mathcal{C} = \beta_0 u_1^{n_1 - 1} \cdots u_g^{n_g - 1} - \sum_{\alpha} c_\alpha v_1^{\alpha_1} \cdots v_{g-1}^{\alpha_{g-1}} u_0^{m_0(\alpha)} \cdots u_g^{m_g(\alpha)}$$

where each α_i is 0 or 1 and $(m_1(\alpha), \dots, m_g(\alpha)) \neq (n_1 - 1, \dots, n_g - 1)$.

Lemma 2.1. *Giving to the variable u_i the weight $\overline{\beta_i}$ and to v_j the (negative) weight $\beta_j - \beta_{j+1}$, the polynomial $\mathcal{C}$ is homogeneous of degree $\sum_{i=1}^{g} (n_i - 1)\overline{\beta}_i$.*

Proof. The statement follows directly from the homogeneity of the polynomials F_i and the computation of $\mathcal{C}$ as a jacobian determinant. □

Let us denote by $I_{\mathcal{C}}$ the ideal of $\mathbf{C}[u_0, \dots, u_g, t_0, \dots, t_g, v_1, \dots, v_{g-1}]$ generated by $(u_0 - t_0, F_1 - t_1, \dots, F_g - t_g, \mathcal{C})$. The generators constitute a regular sequence since their initial forms involve different variables.

Let us consider the $\mathbf{C}[t_0, \dots, t_g, v_1, \dots, v_{g-1}]$-module

$$\mathcal{O}_{\mathcal{C}} = \mathbf{C}[u_0, \dots, u_g, t_0, \dots, t_g, v_1, \dots, v_{g-1}]/I_{\mathcal{C}}.$$

Lemma 2.2. *The* $\mathbf{C}[t_0, \dots, t_g, v_1, \dots, v_{g-1}]$*-module*

$$\mathbf{C}[u_0, \dots, u_g, t_0, \dots, t_g, v_1, \dots, v_{g-1}]/(u_0 - t_0, F_1 - t_1, \dots, F_g - t_g)$$

is free and generated by the β_0 *images of the monomials* $u_1^{i_1} \cdots u_g^{i_g}$ *with* $0 \leq i_k \leq n_k - 1$.

Proof. It follows directly for the form of the equations because each of the equations expresses the corresponding $u_i^{n_i}$ as a linear combination with coefficients in $\mathbf{C}[t_0, \dots, t_g, v_1, \dots, v_{g-1}]$ of our generating monomials. □

If we identify t_0 and u_0 and set

$$N = \mathbf{C}[t_0, \dots, t_g, v_1, \dots, v_{g-1}, u_1, \dots, u_g]/(F_1 - t_1, \dots, F_g - t_g),$$

the $\mathbf{C}[t_0, \dots, t_g, v_1, \dots, v_{g-1}]$-module $\mathcal{O}_{\mathcal{C}}$ is the cokernel of the map of multiplication by $\mathcal{C}$ in N. By [6] again we have:

Proposition 2.3. *1. The discriminant* $\operatorname{Disc}(\phi)$ *of the morphism* ϕ *is (up to multiplication by a nonzero constant) the determinant of the matrix* M *of the multiplication in the free* $\mathbf{C}[t_0, \dots, t_g, v_1, \dots, v_{g-1}]$*-module* N *by the equation* $\mathcal{C}$ *of the critical subspace.*

2. Giving to the variable t_j *the weight* $n_j\overline{\beta_j}$ *and to* v_k *the (negative) weight* $\beta_k - \beta_{k+1}$*, the polynomial* $\Delta = \operatorname{Disc}(\phi) \in \mathbf{C}[t_0, \dots, t_g, v_1, \dots, v_{g-1}]$ *is homogeneous of degree*

$$\deg\Delta = \overline{\beta_0}(\sum_{i=1}^{g}(n_i - 1)\overline{\beta_i}).$$

Proof. The first part of the assertion follows directly from §1 of [6]. For the second part, first note that if we give to u_i the weight $\overline{\beta}_i$, the free $\mathbf{C}[t_0, \dots, t_g, v_1, \dots, v_{g-1}]$-module N is graded when the variables are given the weights of the proposition since the equations F_i are homogeneous.

We now apply Lemma 1 of §1 of [6]; in view of Lemma 2.1, if we want the morphism of multiplication by $\mathcal{C}$ to be homogeneous of degree 0, setting $A = \mathbf{C}[t_0, \dots, t_g, v_1, \dots, v_{g-1}]$ and $d_{i_1,\dots,i_g} = \sum_{k=1}^{g} i_k\overline{\beta}_k$, we may write the first copy of N as

$$N = \bigoplus_{i_1,\dots,i_g} A[d_{i_1,\dots,i_g}],$$

where $A[s]$ is the A-module A regraded (shifted) by giving 1 the degree s, and then we must write the second copy of N as

$$N = \bigoplus_{i_1,\dots,i_g} A[d_{i_1,\dots,i_g} - \sum_{k=1}^{g}(n_k - 1)\overline{\beta}_k]$$

where $0 \leq i_k \leq n_k - 1$. The result follows immediately from *loc.cit.* which states that the degree of the determinant is the sum of the differences of the shifts in the first and second copies over all values of $i_1, \ldots, i_g$. $\square$

Remark 2.4. • In what follows, we shall constantly use the fact that the Fitting image definition of the discriminant commutes with base change and in particular with restriction over subspaces of the target space (see [6]).

• We denote by τ_i the exponent of t_i in a monomial, and by υ_j the exponent of v_j; then Proposition 2.3 means that all the monomials appearing in the equation of the discriminant satisfy (setting $n_0 = 1$):

$$\sum_{i=0}^{g} n_i \overline{\beta}_i \tau_i + \sum_{j=1}^{g-1} (\beta_j - \beta_{j+1}) \upsilon_j = \overline{\beta}_0 \Big(\sum_{i=1}^{g} (n_i - 1) \overline{\beta}_i\Big).$$

3 Curves with Two Characteristic Pairs

The purpose of this section is the computation of the Newton polyhedron in the coordinates (t_0, t_1, t_2) of the discriminant of the morphism ϕ in the case of two characteristic pairs, both for $v = 0$ and v nonzero.

If $g = 2$ the morphism ϕ is defined by the equations

$$u_0 - t_0 = 0\,, \quad u_1^{n_1} - u_0^{\ell_0^{(1)}} - v u_2 - t_1 = 0\,, \quad u_2^{n_2} - u_0^{\ell_0^{(2)}} u_1^{\ell_1^{(2)}} - t_2 = 0\,. \qquad (1)$$

Identifying u_0 with t_0, we have the equations

$$u_1^{n_1} - t_0^{\ell_0^{(1)}} - v u_2 - t_1 = 0\,, \quad u_2^{n_2} - t_0^{\ell_0^{(2)}} u_1^{\ell_1^{(2)}} - t_2 = 0\,,$$

and the equation of the critical subspace is

$$\mathcal{C} = \beta_0 u_1^{n_1 - 1} u_2^{n_2 - 1} - \ell_1^{(2)} v t_0^{\ell_0^{(2)}} u_1^{\ell_1^{(2)} - 1} = 0\,.$$

In view of Proposition 2.3, we have to compute the matrix of multiplication by $\mathcal{C}$ in the basis $e_{i,j} = u_1^i u_2^j$, $0 \leq i \leq n_1 - 1$, $0 \leq j \leq n_2 - 1$ for the $\mathbf{C}[t_0, t_1, t_2, v]$-module $N = \mathbf{C}[t_0, t_1, t_2, v, u_1, u_2]/(F_1 - t_1, F_2 - t_2)$.

Remark 3.1. If $\ell_1^{(2)} = 0$, which is the case for example if $\Gamma = \langle 6, 8, 27 \rangle$, the critical subspace is $\mathcal{C} = \beta_0 u_1^{n_1 - 1} u_2^{n_2 - 1}$, and the computation is simpler but has to be conducted a little differently, introducing $B = t_0^{\ell_0^{(2)}} + t_2$ and the conclusion is the same. *We will present the computations in the case where* $\ell_1^{(2)} \geq 1$.

In order to write down the matrix of a presentation of the $\mathbf{C}\{t_0, t_1, t_2, v\}$-module $\mathcal{O}_{\mathcal{C}}$, we have to compute modulo the ideal $(F_1 - t_1, F_2 - t_2)$ the effect of the multiplication by $\mathcal{C}$ on the generators $e_{i,j}$. The matrix can be presented by blocks, each block corresponding to a fixed value of j. Our matrix is constructed as an $n_2 \times n_2$ matrix of blocks of size n_1; when j is fixed and we fix also the block j' in which we look at the relations, the situation can be represented by an $n_1 \times n_1$-matrix $M_{j,j'}$ whose elements are indexed by (i, i').

For $j = 0$, if $i = 0$, the relation is the equation $\mathcal{C} = 0$, which we write as

$$e_{n_1-1,n_2-1} - (1-c)vT_0 e_{\ell_1^{(2)}-1,0} = 0,$$

where $c = 1 - \frac{\ell_1^{(2)}}{\beta_0}$ and we set for simplicity $T_0 = t_0^{\ell_0^{(2)}}$.

For $j = 0$ and $1 \leq i \leq n_1 - 1$, we have two cases: if $i < n_1 - \ell_1^{(2)} + 1$, we obtain the relation:

$$Ae_{i-1,n_2-1} + vt_2 e_{i-1,0} + cvT_0 e_{\ell_1^{(2)}+i-1,0} = 0,$$

where we set for simplicity $A = t_0^{\ell_0^{(1)}} + t_1$.

If $i \geq n_1 - \ell_1^{(2)} + 1$, we obtain the relation:

$$Ae_{i-1,n_2-1} + vt_2 e_{i-1,0} + cvT_0 A e_{\ell_1^{(2)}+i-1-n_1,0} + cv^2 T_0 e_{\ell_1^{(2)}+i-1-n_1,1} = 0.$$

This gives us our first line of blocks: For $j = 0$ the relations involve elements in the blocks $j' = 0$, $j' = 1$ and $j' = n_2 - 1$.

For $j' = 0$ the matrix is:

$$\mathbf{M_{0,0}} = \begin{pmatrix}
0 & 0 & \dots & -(1-c)vT_0 & 0 & 0 & \dots & 0 \\
vt_2 & 0 & \dots & & cvT_0 & 0 & \dots & 0 \\
0 & vt_2 & \dots & 0 & & cvT_0 & \dots & \\
\vdots & & & & \vdots & & \vdots & \\
 & & & & & & & cvT_0 \\
cvAT_0 & 0 & \dots & vt_2 & \dots & & 0 & 0 \\
\vdots & & & & & & \vdots & \vdots \\
0 & cvAT_0 & 0 & \dots & \dots & vt_2 & 0 & 0 \\
0 & \dots & cvAT_0 & \dots & 0 & \dots & vt_2 & 0
\end{pmatrix}$$

where the nonzero elements are aligned on parallels to the second diagonal, the first nonzero element of the first line is in column $\ell_1^{(2)}$ and the last nonzero element of the first column is in line $n_1 - \ell_1^{(2)} + 2$.

For $j' = 1$ the matrix is:

$$\mathbf{M_{0,1}} = \begin{pmatrix} 0 & 0 & \dots & & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & & 0 & \dots & \\ \vdots & & & & \vdots & & \vdots & \\ & & & & & & & 0 \\ cv^2T_0 & 0 & 0 & \dots & & 0 & 0 & \\ \vdots & & & & & & \vdots & \vdots \\ 0 & cv^2T_0 & 0 & \dots & \dots & \dots & 0 & 0 \\ 0 & \dots & cv^2T_0 & \dots & 0 & \dots & 0 & 0 \end{pmatrix}$$

where the nonzero elements are aligned on parallels to the second diagonal, the first and only nonzero element of the last line is in column $\ell_1^{(2)} - 1$ and the last nonzero element of the first column is in line $n_1 - \ell_1^{(2)} + 2$.

For $j' = n_2 - 1$ the matrix is

$$\mathbf{M_{0,n_2-1}} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ A & 0 & \dots & & 0 & 0 & \dots & 0 \\ 0 & A & \dots & 0 & & 0 & \dots & \\ \vdots & & & & \vdots & & \vdots & \\ & & & & & & & 0 \\ 0 & 0 & \dots & A & \dots & & 0 & 0 \\ \vdots & & & & & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & A & 0 & 0 \\ 0 & \dots & 0 & \dots & 0 & \dots & A & 0 \end{pmatrix}.$$

For $1 \leq j < n_2 - 1$, the relations involve elements in the blocks $j' = j - 1$, $j' = j$ and $j' = j + 1$. They are as follows:

- For $i = 0$, we have

$$t_2 e_{n_1-1,j-1} + cvT_0 e_{\ell_1^{(2)}-1,j} + AT_0 e_{\ell_1^{(2)}-1,j-1} = 0.$$

- For $0 < i < n_1 - \ell_1^{(2)} + 1$, we have

$$At_2 e_{i-1,j-1} + vt_2 e_{i-1,j} + AT_0 e_{i+\ell_1^{(2)}-1,j-1} + cvT_0 e_{i+\ell_1^{(2)}-1,j} = 0.$$

- For $i \geq n_1 - \ell_1^{(2)} + 1$, we have:

$$\begin{aligned} At_2 e_{i-1,j-1} + vt_2 e_{i-1,j} + A^2 T_0 e_{i+\ell_1^{(2)}-n_1-1,j-1} \\ +(1+c)vAT_0 e_{i+\ell_1^{(2)}-n_1-1,j} + cv^2 T_0 e_{i+\ell_1^{(2)}-n_1-1,j+1} = 0. \end{aligned}$$

For $j' = j-1$, the matrix is:

$$\mathbf{M_{j,j-1}} = \left(\begin{array}{cccccccc}
0 & 0 & \dots & AT_0 & 0 & \dots & 0 & t_2 \\
At_2 & 0 & \dots & & AT_0 & 0 & \dots & 0 \\
0 & At_2 & \dots & 0 & & AT_0 & \dots & \\
\vdots & & & & \vdots & & \vdots & \\
 & & & & & & & AT_0 \\
A^2T_0 & 0 & \dots & At_2 & \dots & & 0 & \\
\vdots & & & & & & & \\
0 & & 0 & \dots & \dots & At_2 & 0 & 0 \\
0 & \dots & A^2T_0 & \dots & 0 & & At_2 & 0
\end{array}\right)$$

where the nonzero elements are aligned on parallels to the second diagonal, the first nonzero element of the first line is in column $\ell_1^{(2)}$ and the last nonzero element of the first column is in line $n_1 - \ell_1^{(2)} + 2$.

For $j' = j$, the matrix is:

$$\mathbf{M_{j,j}} = \left(\begin{array}{cccccccc}
0 & 0 & \dots & cvT_0 & 0 & 0 & \dots & 0 \\
vt_2 & 0 & \dots & & cvT_0 & 0 & \dots & 0 \\
0 & vt_2 & \dots & 0 & & cvT_0 & \dots & 0 \\
\vdots & & & & \vdots & & \vdots & \vdots \\
 & & & & & & & cvT_0 \\
(c{+}1)vAT_0 & 0 & \dots & vt_2 & \dots & & 0 & 0 \\
\vdots & & & & & & & \vdots \\
0 & (c{+}1)vAT_0 & 0 & \dots & \dots & vt_2 & 0 & 0 \\
0 & \dots & (c{+}1)vAT_0 & \dots & 0 & & vt_2 & 0
\end{array}\right)$$

where the nonzero elements are aligned on parallels to the second diagonal, the first nonzero element of the first line is in column $\ell_1^{(2)}$ and the last nonzero element of the first column is in line $n_1 - \ell_1^{(2)} + 2$.

For $j' = j+1$, the matrix is:

$$\mathbf{M_{j,j+1}} = \left(\begin{array}{cccccccc}
0 & 0 & \dots & & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & & 0 & \dots & \\
\vdots & & & & \vdots & & \vdots & \\
 & & & & & & & 0 \\
cv^2T_0 & 0 & \dots & 0 & \dots & & 0 & 0 \\
\vdots & & & & & & & \\
0 & cv^2T_0 & 0 & \dots & \dots & \dots & 0 & 0 \\
0 & \dots & cv^2T_0 & \dots & 0 & & 0 & 0
\end{array}\right)$$

where the nonzero elements are aligned on parallels to the second diagonal, the first and only nonzero element of the last line is in column $\ell_1^{(2)} - 1$.

For $j = n_2 - 1$, the relations involve elements in the blocks $j' = n_2 - 1$, $j' = n_2 - 2$, $j' = 0$ and $j' = 1$. They are as follows:

- For $i = 0$, we have

$$t_2 e_{n_1-1,n_2-2} + cvT_0 e_{\ell_1^{(2)}-1,n_2-1} + AT_0 e_{\ell_1^{(2)}-1,n_2-2} = 0.$$

- For $i > 0$ and $i \leq n_1 - \ell_1^{(2)}$, we have:

$$At_2 e_{i-1,n_2-2} + AT_0 e_{\ell_1^{(2)}+i-1,n_2-2} + vt_2 e_{i-1,n_2-1} + cvT_0 e_{\ell_1^{(2)}+i-1,n_2-1} = 0.$$

- For $n_1 - \ell_1^{(2)} + 1 \leq i \leq 2(n_1 - \ell_1^{(2)})$, we have:

$$\begin{aligned} At_2 e_{i-1,n_2-2} \,&+ A^2 T_0 e_{\ell_1^{(2)}+i-n_1-1,n_2-2} + (1+c)AT_0 v e_{\ell_1^{(2)}+i-n_1-1,n_2-1} \\ &+ vt_2 e_{i-1,n_2-1} + cv^2 t_2 e_{\ell_1^{(2)}+i-n_1-1,0} + cv^2 T_0 e_{2\ell_1^{(2)}+i-n_1-1,0} = 0. \end{aligned}$$

- For $i \geq 2(n_1 - \ell_1^{(2)}) + 1$, we have:

$$\begin{aligned} At_2 e_{i-1,n_2-2} \,&+ A^2 T_0 e_{\ell_1^{(2)}+i-n_1-1,n_2-2} + (1+c)vAT_0 e_{\ell_1^{(2)}+i-n_1-1,n_2-1} \\ &+ vt_2 e_{i-1,n_2-1} + cv^2 t_2 e_{\ell_1^{(2)}+i-n_1-1,0} + cv^2 AT_0 e_{2\ell_1^{(2)}+i-2n_1-1,0} \\ &+ cv^3 T_0 e_{2\ell_1^{(2)}+i-2n_1-1,1} = 0. \end{aligned}$$

For $j' = 0$, the matrix is:

$$\mathbf{M_{n_2-1,0}} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & & \dots \\ & & & 0 & & \\ cv^2 t_2 & 0 & \dots & & cv^2 T_0 & 0 \\ \vdots & cv^2 t_2 & & & \vdots & \vdots \\ 0 & \dots & & & & cv^2 T_0 \\ cv^2 AT_0 & & & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & & \dots \\ 0 & cv^2 AT_0 & \dots & cv^2 t_2 & & 0 \end{pmatrix}$$

where the nonzero elements are aligned on parallels to the second diagonal, the first line with nonzero entries is the line with number $n_1 - \ell_1^{(2)} + 2$, and its last nonzero element is in column $\ell_1^{(2)}$. The last nonzero element of the first column is in line $2(n_1 - \ell_1^{(2)} + 1)$.

For $j' = 1$, the matrix is:

$$\mathbf{M_{n_2-1,1}} = \begin{pmatrix} 0 & 0 & \dots & & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & & 0 & \dots & \\ \vdots & & & & \vdots & & \vdots & \\ & & & & & & & 0 \\ cv^3T_0 & 0 & \dots & 0 & \dots & & 0 & 0 \\ \vdots & & & & & & & \\ 0 & cv^3T_0 & 0 & \dots\dots\dots & & & 0 & 0 \\ 0 & \dots & cv^3T_0 & \dots & 0 & & 0 & 0 \end{pmatrix}$$

where the first nonzero element of the first column is on line $2(n_1 - \ell_1^{(2)} + 1)$.
For $j' = n_2 - 2$, the matrix is:

$$\mathbf{M_{n_2-1,n_2-2}} = \begin{pmatrix} 0 & 0 & \dots & AT_0 & 0 & \dots & 0 & t_2 \\ At_2 & 0 & \dots & & AT_0 & 0 & \dots & 0 \\ 0 & At_2 & \dots & 0 & & AT_0 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots & \vdots \\ & & & & & & & AT_0 \\ A^2T_0 & 0 & \dots & At_2 & \dots & & 0 & 0 \\ \vdots & & & & & & \vdots & \\ 0 & A^2T_0 & 0 & \dots & \dots & At_2 & 0 & 0 \\ 0 & \dots & A^2T_0 & \dots & 0 & 0 & At_2 & 0 \end{pmatrix}$$

where the first nonzero element of the first line is in column $\ell_1^{(2)}$ and the last nonzero element of the first column is in line $n_1 - \ell_1^{(2)} + 2$.
For $j' = n_2 - 1$, the matrix is:

$$\mathbf{M_{n_2-1,n_2-1}} = \begin{pmatrix} 0 & 0 & \dots & cvT_0 & & 0 & \dots & 0 \\ vt_2 & 0 & \dots & & cvT_0 & 0 & \dots & 0 \\ 0 & vt_2 & \dots & 0 & & cvT_0 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots & \\ & & & & & & & cvT_0 \\ (1+c)vAT_0 & 0 & \dots & vt_2 & \dots & & 0 & \\ \vdots & \vdots & & & & & & \\ 0 & (1+c)vAT_0 & 0 & \dots & \dots & vt_2 & 0 & 0 \\ 0 & \dots & (1+c)vAT_0 & \dots & 0 & & vt_2 & 0 \end{pmatrix}$$

where the first nonzero element of the first line is in column $\ell_1^{(2)}$ and the last nonzero element of the first column is in line $n_1 - \ell_1^{(2)} + 2$.

Finally, the matrix M of the presentation of the $\mathbf{C}[t_0, t_1, t_2, v]$-module $\mathcal{O}_{\mathcal{C}}$ is described by the blocks $M_{j,j'}$:

$$\mathbf{M} = \begin{pmatrix} M_{0,0} & M_{0,1} & 0 & \ldots & & 0 & 0 & M_{0,n_2-1} \\ \vdots & \vdots & \ldots & \vdots & & \ldots & & \vdots \\ \vdots & \vdots & \ldots & \vdots & & \ldots & & \vdots \\ 0 & & M_{j-1,j-2} & M_{j-1,j-1} & M_{j-1,j} & & & 0 \\ 0 & & & M_{j,j-1} & M_{j,j} & M_{j,j+1} & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & & \ldots & & \vdots \\ \vdots & \vdots & \ldots & \vdots & & \ldots & & \vdots \\ M_{n_2-1,0} & M_{n_2-1,1} & 0 & \ldots & 0 & & M_{n_2-1,n_2-2} & M_{n_2-1,n_2-1} \end{pmatrix}.$$

We are going to get information about the determinant of the matrix of relations between the generators $e_{i,j}$ using this decomposition into blocks.

Lemma 3.2. *For $v = 0$, the determinant of the matrix M is given by:*

$$\det M = A^{n_1-1} \left(\det M_{j,j-1}\right)^{n_2-1} .$$

Proof. For $v = 0$ the only nonzero blocks are M_{0,n_2-1} and the $M_{j,j-1}$ which are all equal. By expanding the determinant of M successively along the last n_1 columns, we find that it is equal to A^{n_1-1} times the determinant of the matrix $\overline{M}$ of size $\beta_0 - n_1$ obtained by deleting the first n_1 lines and the last n_1 columns of M. That matrix is subdivided into blocks $M_{j,j'}$, among which the only nonzero ones are the $n_2 - 1$ blocks $M_{j,j-1}$, which are all equal; therefore they commute and we can compute the determinant of $\overline{M}$ as the product of the determinants of the blocks (see [1], §9, Lemme 1). The result follows. $\square$

Lemma 3.3. *For $v = 0$ and $t_2 = 0$, the discriminant is equal to*

$$T_0^{n_1(n_2-1)} A^{n_2(n_1-1)+\ell_1^{(2)}(n_2-1)}.$$

As a consequence, the Newton polyhedron of the discriminant of ϕ contains as an edge the segment joining the two points

$$P_1 = (n_1(n_2-1)\ell_0^{(2)}, n_2(n_1-1) + \ell_1^{(2)}(n_2-1), 0)$$

and

$$P_2 = ((n_1-1)\overline{\beta}_1 + (n_2-1)\overline{\beta}_2, 0, 0).$$

Proof. The first part of the statement follows directly from Lemma 3.2. Since the Newton polyhedron of the discriminant is necessarily entirely on one side of any of the coordinate hyperplanes $\tau_i = 0$ or $v = 0$, its intersection with one of them is necessarily a face. If, upon intersecting with the other hyperplane, we find a segment, that segment is necessarily an edge. We apply this to $v = 0$ and $\tau_2 = 0$ and use the expressions of the $n_i\overline{\beta}_i$ in terms of $\overline{\beta}_k$ with $k < i$ to compute the coordinates of the points P_i. This proves the second part. $\square$

Lemma 3.4. *For $v = 0$ the Newton polyhedron of the discriminant of $\phi|_{v=0}$ lies entirely on one side of ("above") the hyperplane*

$$n_2\tau_0 + \overline{\beta}_1\tau_1 = (n_1 - 1)n_2\overline{\beta}_1\,.$$

Proof. Using the result of Lemma 3.2 and the expression of $M_{j,j-1}$, we see that it suffices to show that the exponents (τ_0, τ_1) of the monomials appearing in the determinant of M and arising from the term $A^{n_1-1}t_2^{n_2}$ in the determinant of $M_{j,j-1}$ satisfy the inequality $n_2\tau_0+\overline{\beta}_1\tau_1 \geq (n_1-1)n_2\overline{\beta}_1$. Indeed all the other monomials contain higher exponents of t_0 or t_1. This amounts to studying the exponents of t_0 and t_1 appearing in the expansion of $A^{n_1-1}A^{(n_1-1)(n_2-1)} = A^{(n_1-1)n_2}$. But these are terms $t_0^{i\ell_0^{(1)}}t_1^j$ with $i + j = (n_1 - 1)n_2$. Substituting in the equation of our hyperplane and remembering that by definition $n_2\ell_0^{(1)} = \overline{\beta}_1$ gives the result. □

Lemma 3.5. *The Newton polyhedron of the discriminant of ϕ contains as an edge the segment joining the two points*

$$P_2 = ((n_1 - 1)\overline{\beta}_1 + (n_2 - 1)\overline{\beta}_2, 0, 0) \quad \text{and} \quad P_3 = ((n_1 - 1)\overline{\beta}_1, 0, n_1(n_2 - 1)).$$

Proof. Since the Newton polyhedron of the discriminant is contained in the hyperplane of homogeneity, its intersection with the coordinate plane $\tau_1 = 0, v = 0$ is contained in a line. By convexity this line is a segment. By the same argument as above it is an edge of the Newton polyhedron of the discriminant. We are going to determine its extremities by seeking the points of maximum and minimum value of τ_0. We apply this to $v = 0$ and $\tau_1 = 0$, which means that we compute the expression of the discriminant for $v = 0$ and $t_1 = 0$ using lemma 3.2 and the expression of $M_{j,j-1}$ to seek the maximum value of the exponent of t_0, which is obtained by taking the product of the AT_0 and A^2T_0 in the expansion of the determinant of $M_{j,j-1}$, and its minimum value, obtained by taking the term $A^{n_1-1}t_2^{n_1}$ in that expansion. Finally we use the expressions of the $n_i\overline{\beta}_i$ in terms of $\overline{\beta}_k$ with $k < i$ to compute the coordinates of the points P_i. □

Lemma 3.6. *The Newton polyhedron of the discriminant of ϕ contains as an edge the segment $\overline{P_3P_4}$, where $P_4 = (0, n_2(n_1 - 1), n_1(n_2 - 1))$.*

Proof. Again, use Lemma 3.2 and observe that for $v = t_0 = 0$ the determinant of $M_{j,j-1}$ is equal to the monomial $t_1^{n_1-1}t_2^{n_1}$. Use the expressions of the $n_i\overline{\beta}_i$ in terms of $\overline{\beta}_k$ with $k < i$ to compute the coordinates of the point P_4. On the other hand, it follows from Lemma 3.2 that the Newton polyhedron for $v = 0$ is entirely on one side of the hyperplane $\tau_2 = n_1(n_2 - 1)$. It meets it in the two points P_3, P_4 which are in different coordinate planes, therefore along the edge $\overline{P_3P_4}$. □

Lemma 3.7. *The segment $\overline{P_1P_4}$ is an edge of the Newton polyhedron of ϕ and the plane containing it and parallel to the τ_1-axis supports a non-compact face of the Newton polyhedron.*

Proof. The equation of the hyperplane parallel to the t_1-axis and containing $\overline{P_1P_4}$ is:
$$\mathcal{H} : \tau_0 + \ell_0^{(2)}\tau_2 = n_1(n_2-1)\ell_0^{(2)}.$$
The expression given above for the matrix M shows that the products appearing in the expansion of its determinant are all up to a constant factor of the form $v^\delta A^\alpha T_0^\beta t_2^\gamma$ with $\beta+\gamma \geq n_1(n_2-1)$. The result follows because this implies by a direct computation that the Newton polyhedron of the discriminant of ϕ is entirely on one side of $\mathcal{H}$. $\square$

Proposition 3.8. *The Newton polyhedron of the discriminant of the map ϕ restricted to $v = 0$ has one compact face which is is the convex hull of the points $P_1, P_2.P_3, P_4$ and two non compact faces, the plane parallel to the τ_1 axis and containing the segment $\overline{P_1P_4}$ and the plane parallel to the τ_2-axis and containing the segment $\overline{P_3P_4}$.*

Proof. It follows from the previous lemmas since we know by Proposition 2.3 that the compact face of the Newton polyhedron for $v = 0$ is contained in the plane $\overline{\beta}_0\tau_0 + n_1\overline{\beta}_1\tau_1 + n_2\overline{\beta}_2\tau_2 = \overline{\beta}_0((n_1-1)\overline{\beta}_1 + (n_2-1)\overline{\beta}_2)$. $\square$

Lemma 3.9. *The convex hull of the points P_3, P_4, P_5, where $P_5 = (0, 0, \overline{\beta}_0-1)$ is a face of the Newton polyhedron of ϕ.*

Proof. Taking $t_0 = t_1 = 0$, the determinant of the matrix M reduces to the monomial $v^{(n_1-1)n_2}t_2^{\overline{\beta}_0-1}$, which corresponds to P_5. To prove the lemma it suffices to show that the Newton polyhedron is entirely on one side of ("above") the hyperplane $\overline{\beta}_0\tau_0 + n_1\overline{\beta}_1\tau_1 + \overline{\beta}_0\overline{\beta}_1\tau_2 - \overline{\beta}_0(\overline{\beta}_0-1)\overline{\beta}_1 = 0$ determined by the points P_3, P_4, P_5.

Given a point P with coordinates $(\tau_0, \tau_1, \tau_2, v)$ satisfying the relation of homogeneity
$$\overline{\beta}_0\tau_0 + n_1\overline{\beta}_1\tau_1 + n_2\overline{\beta}_2\tau_2 + (\beta_1-\beta_2)v - \overline{\beta}_0((n_1-1)\overline{\beta}_1 + (n_2-1)\overline{\beta}_2) = 0$$
of Proposition 2.3, we must check that it gives a positive value to
$$H_2(\tau_0, \tau_1, \tau_2) = \overline{\beta}_0\tau_0 + n_1\overline{\beta}_1\tau_1 + \overline{\beta}_0\overline{\beta}_1\tau_2 - \overline{\beta}_0(\overline{\beta}_0-1)\overline{\beta}_1.$$
A short computation using the identities between the β_j and $\overline{\beta}_k$ after eliminating τ_0 and τ_1 by substracting the homogeneity relation from H_2 and rewriting (modulo that relation)
$$\begin{aligned} H_2(\tau_0, \tau_1, \tau_2) \;=\; & \overline{\beta}_0((n_1-1)\overline{\beta}_1 \,+\, (n_2-1)\overline{\beta}_2) - \overline{\beta}_0(\overline{\beta}_0-1)\overline{\beta}_1 \\ & + (\beta_2-\beta_1)v + (\overline{\beta}_0\overline{\beta}_1 - n_2\overline{\beta}_2)\tau_2\,, \end{aligned}$$

shows that for such a point we have the equality

$$H_2(\tau_0, \tau_1, \tau_2) = (\beta_2 - \beta_1)(\overline{\beta}_0(n_2 - 1) + \upsilon - n_2\tau_2),$$

so that we have the required inequality if and only if $n_2\tau_2 - \upsilon \leq \overline{\beta}_0(n_2 - 1)$.

But this is always true since the greatest possible exponent of t_2 in the determinant of M is $\overline{\beta}_0 - 1$, corresponding to the terms in the subdiagonal of M, and by looking again at the matrix one sees that $\tau_2 - \upsilon \leq n_2 - 1$ since there are only $n_2 - 1$ occurrences of t_2 without a factor υ. ☐

Proposition 3.10. *The Newton polyhedron of the discriminant of the map ϕ for a fixed $\upsilon \neq 0$ has two compact faces, which are respectively the convex hulls of $P_1, P_2.P_3, P_4$ and of P_3, P_4, P_5 and one non compact face, which is that of Lemma 3.7.*

Proof. The statement follows from the previous lemmas. ☐

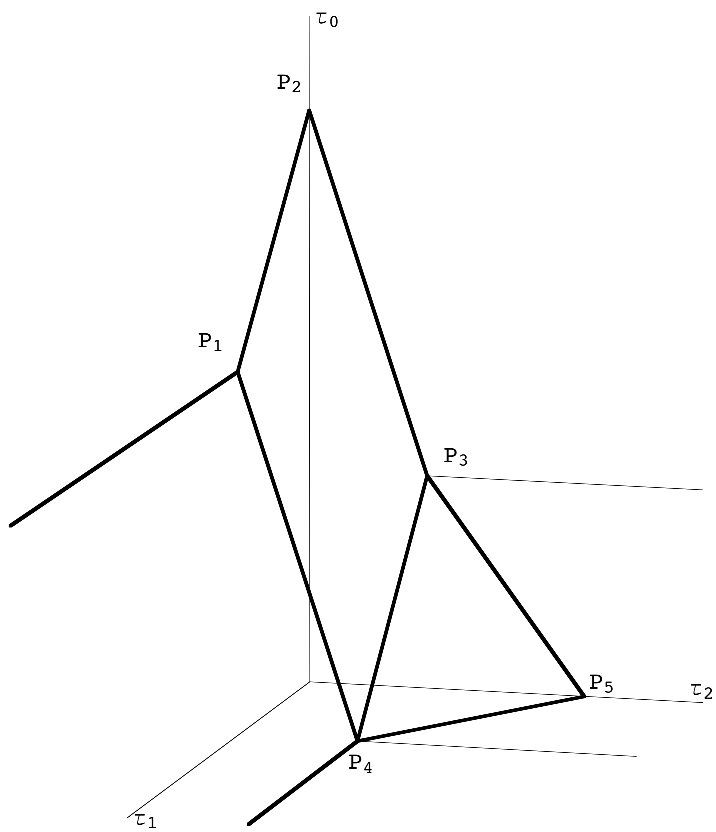

Figure 1: Newton polyhedron of the discriminant of ϕ.

Figure 1 gives an idea for the shape of the Newton polyhedron for $v \neq 0$. The non-compact face which appears for $v = 0$ is suggested in thinner lines.

The intersection with $\tau_1 = 0$ is the jacobian Newton polygon of the plane branch.

1. $P_1 = (n_1(n_2-1)\ell_0^{(2)},\ n_2(n_1-1)+\ell_1^{(2)}(n_2-1),\ 0)$,
2. $P_2 = ((n_1-1)\overline{\beta}_1 + (n_2-1)\overline{\beta}_2,\ 0,\ 0)$,
3. $P_3 = ((n_1-1)\overline{\beta}_1,\ 0,\ n_1(n_2-1))$,
4. $P_4 = (0,\ n_2(n_1-1),\ n_1(n_2-1))$,
5. $P_5 = (0,\ 0,\ \overline{\beta}_0 - 1)$.

4 A Question of Genericity

The linear form u_0 is *not* general with respect to the monomial curve defined by the vanishing of $f_1 = u_1^{n_1} - u_0^{\ell_0^{(1)}}$ and $f_2 = u_2^{n_2} - u_0^{\ell_0^{(2)}} u_1^{\ell_1^{(2)}}$; this is attested by the fact that the critical space of the map (u_0, f_1, f_2) is not reduced, contradicting a known result on polar varieties (see [8], Chap. IV). Therefore it could be that the Newton polyhedron that we obtain for $v = 0$ is not really the jacobian Newton polyhedron of the monomial curve. We are going to verify that in fact it is.

The method is to check that considering the critical subspace with respect to a general linear form $u_0 + \sigma u_1 + \tau u_2$ affects the matrix of our presentation only by adding terms whose effect on the determinant is to possibly add exponents which can be seen to be above the Newton polyhedron computed for u_0. Therefore those terms do not modify the Newton polyhedron.

A direct computation shows that modulo the equation $f_1 = 0$ we have

$$\begin{aligned} df_1 \wedge df_2 = {} & n_1 n_2 u_1^{n_1-1} u_2^{n_2-1} du_1 \wedge du_2 - \ell_0^{(1)} n_2 u_0^{\ell_0^{(1)}-1} u_2^{n_2-1} du_0 \wedge du_2 \\ & + (n_1 \ell_0^{(2)} + \ell_0^{(1)} \ell_1^{(2)}) u_0^{\ell_0^{(1)}+\ell_0^{(2)}-1} u_1^{\ell_1^{(2)}-1} du_0 \wedge du_1 . \end{aligned}$$

In fact, the computation gives

$$\begin{aligned} df_1 \wedge df_2 = {} & n_1 n_2 u_1^{n_1-1} u_2^{n_2-1} du_1 \wedge du_2 - \ell_0^{(1)} n_2 u_0^{\ell_0^{(1)}-1} u_2^{n_2-1} du_0 \wedge du_2 \\ & + u_0^{\ell_0^{(2)}-1} u_1^{\ell_1^{(2)}-1} (n_1 \ell_0^{(2)} u_1^{n_1} + \ell_0^{(1)} \ell_1^{(2)} u_0^{\ell_0^{(1)}}) du_0 \wedge du_1, \end{aligned}$$

but modulo f_1, we can replace $u_1^{n_1}$ by $u_0^{\ell_0^{(1)}}$. From this it follows, using the definitions of the $\ell_k^{(j)}$, that

$$\begin{aligned} df_1 \wedge df_2 \wedge du_0 &= \overline{\beta}_0 u_1^{n_1-1} u_2^{n_2-1} du_0 \wedge du_1 \wedge du_2, \\ df_1 \wedge df_2 \wedge du_1 &= \overline{\beta}_1 u_0^{\ell_0^{(1)}-1} u_2^{n_2-1} du_0 \wedge du_1 \wedge du_2, \\ df_1 \wedge df_2 \wedge du_2 &= \overline{\beta}_2 u_0^{\ell_0^{(1)}+\ell_0^{(2)}-1} u_1^{\ell_1^{(2)}-1} du_0 \wedge du_1 \wedge du_2, \end{aligned}$$

the last equation being read mod.f_1. The equation of our critical subspace with respect to a general linear form therefore now reads, modulo f_1,

$$\overline{\beta}_0 u_1^{n_1-1}u_2^{n_2-1} + \sigma\overline{\beta}_1 u_0^{\ell_0^{(1)}-1}u_2^{n_2-1} + \tau\overline{\beta}_2 u_0^{\ell_0^{(1)}+\ell_0^{(2)}-1}u_1^{\ell_1^{(2)}-1} = 0.$$

Since σ and τ are now assumed to be “general” constants, we may simplify this to

$$u_1^{n_1-1}u_2^{n_2-1} + \sigma u_0^{\ell_0^{(1)}-1}u_2^{n_2-1} + \tau u_0^{\ell_0^{(1)}+\ell_0^{(2)}-1}u_1^{\ell_1^{(2)}-1} = 0.$$

Since $u_0 = t_0$, this means that we have to study which effect adding the multiplication by $\sigma t_0^{\ell_0^{(1)}-1}u_2^{n_2-1} + \tau t_0^{\ell_0^{(1)}+\ell_0^{(2)}-1}u_1^{\ell_1^{(2)}-1}$ has on our matrix and its determinant for $v = 0$.

Using the same method as above, we see that the submatrices $M_{j,j'}$ which are affected are M_{0,n_2-1}, which becomes

$$\tilde{\mathbf{M}}_{\mathbf{0,n_2-1}} = \begin{pmatrix}
\sigma t_0^{\ell_0^{(1)}-1} & 0 & \dots & 0 & 0 & 0 & \dots & 1 \\
A & \sigma t_0^{\ell_0^{(1)}-1} & \dots & & 0 & 0 & \dots & 0 \\
0 & A & \sigma t_0^{\ell_0^{(1)}-1} & 0 & & 0 & \dots & \\
\vdots & & & & \vdots & & \vdots & \\
 & & & & & & & 0 \\
0 & 0 & \dots & A & \dots & & 0 & 0 \\
\vdots & & & & & & \vdots & \vdots \\
0 & 0 & 0 & \dots & \dots & A & \sigma t_0^{\ell_0^{(1)}-1} & 0 \\
0 & \dots & 0 & \dots & 0 & \dots & A & \sigma t_0^{\ell_0^{(1)}-1}
\end{pmatrix},$$

and $M_{j,j}$, for $j \geq 0$, which becomes (remember that $v = 0$)

$$\tilde{\mathbf{M}}_{\mathbf{j,j}} = \tau \begin{pmatrix}
0 & 0 & \dots & t_0^{\ell_0^{(1)}+\ell_0^{(2)}-1} & 0 & \dots & 0 \\
0 & 0 & \dots & & t_0^{\ell_0^{(1)}+\ell_0^{(2)}-1} & \dots & 0 \\
\vdots & & & & \vdots & \vdots & \vdots \\
 & & & & & & t_0^{\ell_0^{(1)}+\ell_0^{(2)}-1} \\
At_0^{\ell_0^{(1)}+\ell_0^{(2)}-1} & 0 & \dots & 0 & \dots & 0 & 0 \\
\vdots & & & & & & \vdots \\
0 & \dots & 0 & \dots & \dots & 0 & 0 \\
0 & \dots At_0^{\ell_0^{(1)}+\ell_0^{(2)}-1} & & \dots & 0 & 0 & 0
\end{pmatrix}.$$

(the nonzero elements are aligned on parallels to the second diagonal, the first nonzero element in the first line is in column $\ell_1^{(2)}$, and the first nonzero element in the first column is in line $n_1 - \ell_1^{(2)} + 2$), and, finally, the matrix $M_{j,j-1}$, which becomes (setting $U_0 = t_0^{\ell_0^{(1)}-1}$)

$$\tilde{\mathbf{M}}_{\mathbf{j,j-1}} = \begin{pmatrix} \sigma U_0 t_2 & 0 & \dots & AT_0 & \sigma T_0 U_0 & \dots & 0 & t_2 \\ At_2 & \sigma U_0 t_2 & \dots & & AT_0 & \sigma T_0 U_0 & \dots & 0 \\ 0 & At_2 & \dots & 0 & & AT_0 & \dots & \\ \vdots & & & & \vdots & & \vdots & \sigma T_0 U_0 \\ \sigma t_0^{\ell_0^{(1)}} T_0 U_0 & & & & & & & AT_0 \\ A^2 T_0 & \sigma t_0^{\ell_0^{(1)}} T_0 U_0 & \dots & At_2 & \dots & & 0 & \\ \vdots & & & & & & & \\ 0 & & 0 & \dots & \dots & At_2 & \sigma U_0 t_2 & 0 \\ 0 & \dots & A^2 T_0 & \sigma t_0^{\ell_0^{(1)}} T_0 U_0 & 0 & & At_2 & \sigma U_0 t_2 \end{pmatrix}.$$

The matrix $\tilde{M}(0)$ corresponding to a general linear form and $v = 0$ has the following structure: it is described by the blocks $\tilde{M}_{j,j'}$:

$$\tilde{\mathbf{M}}(\mathbf{0}) = \begin{pmatrix} \tilde{M}_{0,0} & 0 & 0 & \dots & & 0 & 0 & \tilde{M}_{0,n_2-1} \\ \tilde{M}_{1,0} & \tilde{M}_{1,1} & 0 & 0 & & \dots & & 0 \\ \vdots & \vdots & \dots & \vdots & & \dots & & \vdots \\ 0 & & \tilde{M}_{j-1,j-2} & \tilde{M}_{j-1,j-1} & 0 & & & 0 \\ 0 & & & \tilde{M}_{j,j-1} & \tilde{M}_{j,j} & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & & \dots & & \vdots \\ \vdots & \vdots & \dots & \vdots & & \dots & & \vdots \\ 0 & 0 & 0 & \dots & 0 & & \tilde{M}_{n_2-1,n_2-2} & \tilde{M}_{n_2-1,n_2-1} \end{pmatrix}.$$

Let us set $E = \det \tilde{M}_{0,n_2-1}$ and, using the fact that the matrices $\tilde{M}_{j,j}$ and $\tilde{M}_{j,j-1}$ are in fact independant of j, write $D = \det \tilde{M}_{j,j}$ and $S = \det \tilde{M}_{j,j-1}$. We can now use the Laplace expansion (see [1, §8]) of the determinant with respect to the last n_1 columns, we obtain (neglecting signs) an expression $\pm \det \tilde{M} = E \det M_1 \pm D \det N_1$. Then we notice that we can again use the Laplace expansion with respect to the last n_1 lines or n_1 columns, and we obtain

$$\pm \det \tilde{M}(0) = ES^{n_2-1} \pm D^{n_2}.$$

The discriminant D is easy to compute and equal to

$$D = \tau^{n_1} A^{\ell_1^{(2)}-1} t_0^{n_1(\ell_0^{(1)}+\ell_0^{(2)}-1)}.$$

We can check that the exponent of t_0 appearing in D^{n_2} is larger than an exponent already appearing in the discriminant for $\sigma = \tau = 0$. Therefore, it does not affect the Newton polyhedron.

In the expression that we have seen above in Lemma 3.2 for $\ell = u_0$ and $v = 0$, the power of t_0 which appears is

$$(n_1 - 1)\ell_0^{(1)} + (n_2 - 1)\big(n_1(\ell_0^{(1)} + \ell_0^{(2)}) + (\ell_1^{(2)} - 1)\ell_0^{(1)}\big).$$

We only have to prove the inequality

$$
\begin{aligned}
&n_1n_2(\ell_0^{(1)}+\ell_0^{(2)}-1)+n_2(\ell_1^{(2)}-1)\ell_0^{(1)}\\
&\qquad\geq (n_1-1)\ell_0^{(1)}+(n_2-1)\big(n_1(\ell_0^{(1)}+\ell_0^{(2)})+(\ell_1^{(2)}-1)\ell_0^{(1)}\big).
\end{aligned}
$$

After some rewriting, it comes down to

$$\ell_1^{(2)}\ell_0^{(1)}+n_1\ell_0^{(2)}-n_1n_2\geq 0.$$

But if we remember that we have the equality

$$\ell_0^{(2)}\overline{\beta}_0=n_2\overline{\beta}_2-\ell_1^{(2)}\overline{\beta}_1=(n_2-1)\overline{\beta}_2+\overline{\beta}_2-\ell_1^{(2)}\overline{\beta}_1$$

and the fact that $\ell_1^{(2)}<n_1$, we get $\ell_0^{(2)}\geq n_2$, and this suffices to prove our inequality.

Let us now deal with ES^{n_2-1}: The exponent of the diagonal term in E, equal to $\sigma^{n_1}t_0^{n_1(\ell_0^{(1)}-1)}$, is larger than the exponent of $t_0^{(n_1-1)\ell_0^{(1)}}$ which appears in A^{n_1-1}, because $\ell_0^{(1)}>n_1$. So we can forget about that diagonal term in E.

Next, let us consider S^{n_2-1}: Our polyhedron for $v=0$ is bounded by the three hyperplanes

1. $\overline{\beta}_0\tau_0+n_1\overline{\beta}_1\tau_1+n_2\overline{\beta}_2\tau_2=\overline{\beta}_0((n_1-1)\overline{\beta}_1+(n_2-1)\overline{\beta}_2)$
2. $n_2\tau_0+\overline{\beta}_1\tau_1=(n_1-1)n_2\overline{\beta}_1$
3. $\tau_0+\ell_0^{(2)}\tau_2=n_1(n_2-1)\ell_0^{(2)}$

Calling L_1,L_2,L_3 the linear forms appearing in the left-hand side of these three equations, for each L_i we seek successively in each column of the matrix $\tilde{M}_{j,j-1}$ the terms which give it the lowest value and which it is possible to choose in the expansion of the discriminant, and then check that such a choice gives rise in ES^{n_2-1} to exponents which are above the corresponding support hyperplane of our polyhedron.

For example, the linear form L_1 takes as minimum value in the first $\ell_1^{(2)}$ columns the value $\overline{\beta}_0(2\ell_0^{(1)}-1+\ell_0^{(2)})$ which corresponds to $\sigma t_0^{\ell_0^{(1)}}T_0U_0$, and on the last $n_1-\ell_1^{(2)}$ columns the minimal value $\overline{\beta}_0(\ell_0^{(1)}-1+\ell_0^{(2)})$ which corresponds to σT_0U_0. This gives us a term $t_0^{n_1(\ell_0^{(1)}-1+\ell_0^{(2)})+\ell_1^{(2)}\ell_0^{(1)}}$ in S and, therefore, exponents $\big((n_2-1)(n_1(\ell_0^{(1)}-1+\ell_0^{(2)})+\ell_1^{(2)}\ell_0^{(1)})+(n_1-1-i)\ell_0^{(1)},i,0\big)$ in the expansion of $S^{n_2-1}A^{n_1-1}$.

Since $n_1\overline{\beta}_1=\ell_0^{(1)}\overline{\beta}_0$, it suffices to check the inequality on L_1 for $i=0$. This means to verify the inequality

$$
\begin{aligned}
&(n_2-1)\big((n_2-1)(n_1(\ell_0^{(1)}-1+\ell_0^{(2)})+\ell_1^{(2)}\ell_0^{(1)})\big)+(n_1-1)\ell_0^{(1)}\\
&\qquad\qquad\geq (n_1-1)\overline{\beta}_1+(n_2-1)\overline{\beta}_2\,.
\end{aligned}
$$

We can now use the equalities $\overline{\beta}_2 = n_1\ell_0^{(2)} + \ell_0^{(1)}\ell_1^{(2)}$ and $n_2\ell_0^{(1)} = \overline{\beta}_1$ which follow from the definitions to rearrange the terms on the left into

$$(n_2-1)\overline{\beta}_2 + (n_1-1)\overline{\beta}_1 + (n_2-1)(\ell_0^{(1)} - n_1 + 2)$$

and prove that the inequality follows from $\ell_0^{(1)} > n_1$.

If we now take L_2, the term giving the minimal value in each column of $\tilde{M}_{j,j-1}$ is $\sigma U_0 t_2$. This gives a term with $\tau_0 = n_1(n_2-1)(\ell_0^{(1)}-1) + (n_1-1)\ell_0^{(1)}$, which again gives the same value to L_2 as all the other terms coming from A^{n_1-1}. So we have to prove the inequality

$$n_1n_2(n_2-1)(\ell_0^{(1)}-1) + (n_1-1)n_2\ell_0^{(1)} \geq (n_1-1)n_2\overline{\beta}_1.$$

Again, using $n_2\ell_0^{(1)} = \overline{\beta}_1$, we can rearrange the left-hand side of this inequality into $(n_1-1)n_2\overline{\beta}_1 + (\ell_0^{(1)} - n_1)n_2(n_2-1)$, and the result then follows from $\ell_0^{(1)} > n_1$. The last case is left to the reader. From these computations one finally deduces that the Newton polyhedron with respect to the linear form u_0 is indeed the general one for $v = 0$.

5 The Information is Constant

To conclude let us check that the Newton polyhedra for $v = 0$ and for $v \neq 0$ both contain the same information, namely the semigroup of the plane branch, or equivalently its Puiseux characteristic, its equisingularity type, or its topological type.

First, it follows from the description of the polyhedra that they are both determined by the generators of the semigroup; the numbers $\overline{\beta}_i$, n_1, n_2 and $\ell_k^{(j)}$ are all determined by the semigroup. The Newton polyhedron for $v \neq 0$ contains as a plane section the jacobian Newton polyhedron of the plane branch which is known to determine the equisingularity type, so that its datum is equivalent to that of the equisingularity type, or the semigroup. It is also easy to check directly that its knowledge gives us the generators of the semigroup: the point P_5 gives us $\overline{\beta}_0 = n_1n_2$, so that from the homogeneity relation of Proposition 2.3 we know $n_1\overline{\beta}_1$ and $n_2\overline{\beta}_2$. But once we know $\overline{\beta}_0$ the coordinates of the point P_3 give us n_1 and n_2, and we are done.

It remains to verify that no information is lost when $v = 0$. Let us collect the information that we have: First we have the homogeneity relation for $v = 0$:

$$\overline{\beta}_0\tau_0 + n_1\overline{\beta}_1\tau_1 + n_2\overline{\beta}_2\tau_2 = \overline{\beta}_0((n_1-1)\overline{\beta}_1 + (n_2-1)\overline{\beta}_2).$$

It gives us the coefficients up to a multiplicative rational factor.

The point P_4 gives us $n_1 - n_2 = d$ by difference of its second and third coordinates. Substituting in the second coordinate we find that it is equal

to $n_2(n_2+d-1)$, so that we know the product and the difference of n_2 and n_2+d-1. Since d is known, we now know n_2, hence also n_1 and their product $\overline{\beta}_0$. From the homogeneity equation we can finally deduce $\overline{\beta}_1$ and $\overline{\beta}_2$.
So the information is indeed constant, with two different encodings.

Questions: It is to be hoped that for any number of characteristic pairs, the Newton polyhedron for $v \neq 0$ has exactly g compact faces, which intersect the plane $\tau_1 = \cdots = \tau_{g-1} = 0$ along the jacobian Newton polygon of the plane branch, and that the information contained in the Newton polyhedron for $v = 0$ is still equivalent to the knowledge of the semigroup of the branch.

More generally, one can hope that given a branch, plane or not, such that the monomial curve with the same semigroup is a complete intersection, the jacobian Newton polyhedron associated to the map defined by the equations of the branch and a general linear form encodes the semigroup.

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