COMPLEX CURVE SINGULARITIES: A BIASED INTRODUCTION

Bernard TEISSIER

Institut mathématique de Jussieu,
UMR 7586 du C.N.R.S.,
Case 7012, 2 Place Jussieu,
75251 Paris Cedex 05, France
E-mail: teissier@math.jussieu.fr

The goal of this text is to provide an introduction to the *local* study of singular curves in complex analytic geometry. It contains resolution of singularities, the Newton polygon and the Newton parametrization, the classical Newton-Puiseux invariants, the semigroup associated to a branch as well as the specialization to the corresponding monomial curve, and a proof of Bézout's theorem. It ends with the computation of the class of a projective plane curve. Some familiarity with the basic concepts of commutative algebra and complex analytic geometry is assumed.

Keywords: curve singularities, Newton-Puiseux, embedded resolution, monomial curves, Bézout Theorem, Plücker formula.

1. What is a curve?

In these lectures[a] I will discuss singular points of complex analytic curves. A complex curve may locally be regarded as a family of points in complex affine space $\mathbf{A}^d(\mathbf{C})$ depending algebraically or analytically on one complex parameter.

• The dependence may be *explicit*, which means that the coordinates of our points depend explicitely on one parameter, as in:

$$\begin{aligned} z_1 &= z_1(t) \\ z_2 &= z_2(t) \\ \vdots\ &= \vdots \\ z_d &= z_d(t) \end{aligned}$$

[a]This text is an expansion of the notes of a course given at the CIMPA-LEBANON school in Beyrouth in July 2004 and of lectures at the ICTP College on Singularities.

where the $z_i(t)$ may be any functions $\mathbf{C} \to \mathbf{C}$ although here, since we said we would consider families of points depending analytically on the parameter, we shall consider only polynomials or convergent power series. In this case the functions $z_i(t)$ may be defined only in a neighborhood of some point, which we will usually assume to be the origin $t = 0$ and it is convenient to assume that $z_i(0) = 0$ for all i; one may reduce to this case by a translation on the coordinates z_i and t.
The curves which appear naturally are usually finite unions of parametrized curves as above. The parametrized curves are then the *irreducible components* or *branches* of the union. A curve is non singular at the origin if and only if it has only one component, for which the minimum of the t-adic orders of the $z_i(t)$ is equal to one. By the implicit function theorem, this means that the curve is locally analytically isomorphic to the complex line.

A *germ of curve at a point* (which we take to be the origin) is an equivalence class of curves given parametrically or by equations in an open neighborhood of the origin. Two such objects defined respectively on U and U' are equivalent if their restrictions to a third neighborhood $U'' \subset U \cap U'$ of the origin coincide. Of course when we talk of germs we think of representatives in some "sufficiently small" neighborhood of 0. Because of analyticity, to give a germ is equivalent to giving the convergent power series parametrizing the branches of the curve at the origin in some coordinate system.

• A curve may also be given *implicitely*, which means that it is given by equations.

Here the fact that we are dealing with a curve should manifest itself in the fact that we have $d-1$ equations in d variables, so that implicitely all the variables depend on one of them. However, the situation is not so simple in general, and a curve may need more than $d-1$ equations.

It is a fundamental fact of the theory of analytic curves that each germ can be decomposed uniquely as a union of irreducible germs, and that on an irreducible germ all coordinates can be expressed as convergent power series in the n-th root of one of them, for some integer m. This is often called the Newton-Puiseux theorem.
The simplest case is that of a *plane algebraic curve* in the 2-dimensional affine space $\mathbf{A}^2(\mathbf{C})$, defined by an equation $f(x,y) = 0$, where $f \in \mathbf{C}[x,y]$ is a polynomial:

$$f(x,y) = a_0(x)y^n + a_1(x)y^{n-1} + \cdots + a_n(x),$$

with $a_i(x) \in \mathbf{C}[x],\ a_0(x) \neq 0$. Recall that $\mathbf{C}[z_1,\ldots,z_d]$, $\mathbf{C}\{z_1,\ldots,z_d\}$ and

$\mathbf{C}[[z_1,\ldots,z_d]]$ denote respectively the ring of polynomials in d variables with complex coefficients, the ring of convergent complex power series in d variables and the ring of formal power series in d variables with complex coefficients.
The degree n in y of the polynomial is the number of solutions in y (counted with multiplicities) for any fixed value x_0 of x which is "sufficiently general" in the sense that $a_0(x_0) \neq 0$. The curve is non-singular at the origin if at least one of the derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ does not vanish at the origin. All this remains valid locally if $f(x,y)$ is a convergent power series in x, y such that $f(0,0) = 0$, thanks to the *Weierstrass preparation theorem* which asserts that, if $f(0,y)$ is not identically zero, which we can always assume at the cost of a linear change of variables, then $f(x,y)$ can be written

$$f(x,y) = u(x,y)(y^n + a_1(x)y^{n-1} + \cdots + a_n(x)),$$

with $a_i(x) \in \mathbf{C}\{x\}$ where $u(x,y)$ is a convergent series which does not vanish in a neighborhood of the origin. This means that $u(x,y)$ is an invertible element in the ring $\mathbf{C}\{x,y\}$ and also that the curve defined by $f(x,y) = 0$ is just as well defined by the polynomial in y, in a sufficiently small neighborhood of the origin.

We shall see later an interpretation for the total degree $\max(i+j)$ for monomials x^iy^j appearing in a polynomial $f(x,y)$). To a parametrized plane curve $x(t), y(t)$, one can associate a series $f(x,y)$ such that $f(x(t),y(t)) \equiv 0$, by a process of *elimination*, to which we shall come back later. If the exponents of t appearing in the two series have no common divisor, this series is an *irreducible* element in the ring $\mathbf{C}\{x,y\}$; it is not a product of two other series vanishing at the origin. Conversely the set of zeroes $f(x,y) = 0$ of a convergent power series in two variables defines coincides with a finite number of parametrized analytic curves in a small neighborhood of the origin of the plane, (the *branches* of the curve defined by the equation). The construction will be detailed below. The number r of these branches is given by the decomposition of $f(x,y)$ into a product $f_1^{a_1}\ldots f_r^{a_r}$ of powers of irreducible elements in the ring $\mathbf{C}\{x,y\}$.

We shall deal here with *reduced* curves, which means that all the a_i are equal to one or, as we shall see that the exponents appearing in the power series $x(t), y(t)$ of a parametrization of each branch, taken all together, have no non trivial common divisor.

The description of a curve in $\mathbf{A}^d(\mathbf{C})$ for $d \geq 3$ is much more complicated in general. One can prove that it requires at least $d-1$ equations, but it may require much more, and then what is important is the *ideal* I which these equations generate in the ring $\mathbf{C}[z_1, \ldots, z_d]$ or the ring $\mathbf{C}\{z_1, \ldots, z_d\}$. In fact, it is the quotient ring $\mathbf{C}[z_1, \ldots, z_d]/I$, or $\mathbf{C}\{z_1, \ldots, z_d\}/I$ which we consider, and the fact that we a re dealing with reduced complex curves is translated into the fact that we suppose that it is a reduced $\mathbf{C}$-algebra with Krull dimension 1.

In the case of plane curves, this ideal is a *principal* ideal so we have one equation only; more precisely, in the case of polynomials, this equation is unique up to multiplication by a nonvanishing polynomial, i.e., a nonzero constant. In fact, it is the quotient ring $\mathbf{C}[z_1, \ldots, z_d]/I$, or $\mathbf{C}\{z_1, \ldots, z_d\}/I$ which we consider.

Assuming that we are in the analytic (as opposed to algebraic) case and the ring $\mathcal{O} = \mathbf{C}\{z_1, \ldots, z_d\}/I$ is an integral domain of dimension one, we can bring together the equational and parametric representations together in the following diagram of $\mathbf{C}$-algebras:

$$\mathbf{C}\{z_1, \ldots, z_d\} \to \mathcal{O} \subset \mathbf{C}\{t\},$$

where the first map is the surjection with kernel I describing $\mathcal{O}$ as a quotient, and the second one is determined by: $z_i \mapsto z_i(t) \quad 1 \leq i \leq d$.

The conclusion is that a complex curve, locally, should be thought of as a one-dimensional reduced local $\mathbf{C}$-algebra $\mathcal{O}$, which is the localization of a $\mathbf{C}$-algebra of finite type (algebraic case), or is a quotient of a convergent power series ring $\mathbf{C}\{z_1, \ldots, z_d\}$ (analytic case) or of a formal power series ring $\mathbf{C}[[z_1, \ldots, z_d]]$ (formal, or algebroid, case).

Then we can consider it as given by an ideal in a regular local ring, namely the kernel of the map defining it as a quotient of a localization of a polynomial ring, or as a subring of a regular one-dimensional semi-local ring (its normalization) $\overline{\mathcal{O}}$, which in the analytic or formal case corresponds to the parametrization.

It is a theorem that in the cases considered here, the normalization $\overline{\mathcal{O}}$ is a finitely generated $\mathcal{O}$-module.

2. What does one do with curves?

Real curves (often analytic) appear everywhere in mechanics as trajectories, and complex curves appear everywhere in Mathematics as soon as points depend on one parameter; for example given a family of square complex matrices depending on a complex parameter, the family of the eigenvalues lies on a complex curve. In order to study a real analytic curve, it is often useful to look at its complexification.

If a complex algebraic group G acts algebraically on a variety X, one may study the action by restricting it to one-parameter subgroups $\mathbf{C}^* \subset G$ and the the orbit of each point $x \in X$ is a curve. Its closure in X is in general singular.

More generally, the closure of a non singular curve will in general have singularities.

We have seen also non singular algebraic curves which are themselves algebraic groups; elliptic curves. However, if we want to understand the totality of elliptic curves, we must also consider their singular limits.

Curves also appear naturally in inductive steps in algebraic geometry: non-singular surfaces are studied in large part through the families of curves which they contain, and singular curves *must* appear in these families.

To study the geometry of a non singular surface S, it is natural to project it to a non-singular space of the same dimension. The set of points of S where the projection $p\colon S \to \mathbf{P}^2$ is not a local isomorphisme, the *critical locus* of the projection, is a curve (if it is not empty) which contains much information about the geometry of S. The critical locus may be a non singular curve, but its image under projection p will in general be singular.

More generally, even if one is interested in non singular curves only, their plane projections will in general be singular, having at least ordinary double points or *nodes*

In these lectures, I will mostly study different ways to transform curves into other curves, by deformation either of the parametric representation or of the equations, by taking the transforms of curves in $\mathbf{A}^d_k$ under maps $Z \to \mathbf{A}^d_k$, and by projection (in the case of non plane curves).

Since the passage from the implicit presentation to the parametrization uses some form of the implicit function theorem, I will mostly work in the context of complex analytic functions, for which there is an implicit function theorem which does not exist for polynomials.
If one wishes to work over a field different from $\mathbf{C}$, one could replace convergent power series with formal power series with coefficients in a field, keeping most of the algebra but losing a lot of geometry or one could work in the henselizations of the polynomial ring and its quotients, keeping just about everything. If the field has positive characteristic new phenomena appear: Newton-Puiseux's theorem is no longer true (see section 3 below).

Let us call *analytic algebra* any $\mathbf{C}$-algebra which is a quotient of a convergent power series ring $\mathbf{C}\{z_1, \ldots, z_d\}$. To any localization R_m of a finitely generated local algebra R over the field of complex numbers at one of its maximal ideals is associated in a unique way (up to unique isomorphism) an analytic algebra R^h_m, which has the property that any $\mathbf{C}$-algebra morphism $R_m \to A$ from R_m to an analytic algebra A factors in a unique manner $R_m \to R^h_m \to A$ where $R^h_m \to A$ is a morphism of analytic algebras.

The candidate for R^h is simple to see: write a presentation $R = \mathbf{C}[T_1, \ldots, T_n]/I$, where I is generated by finitely many polynomials. The maximal ideal m of R is the image of a maximal ideal $\tilde{m}$ of $\mathbf{C}[T_1, \ldots, T_d]$. By the nullstellensatz, the ideal $\tilde{m}$ corresponds to a point $(a_1, \ldots, a_d)$ of the affine space $\mathbf{A}^d(\mathbf{C})$. Set $R^h = \mathbf{C}\{T_1 - a_1, \ldots, T_d - a_d\}/I^h$, where I^h is the ideal generated in $\mathbf{C}\{T_1 - a_1, \ldots, T_d - a_d\}$ by the polynomials which generate I.

In the case of curves, the first serious difficulty comes from the fact that an irreducible polynomial $P(x, y) \in \mathbf{C}[x, y]$ may well become reducible in $\mathbf{C}\{x, y\} = \mathbf{C}[x, y]^h_{(x,y)}$. In other words, the analytization of a local integral $\mathbf{C}$-algebra may not be integral.
Consider for example the nodal cubic with equation

$$y^2 - x^2 - x^3 = 0;$$

it is an irreducible affine plane curve, but the image of its equation in $\mathbf{C}\{x, y\}$ by the natural injection $\mathbf{C}[x, y] \subset \mathbf{C}\{x, y\}$ factors as

$$y^2 - x^2 - x^3 = (y + x\sqrt{1 + x})(y - x\sqrt{1 + x}).$$

The interaction between the global invariants of a plane projective curve and its singularities is also an important theme requiring the local study of singularities:
- We know how to compute the class and the genus of a non singular plane projective curve of degree d. Assume now that it is singular; how does it affect the formulas for the class and genus?
- The single most important result about plane curves is Bézout's theorem, which is the generalization of the fundamental theorem of algebra:
Given two plane projective curves C and C' of degrees d and d' having no common component, the number of their points of intersection counted with multiplicity is the product dd'.

At the points of intersection where both curves are not regular and meeting transversally, how does one properly count the intersection multiplicity? can one effectively count it, given the equations of the two curves? In fact, as we shall see, the best situation to compute the intersection multiplicity is to have one curve given parametrically and the other given implicitely, although if both are given parametrically, there is a formula, due to Max

Noether (see [M]). What is the geometric meaning of intersection multiplicity?

We know that we can normalize an algebraic curve to obtain a non singular algebraic curve, and it can be shown that the same is true in complex analytic geometry. The parametrization of an analytic curve mentioned above is in fact its normalization, provided we take care that the powers of t appearing in the series are coprime; if this is the case, the inclusion $\mathcal{O} \subset \mathbf{C}\{t\}$ induces an isomorphism of fraction fields and is therefore the normalization of $\mathcal{O}$. The normalization, however, is *a priori* difficult to compute from the equations of our curve.

An algorithm to do precisely this, in fact to compute a parametrization from an equation of a plane curve, was given by Newton; it is based on the *Newton polygon.*

3. Newton's study of plane curve singularities

Let us recall that the ring of power series in fractional powers with fixed denominator m of a variable x with fixed denominator m is defined as $\mathbf{C}[[x^{\frac{1}{m}}]] = \mathbf{C}[[x]][T]/(T^m - x)$, and similarly for $\mathbf{C}\{x^{\frac{1}{m}}\}$. The notation $x^{\frac{1}{m}}$ describes a multi-valued function of x, defined as a function not on $\mathbf{C}$ but on an m-fold ramified covering of $\mathbf{C}$. The various "determinations" are exchanged by multiplying one of them by an m-th root of unity. The work of Newton and Puiseux shows that functions on a branch can be viewed as functions of $x^{\frac{1}{m}}$ for some m.

Let $f(x,y) \in \mathbf{C}[[x,y]]$ be a formal power series without constant term. We seek series $y(x)$ without constant term such that $f(x,y(x)) = 0$.

Let us first eliminate a marginal case; if $f(0,y) = 0$, it means that $f(x,y)$ is divisible by some power of x; let a be the maximum power of x dividing $f(x,y)$, and let us set $f(x,y) = x^a f'(x,y)$. Geometrically, the equality $f(0,y) = 0$ means that the curve $f(x,y) = 0$ contains the y-axis, and the equality above means that this axis should be counted a times in the curve. This component may be parametrized by $x = 0, y = t$ and we are left with the problem of parametrizing the rest of the curve, which is defined by $f'(x,y) = 0$. We now have $f'(0,y) \neq 0$, and we may thus reduce to the case $f(0,y) \neq 0$. From now on we shall assume that $f(0,y) \neq 0$.
We may then write, since $f(0,y)$ is a formal power series in y, $f(0,y) = y^n g(y)$, with $g(0) \neq 0$.

The proof of the existence of parametrizations proceeds by induction on the integer n. If $n = 1$, we have $\frac{\partial f}{\partial y}(0,0) \neq 0$, and by the implicit function

theorem there exists a unique formal power series $y(x) \in \mathbf{C}[[x]]$ such that $y(0) = 0$ and $f(x, y(x)) = 0$. We now assume that $n > 1$.
Considering series $f(x, y)$ of the form $y^n - x^q$ with $n, q > 1$ and $(n, q) = 1$ shows that one cannot hope to find series in powers of x. Newton's idea is to seek solutions which are *fractional power series* in x, that is, he seeks series in $x^{\frac{1}{m}}$ for some integer m, say $\phi(x^{\frac{1}{m}}) \in \mathbf{C}[[x^{\frac{1}{m}}]]$ such that $f(x, \phi(x^{\frac{1}{m}})) = 0$. More precisely he seeks solutions of the form:

$$y = x^{\nu}(c_0 + \phi_0(x^{\frac{1}{m}}))$$

with $c_0 \neq 0, \quad \nu \in \mathbf{Q}_+, \ \phi_0$ without constant term. If we write

$$f(x, y) = \sum_{i,j \in \mathbf{N}} a_{i,j} x^i y^j \qquad \text{with } a_{0,0} = 0$$

and substitute, we get

$$\sum_{i,j} a_{i,j} x^{i+\nu j}(c_0 + \phi_0(x^{\frac{1}{m}}))^j$$

and we seek ν, $c_0 \neq 0$ and a series $\phi_0(x^{\frac{1}{m}})$ such that this series is zero. In particular, its lowest order terms in x must be zero. Since ϕ_0 has no constant term, if we denote by μ the minimum value of $i + \nu j$ for (i, j) such that $a_{i,j} \neq 0$, we have

$$\sum_{i,j} a_{i,j} x^{i+\nu j}(c_0 + \phi_0(x^{\frac{1}{m}}))^j = x^{\mu} \sum_{i+\nu j = \mu} a_{i,j} c_0^j + x^{\mu} h(x^{\frac{1}{m}})$$

where h has no constant term. So c_0 must satisfy

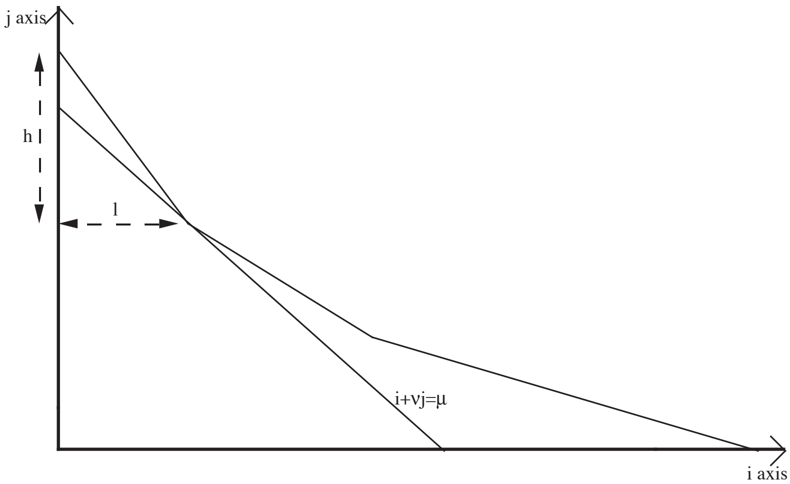
$$\sum_{i+\nu j=\mu} a_{i,j} c_0^j = 0$$

For this equation to have a non-zero root in $\mathbf{C}$, it is necessary and sufficient that the sum has more than one term.

Let us consider in the (i, j)-plane the set of points (i, j) such that $a_{i,j} \neq 0$. It is a subset $\mathcal{N}(f)$ of the first quadrant

$$\mathbf{R}_0^2 = \quad \{(i, j) \ / i \geq 0, \ j \geq 0\},$$

called the *Newton cloud* of the series $f(x, y)$. Any two subset A and B of $\mathbf{R}^d$ can be added coordinate-wise, to give the Minkowski sum $A + B = \{a + b, a \in A, \ b \in B\}$ of A and B. Let us consider the subset $\mathcal{N}_+(f) = \ \mathcal{N}(f) + \mathbf{R}_0^2$ of $\mathbf{R}_0^2$; its boundary is a sort of staircase with possibly infinite vertical or horizontal parts. The Newton polygon $\mathcal{P}(f)$ of $f(x, y)$ is defined as the boundary of the convex hull of $\mathcal{N}_+(f)$. It is a broken line with infinite horizontal and vertical sides, possibly different from the coordinate axis.



Just above is a picture of a Newton polygon in the case where the infinite sides do coincide with the coordinate axis, or equivalently where the area bounded by the polygon is finite.

Recall that the convex hull of a subset of $\mathbf{R}^d$ can be defined as the intersection of the half-spaces which contain it. A half-space is the set of points situated on one side of an affine hyperplane. Thus, the number

$$\mu = \min_{a_{i,j} \neq 0}\{i + \nu j\}$$

is the minimal abscissa of the intersection points with the i-axis of the lines with slope $\frac{-1}{\nu}$ meeting $\mathcal{N}_+(f)$. Let us denote by L_ν the line which gives this minimum; an example in drawn on the picture.
So the polynomial

$$\sum_{i+\nu j=\mu} a_{i,j} c_0^j$$

corresponds to the sum of the terms $a_{i,j}x^i y^j$ such that (i,j) lies on the intersection of the line L_ν with the Newton polygon.
A necessary and sufficient condition for this polynomial to have more than one term is that $\frac{-1}{\nu}$ is the slope of one of the sides of the Newton polygon. For simplicity of notation, let us call ν the *inclination* of the line of slope $\frac{-1}{\nu}$. Let us denote by $\frac{l}{h}$ the inclination of the "first side" of the Newton polygon of f, that is, the side with the smallest inclination. Let c_0 be a non zero root of the corresponding equation, and let us make the change of

variables

$$x = x_1^l$$
$$y = x_1^h(c_0 + y_1)$$

The substitution in $f(x, y)$ gives

$$f(x_1^h, x_1^\ell(c_0 + y_1)) = \sum a_{i,j} x_1^{hi+\ell j}(c_0 + y_1)^j \ .$$

By definition of μ, for each $a_{i,j} \neq 0$, we have $hi + \ell j \geq \mu h$, so we may factor the series above as

$$x_1^{\mu h} f_1(x_1, y_1) \text{ , where } f_1(x_1, y_1) = \sum a_{i,j} x_1^{hi+\ell j-\mu h}(c_0 + y_1)^j \ .$$

We remark that

$$f_1(0, y_1) = \sum_{i+\nu j=\mu} a_{i,j}(c_0 + y_1)^j \ ,$$

and since $a_{0,k} \neq 0$ by definition of n, *the order in* y_1 *of* $f_1(0, y_1)$ *is* $\leq n$.

Since c_0 has been chosen as a root of the polynomial $\sum_{i+\nu j=\mu} a_{i,j} c_0^j$, this order is ≥ 1. We remark that
The order in y_1 *of* $f_1(0, y_1)$ *is equal to* n *if and only if* c_0 *is a root of multiplicity* n *of the polynomial* $\sum_{i+\nu j=\mu} a_{i,j} T^j = 0$
But then we must have an equality

$$\sum_{i+\nu j=\mu} a_{i,j} T^j = a_{0,n}(T - c_0)^n$$

which implies by the binomial formula and since $\mathbf{C}$ is a field of characteristic zero, that the term in T^{n-1} is not zero; this is possible only if ν is an integer and then the equality above shows that the "first side of the Newton polygon" meets the horizontal axis at the point $(\nu n, 0)$, which corresponds to the monomial $x^{\nu n}$, which has the non zero coefficient $(-1)^n a_{0,n} c_0^n$, so it is actually the only finite side of the Newton polygon of $f(x, y)$, which means that we may write in this case

$$f(x, y) = a_{0,n}(y - c_0 x^\nu)^n + \sum_{i+\nu j>\mu} a_{i,j} x^i y^j \quad \text{with } \nu \in \mathbf{N},\ \mu = \nu n.$$

Making the change of variables

$$x = x_1$$
$$y = y_1 + c_0 x_1^\nu$$

the series $f(x, y)$ becomes

$$f'(x, y) = a_{0,k} y_1^n + \sum_{i+\nu j>\mu} a_{i,j} x_1^i (y_1 + c_0 x_1^\nu)^j$$

The monomials which appear are of the form $x_1^{i+\nu l}y_1^{j-l}$, so that they all satisfy $i+\nu l+\nu(j-l)=i+\nu j>\nu n$. This means that *if the order of $f_1(0,y_1)$ is n, the Newton polygon of $f_1(x_1,y_1)$ still contains the point $(0,n)$ and the inclination ν_1 of its first side is strictly greater than ν.*

The proof now proceeds as follows, :

a) If the order in y_1 of $f_1(0,y_1)$ is less than n, by the induction hypothesis, there exist an integer m_1 and a series $\phi_1(x_1^{\frac{1}{m_1}})\in \mathbf{C}[[x_1^{\frac{1}{m_1}}]]$ such that

$$f_1(x_1,\phi_1(x_1^{\frac{1}{m_1}}))=0$$

By the definition of f_1, this implies that

$$f(x_1^h,x_1^\ell(c_0+\phi_1(x_1^{\frac{1}{m_1}}))=0$$

If we set $m=m_1h$ and $\phi(x^{\frac{1}{m}})=x^{\frac{l}{h}}(c_0+\phi_1(x^{\frac{1}{m}}))\in \mathbf{C}[[x^{\frac{1}{m}}]]$, we have $f(x,\phi(x^{\frac{1}{m}}))=0$ and the result in this case.

b) If the order in y_1 of $f(0,y_1)$ is still equal to n, we saw that ν is an integer and the inclination of the first side of the Newton polygon of the function $f_1(x_1,y_1)$ obtained from $f(x,y)$ as above is strictly greater than ν.

We now set $\nu_0=\nu\in\mathbf{N}$ and repeat the same analysis for f_1, defining a function $f_2(x_2,y_2)$. If again the order of $f_2(0,y_2)$ is n, the slope of the first side of the Newton polygon of $f_1(x_1,y_1)$ is an integer $\nu_1>\nu_0$ and after the change of variables $x=x_2, y=y_2+c_0x_2^{\nu_0}+c_1x^{\nu_1}$ the slope of the Newton polygon has become greater than ν_1.

There are two possibilities;

– either after a finite number of such steps we get a function $f_p(x_p,y_p)$ such that $f(0,y_p)$ is of order $<n$, and by the induction hypothesis we have a series $\phi_p(x^{\frac{1}{m_p}})\in\mathbf{C}[[x^{\frac{1}{m_p}}]]$ such that $f_p(x,\phi_p(x^{\frac{1}{m_p}}))=0$, and so a series

$$y=c_0x^{\nu_0}+c_1x^{\nu_1}+\cdots+c_{p-1}x^{\nu_{p-1}}+\phi_p(x^{\frac{1}{m_p}})$$

such that $f(x,y(x))=0$;

Or the order remains indefinitely equal to n and we have an infinite increasing sequence of integers

$$\nu_0\ <\nu_1\ <\ldots<\nu_p<\ \ldots$$

and a formal power series

$$\phi_\infty(x)=c_0x^{\nu_0}+c_1x^{\nu_1}+\cdots+c_px^{\nu_p}+\cdots\in\mathbf{C}[[x]]$$

such that the Newton polygon of the function $f_\infty(x_\infty,y_\infty)$ obtained from $f(x,y)$ by the change of variables $x=x_\infty\ ,\ y=y_\infty+\phi_\infty(x)$ has a Newton polygon containing the point $(0, n)$ and with inclination 0. This means that $f_\infty(x_\infty, y_\infty)$ is divisible by y_∞^n, so we may write

$$f_\infty(x_\infty, y_\infty) = y_\infty^n g(x_\infty, y_\infty)$$

This implies that the order of $g(0, y_\infty)$ is zero, so $g(0,0) \neq 0$. Geometrically, our curve is the non singular curve $y = \phi_\infty(x)$ counted n times. Indeed, for each integer p, we have

$$f(x, c_0x^{\nu_0} + c_1x^{\nu_1} + \cdots + c_px^{\nu_p}) = x^{\nu_0+\nu_1+\cdots+\nu_p} f_p(x, 0),$$

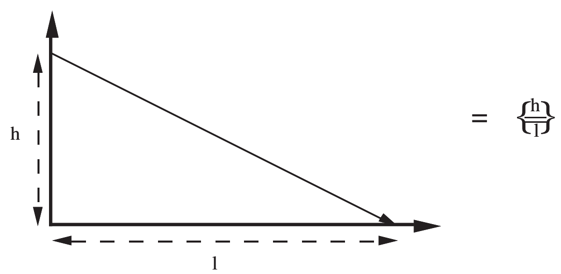
so that by Taylor's expansion theorem, $f(x, \phi(x)) = 0$. This completes in the formal case the proof of the existence of a fractional power series such that $f(x, y(x)) = 0$.

In order to describe all the solutions of the equation $f(x, y) = 0$, it is convenient to develop a little more the formalism of the Newton polygon. Let $\mathcal{P}$ and $\mathcal{P}'$ be two Newton polygons; we can define their sum $\mathcal{P} + \mathcal{P}'$ as the boundary of the convex hull of the Minkowski sum of the convex domains in $\mathbf{R}_+^2$ bounded by $\mathcal{P}$ and $\mathcal{P}'$ respectively. It is easy to verify that the following equality holds for $f,\ f' \in \mathbf{C}[[x, y]]$

$$\mathcal{P}(ff') = \mathcal{P}(f) + \mathcal{P}(f').$$

Any Newton polygon has a *length* and an *height* which are the length of the horizontal and vertical projections of its finite part, respectively.

We say that a Newton polygon is *elementary* if it has only one finite side. If it bounds a finite area, it is then uniquely determined by its length and height. We use the following notation for such an elementary Newton polygon.



We also need a little more algebra, beginning with the following fundamental theorem:
One says that a holomorphic function $f(x_1,\dots,x_d,y)$ defined on a neighborhood of 0 in $\mathbf{C}^d\times\mathbf{C}$ is y-regular (of order n) if $f(0,y)$ has a zero of finite order n at $0\in\{0\}\times\mathbf{C}$. Geometrically this means that if we consider the germ of hypersurface $(W,0)\subset\mathbf{C}^d\times\mathbf{C}$ defined by $f(x_1,\dots,x_d,y)=0$ and the first projection $p\colon W\to\mathbf{C}^d$, then for a small enough representative, if W is not empty (i.e., $n\geq 1$), the fiber $p^{-1}(0)$ is the single point 0. In other words, the fiber is a finite subset of $\{0\}\times\mathbf{C}$. The general idea of the avatars of the Weierstrass preparation theorem is that finiteness of the fiber over one point x in an analytic map implies finiteness of the fibers above points sufficiently close to x.

Theorem 3.1. *(Weierstrass preparation Theorem) If $f(x_1,\dots,x_d,y)$ is regular of order n in y, there exist a unique polynomial of the form*

$$P(x_1,\dots,x_d,y)=y^n+a_1(x_1,\dots,x_d)y^{n-1}+\cdots+a_n(x_1,\dots,x_d)$$

with $a_i\in\mathbf{C}\{x_1,\dots,x_d\}$ and a convergent series $u(x_1,\dots,x_d)$ with $u(0)\neq 0$, i.e., invertible in $\mathbf{C}\{x_1,\dots,x_d\}$ such that we have the equality of convergent series

$$f(x_1,\dots,x_d,y)=u(x_1,\dots,x_d,y)P(x_1,\dots,x_d,y).$$

The polynomial P is said to be *distinguished* in y, or to be a Weierstrass polynomial.

If we start with any power series f, we have the same result but in the ring of formal power series.
It can be shown that, given a function f, for almost every choice of coordinates in $C^n\times\mathbf{C}$, the function f is distinguished with respect to the last coordinate.

It follows from the Weierstrass preparation theorem that provided we have chosen coordinates such that $f(0,y)\neq 0$, say $f(0,y)=a_qy^q+\cdots$ with $a_q\neq 0$, it is equivalent to seek solutions of $f(x,y)=0$ and of $P(x,y)=0$, where $P(x,y)$ is the Weierstrass polynomial

$$u^{-1}(x,y)f(x,y)=y^q+a_1(x)y^{q-1}+\cdots+a_q(x)=0 \quad\text{with } a_i(x)\in\mathbf{C}[[x]]$$

Now from an algebraic point of view, we must consider the field of fractions $\mathbf{C}((x))$ of the integral domain $\mathbf{C}[[x]]$; the irreducible polynomial $T^m-x\in\mathbf{C}((x))[T]$ defines an algebraic extension of degree m of $\mathbf{C}((x))$, denoted by $\mathbf{C}((x^{\frac{1}{m}}))$ which is a Galois extension with Galois group equal to the group

μ_m of m-th roots of unity in $\mathbf{C}$. The action of μ_m is exactly the change in determination of $x^{\frac{1}{m}}$, determined by $x^{\frac{1}{m}} \mapsto \omega x^{\frac{1}{m}}$ for $\omega \in \mu_m$. A series of the form $y = \sum a_i x^{\frac{i}{m}}$ such that the greatest common divisor of m and all the exponents i which effectively appear is 1 gives m different series as ω runs through μ_m.

Suppose now that our function f is an irreducible element of $\mathbf{C}[[x,y]]$, and that the order in y of $f(0,y)$ is $n < \infty$. Then the construction described above provides a series $y(x^{\frac{1}{m}})$ with $m \leq n$ such that $f(x, y(x^{\frac{1}{m}})) = 0$. In fact $m = n$ since f is irreducible.The product

$$\Pi_{\omega \in \mu_m}(y - y(\omega x^{\frac{1}{m}}))$$

is a polynomial $Q(x,y) \in \mathbf{C}[[x]][y]$ which, by the algorithm of division of polynomials in $\mathbf{C}((x))[y]$, divides $P(x,y)$; the rest of the division of P by Q is a polynomial of degree $< n-1$ with n different roots; it is zero.
We have therefore $Q(x,y) = P(x,y)$ and $m = n$ in this case.
We remark that the expansions $y(\omega x^{\frac{1}{n}})$ all have the same initial exponent $\frac{l}{h}$, here $\frac{l}{n}$, and by the definition of $Q(x,y)$, only monomials $x^i y^j$ with $\frac{i}{\ell} + \frac{j}{h} \geq \frac{\mu}{\ell}$ appear, and the monomial x^h actually appears. So we have verified:

Proposition.- *The Newton polygon of an irreducible series is elementary, and of the form* $\{\frac{p}{n}\}$*, where* n *is the order of* $f(0,y)$.

Now it is known that rings such as $k[[x,y]]$, where k is a field, or $\mathbf{C}\{x,y\}$ are *factorial* ; each element has a decomposition $f = f_1^{a_1} \ldots f_r^{a_r}$ where f_i is irreducible, which means that it cannot be factored again as a product $f_i = gh$ in a non trivial way, that is, without g or h being an invertible element in $k[[x,y]]$, (= a series with a non zero constant term).
My aim now is to prove the following

Theorem 3.2. .- *a) Let* k *be an algebraically closed field of characteristic zero, and let* $f \in k[[x,y]]$ *be a power series without constant term such that* $f(0,y) \neq 0$. *Consider the decomposition* $f = uf_1^{a_1} \ldots f_r^{a_r}$ *of* f *into irreducible Weierstrass polynomials* $f_i^{a_i}$*, with a factor* u *which is invertible in* $k[[x,y]]$. *For each index* i, $1 \leq i \leq r$*, there are power series without constant term* $x_i(t), y_i(t) \in k[[t]]$ *such that* $f(x_i(t), y_i(t)) \equiv 0$*; we may choose* $x_i(t) = t^{m_i}$ *where* m_i *is the degree of the Weierstrass polynomial* f_i*, and* $y_i(t)$ *is then uniquely determined. Moreover if we then write* $y_i(t) = c_i t^{l_i} + \ldots$ *with* $c_i \in k^*$*, then the Newton polygon of* f *in the coordinates*

(x, y) is the sum

$$\mathcal{N}(f) = \sum_1^r \{\frac{a_i m_i}{a_i l_i}\}.$$

Here we have to allow the case where for some i, $y_i(t) \equiv 0$, that is $l_i = \infty$.
b) If $k = \mathbf{C}$ and $f \in \mathbf{C}\{x, y\}$ is a convergent power series, the series $x_i(t)$ and $y_i(t)$ are also convergent.

Remark: if we do not assume $f(0, y) \neq 0$, a similar result holds, but we may no longer apply Weiertrass' theorem and we have to allow expansions of the form $x = 0,\ y = t$ and the corresponding Newton polygons appears as summands in $\mathcal{N}(f)$.

The geometric interpretation of this result is that if we take any reduced analytic plane curve $f = uf_1 \dots f_r$ with f_i irreducible, i.e., all $a_i = 1$, the curve defined by $f = 0$ is a sufficiently small neighborhood of the origin is the analytic image of a representative of a complex-analytic map-germ

$$\bigsqcup_{i=1}^r (\mathbf{C}, 0)_i \longrightarrow (\mathbf{C}^2, 0)$$

which we can explicitely build by using Newton's method.
Remark The Newton polygon depends upon the coordinates. One usually chooses the coordinates (x, y) in such a way that the degree of the Weierstrass polynomial is equal to the order of the equation $f(x, y)$. I leave it as an exercize to show that if one writes the series f as a sum of homogeneous polynomials

$$f(x, y) = f_n(x, y) + f_{n+1}(x, y) + \cdots,$$

where f_i is homogeneous of degree i, this condition is equivalent to: $f_n(0, y) \neq 0$.

Conversely, given two power series $x(t), y(t) \in k[[t]]$ without constant term, one may *eliminate* t between them to produce an equation $f(x, y) = 0$ with the property that $f(x(t), y(t) = 0$. Indeed, by using the "natural" elimination process (see[T1]) we may do this in such a way that eliminating t beween $x(t^q), y(t^q)$ produces the equation $f^q(x, y)$, so that we may even represent parametrically a non-reduced equation.

There are several ways to prove this theorem; one is to prove the convergence first, either directly by providing bounds for the coefficient of the series produced by Newton's method, which works but is inelegant, or by considering the analytic curve $f(t^m, y) = 0$, and proving that it is a ramified analytic covering of the t-axis; it is also the union of m non singular curves, so each of them is analytic, and this proves the convergence of the series. (see [L], II.6).

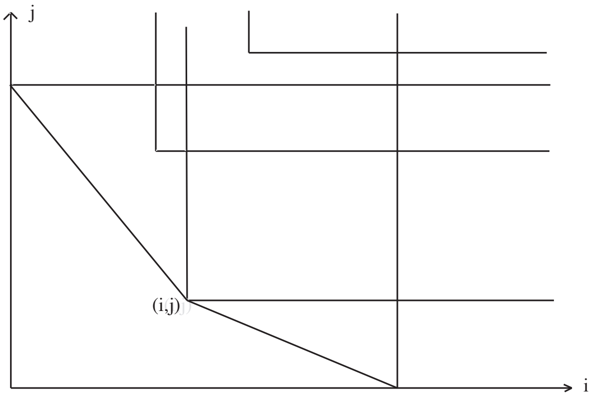
These proofs give no basis for generalizations to higher dimension, so I chose to present a geometric method of constructing the analytic map

$$\bigsqcup_{i=1}^{r} (\mathbf{C}, 0)_i \longrightarrow (\mathbf{C}^2, 0).$$

This method was perfected by Hironaka and is the basis for his method of resolution in all dimensions over a field of characteristic zero.

Remark In the study of analytic functions of one variable near one of their zeroes, a basic fact is that given two monomials $x^a,\ x^b$, one must divide the other in $\mathbf{C}\{x\}$, $\mathbf{C}[[x]]$, or even $\mathbf{C}[x]$. This allows us to write any series $f(x) = x^a u(x)$ with $u(0) \neq 0$, in $\mathbf{C}\{x\}$, and the local behavior of f is determined by the integer a. It is no longer true that given two monomials in (x, y), one must divide the other; the typical example is the pair of monomials y^n, x^q. In particular, the ideal of $\mathbf{C}\{x, y\}$ generated by all the monomials appearing in the expansion $f(x, y) = \sum a_{ij} x^i y^j$ is no longer principal. However, since $\mathbf{C}\{x, y\}$ is a nœtherian ring, this ideal is finitely generated. If we plot the quadrant $R_{ij} = (i, j) + \mathbf{R}^2_+$ for each monomial $x^i y^i$ appearing in our series, and observe that the integral points in this quadrant correspond to the monomials which are multiples of $x^i y^j$, we have a graphic way of representing the generators of the ideal generated by all the monomials appearing in the series f: Consider the union of all the R_{ij} for $(i, j)/a_{ij} \neq 0$; its boundary is a sort of staircase. Our generators correspond to the insteps of the staircase. The convex hull of the union is the Newton polygon.

Note finally that from the viewpoint of considering lines $L_\nu \ : i + j\nu = c$ as above, and where they meet the staircase, it is the convex hull which is relevant.



4. Puiseux exponents

Let $f(x,y) \in \mathbf{C}\{x,y\}$ be such that $f(0,y) = y^n u(y)$ with $u \in \mathbf{C}\{y\},\ o(0) \neq 0$. As we have just seen, it is equivalent to find solutions $y(x)$ for f and to find roots of the Weierstass polynomial

$$P(x,y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$$

corresponding to f.

If the element $f(x,y) \in \mathbf{C}\{x,y\}$ is irreducible, so is the Weierstrass polynomial in $\mathbf{C}\{x\}[y]$.

Newton's theorem tells us that such an irreducible polynomial has all its roots of the form

$$y = \sum_{i=1}^{\infty} a_i \omega^i x^{\frac{i}{n}}$$

where ω runs through the n-th roots of unity in $\mathbf{C}$.

This is equivalent to the statement that an analytically irreducible curve as above can be parametrized in the following manner:

$$x = t^n$$
$$y = \sum_{i=1}^{\infty} a_i t^i$$

In particular, this shows that the polynomial P determines a Galois extension of the field of fractions $\mathbf{C}\{\{x\}\}$ of $\mathbf{C}\{x\}$, and of the field of fractions $\mathbf{C}((x))$ of $\mathbf{C}[[x]]$, with Galois group μ_n.
A direct consequence of this is the:

Theorem 4.1. *(Newton-Puiseux Theorem). The algebraic closure of the field* $\mathbf{C}\{\{x\}\}$ *(resp.* $\mathbf{C}((x))$ *) is the field* $\bigcup_{n\geq 1}\mathbf{C}\{\{x^{\frac{1}{n}}\}\}$ *(resp.* $\bigcup_{n\geq 1}\mathbf{C}((x^{\frac{1}{n}}))$ *)*

This result is the algebraic counterpart of the fact that the fundamental group of a punctured disk is $\mathbf{Z}$.

A linear projection onto $(\mathbf{C}, 0)$ of small representative of a germ of complex analytic curve $(X, 0) \subset (\mathbf{C}^2, 0)$ which is finite (the curve does not contain the kernel of the linear projection as one of its components) can be restricted over a small punctured disk $\mathbf{D}^*_\eta = \mathbf{D}_\eta \setminus \{0\}$ to give a finite covering of $\mathbf{D}^*_\eta$, which is connected for all sufficiently small η if and only if $(X, 0)$ is analytically irreducible at 0. It can be shown that conversely the connected coverings of a punctured disk correspond to irreducible curves as above, and that in this correspondence, the Galois group of the covering, which is of the form $\mathbf{Z}/k\mathbf{Z}$ since the fundamental group of the disk is $\mathbf{Z}$, corresponds to the Galois group of the extension of the field $\mathbf{C}\{\{x\}\}$ or of the field $\mathbf{C}((x))$ defined as above by the curve.

Let

$$f(x, y) = 0 \text{ with } f(x, y) \in \mathbf{C}\{x, y\}$$

be an equation for a branch $(X, 0) \subset (\mathbf{C}^2, 0)$, which means that the series f is an irreducible element of $\mathbf{C}\{x, y\}$.

As we saw, we may assume thanks to the Weierstrass preparation theorem that f is of the form

$$f(x, y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$$

where n is the intersection multiplicity at 0 of the branch C with the axis $x = 0$.

As we saw, after possibly a change of coordinates to achieve that $x = 0$ is transversal to it at 0,the branch X can be parametrized near 0 as follows

$$\begin{aligned} x(t) &= t^n \\ y(t) &= a_m t^m + a_{m+1} t^{m+1} + \cdots + a_j t^j + \cdots \qquad \text{with } m \geq n \end{aligned}$$

Let us now consider the following grouping of the terms of the series $y(t)$: set $\beta_0 = n$ and let β_1 be the smallest exponent appearing in $y(t)$ which is not divisible by β_0. If no such exponent exists, it means that y is a power series in x, so that our branch is analytically isomorphic to $\mathbf{C}$, hence non singular. Let us suppose that this is not the case, and set $e_1 = (n, \beta_1)$, the greatest common divisor of these two integers. Now define β_2 as the smallest exponent appearing in $y(t)$ which is not divisible by e_1. Define $e_2 = (e_1, \beta_2)$; we have $e_2 < e_1$, and we continue in this manner. Having defined $e_i = (e_{i-1}, \beta_i)$, we define β_{i+1} as the smallest exponent appearing in $y(t)$ which is not divisible by e_i. Since the sequence of integers

$$n > e_1 > e_2 > \cdots > e_i > \cdots$$

is strictly decreasing, there is an integer g such that $e_g = 1$. At this point, we have structured our parametric representation as follows:

$$\begin{aligned} x(t) &= t^n \\ y(t) &= a_n t^n + a_{2n} t^{2n} + \cdots + a_{kn} t^{kn} + \\ &\quad + a_{\beta_1} t^{\beta_1} + a_{\beta_1 + e_1} t^{\beta_1 + e_1} + \cdots + a_{\beta_1 + k_1 e_1} t^{\beta_1 + k_1 e_1} \\ &\quad + a_{\beta_2} t^{\beta_2} + a_{\beta_2 + e_2} t^{\beta_2 + e_2} + \cdots + a_{\beta_q} t^{\beta_q} + a_{\beta_q + e_{q-1}} t^{\beta_q + e_{q-1}} + \cdots \\ &\quad + a_{\beta_g} t^{\beta_g} + a_{\beta_g + 1} t^{\beta_g + 1} + \cdots \end{aligned}$$

where by construction the coefficients of the $t^{\beta_i}; i \geq 1$ are not zero. Let us define integers n_i and m_i by the equalities

$$e_{i-1} = n_i e_i, \quad \beta_i = m_i e_i \qquad \text{for } 1 \leq i \leq g$$

and note that we may rewrite the expansion of y into powers of t as an expansion of y into fractional powers of x as follows:

$$\begin{aligned} y = {} & a_n x + a_{2n} x^2 + \cdots + a_{kn} x^k + \\ & + a_{\beta_1} x^{\frac{m_1}{n_1}} + a_{\beta_1 + e_1} x^{\frac{m_1 + 1}{n_1}} + \cdots + a_{\beta_1 + k_1 e_1} x^{\frac{m_1 + k_1}{n_1}} \\ & + a_{\beta_2} x^{\frac{m_2}{n_1 n_2}} + a_{\beta_2 + e_2} x^{\frac{m_2 + 1}{n_1 n_2}} + \cdots + a_{\beta_q} x^{\frac{m_q}{n_1 n_2 \cdots n_q}} + a_{\beta_q + e_{q-1}} x^{\frac{m_q + 1}{n_1 n_2 \cdots n_q}} + \cdots \\ & + a_{\beta_g} x^{\frac{m_g}{n_1 n_2 \cdots n_g}} + a_{\beta_g + 1} x^{\frac{m_g + 1}{n_1 n_2 \cdots n_g}} + \cdots \end{aligned}$$

The set of pairs of coprime integers (m_i, n_i) are sometimes also called the Puiseux characteristic pairs. Their datum is obviously equivalent to that of the characteristic exponents β_i. The sequence of integers $B(X) =$

$(\beta_0, \beta_1, \ldots, \beta_g)$, where $\beta_0 = n$, may be characterized algebraically as follows: let μ_n denote the group of n-th roots of unity. For $\omega \in \mu_n$ let us compute the order in t of the series $y(t) - y(\omega t)$. If we write $\omega = e^{\frac{2\pi ik}{n}}$, we have

$$y(\omega t) = a_n \omega^n t^n + \cdots + a_{\beta_1} \omega^{\beta_1} t^{\beta_1} + \cdots$$

and we see that multiplying t by ω does not affect the terms in t^{jn}. The term in t^{β_1} is unchanged if and only if $\omega^{\beta_1} = 1$, that is $\frac{k\beta_1}{n}$ is an integer, i.e., $k\beta_1 = ln$ or $km_1 = ln_1$ with the notations introduced above. Since n_1 and m_1 are coprime, this means that k is a multiple of n_1, which is equivalent to saying that ω belongs to the subgroup $\mu_{\frac{n}{n_1}}$ of μ_n consisting of $\frac{n}{n_1} = n_2 \cdots n_g$-th roots of unity. If this is the case, then the coefficients of all the terms of the form $t^{\beta_1 + je_1}$ in the Puiseux expansion are also unchanged when t is multiplied by ω, and the first term which may change is $a_{\beta_2} t^{\beta_2}$. An argument similar to the previous one shows that if $\omega \in \mu_{\frac{n}{n_1}}$, then $\omega^{\beta_2} = 1$ if and only if $\omega \in \mu_{\frac{n}{n_1 n_2}}$, and so on.

Finally, if we denote by v the order in t of an element of $\mathbf{C}\{t\}$, we see that

$$v(y(t) - y(\omega t)) = \beta_i \ \text{ if and only if } \ \omega \in \mu_{\frac{n}{n_1 \cdots n_{i-1}}} \setminus \mu_{\frac{n}{n_1 \cdots n_i}} \ \text{ for } \ 1 \le i \le g$$

This provides an algebraic characterization, and a sequence of cyclic subextensions

$$\mathbf{C}\{x\} \subset \mathbf{C}\{x^{\frac{1}{n_1}}\} \subset \mathbf{C}\{x^{\frac{1}{n_1 n_2}}\} \subset \cdots \subset \mathbf{C}\{x^{\frac{1}{n_1 n_2 \cdots n_i}}\} \subset \cdots \subset \mathbf{C}\{x^{\frac{1}{n}}\}$$

corresponding to the nested subgroups $\mu_{\frac{n}{n_1 \cdots n_i}}$ of th group μ_n.

This shows that the sequence $(\beta_0, \beta_1, \ldots \beta_g)$ depends only upon the ring inclusion $\mathbf{C}\{x\} \subset \mathcal{O}_{X,0}$.

We shall see later in a different way that this sequence does not depend upon the choice of coordinates (x, y) in which we write the Puiseux expansion *as long as the curve $x = 0$ is transversal to X.* If this is not the case, one still obtains other characteristic exponents, which are related to the transversal ones by the *inversion formula* which I leave as an exercise (or see [PP] and [GP]).
As an example consider the curve with equation $y^3 - x^2 = 0$.

Remark The Newton-Puiseux theorem is strictly a characteristic zero fact. It implies in particular that if a fractional power series in x is a solution of an algebraic equation with coefficients in $\mathbf{C}[x]$ or $\mathbf{C}\{x\}$ or $\mathbf{C}[[x]]$, the denominators of the exponents of x appearing in that power series are bounded. Let k be a field of characteristic p, and consider the series where the exponents have unbounded denominators:

$$y = \sum_{i=1}^{\infty} x^{1-\frac{1}{p^i}};$$

it is a solution of the algebraic equation

$$y^p - x^{p-1}(1+y),$$

as one can check directly. It is an *Artin-Schreier* equation.
The corresponding result in positive characteristic has been proved recently by Kiran S. Kedlaya (see [Ke]) and is quite a bit more delicate.

5. From parametrizations to equations

We have just seen an algorithm to produce local parametrizations of the branches of a complex analytic plane curve from its equation. To go in the other direction is to *eliminate* for each parametrized branch the variable t between the equations $x - x(t) = 0,\ y - y(t) = 0$, and then make the product of the equations obtained.
Elimination is in general computationally arduous. Is this special case, we have a direct method as follows: write our parametrization in the form $x = t^n, y = \xi(t) = \sum_i a_i t^i$. The product

$$\Pi_{\omega\in\mu_n}(y - \xi(\omega t))$$

is invariant under the action of μ_n by $t \mapsto \omega t$; it is a series $f(t^n, y)) = f(x, y)$ which has the property that $f(x, y) = 0$ is an equation for our curve. However this method does not work for curves in 3-dimensional space, or in positive characteristic. Here is my favourite method (see [T5]) to compute images, explained in this special case.

5.1. *Fitting ideals*

Let M be a finitely generated module over a commutative nœtherian ring A; then we have a presentation, which is an exact sequence of A-modules

$$A^q \to A^p \to M \to 0$$

The map A-linear map $A^q \to A^p$ is represented, in the canonical basis, by a matrix with entries in A. For each integer j, consider the ideal $F_j(M)$ of A generated by the $(p-j)\times(p-j)$ minors of that matrix. Note that if $j \geq p$,

then $F_j(M) = A$ (the empty determinant is equal to 1), and if $p - j > q$, then $F_j(M) = 0$ (the ideal generated by the empty set is (0)).
It is not very difficult to check that $F_j(M)$ depends only on the A-module M, and not on the choice of presentation. Moreover, if $A \to B$ is a morphism of commutative rings, the sequence

$$B^q \to B^p \to M \otimes_A B \to 0$$

is a presentation of the B-module $B \otimes_A M$, with the same matrix; therefore $F_j(M \otimes_A B) = F_j(M).B$. One says that *The formation of Fitting ideals commutes with base change.*

The most important feature of Fitting ideals, is as follows:

Proposition 5.1. *A maximal ideal m of A contains $F_j(M)$ if and only if*

$$dim_{A/m} M \otimes_A A/m > j.$$

Proof. Tensoring with A/m the presentation of M gives for each maximal ideal of A an exact sequence of A/m-vector spaces

$$(A/m)^q \to (A/m)^p \to M \otimes_A A/m \to 0.$$

the dimension of the cokernel is $> j$ if and only if the rank of the matrix describing the map $(A/m)^q \to (A/m)^p$ is $< p - j$, which means that all the $p - j$ minors are 0 modulo m, which means that $F_j(M) \subset m$. $\square$

Let me explain what this has to do with elimination: suppose that we have a branch parametrized by $x(t), y(t)$. This gives a map $\mathbf{C}\{x, y\} \to \mathbf{C}\{t\}$. Observe that this map of $\mathbf{C}$-algebras gives $\mathbf{C}\{t\}$ the structure of a finitely generated $\mathbf{C}\{x, y\}$-module. Indeed, since $x \mapsto t^n$ say, it is even a finitely generated $\mathbf{C}\{x\}$-module, generated by $(1, t, \ldots, t^{n-1})$.
We can therefore write a presentation of $\mathbf{C}\{t\}$ as $\mathbf{C}\{x, y\}$-module:

$$\mathbf{C}\{x, y\}^q \to \mathbf{C}\{x, y\}^p \to \mathbf{C}\{t\} \to 0.$$

Now it is a theorem of commutative algebra that since $\mathbf{C}\{x, y\}$ is a 2-dimensional regular local ring, for every finitely generated $\mathbf{C}\{x, y\}$-module M, if we begin to write a free resolution by writing M as a quotient of a finitely generated free $\mathbf{C}\{x, y\}$-module, $\mathbf{C}\{x, y\}^p \to M$, then writing the kernel of that map as a quotient of a finitely generated free $\mathbf{C}\{x, y\}$-module, and so on, this has to stop after 2 steps. This means that the kernel above is already free, so that in fact, in our case, we have an exact sequence

$$0 \to \mathbf{C}\{x, y\}^q \xrightarrow{\phi} \mathbf{C}\{x, y\}^p \to \mathbf{C}\{t\} \to 0.$$

This immediately implies that we have $q \leq p$. On the other hand, the $\mathbf{C}\{x,y\}$-module $\mathbf{C}\{t\}$ must be a *torsion* module, which means that it must be annihilated by some element of $\mathbf{C}\{x,y\}$; intuitively this means that the image of our parametrization has an equation: an element $f \in \mathbf{C}\{x,y\}$ such that $f\mathbf{C}\{t\} = f(x(t),y(t)) = 0$. If this was not the case, there would be an ideal $T \subset \mathbf{C}\{t\}$ consisting of the elements which are annihilated by multiplication by some non zero element of $\mathbf{C}\{x,y\}$. If we assume that $T \neq \mathbf{C}\{x,y\}$ and remark that by construction our map of algebras $\mathbf{C}\{x,y\} \to \mathbf{C}\{t\}$ induces an injection $\mathbf{C}\{x,y\} \subset \mathbf{C}\{t\}/T$, then either $T \neq 0$ and we have an injection of $\mathbf{C}\{x,y\}$ in a finite-dimensional vector space over $\mathbf{C}$, which is absurd, or $T = 0$ and we have an injection $\mathbf{C}\{x,y\} \subset \mathbf{C}\{t\}$, but since $\mathbf{C}\{t\}$ is a finitely generated $\mathbf{C}\{x,y\}$-module, the two rings should have the same dimension, by the third axiom of dimension theory (see [Ei], 8.1), which is absurd. So $\mathbf{C}\{t\}$ is a torsion $\mathbf{C}\{x,y\}$-module, which implies that the map induced by ϕ after tensorization of our exact sequence by the field of fractions $\mathbf{C}\{\{x,y\}\}$ of $\mathbf{C}\{x,y\}$ is surjective, hence $q \geq p$ and finally $q = p$.

Now we know that q must be equal to p, and that we have an exact sequence

$$0 \to \mathbf{C}\{x,y\}^p \xrightarrow{\phi} \mathbf{C}\{x,y\}^p \to \mathbf{C}\{t\} \to 0.$$

Proposition 5.2. *The 0-th Fitting ideal of the $\mathbf{C}\{x,y\}$-module $\mathbf{C}\{t\}$ is principal and generated by the determinant of the matrix encoding the homomorphism ϕ in the canonical basis.*

Example 5.1. Consider the parametrization $x = t^2, y = t^3$; it makes $\mathbf{C}\{t\}$ into a $\mathbf{C}\{x,y\}$-module generated by $(e_0 = 1, e_1 = t)$. The relations are $-ye_0 + xe_1, x^2e_0 - ye_1$. In this case, $p = 2$ and the matrix ϕ has entries

$$\begin{pmatrix} -y & x \\ x^2 & -y \end{pmatrix}$$

Exercise: 1) For any integer k, consider the curve parametrized by $x = t^{2k}, y = t^{3k}$. Show that the Fitting ideal is generated by $(y^2 - x^3)^k$.
2) Consider the curve in $\mathbf{C}^3$ parametrized by $x = t^3, y = t^4, z = 0$. In this case, the $\mathbf{C}\{x,y,z\}$-module $\mathbf{C}\{t\}$ is generated by $1, t, t^2$. Of course we can no longer hope to have $q = p$ in its presentation, but one can compute a presentation (see [T5], 3.5.2)

$$\mathbf{C}\{x,y,z\}^6 \to \mathbf{C}\{x,y,z\}^3 \to \mathbf{C}\{t\} \to 0.$$

and find that the Fitting ideal

$$F_0(\mathbf{C}\{t\}) = (y^3 - x^4, z^3, zx^2, zy^2, z^2y, z^2x)\mathbf{C}\{x,y,z\}$$

It defines the plane curve $y^3 - x^4 = 0, z = 0$, plus a 0-dimensional (embedded) component sticking out of the $z = 0$ plane.

The appearance of this embedded component corresponds to the fact that we can embed our plane curve singularity in a family of space curve singularities, say $x = t^3, y = t^4, z = vt^5$ for example, where v is a deformation parameter. Now if we compute the Fitting ideal of the image of the map $\mathbf{C} \times \mathbf{C} \to \mathbf{C}^3 \times \mathbf{C}$ given by $x = t^3, y = t^4, z = vt^5, v = v$, and then set $v = 0$, we must find the Fitting ideal defining the image of our original map $\mathbf{C} \to \mathbf{C}^3$. This is one of the basic properties of Fitting ideals. On the other hand, by a classical result, the embedding dimension in a family of singularities can only increase by specialization. Since for $v \neq 0$ the image computed by the Fitting ideal has embedding dimension 3, it must also be 3 for $v = 0$, so the Fitting ideal must define something which contains our plane curve but has embedding dimension 3. Our zero-dimensional component increases the embedding dimension from 2 to 3.

Now we must prove that the generator of the 0-th Fitting ideal is an acceptable equation of the image of our parametrization.

Given our map $\pi\colon \mathbf{C}\{x, y\} \to \mathbf{C}\{t\}$ corresponding to the curve parametrization, we could say that the equation of the parametrized curve is given simply by the kernel K of this map of algebras. We are going to prove that the kernel K and the Fitting ideal have the same radical, and so define the same underlying set, but they are not equal in general, and the formation of the kernel *does not* commute with base extension while the formation of the Fitting ideal does. First, we must check that, with the notations of the definition of Fitting ideals, we have $F_0(M).M = 0$, which means that the Fitting ideal is contained in the kernel. In our case, where $q = p$, it follows directly from Cramer's rule if you interpret the statement as: $\det\phi.\mathbf{C}\{x, y\}^p \in \mathrm{Image}(\phi)$. Note that this is true in the general situation of a finitely generated A-module; the Fitting ideal is contained in the annihilator of M. Secondly we must prove that K is contained in the radical of $F_0(M)$. Take a non zero element $h \in K$; we have $hM = 0$, so that applying the rule of base extension to the map $A \to A[h^{-1}]$, with $A = \mathbf{C}\{x, y\}$ in this case, we get $F_0(M)A[h^{-1}] = A[h^{-1}]$, and since $F_0(M)$ is finitely generated, this implies that there exists an integer s such that $h^s \in F_0(M)$, and the result.

So we have proved the inclusions

$$F_0(M) \subseteq K \subseteq \sqrt{F_0(M)}.$$

5.2. *A proof of Bézout's theorem (after [T5], §1)*

We begin with a Fitting definition of the resultant of two polynomials in one variable. Let A be a commutative ring and

$$\begin{aligned} P &= p_0 + p_1 X + \cdots + p_n X^n \\ Q &= q_0 + q_1 X + \cdots + q_m X^m \end{aligned}$$

be two polynomials in $A[X]$ Let us assume that p_n and q_m are invertible in A. The natural ring in which to treat the resultant is

$$\mathcal{A} = \mathbf{Z}[p_0, \dots, p_n, q_0, \dots, q_m, p_n^{-1}, q_m^{-1}],$$

considering the two polynomials

$$\begin{aligned} \mathcal{P} &= p_0 + p_1 X + \cdots + p_n X^n \\ \mathcal{Q} &= q_0 + q_1 X + \cdots + q_m X^m \end{aligned}$$

now with coefficients in $\mathcal{A}$. The difference is that now the p_i, q_j have become indeterminates.

Given any ring A and two polynomials P, Q as above with coefficients in A and highest coefficients invertible in A, there is a unique homomorphism $ev\colon \mathcal{A} \to A$ such that $ev(\mathcal{P}) = P,\ ev(\mathcal{Q}) = Q$; it sends the indeterminate p_i (resp. q_j) to the coefficient of X^i in P (resp. X^j in Q).
The $\mathcal{A}$-module $\mathcal{A}[X]/\mathcal{P}$ is a free $\mathcal{A}$-module of rank n, and multiplication by $\mathcal{Q}$ (which is injective since the p_i, q_j are indeterminates, gives us an exact sequence

$$0 \to \mathcal{A}[X]/\mathcal{P} \xrightarrow{\phi} \mathcal{A}[X]/\mathcal{P} \to \mathcal{A}[X]/(\mathcal{P}, \mathcal{Q}) \to 0.$$

This allows us to compute the 0-th Fitting ideal of the $\mathcal{A}$-module $\mathcal{A}[X]/(\mathcal{P}, \mathcal{Q})$ as the determinant of the matrix of ϕ.

Definition 5.1. A universal resultant $\mathcal{R}(\mathcal{P}, \mathcal{Q})$ of the universal polynomials $\mathcal{P}$ and $\mathcal{Q}$ is a generator with coprime integer coefficients of the 0th Fitting ideal of the $\mathcal{A}$-module $\mathcal{A}[X]/(\mathcal{P}, \mathcal{Q})$.

Given a ring A and two polynomials as above, the resultant of P and Q is the image $ev(\mathcal{R}(\mathcal{P}, \mathcal{Q}) \in A$. It may be the zero element.

Note that the ring $\mathcal{A}$ has a grading given by $\deg p_i = n - i,\ \deg q_j = m - j$. If we give X the degree 1, the polynomials $\mathcal{P}$ and $\mathcal{Q}$ are homogeneous of degree n and m respectively for the corresponding grading of $\mathcal{A}[X]$.

In order to deal with graded free modules, it is convenient to introduce the following notation: If A is a graded ring, for any integer e, denote by $A(e)$ the free graded A-module of rank one consisting of the ring A where the degree of an element of degree i in A is of degree $i+e$ in $A(e)$. Any free graded A-module is a sum of $A(e_i)$. The proof of Bézout's theorem relies on the Fitting definition of the resultant and the following two lemmas:

Lemma 5.1. *Let A be a graded ring; for any homogeneous homomorphism of degree zero between free graded A-modules*

$$\Psi\colon \bigoplus_{i=1}^{p} A(e_i) \to \bigoplus_{j=1}^{p} A(f_j),$$

setting $M = \mathrm{coker}\Psi$, the Fitting ideals $F_k(M)$ are homogeneous and moreover

$$\deg F_0(M) = \deg(\det\Psi) = \sum_{i=1}^{p} e_i - \sum_{j=1}^{p} f_j.$$

Proof. To say that the morphism is of degree zero means that it sends an homogeneous element to an homogeneous element of the same degree. This implies that the entries of the matrix of Ψ satisfy

$$\deg\Psi_{ij} = e_i - f_j,$$

and this suffices to make the minors homogeneous; let us check it for the determinant.

Each term in its expansion is a product $\psi_{i_1j_1}\dots\psi_{i_pj_p}$ where each i and j appear exactly once. It is homogeneous of degree

$$\sum e_{i_k} - \sum f_{j_k} = \sum_{i=1}^{p} e_i - \sum_{j=1}^{q} f_j.$$

□

We can now compute the degree of $\mathcal{R}(\mathcal{P},\mathcal{Q}) \in \mathcal{A}$. If we use the presentation given above, we find that the homomorphism ϕ is of degree zero if we give each X^i in the first copy of $\mathcal{A}[X]/\mathcal{P}$ the degree $i+m$ and keep X^j of degree j in the second.

Thus we find

Lemma 5.2. *We have the equality*

$$\deg\mathcal{R}(\mathcal{P},\mathcal{Q}) = \sum_{i=1}^{n}(m+i) - \sum_{j=1}^{n} j = mn.$$

Remark 5.1. 1) There are other presentations for the $\mathcal{A}$-module $\mathcal{A}[X]/(\mathcal{P},\mathcal{Q})$. For example

$$\begin{array}{c} 0 \to \mathcal{A}[X]/(X^n) \oplus \mathcal{A}[X]/(X^m) \to \mathcal{A}[X]/(\mathcal{P}.\mathcal{Q}) \to \mathcal{A}[X]/(\mathcal{P},\mathcal{Q}) \to 0 \\ (\bar{a},\bar{b}) \mapsto \overline{a\mathcal{Q}+b\mathcal{P}} \end{array}$$

or

$$\begin{array}{ccccccc} 0 \to \mathcal{A}[X]/(\mathcal{P}.\mathcal{Q}) & \to & \mathcal{A}[X]/(X^n) \oplus \mathcal{A}[X]/(X^m) & \to & \mathcal{A}[X]/(\mathcal{P},\mathcal{Q}) \to 0 \\ \overline{a} & & \mapsto (\overline{a},\overline{a}) \qquad (\overline{a},\overline{b}) \mapsto & & (\overline{a}-\overline{b}) \end{array}$$

The first of these two gives the usual Sylvester determinant, of size $m+n$. The second follows from from the chinese remainder theorem.
2) The total degree of a polynomial defining an affine plane curve is equal to the degree of the homogeneous polynomial in three variables defining the projective plane curve defining the closure of the affine curve in projective space; it is the degree of the curve.

The other lemma is of the same nature and shows that the Fitting ideal locally computes the image of an intersection of two curve with a multiplicity equal to the intersection multiplicity of the two curves.

Lemma 5.3. *Let R be a discrete valuation ring containing a representative of its residue field k, and let v be its valuation. Let*

$$\Psi\colon R^p \to R^p$$

be an homomorphism of free R-modules whose cokernel M is of finite length, i.e., a finite-dimensional vector space over k. Then we have the equality

$$v(\det\Psi) = \dim_k M.$$

Proof. A discrete valuation ring is a principal ideal domain. By the main theorem on principal ideal domains we can find bases for both R^p such that the matrix representing Ψ is diagonal, with entries $a_1,\dots,a_p$ on the diagonal, say. Then clearly $v(\det\Psi) = \sum_{i=1}^p v(a_i)$ and $\dim_k M = \sum_{i=1}^p \dim_k R/a_iR$. Thus it suffices to consider the case where $p=1$. Then we have $a = u\pi^s$ where π is a generator of the maximal ideal of R, and u is invertible in R. Then $v(a) = s$ while R/aR is the k-vector space freely generated by the images of $1, \pi \dots, \pi^{s-1}$. $\square$

This applies to $R = \mathbf{C}\{t\}$ or $R = \mathbf{C}[t]_{(t)}$, the valuation being the t-adic order.

Now let us begin the proof of Bézout's theorem.
Let A be the graded ring $\mathbf{C}[x_1,x_2]$ and let $P,Q \in \mathbf{C}[x_0,x_1,x_2]$ be two homogeneous polynomials defining the curves C and D in the complex projective plane, of respective degrees m and n. We can write

$$\begin{array}{ll} P = \sum_{i=1}^n p_i(x_1,x_2)x_0^i & \deg p_i = n-i \\ Q = \sum_{j=1}^m q_j(x_1,x_2)x_0^j & \deg q_j = m-j. \end{array}$$

After a change of coordinates, we may assume that the constants p_n and q_m are non zero, hence invertible in A. Geometrically this means that the point with homogeneous coordinates $(1, 0, 0)$ does not lie on either of the curves C and D.

As we saw above, there exists a homogeneous morphism of degree zero $ev\colon \mathcal{A} \to A$ such that $ev(\mathcal{P}) = P,\ ev(\mathcal{Q}) = Q$, and if the resultant $\mathcal{R}(P, Q)$ is not zero, it is of degree mn by Lemma 5.2. I leave it as an exercise to check, using the factoriality of polynomial rings over $\mathbf{C}$, that $\mathcal{R}(P, Q) = 0$ if and only if C and D have a common component.

Let us now consider the projection $\pi\colon \mathbf{P}^2(\mathbf{C}) \setminus (1, 0, 0) \to \mathbf{P}^1(\mathbf{C})$ given by $(x_0, x_1, x_2) \mapsto (x_1, x_2)$. It induces a well defined projection on C and on D since meither of them contains $(1, 0, 0)$. For each point $x \in \mathbf{P}^1(\mathbf{C})$ there are finitely many points $y \in C \cap D$ such that $\pi(y) = x$. By the definition of the resultant and the fact that the formation of the Fitting ideal commutes in particular with localization, we have the following equality:

$$\mathcal{R}(P, Q)\mathcal{O}_{\mathbf{P}^1,x} = F_0\big(\bigoplus_{\pi(y)=x} \mathcal{O}_{\mathbf{P}^2,y}/(P, Q)\mathcal{O}_{\mathbf{P}^2,y}\big).$$

$$v_x(\mathcal{R}(P, Q)) = \sum_{\pi(y)=x} \dim_{\mathbf{C}}\mathcal{O}_{\mathbf{P}^2,y}/(P, Q)\mathcal{O}_{\mathbf{P}^2,y}.$$

Since, if we assume that C and D have no common component, the resultant $\mathcal{R}(P, Q)$ is a homogeneous polynomial of degree mn in (x_1, x_2), it follows from the fundamental theorem of algebra applied to the homogeneous polynomial $\mathcal{R}$ that

$$mn = \sum_{x\in\mathbf{P}^1} v_x(\mathcal{R}(P, Q)) = \sum_{y\in C\cap D} \dim_{\mathbf{C}}\mathcal{O}_{\mathbf{P}^2,y}/(P, Q)\mathcal{O}_{\mathbf{P}^2,y}.$$

This is Bézout's theorem if we agree that the intersection multiplicity of C and D at y is equal to

$$(C, D)_y = \dim_{\mathbf{C}}\mathcal{O}_{\mathbf{P}^2,y}/(P, Q)\mathcal{O}_{\mathbf{P}^2,y}.$$

We shall see later that there are many reasons to do that.
Finally we get

Theorem 5.1. *(Bézout) For closed algebraic curves in* $\mathbf{P}^2(\mathbf{C})$ *without common component, we have*

$$\deg C.\deg D = \sum_{y\in C\cap D} (C, D)_y.$$

6. Resolution of plane curves

Let us consider the projective space $\mathbf{P}^n(\mathbf{C})$ as the space of lines through the origin in $\mathbf{C}^{n+1}$. If we choose coordinates $x_0, \dots, x_n$ on $\mathbf{C}^{n+1}$ the projective space is covered by affine charts U_i, the points of each U_i corresponding to the lines contained in the open set $x_i \neq 0$. It is customary to take homogeneous coordinates $(u_0 : \cdots : u_n)$ on the projective space, corresponding to the lines given parametrically by $x_i = u_i t$, or by the equations $x_i u_j - x_j u_i = 0$, where it is enough to take the n equations for which $j = i + 1$ and $i < n$. The term "homogeneous coordinates " means that for any $\lambda \in k*$ the coordinates $(u_0 : \cdots : u_n)$ and $(\lambda u_0 : \cdots : \lambda u_n)$ define the same point.

Now consider the subvariety Z of the product space $\mathbf{C}^{n+1} \times \mathbf{P}^n$ defined by these n equations. It is a nonsingular algebraic variety of dimension $n+1$ and the first projection induces an algebraic morphism $B_0 \colon Z \to \mathbf{C}^{n+1}$.

The fiber $B_0^{-1}(0)$ is the entire projective space $\mathbf{P}^n(k)$ since when all x_i are zero, all the equations between the u_j vanish,while the fiber $B_0^{-1}(x)$ for a point $x \neq 0$ consists of a unique point because then the coordinates x_i determine uniquely the ratios of the u_j which means a point of $\mathbf{P}^n(k)$. Blowing up a point "replaces the observer at the point by what he sees", because the observer essentially sees a projective space (in fact a sphere, if we think of a real observer, but this is just a metaphor).

A basic properties of blowing up is that it separates lines: in fact consider the algebraic map $\delta \colon \mathbf{C}^{n+1} \setminus \{0\} \to \mathbf{P}^n$ which to a point outside the origin associates the line joining the origin to this point. Of course we cannot extend the definition of this map through the origin; The graph of δ however, is an algebraic subvariety of $(\mathbf{C}^{n+1} \setminus \{0\}) \times \mathbf{P}^n$, and we may take the closure (for the strong topology if $k = \mathbf{C}$, or for the Zariski topology) of this graph. It is a good exercise to check that this closure coincides with Z as defined above. A point of $B_0^{-1}(0)$ is precisely a direction of line, so the map $\delta \circ B_0$ can be defined there as the map which to this point associates the direction: in Z we have separated all the lines meeting at the origin.

Let us consider in more detail the case $n = 1$. Then Z is a surface covered by two affine charts corresponding to the charts of the projective space: for convenience of notation set $u_0 = u,\ u_1 = v,\ x_0 = x,\ x_1 = y$ so that Z is defined by $vx - uy = 0$. On the open set U of Z where $u \neq 0$ we may taxe as coordinates $x_1 = x,\ y_1 = \frac{v}{u}$ and then the map induced by B_0

on U is described in these coordinates by

$$\begin{aligned} x \circ B_0 &= \quad x_1 \\ y \circ B_0 &= x_1 y_1 \end{aligned}$$

and similarly on the open set V defined by $v \neq 0$, we take as coordinates $x_1 = \frac{u}{v},\ y_1 = y$ and the map B_0 is described by

$$\begin{aligned} x \circ B_0 &= x_1 y_1 \\ y \circ B_0 &= \quad y_1 \end{aligned}$$

Remark that in the first chart the projective space $B_0^{-1}(0)$ is defined by $x_1 = 0$ and in the second by $y_1 = 0$ (remember that they are coordinates on two distinct charts, and on the intersection of the two charts they define the same subspace). It is a crucial property of blowing up that it transforms the blown-up subspace (here the origin) into a subspace defined locally by one equation (called a divisor); it is a good exercise to check that this is the case in any dimension. The space $B_0^{-1}(0)$ is called the *exceptionnal divisor*. We are now able to study the effect on a function $f(x, y)$ (formal or convergent) of its composition with the map B_0. Consider the expansion of f as a sum of homogeneous polynomials

$$f(x, y) = f_m(x, y) + f_{m+1}(x, y) + \cdots + f_{m+k}(x, y) + \cdots \ ,$$

where f_j is homogeneous of degree j. In the chart U, we may write

$$\begin{aligned} &f \circ B_0 = f(x_1, x_1 y_1) = \\ &x_1^m \big(f_m(1, y_1) + x_1 f_{m+1}(1, y_1) + \cdots + x_1^k f_{m+k}(1, y_1) + \cdots\big) \end{aligned}$$

and there is a similar expansion in the other chart. Now if we look at the zero set of $f \circ B_0$ we see that in each chart it contains the exceptionnal divisor counted m times. If we remove this exceptionnal divisor as many times as possible, i.e., divide $f \circ B_0$ by x_1^m in the first chart and by y_1^m in the second, we obtain the equation of a curve on the surface Z, either formal or defined near $B_0^{-1}(0)$, which no longer contains the exceptionnal divisor. This curve is called the *strict transform* of the original curve. We also say that the equation obtained in this way is the strict transform of f). In the first chart it is $x_1^{-m} f(x_1, x_1 y_1)$, and in the second $y_1^{-m} f(x_1 y_1, y_1)$.

By construction, the strict transform meets the exceptionnal divisor only in finitely many points; let us determine them: in the first chart they are given by $f_m(1, y_1) = 0$ and in the second, by $f_m(1, y_1) = 0$. By construction of the projective space the points we seek are therefore the points in the projective line defined by the homogeneous equation $f_m(u, v) = 0$. The homogeneous polynomial f_m of lowest degree appearing in $f(x, y)$ is called the *initial form* and $f_m(x, y) = 0$ is a union of m lines (counted with multiplicity) called the *tangent cone* of f at the point 0. So we see that the strict transform of f meets the exceptionnal divisor at the points in this projective space corresponding to the lines which are in the tangent cone at 0 of our curve.

In particular, if our curve has two components with tangent cones meeting only at the origin, their strict transforms are disjoint. Consider for example $f(x, y) = (y^2 - x^3)(y^3 - x^2)$.

In order to analyze in more detail what goes on, we have to *assume that k is algebraically closed, which we will do from now on*, and introduce the concept of *intersection number* of two curves at a point. The simplest definition (but not the most useful for computations) is the following:
Let $f, h \in k[[x, y]]$ be series without constant term and without common irreducible factor. Let (f, h) denote the ideal generated by f and h in $k[[x, y]]$. Then the dimension

$$\dim k[[x, y]]/(f, h)$$

is finite and is by definition the intersection number of the two curves at 0. If $k = \mathbf{C}$ and f, h are in $\mathbf{C}\{x, y\}$, then the dimension above is also

$$\dim \mathbf{C}\{x, y\}/(f, h)$$

where now (f, h) is the ideal generated in $\mathbf{C}\{x, y\}$.

To prove the finiteness we first remark that it is sufficient to prove it after replacing k by its algebraic closure and then we may use the Hilbert nullstellensatz which tells us that since $f = 0,\ h = 0$ meet only at the origin, the ideal (f, h) contains a power of the maximal ideal $\mathbf{m} = (x, y)$ say $\mathbf{m}^N$. This implies the finiteness since $k[[x, y]]/(f, h)$ is then a quotient vector space of $k[[x, y]]/\mathbf{m}^N$ and also shows that we may without changing the ideal assume that f, h are polynomials of degree $< N$, so that for example if f, h are convergent power series the vector spaces $\mathbf{C}\{x, y\}/(f, h)$ and $\mathbf{C}[[x, y]]/(f, h)$ are equal.

The definition of intersection multiplicity at the point 0, of the two curves $f = 0,\ h = 0$, say in the analytic case is then

$$\big(f, h\big)_0 = \dim_{\mathbf{C}} \mathbf{C}\{x, y\}/(f, h).$$

Note that we use large parentheses for the intersection number, small ones for the ideal generated by f, g.

In any case this definition of the intersection multiplicity has the advantage to suggest the following intuitive interpretation :

Consider a 1-parameter deformation of one of the two functions, say $f+\epsilon$; it is possible to show that if f, h converge in a nice neighborhood U of 0, for small enough ϵ, then the two curves $h=0, f+\epsilon=0$ meet in U transversally at points which are non-singular on each. Moreover, these points tend to 0 as ϵ tends to 0, and the number of these points is $\dim \mathbf{C}\{x,y\}/(f,h)$. So this number may be thought of as the number of ordinary intersections (i.e., transverse intersection of non-singular curves) which are concentrated at 0.

There is another way to present this intersection number, which is very useful for computations:
Suppose that $h(x,y) = uh_1^{e_1}\dots h_r^{e_r}$ with $u(0)\neq 0$. For each i, $1\leq i\leq r$, let us parametrize the curve $h_i(x,y)=0$ by $x(t_i), y(t_i)$. Now substitute these power series in $f(x,y)$; we get a series in t_i, the order of which we denote by I_i. Then we have

$$I_i = \dim_{\mathbf{C}} \mathbf{C}\{x,y\}/(f,h_i) \ ,$$

and

$$\big(f,h\big)_0 = \sum_1^r e_i I_i .$$

Remark: Given a germ of curve $f=0$, where

$$f = f_m + f_{m+2} + \cdots \ ,$$

its multiplicity at the origin may be defined as the smallest degree m of a monomial appearing in the series f. A better definition is to say that the multiplicity is the intersection number $\big(f,\ell\big)_0$ for a sufficiently general linear form ℓ. In fact, we have

$$m \leq \big(f,\ell\big)_0$$

with equality if and only if the line $\ell(x,y)=0$ is not in the tangent cone defined by $f_m(x,y)=0$.

Indeed, we may parametrize the line $\ell=0$ by $x=\alpha t,\ y=\beta t$; then we substitute in f:

$$f(\alpha t,\beta t) = f_m(\alpha,\beta)t^m + f_{m+1}(\alpha,\beta)t^{m+1} + \cdots$$

is of order $\geq m$, and of order m exactly if and only if $f_m(\alpha,\beta)\neq 0$.

It is convenient, given a curve $f(x,y)$ and a point z in the plane, to define the *multiplicity of f at z* as follows: take coordinates $(x',y,)$ centered at z, which means that they vanish at z; if $z=(a,b)$ we may take $x'=x-a, y'=$

$y-b$. Then expand f in those coordinates (of course we assume that z is in the domain of convergence of f).

We get $f'(x',y') = f(a+x', b+y')$. Then we compute the lowest degree terms appearing in the expansion of f' and denote this by $m_z(f)$ or, if X is the curve $f(x,y)=0$, by $m_z(X)$. We see that $m_z(f)=0$ unless $f(z)=0$, and that if ℓ is a line through z, we have $m_z(X) \leq \big(X,\ell\big)_z$ with equality except if ℓ is in the tangent cone of X at z.

Let us apply this, in our blowing up as described above, to the line $x_1=0$ (the exceptionnal divisor) and the strict transform $f_1(x_1,y_1)=0$, at a point x' with coordinates $x_1=0, y_1=t_1)$ where $f_m(1,t_1)=0$ i.e., a point of intersection of the strict transform with the exceptionnal divisor. We have

$$f_1 = f_m(1,y_1) + x_1 f_{m+1}(1,y_1) + \cdots$$

and if we denote by $e_{x'}$ the multiplicity of t_1 as a root of the polynomial $f_m(1,Y)$, it follows from what we saw above that we have

$$e_{x'} \geq m_{x'}(f_1)$$

with equality unless the curve $f_1(x_1,y_1)=0$ is tangent to the exceptionnal divisor at the point x', in the sense that the tangent at x' to the exceptionnal divisor is in the tangent cone of $f_1=0$ at the point x'. Since the multiplicity of f_1 is zero at points where $f_m(1,y_1)$ does not vanish, we see that if we look at all the points x' in the blown up surface Z which are mapped to our origin by the projection $Z \to \mathbf{C}^2$, which we denote by $x' \to 0$, we have

$$\sum_{x'\to 0} m_{x'}(f_1) \leq \sum_{x'\to 0} e_{x'} = m \ ,$$

so that in particular, if *there is a point x' of the strict transform X' of X which is mapped to* 0 *and is of multiplicity m on $f_1=0$, then it is the only point of X' mapped to* 0 *and X' is transversal to the exceptionnal divisor at x'*. This fact and its generalizations play a crucial role in Hironaka's proof of the resolution of singularities.

In order to show that the situation which we have just described cannot persist indefinitely in a sequence of blowing ups, we have to use the intersection number in another manner, according to Hironaka:

Given a germ of a plane curve (X,x) with r branches $(X_i,x)_{1\leq i\leq r}$ and a nonsingular curve W through the point x, define the *contact exponent* of W with X at x as follows:

$$\delta_x(W,X) = \min_{i=1}^{r}\left(\frac{\big(X_i,W\big)_x}{m_x(X_i)}\right)$$

and the *contact exponent* of X at x as follows[b]

$$\delta_x(X) = \max_W \delta_x(W, X),$$

where W runs through the set of germs at x of non-singular curves.

Lemma 6.1. *Let $f(x, y) = 0$ be an equation for X. If the coordinates (x, y) are chosen in such a way that $x = 0$ is not tangent to X at x and W is defined by $y = 0$, the rational number $\delta_x(W, X)$ is the inclination of the first side of the Newton polygon of $f(x, y)$.*

By definition of $\delta_x(W, X)$ is enough to prove that for an irreducible f, the inclination of the only side of it Newton polygon is $\frac{(X,W)_x}{m_x(X)}$, but if we parametrize X by $x = t^m,\ y = t^q + \cdots$, we find that the transversality condition implies $m \leq q$, and we have $(X, W)_x = q$; the result follows.

Lemma 6.2. *Assume that W is the curve $y = 0$ and that $f(x, y)$ is in Weierstrass form, i.e.,*

$$f(x, y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x) \qquad a_i(x) \in \mathbf{C}\{x\},$$

then the inclination of the first side of its Newton polygon is

$$\delta_x(W, X) = min_{1\leq i\leq n-1} \frac{\nu_0(a_i)}{i}.$$

Here as usual $\nu_0(a(x))$ denotes the order of vanishing at the origin of the series $a(x)$.

Indeed, the point $(0, n)$ is a vertex of the first side of the Newton polygon, and the lemma is just the observation that if we write $a_i(x) = \alpha_i x^{c_i} + \cdots$, the other vertices of the Newton polygon are among the points $(c_i, n - i)$, which follows directly from the definition.

A nonsingular curve W such that $\delta_x(W, X) = \delta_x(X)$ is said to have *maximal contact* at x. non singular curves with maximal contact are the nonsingular curves which it is hardest to separate from X by a succession of blowing ups (in the sense of separating strict transforms), and so when they eventually separate, something nice should happen; indeed once they separate, there is no point of multiplicity $m_x(X)$ in the iterated strict transform mapping to x. As one says, "the multiplicity has dropped". Hironaka's

[b] This $\delta_x(X)$ is not related to the δ invariant of a singularity which measures the diminution of genus due to the presence of the singularity, and which is denoted in the same way. These are the classical notations, however.

approach to resolution uses the existence of varieties with maximal contact to build an induction on the dimension.

The next step is to prove the existence of curves with maximal contact.

Assume that a non singular curve W defined by $y = 0$ does not have maximal contact with X at x. We way assume that the curve $x = 0$ is transversal to $f(x, y) = 0$, which means that $f(0, y) = a_{0,m}y^m + \cdots$, where m is the multiplicity of f at 0. By a change of variable $y = (a_{0,m})^{\frac{1}{m}} y'$, which does not change the contacts, we may assume that $a_{0,m} = 1$. To say that $\delta_x(W, X) < \delta_x(X)$ means that there is a series $A(x)$ such that the contact of the curve $f(x, y) = 0$ with $y - A(x) = 0$ is greater than its contact with $y = 0$. By a change of the coordinate x which does not affect the contacts, we may assume that $A(x) = \xi x^d$ for some integer d and $\xi \in \mathbf{C}^*$. Let us now compute the power series expansion in the coordinates $x' = x, y' = y - A(x)$;

$$f'(x, y') = \sum_{\frac{i}{\delta}+j\geq m} a_{i,j}x^i(y' + \xi x^d)^j = \sum_{\frac{k}{\delta'}+\ell\geq m} a'_{k,\ell}x^k {y'}^{\ell}.$$

By expanding the powers of $y' + \xi x^d$ we get, for each (i, j), and $k \leq j$ the inequality $\frac{i+kd}{\delta} + j - k \geq m$ but we know that $\frac{i}{\delta} + j \geq m$. From this follows the inequality $d \geq \delta$.
Isolating the terms which lie on the first side of the Newton polygon, we get:

$$(*) \sum_{\frac{i}{\delta}+j=m} a_{i,j}x^i y^j + \sum_{\frac{i}{\delta}+j>m} a_{i,j}x^i y^j = \sum_{\frac{k}{\delta'}+\ell\geq m} a'_{k,\ell}x^k {y'}^{\ell},$$

and the slope of the first side of the Newton polygon of the right-hand side is $\delta' > \delta$. Let us first assume that $\delta = 1$. Remark that all the terms $x^k {y'}^{\ell}$ with $\frac{k}{\delta'} + \ell \geq m$ except ${y'}^m$ are in the ideal $(x, y')^{m+1}$. Therefore we must have the equality

$$\sum_{\frac{i}{\delta}+j=m} a_{i,j}x^i y^j = {y'}^m \ \mathrm{mod}.(x, y)^{m+1}$$

so that the left hand side is the m-th power of $y - \xi x^d$. This implies that $d = 1 = \delta$ since the left hand side is homogeneous.

If $\delta > 1$ we follow the same method. Since we know that $d \geq \delta$, it is easy to check that the ideal of $k[[x, y]]$ generated by the monomials $x^k {y'}^{\ell}$, $\frac{k}{\delta'} + \ell \geq m$, $k \neq 0$ is contained in the ideal I generated by the monomials

x^iy^j , $\frac{i}{\delta}+j>m$. Looking at the equation $(*)$ modulo I gives us

$$\sum_{\frac{i}{\delta}+j=m} a_{i,j}x^iy^j = {y'}^m \text{ mod.}I$$

which again by homogeneity shows that $d=\delta$ and the sum on the left hand side is $(y-\xi x^d)^m$.

Note that this argument also works if $\delta_x(X)=\infty$. So there are two possibilities:

1) We have $\delta_x(W,X)<\delta_x(X)$; in this case the sum of the terms of $f(x,y)$ lying on the first side of the Newton polygon is of the form $(y-\xi x^d)^m$.
2) The sum of the terms of $f(x,y)$ lying on the first side of the Newton polygon is not of the form $(y-\xi x^d)^m$.
In the first case, as we have seen, $d=\delta_x(W,X)$. We make the change of variables $x'=x$; $y'=y-\xi x^d$ and in the new coordinates x',y', if W' is the curve $y'=0$, we have $\delta_x(W',X)>\delta_x(W,X)$. This follows easily from the computation we have just made; an effect of the change of variables is that all the terms lying on the first side of the Newton polygon, of inclination d, are transformed into the single term monomial ${y'}^m$. So the inclination of the new Newton polygon has to be $>d$; but we know this inclination to be $\delta_x(W',X)$. If we have not reached $\delta_x(X)$, we continue the same procedure, and after possibly infinitely many steps, i.e., after a change of variables of the form

$$x'=x\ ;\ y'=y-\xi_1x^{d_1}-\xi_2x^{d_2}-\cdots-\xi_rx^{d_r}-\cdots$$

we reach the stage where the sum of terms on the first side of the Newton polygon is not a $m-th$ power, so $\delta_x(W_s,X)=\delta_x(X)$, with s possibly infinite. Since the denominators of the $\delta_x(W,X)$'s are bounded, the series is infinite only in the case where $\delta_x(X)=\infty$. At least formally this series converges, since we have $d_1>d_2>\cdots>d_r>\ldots$, but we can omit the proof of convergence if we work in $\mathbf{C}\{x,y\}$ since the equality $\delta_x(X)=\infty$ means that in some coordinates $f(x,y)$ is of the form $u(x,y)y^m$ where u is an invertible element in $k[[x,y]]$; indeed for any other case, we see from the definition that $\delta_x(X)<\infty$. But the Weierstrass preparation theorem tells us that if such a presentation exists with formal power series, it also exists with convergent power series, so that the series defining our final coordinates converges.

So in all cases, we can find a nonsingular curve W which has *maximal contact* with X at x, i.e., such that $\delta_x(W,X)=\delta_x(X)$. Remark that all the discussion above is valid on a germ of a non singular surface, since it is analytically isomorphic to the plane. The definition of the blowing up is independant of the choice of coordinates, and makes sense on any nonsingular surface.

The next step is to study the behavior of the contact under blowing up of the origin. I will leave the proof of this as an exercise, since it is a direct application of what we have just seen and the definition of blowing up:

Theorem 6.1. *(Hironaka) Let m be an integer, let $f(x,y)=0$ define a germ of a plane curve, $(X,0)\subset(\mathbf{C},0)$ of multiplicity m and let $(W,0)\subset(\mathbf{C},0)$ be a non singular curve with maximal contact with X at 0. If, after blowing up the point 0 by the map $B_0\colon Z\to\mathbf{C}^2$, there is a point $x'\in X'$ of multiplicity m in the strict transform $X'\subset Z$ of X, then*
1) The point x' is the only point of X' mapped to 0 by B_0,
2) The strict transform W' of W by B_0 contains the point x', and W' has maximal contact with X' at x',
3) We have the equality $\delta_{x'}(W',X')=\delta_x(W,X)-1$.

Corollary 6.1. *The maximal length of a sequence of infinitely near points of multiplicity m on the strict transforms of X, each mapping to its predecessor in successive blowing ups*

$$\cdots\to Z^{(r)}\to Z^{(r-1)}\to\cdots\to Z^{(2)}\to Z^{(1)}\to\mathbf{C}^2$$

is equal to the integral part $[\delta_x(X)]$.

This suffices to show that unless the curve is of the form $y^m=0$, the multiplicity of its strict transform in the sequence of blowing ups obtained by blowing up at each step the points of maximal multiplicity drops after a finite number of steps. By induction on the multiplicity, this proves the resolution of the singularity of X at 0 by a finite number of blowing ups of points on non singular surfaces.We should remark that the map $X'\to X$ of the strict transform of X to X is defined by itself, without any reference to an embedding $(X,0)\subset(\mathbf{C}^2,0)$ (see [K]).

We have proved a local result, but if now we consider any algebraic or analytic curve, it has finitely many singular points, and the local resolution processes at each point are independant, so we have:

Theorem 6.2. *Given an algebraic or analytic plane curve X there exists a finite sequence of point blowing ups such that in the composed map $X'\to X$ the curve X' has no singularities.*

Actually we can get, by the same method, a better result, known as *embedded resolution* and originally due to Max Nœther, as follows:

Theorem 6.3. *Given a curve X on a non singular surface S, there exists a finite sequence of blowing ups of points*

$$S^{(r)} \to \cdots \to S^{(1)} \to S$$

such that if we denote by $\pi\colon S^{(r)} = S' \to S$ their compositum, then the inverse image of the singular points of X (the exceptionnal divisor) is a union of non singular curves (each isomorphic to $\mathbf{P}^1(\mathbf{C})$) meeting transversally on the non singular surface S', and the strict transform X' of X by π is a non singular curve meeting transversally these curves.

In analytic terms, if $f(x,y) = 0$ is a local equation for X in S, then $f \circ \pi$ is, at every point x' of S', of the form $(f \circ \pi)_{x'} = u^a v^b$ for suitable local coordinates of S' at x'. Of course a and b will be zero unless we have $x' \in \pi^{-1}(X)$.The induced map $\pi\colon X' \to X$ is a resolution of singularities of X. If we fix a singular point $x \in X$, let r be the number of analytically irreducible components of the germ (X,x). The number of points in $\pi^{-1}(x)$ is equal to r and for a small enough representative X_x of the germ (X,x), the part $\pi^{-1}(X_x)$ of X' lying over X_x consists of r non singular curves $\mathbf{D}_i$, each marked with one of the points of $\pi^{-1}(x)$. The image by π of each of these non singular curves $\mathbf{D}_i$ is one of the irreducible components of X_x.

If we choose for each non singular curve $\mathbf{D}_i$ a coordinate t_i vanishing at the only point z_i of $\mathbf{D}_i$ lying over x, then $\mathbf{D}_i$ is described parametrically, in local coordinates (u,v) on S' centered at z_i, by *convergent* power series $u(t_i), v(t_i)$, because of the implicit function theorem. Since the map $\pi\colon S' \to S$ is a composition of algebraic maps, $x \circ \pi$ and $y \circ \pi$ are at worst convergent power series in (u,v), so when we restrict them to $\mathbf{D}_i$, we get convergent power series in t_i. This shows that each branch of our curve has a convergent parametrization, and from this we deduce that the formal parametrization constructed by Newton's method converges.

Note that this convergence argument works equally well with the first resolution theorem. The new fact in the resolution result above with respect to the resolution theorem is the transversality of the strict transform with the exceptionnal divisor, which is not part of the resolution theorem as we have stated it above. The proof of this improvement is not difficult: it amounts to resolving singularities, by a sequence of points blowing up, of the union of the strict transform and the exceptional divisor of the map which resolves the singularities of X.

As an example, given an integer $m > 1$, after one blowing up the strict transform of a curve with equation $y^m - x^{m+1} = 0$, is non singular, but it is not transversal to the exceptionnal divisor.

It is the first example of a fundamental fact of analytic or algebraic geometry: *you can make spaces (in fact, their strict transforms) transversal by well chosen sequences of blowing ups.*

7. Resolution of space curves

7.1. *Integral dependance*

To prove a resolution theorem for space curves, one meets the difficulty that their equations may be complicated (for example to define a curve in $\mathbf{C}^n$ one may need more that $n-1$ equations; those for which $n-1$ equations suffice are called *complete intersections*, and also that rather different looking sets of equations may generate the same ideal in $\mathbf{C}\{x_1, \ldots, x_n\}$ and therefore define the same curve. In the proofs above we have used constructions which depend heavily on the equation. Moreover, even to show that a germ of a complex curve in $\mathbf{C}^d$ has a finite number of irreducible components, which are analytic germs, is not completely trivial (see [L], II.5). There are two possibilities: we can conceptualize and abstract the proof for plane curves to make it less dependent on the equation, or try to reduce to the plane curve case. As it happens, the two methods are not so different, at least for one of the ways of abstracting the ideas.
To reduce to the plane curve case, the natural idea is to project the space curve X to a plane curve X_1. One can then show that a resolution of X_1 has to map to X, and that this map is a resolution of the singularities of X!.

The key idea is that of *normalization.* The Italian geometers called *normal* a projective variety $Z \in \mathbf{P}^n$ having the property that any map $Z' \to Z$ presenting Z as a "general" projection by a linear map $\mathbf{P}^{n'} \setminus L \to \mathbf{P}^n$ of an algebraic variety $Z' \subset \mathbf{P}^{n'}$ had to be an isomorphism. A typical non normal surface in $\mathbf{P}^3$ is therefore a general projection of a non singular surface in $\mathbf{P}^4$ which cannot be embedded in $\mathbf{P}^3$; such a projection has a curve of double points, on which are finitely many more complicated singular points, the "pinch points". Here the meaning of "general" has to be made precise;
The variety Z is normal if any map $\pi \colon Z' \to Z$ which
a) is finite-to-one and
b) induces an isomorphisme $Z' \setminus \pi^{-1}(U) \to U$, over the complement U of a closed algebraic or analytic subset of Z of smaller dimension,

is an isomorphism.

The resolution theorem we saw above shows that a singular curve in $\mathbf{P}^2$ cannot be normal.

The concept of normalization was "localized" and transfigured into a concept of commutative algebra as follows: Recall that the total ring of quotients of a ring A is the ring of equivalence classes of couples (a, b) of elements of A, where b is not a zero divisor in A with addition $(a, b)+(a', b') = (ab' + ba', bb')$ and component wise multiplication, the equivalence being $(a, b) \equiv (a', b')$ when $ab' - ba' = 0$. The map $a \mapsto (a, 1)$ induces an injection of A in F and we indentify A with its image in F. If A is an integral domain, F is its field of fractions.

Definition Let A be a commutative ring without nilpotent elements, and let F be its total ring of quotients.
Definition.- An element $h \in F$ is *integral* over A if it satisfies an equation

$$h^k + a_1 h^{k-1} + \cdots + a_k = 0 \text{ with } a_i \in A.$$

Example.- Consider the germ of plane curve X in $\mathbf{C}^2$ defined by the equation $y^p - x^q = 0$. The quotient $\mathcal{O}$ of the ring $\mathbf{C}\{x, y\}$ by the ideal generated by $y^p - x^q$ is the ring of germs of analytic functions on the germ X (the restrictions to X of two analytic functions on $\mathbf{C}^2$ coincide if and only if their difference is in the ideal). The ring $\mathcal{O}$ is an integral domain; let K be its field of fractions. If we keep the notations x, y, etc.. for the restrictions to X of functions on $\mathbf{C}^2$, we have $\frac{y}{x} \in F$. I claim that if $p \leq q$, it is integral over $\mathcal{O}$; indeed, we have the relation

$$(\frac{y}{x})^p - x^{q-p} = 0 \ .$$

We can remark that the function $\frac{y}{x}$ is defined and analytic on the strict transform of X by the blowing up of the origin for any sufficiently small representative of the germ X. We remark also that the condition $p \leq q$ is equivalent to saying that the meromorphic function $\frac{y}{x}$ remains bounded on X for any small representative.

Proposition 7.1. *Given a ring A without nilpotent elements, let F be its total ring of fractions; the set of elements of F integral over A is a ring for the operations induced by those of F.*

This ring is called the normalization of A (or the integral closure of A in F) and often denoted by $\overline{A}$. Of course we have $A \subset \overline{A}$; a ring such that

$A = \overline{A}$ is said to be integrally closed. Is is not difficult to check that $\overline{A}$ is integrally closed.

If A is nœtherian and integrally closed, any injective map $A \to B$ to a subring B of the total ring of fractions of A which makes B into a finite A-module is an isomorphism; this is the translation of the original definition of normality. To prove it, check that if h is an element of B, the powers of h cannot all be linearly independant over A, so h satisfies an integral dependance relation, and if A is normal, it is in A!

An important theorem is that if A is an analytic algebra, i.e., a quotient of a convergent power series ring by some ideal, then $\overline{A}$ is a finite sum of integrally closed analytic algebras, and moreover that the injection $A \to \overline{A}$ makes $\overline{A}$ into a *finitely generated* A-module. Taking a common denominator (in F) for a finite set of generators of the A-module $\overline{A}$, we see that the ("conductor") ideal $\mathcal{C} = \{d \in \overline{A},\ d.\overline{A} \subset A\}$ is not zero.
Another important fact is that if the analytic algebra of germs of functions on a curve at a point is normal, the point is non singular on the curve, and the analytic algebra is isomorphic to a convergent power series ring in one variable $\mathbf{C}\{t\}$. ([L], VI.3, Thm.2)

7.2. *The δ invariant of a plane curve singularity*

Let $\mathcal{O}$ be the analytic algebra of a germ of curve $(X, 0)$, plane or not, and let $\overline{\mathcal{O}}$ be its normalization. Since it is an $\mathcal{O}$-module of finite type with the same total ring of quotients, a version of the Hilbert Nullstellensatz shows that the quotient *vector space* over $\mathbf{C}$ is finite dimensional. So we may define an invariant to measure how far $\mathcal{O}$ is from being integrally closed, i.e., regular:

$$\delta_{X,0} = \dim_{\mathbf{C}} \frac{\overline{\mathcal{O}}}{\mathcal{O}}$$

In the case of plane curves, this invariant is known as the *diminution of genus* which the presence of the singularity imposes on the curve:
Take a projective plane curve C of degree d. If it is non singular, it is topologically a differentiable surface, classified by its *genus*. This genus is equal to

$$g(C) = \frac{(d-1)(d-2)}{2}.$$

If the curve has singularities, then its normalization is topologically a differentiable surface, and its genus is

$$g(\overline{C}) = \frac{(d-1)(d-2)}{2} - \Sigma_{x \in C}\delta_{C,x},$$

where the sum on the right is finite since $\delta_{C,x}$ is nonzero only at singular points, and those are finite in number.
The genus of the normalization of C is traditionally called the *geometric genus* of C.
Moreover, the local invariant $\delta_{C,x}$ has a local geometrical interpretation, (see [T4], [T6]) which I will describe only in the case of a branch, for simplicity:
Let $t^n, y(t)$ be a parametrization of our branch X. Consider the product of the normalization of X with itself, with coordinates $(t't')$ and the two curves in $(\mathbf{C}^2, 0) = (\overline{X} \times \overline{X}, 0)$ defined by

$$\frac{t^n - t'^n}{t - t'} = 0 \quad \frac{y(t) - y(t')}{t - t'} = 0$$

The intersection number of these two curves at the origin is equal to $2\delta_X$; if now we perturb slightly the parametrization of X by $t^n + \alpha vt, y(t) + \beta vt$ with two "general " complex numbers $\alpha,\ \beta$, we can see that the two curves now have deformed equations and for small v they now meet transversally in $2\delta_X$ points in $\overline{X} \times \overline{X}$. this means that the curve defined parametrically by $t^n + v\alpha t, y(t) + v\beta t$ has δ_X ordinary double points (two branches meeting transversally), which tend to 0 as v tends to 0. So we can view δ_X as the number of ordinary double points which have coalesced to form the singularity of X at the origin. Of course, for an ordinary double point $\delta = 1$.

In fact this geometric interpretation follows from the fact that the δ invariant plays a key role in understanding which deformations of curves come from deforming the parametrization.

If a germ of plane curve is given parametrically by $x(t), y(t)$, we can define (one parameter) deformations of the parametrization as follows:

$$\begin{aligned} x(t;v) &= x(t) + \textstyle\sum_i a_i(v)t^i, \quad a_i \in \mathbf{C}\{v\},\ a_i(0) = 0 \\ y(t;v) &= y(t) + \textstyle\sum_j b_j(v)t^j, \quad b_j \in \mathbf{C}\{v\},\ b_j(0) = 0. \end{aligned}$$

If on the other hand our curve is given implicitely by an equation $f(x, y) = 0$, then we can define a deformation as

$$f(x, y; v) = f(x, y) + \sum_{(i,j)\neq(0,0)} g_{ij}(v)x^i y^j \qquad g_{ij} \in \mathbf{C}\{v\}, \quad g_{ij}(0) = 0.$$

The elimination process can be performed over $\mathbf{C}\{v\}$ to show that a deformation of the parametrization always give a deformation of the equation (again this follows from the fact that the formation of Fitting ideals commutes with base extension).

Is the converse true in the sense that any deformation of the equation can be represented by a deformation of the parametrization? The answer is NO!

In order to understand what happens, we must reinterpret the problem. To say that a family of curves is obtained by deforming a parametrization is to say that they all have "the same normalization" in some sense. Thus we are led to study how the normalizations vary in an analytic family of reduced plane curves.

Definition 7.1. Let $(C, 0)$ be a germ of a reduced analytic curve, or let C be a closed reduced analytic curve in a suitable open polycylinder of $\mathbf{C}^d$, or an affine curve, with ring $\mathcal{O}_C$. Then its δ invariant is defined by $\delta_0(C) = \dim_{\mathbf{C}}\big(\frac{\overline{\mathcal{O}_{C,0}}}{\mathcal{O}_{C,0}}\big)$, or $\delta(C) = \dim_{\mathbf{C}}\big(\frac{\overline{\mathcal{O}_C}}{\mathcal{O}_C}\big)$.

Since normalization of a sheaf of algebras form a sheaf, we have

$$\delta(C) = \sum_{x \in C} \delta_x(C)$$

where the sum on the right is finite since δ is nonzero only at singular points.

Now let $f\colon (S, 0) \to (\mathbf{C}, 0)$ be a germ of a flat morphism such that $f^{-1}(0)$ is a germ of a reduced analytic curve. Here flatness means that no element of $\mathcal{O}_{S,0)}$ is annihilated by multiplication by an element of $\mathbf{C}\{v\}$ where v is a local coordinate on $(\mathbf{C}, 0)$.
Let $n\colon \overline{S} \to S$ be the normalization of the surface S (a small representative of the germ), and let

$$p = f \circ n\colon (\overline{S}, n^{-1}(0)) \to (S, 0).$$

Let us denote $p^{-1}(0)$ by $(\overline{S})_0$, and to write $\delta((\overline{S})_0) = \sum_{x \in n^{-1}(0)} \delta((\overline{S})_0, x)$. similarly, write $\delta(S_0)$ for $\delta(f^{-1}(0), 0)$ and $\delta(S_y)$ for $\delta(f^{-1}(y)$ when $y \in \mathbf{C} \setminus \{0\}$ in a small enbough representative of f, so that all the singular points of $f^{-1}(y)$ tend to 0 when $y \to 0$, and 0 is the only singular point of $f^{-1}(0)$. Note that $\delta(S_y) = \sum_{z \in S_y} \delta(S_y, z)$.
Then we have:

Proposition 7.2. *(see [T6], and [CH-L] for a beautiful generalization)*
a) The morphism $p = f \circ n\colon (\overline{S}, n^{-1}(0)) \to (\mathbf{C}, 0)$ is a multigerm of a flat mapping.
b) We have the equality

$$\delta((\overline{S})_0) = \delta(S_0) - \delta(S_y),$$

for $y \neq 0$ sufficiently small.

To say that the normalizations of the various fibers $f^{-1}(y)$ glue up into a non singular surface is therefore equivalent to saying that $p^{-1}(0)$ is non singular and this is equivalent to saying that "the δ invariant of the fibers S_y is constant as y varies in $\mathbf{C}$ near 0.

Note that the fiber $f^{-1}(y)$ will in general have several singular points, at which it is not necessarily analytically irreducible even if $f^{-1}(0)$ is irreducible.

This explains what happens when we deform the parametrization by $x(t)+\alpha vt, y(t)+\beta vt$; since it is a deformation of the parametrization, the sum of the δ invariants must be the same for all values of v, while for $v \neq 0$ the curve has only ordinary double points, whose δ invariant is one.

7.3. *projections of space curves*

So this abstract idea, normalization, provides us with a proof of the resolution of singularities of space curves: given $(C,0) \in (\mathbf{C}^d, 0)$, the normalization $\mathcal{O} \to \overline{\mathcal{O}}$ of the (reduced) analytic algebra of germs of functions on C is an analytic algebra which is a product $\Pi_{i=1}^r \mathbf{C}\{t_i\}$ of a finite number of convergent power series rings in one variable. If $x_1, \ldots, x_d$ generate the maximal ideal of $\mathcal{O}$, we get r d-uples of convergent power series expansions $x_j(t_i)$, which are our Newton series in this case. They geometrically correspond to a map

$$\bigsqcup_{i=1}^{r} (\mathbf{C}, 0)_i \to (C, 0)$$

which is our resolution of singularities. However, normalization is geometrically subtle in general, and the finiteness of normalization for the rings one meets in Geometry is a fairly deep theorem of commutative algebra; in addition, we may seek a more geometric proof, as follows

We now turn to the definition of plane projections of a space curve.
Let $(C,0) \in (\mathbf{C}^d, 0)$ be a germ of a (reduced) space curve defined by an ideal $I \subset \mathbf{C}\{x_1, \ldots, x_d\}$. Let us choose a linear projection $p \colon \mathbf{C}^d \to \mathbf{C}^2$. Let M denote the space of all such projections; think of it as a set of $d \times 2$ matrices of rank 2. We endow M with the topology (complex or Zariski) induced by that of the space of matrices. We wish to consider only the projections such that $p|C \colon C \to p(C)$ is finite to one. If that is not the case, the kernel of p, which is a linear subspace of codimension 2 of $\mathbf{C}^d$, contains one of the irreducible components of the curve C; the intersection is analytic, so it is either of dimension 0 or 1. By looking at the equations of C, it is not too

difficult to check that the projections which do not contain a component of C form a dense open set of M. The fact that they are those which induce a finite map $C \to p(C)$ is a consequence of the Weierstrass preparation theorem.
Assume now that the map $C \to p(C)$ is finite. Again by the Weiertrass theorem, it means that the map of analytic algebras $\mathbf{C}\{x, y\} \to \mathcal{O}$ defined by $f \mapsto (f \circ p)|C$ makes $\mathcal{O}$ a $\mathbf{C}\{x, y\}$-module of finite type. Since $\mathbf{C}\{x, y\}$ is nœtherian, as we saw in a preceding section, it means we have a presentation by an exact sequence of $\mathbf{C}\{x, y\}$-modules:

$$\mathbf{C}\{x, y\}^q \to \mathbf{C}\{x, y\}^p \to \mathcal{O} \to 0$$

An argument which we have seen above shows that since C is of dimension 1, we must have $q = p$, so the first map is described by a square matrix with entries in $\mathbf{C}\{x, y\}$. Let $\phi(x, y)$ be the determinant of that matrix. This determinant is, up to an invertible factor, independant of the choice of the presentation. Then the image $p(C)$ is the plane curve with equation $\phi(x, y) = 0$.

On the other hand, let us say that a linear plane projection $p\colon \mathbf{C}^d \to \mathbf{C}^2$ is *general* for the curve $C \subset \mathbf{C}^d$ at the point $0 \in C$ if it has the following property:
For any sequence of couples of points $(a_i, b_i) \in (C \setminus \{0\}) \times (C \setminus \{0\})$ tending to 0, the limit direction of the secant line $\overline{a_i, b_i}$ (for any subsequence) is *not* contained in the kernel of p.

We will see in the next paragraph that all general projections of a given germ $(C, 0)$ of space curve are topologically indistinguishable as germs of plane curves in $\mathbf{C}^2$. In [T2] it is shown that if p is general for $(C, 0)$, then the inclusion of the ring $\mathcal{O}_1 = \mathcal{O}_{X_1,0}$ of the image $X_1 = p(X)$ as defined above into the ring $\mathcal{O} = \mathcal{O}_{C,0}$ (induced by the composition of functions with p) induces an isomorphism of the total rings of fractions of these two rings, and because $\mathcal{O}$ is a finite $\mathcal{O}_1$-module, every element of $\mathcal{O}$ is integral over $\mathcal{O}_1$, as we saw above. Therefore $\mathcal{O}$ is contained in the normalization $\overline{\mathcal{O}_1}$ of $\mathcal{O}_1$.
Therefore $\overline{\mathcal{O}_1}$ is also the normalization of $\mathcal{O}$, and it is a finite $\mathcal{O}$-module for general reasons (see [K]; it suffices to know that the integral closure of $\mathcal{O}_1$ is a finite $\mathcal{O}_1$-module). Now we can use the universal property of blowing ups: in $\overline{\mathcal{O}_1}$ all ideals become principal and generated by a non zero divisor in each $\mathbf{C}\{t_i\}$. By the universal property of blowing up ([L], VII.5) if we blow up the origin in $\mathcal{O}$, the resulting algebra is still contained in $\overline{\mathcal{O}_1}$, and as we repeat blowing up points, we get an increasing sequence

of $\mathcal{O}$-algebras contained in $\overline{\mathcal{O}_1}$, all having the same total ring of fractions. Since $\overline{\mathcal{O}_1}$ is a finite $\mathcal{O}$-module, this sequence stabilizes after finitely many steps. We have to show that this limit algebra is $\overline{\mathcal{O}_1}$. But if this were not the case, the maximal ideal of one of the component local algebras would not be principal, so we could blow it up and get a strictly bigger algebra, contradicting the stability.

In conclusion, we have shown that any space curve singularity can also be desingularized by a finite sequence of point blowing ups.

One can also prove embedded resolution for space curves; it is not much more difficult than in the plane curve case.

8. The semigroup of a branch

There is another natural object associated to the inclusion $\mathcal{O} \to \overline{\mathcal{O}}$; again I will decribe it only in the case of a branch.

Let $\mathcal{O}$ be the analytic algebra of a germ of analytically irreducible curve X, and let $\overline{\mathcal{O}}$ be its normalization; we have an injection $\mathcal{O} \to \overline{\mathcal{O}}$ which makes $\overline{\mathcal{O}}$ an $\mathcal{O}$-module of finite type and $\overline{\mathcal{O}}$ is a subalgebra of the fraction field of $\mathcal{O}$. Since $\overline{\mathcal{O}}$ is isomorphic to $\mathbf{C}\{t\}$, the order in t of the series defines a mapping $\nu\colon \mathbf{C}\{t\} \setminus 0 \to \mathbf{N}$ which satisfies

i) $\nu(a(t)b(t)) = \nu(a(t)) + \nu(b(t))$ and

ii) $\nu(a(t) + b(t)) \geq \min(\nu(a(t)), \nu(b(t)))$ with equality if $\nu(a(t)) \neq \nu(b(t))$;

in other words, ν is a *valuation* of the ring $\mathbf{C}\{t\}$.

We consider the valuations of the elements of the subring $\mathcal{O}$, i.e., the image Γ of $\mathcal{O} \setminus \{0\}$ by ν; in view of i), it is a semigroup contained in $\mathbf{N}$. The fact that $\overline{\mathcal{O}}$ is a finite $\mathcal{O}$-module implies that $\mathbf{N} \setminus \Gamma$ is finite, and in fact (see [Z]) we have for the δ invariant of C the equality

$$\delta_X = \#(\mathbf{N} \setminus \Gamma)$$

Now we seek a minimal set of generators of Γ as a semigroup:

Let $\overline{\beta_0}$ be the smallest non zero element in Γ, let $\overline{\beta_1}$ be the smallest element of Γ which is not a multiple of $\overline{\beta_0}$, let $\overline{\beta_2}$ be the smallest element of Γ which is not a combination with non negative integral coefficients of $\overline{\beta_0}$ and $\overline{\beta_1}$, i.e., is not in the semigroup $\langle \overline{\beta_0}, \overline{\beta_1} \rangle$, and so on. Finally, since $\mathbf{N} \setminus \Gamma$ is finite, we find in this way a minimal set of generators:

$$\Gamma = \langle \overline{\beta_0}, \overline{\beta_1}, \ldots, \overline{\beta_g} \rangle$$

This set is uniquely determined by the semigroup Γ, and of course determines it.

By a theorem of Apéry and Zariski (see [Z]), if $(X,0)$ is a plane branch, the datum of these generators, or of the semigroup, is equivalent to the datum of the Puiseux characteristic of $(X,0)$, or of its topological type.

Let us take the notations introduced for the Puiseux pairs; it is easy to check that if we set $\beta_0 = n$, the multiplicity, then $\overline{\beta_0} = \beta_0 = n, \overline{\beta_1} = \beta_1$. After that it becomes more complicated. Zariski ([Z], Th. 3.9) proved the following formula for $q = 2, \dots, g$:

$$\overline{\beta_q} = (n_1 - 1)n_2 \dots n_{q-1}\beta_1 + (n_2 - 1)n_3 \dots n_{q-1}\beta_3 + \cdots + (n_{q-1} - 1)\beta_{q-1} + \beta_q,$$

which can be summarized in the following recursive formula:

$$\overline{\beta_q} = n_{q-1}\overline{\beta_{q-1}} - \beta_{q-1} + \beta_q$$

The proof relies on a formula of Max Noether which computes the *contact exponent* $\frac{(C,D)_0}{m_0(D)}$ of two analytic branches at the origin in terms of the coincidence of their Puiseux expansions in fractional powers of x.

This fact leads to a very interesting constatation:

Consider the Puiseux expansion of a root $y(x)$ of the Weierstass polynomial defining an analytically irreducible plane curve near the origin, assuming that $x = 0$ is not in the tangent cone of that curve:

$$\begin{aligned} y = {} & a_n x + a_{2n}x^2 + \cdots + a_{kn}x^k + a_{\beta_1}x^{\frac{m_1}{n_1}} + a_{\beta_1+e_1}x^{\frac{m_1+1}{n_1}} + \cdots + a_{\beta_1+k_1e_1}x^{\frac{m_1+k_1}{n_1}} \\ & + a_{\beta_2}x^{\frac{m_2}{n_1n_2}} + a_{\beta_2+e_2}x^{\frac{m_2+1}{n_1n_2}} + \cdots + a_{\beta_q}x^{\frac{m_q}{n_1n_2\cdots n_q}} + a_{\beta_q+e_{q-1}}x^{\frac{m_q+1}{n_1n_2\cdots n_q}} + \cdots \\ & + a_{\beta_g}x^{\frac{m_g}{n_1n_2\cdots n_g}} + a_{\beta_g+1}x^{\frac{m_g+1}{n_1n_2\cdots n_g}} + \cdots \end{aligned}$$

and the following series

$$\begin{aligned} \xi_0 &= x, \\ \xi_1 &= a_n x + a_{2n}x^2 + \cdots + a_{kn}x^k \\ \xi_2 &= a_n x + a_{2n}x^2 + \cdots + a_{kn}x^k + a_{\beta_1}x^{\frac{m_1}{n_1}} + a_{\beta_1+e_1}x^{\frac{m_1+1}{n_1}} + \cdots + a_{\beta_1+k_1e_1}x^{\frac{m_1+k_1}{n_1}} \\ &\vdots \\ \xi_g &= y - (a_{\beta_g}x^{\frac{m_g}{n_1n_2\cdots n_g}} + a_{\beta_g+1}x^{\frac{m_g+1}{n_1n_2\cdots n_g}} + \cdots) \end{aligned}$$

That is, the sequence of truncations of the Puiseux series just before the appearance of a new Puiseux exponent. Each $\xi_j,\ 0 \le j \le g$ is a root of a Weierstrass polynomial Q_j defining a branch C_j. Note that we have $Q_0 = x$ and that $Q_1 = y$ if $y = 0$ has maximal contact with C.

Proposition 8.1. *(Apéry-Zariski), see also [PP] For the semigroup* $\Gamma = \langle \overline{\beta}_0, \overline{\beta}_1, \dots, \overline{\beta}_g \rangle$ *associated to a plane branch and the curves* C_j *just defined, we have the equalities*

$$\overline{\beta}_j = (C, C_j)_0.$$

In fact, this equality remains true if we replace the expansions ξ_j by any series which coincide with the series $y(x)$ until just before the j-th Puiseux exponent; see [PP]. It follows easily from this that the datum of the semigroup is equivalent to the datum of the multiplicity n and the Puiseux exponents β_i of the curve.

The semigroups coming from plane branches are characterized among all semigroups of analytically irreducible germs of curves by the following two properties:

1) $n_i\overline{\beta_i} \in \langle \overline{\beta_0}, \dots, \overline{\beta_{i-1}} \rangle$

2) $n_i\overline{\beta_i} < \overline{\beta}_{i+1}$

That the semigroups of plane branches have these properties follows from the induction formula and the inequalities $\beta_i < \beta_{i+1}$. The converse can be proved by the construction outlined below (see [Z], appendix).

Conversely, given a semigroup Γ in $\mathbf{N}$ with finite complement, we can associate to it an analytic (in fact algebraic) curve, called the *monomial curve* associated to Γ. If $\Gamma = \langle \overline{\beta_0}, \overline{\beta_1}, \dots, \overline{\beta_g} \rangle$, the monomial curve C^Γ is described parametrically by

$$\begin{aligned} u_0 &= t^{\overline{\beta_0}} \\ u_1 &= t^{\overline{\beta_1}} \\ &. \\ &. \\ &. \\ u_g &= t^{\overline{\beta_g}} \end{aligned}$$

If the semigroup Γ comes from a plane branch, the relations 1) above mean that there exist natural numbers $\ell_i^{(j)}$ such that we have

$$\begin{aligned} n_1\overline{\beta_1} &= \ell_0^{(1)}\overline{\beta_0} \\ n_2\overline{\beta_2} &= \ell_0^{(2)}\overline{\beta_0} + \ell_1^{(2)}\overline{\beta_1} \\ &. \\ &. \\ n_j\overline{\beta_j} &= \ell_0^{(j)}\overline{\beta_0} + \cdots + \ell_{j-1}^{(j)}\overline{\beta_{j-1}} \\ &. \\ &. \\ &. \\ n_g\overline{\beta_g} &= \ell_0^{(g)}\overline{\beta_0} + \cdots + \ell_{g-1}^{(g)}\overline{\beta_{g-1}} \end{aligned}$$

These relations translate into equations for the curve $C^\Gamma \subset \mathbf{C}^{g+1}$; since $u_i = t^{\overline{\beta_i}}$, our curve satisfies the g equations

$$u_j^{n_i} - u_0^{\ell_0^{(j)}} u_1^{\ell_1^{(j)}} \dots u_{j-1}^{\ell_{j-1}^{(j)}} = 0, \quad 1 \leq j \leq g,$$

and it can be shown that they actually define $C^\Gamma \subset \mathbf{C}^{g+1}$, so that if Γ is the semigroup of a plane branch, C^Γ is a complete intersection.

Remark that if we give to u_i the weight $\overline{\beta_i}$, the i-th equation is homogeneous of degree $n_i\overline{\beta_i}$.

The connection between a plane curve X having semigroup Γ and the monomial curve is much more precise and interesting than the formal relation we have just seen; by small deformations of the monomial curve one obtains all the branches with the same semigroup. In fact the best way to understand all branches with semigroup Γ is to consider the not necessarily plane curve C^Γ (C^Γ is plane if and only if C has only one characteristic exponent).

By definition of Γ, there are elements $\xi_q \in \mathcal{O}$ with $\nu(\xi_q) = \overline{\beta_q}$. We can write these elements in $\mathbf{C}\{t\}$ as

$$\xi_q = t^{\overline{\beta_q}} + \sum_{j > \overline{\beta_q}} \gamma_{q,j} t^j.$$

Let us consider the one-parameter family of parametrizations

$$\begin{aligned}
&u_0 = t^m \\
&u_1 = t^{\overline{\beta_1}} + \textstyle\sum_{j>\overline{\beta_1}} v^{j-\overline{\beta_1}} \gamma_{1,j} t^j \\
&. \\
&. \\
&u_g = t^{\overline{\beta_g}} + \textstyle\sum_{j>\overline{\beta_g}} v^{j-\overline{\beta_g}} \gamma_{g,j} t^j
\end{aligned}$$

The reader can check that for $v \neq 0$, the curve thus described is isomorphic to our original curve C. (hint: make the change of parameter $t = vt'$ and the change of coordinates $u_j = v^{-\overline{\beta_j}} v'_j$, and remember the definition of the ξ_j). For $v = 0$, we have the parametric description of the monomial curve.

So we have in fact described a map

$$\mathbf{C} \times \mathbf{C} \to \mathbf{C}^{g+1} \times \mathbf{C}$$

which induces the identity on the second factors (with coordinate v). The image of this map is a surface, which is the total space of a deformation of the monomial curve, all of its fibers except the one for $v = 0$ being isomorphic to our plane curve C.

So the monomial curve is a specialization, in this family, of our plane curve. In this specialization the multiplicity and the semigroup remain constant; in a rather precise sense it is an equisingular specialization, or one may say that *the plane curve is an equisingular deformation of the monomial curve with the same semigroup.*

The same phenomenon can be also observed in the language of equations rather than parametrizations. Let us consider a one parameter family of equations for curves in $\mathbf{C}^{g+1}$, of the form

$$\begin{array}{l} u_1^{n_1} - u_0^{\ell_0^{(1)}} - vu_2 = 0 \\ u_2^{n_2} - u_0^{\ell_0^{(2)}} u_1^{\ell_1^{(2)}} - vu_3 = 0 \\ . \\ . \\ u_{g-1}^{n_{g-1}} - u_0^{\ell_0^{(g-1)}} u_1^{\ell_1^{(g-1)}} \ldots u_{g-2}^{\ell_{g-2}^{(g-1)}} - vu_g = 0 \\ u_g^{n_g} - u_0^{\ell_0^{(g)}} u_1^{\ell_1^{(g)}} \ldots u_{g-1}^{\ell_{g-1}^{(g)}} = 0 \end{array}$$

For $v = o$ we get the equations of the monomial curve, and for $v \neq 0$ we get a curve which has semigroup Γ; this is a general heuristic principle of equisingularity: we have added to each equation of the monomial curve, homogeneous of degree $n_i\overline{\beta_i}$, a perturbation of degree $\overline{\beta_{i+1}} > n_i\overline{\beta_i}$, and this should not change the equisingularity class (the perturbation is "small" compared to the equation).

Notice that for each fixed $v \neq 0$ the curve described by the above equations is a plane curve: for simplicity take $v = 1$; then use the first equation to compute $u_2 = u_1^{n_1} - u_0^{\ell_0^{(1)}}$, substitute this in the next equation, and use this to compute u_3 as a function of u_0, u_1, and so on. Finally the last equation gives us the equation of a plane curve of the form

$$\Big(\cdots\big((u_1^{n_1} - u_0^{\ell_0^{(1)}})^{n_2} - u_0^{\ell_0^{(2)}} u_1^{\ell_1^{(2)}}\big)^{n_3} - \cdots\Big)^{n_g} - u_0^{\ell_0^{(g)}} u_1^{\ell_1^{(g)}} (u_1^{n_1} - u_0^{\ell_0^{(1)}})^{\ell_2^{(g)}} \cdots = 0$$

The first consequence (see the appendix to [Z]) is that we can produce explicitely the equation of a plane curve with given characteristic exponents: compute the semigroup and its generators, and then write the equation above.

A more important fact is that one can show (see [*loc. cit*) that any plane curve with a given semigroup appears up to isomorphism as a fiber in a deformation depending on a finite number of parameters: it is a deformation of the monomial curve obtained by adding to the j-th equation a polynomial in the u_i's of order $> n_j\overline{\beta_j}$, and these polynomials can in principle be explicitely computed.

In fact it is shown in [G-T] that we can in this manner produce equations for *all* branches having the same semigroup (or equisingularity type) up to an analytic isomorphism.
The fact that the curve is plane corresponds to the condition that u_{j+1} appears linearly in the deformation of the j-th equation, for $1 \leq j \leq g-1$. Finally, all the plane branches with the same semigroup have "the same" process of resolution of singularities: you have to blow up points according to the same rules, the multiplicities of the strict transforms are the same, and so on. So the resolution of the plane curve described above shows the structure of the resolutions of all the curves with the same semigroup. First you resolve the curve $u_1^{n_1} - u_0^{\ell_0^{(1)}} = 0$; when its strict transform is non singular (after a number of blowing ups which depends on the continued fraction expansion of the ratio $\frac{\ell_0^{(1)}}{n_1}$, you take it as a coordinate axis: then you have one parenthesis less in the equation above (the point is that the form of the equation does not change), and you proceed like this. After g such steps the branch is resolved.

There is however another way to use the structure given by the description of our branch as a deformation of the monomial curve to get embedded resolution; it is the subject of the next paragraph.

9. Resolution of binomials

Let a^1, a^2 be two integral vectors in the first quadrant of $\check{\mathbf{R}}^2$, and assume that their determinant is ± 1. Then they are primitive vectors and they generate the integral lattice $\check{\mathbf{Z}}^2$ of $\check{\mathbf{R}}^2$. Consider the cone

$$\sigma = \langle a^1, a^2 \rangle$$

of their positive linear combinations. It is a rational convex cone (= a convex cone which is the intersection of finitely many half spaces determined by hyperplanes with rational -even integral- equations). Because it is generated by integral vectors which form a basis of the integral lattice $\mathbf{Z}^2$, we say tha it is a *regular* cone. Since it is convex it has a convex dual which is a rational convex cone in $\mathbf{R}^2$:

$$\check{\sigma} = \{m \in \mathbf{R}^2 / m(\ell) \geq 0 \ \ \forall\, \ell \in \sigma\}.$$

The cone $\check{\sigma}$ is also generated by two vectors with determinant ± 1, which therefore generate the integral lattice $\mathbf{Z}^2$ of $\mathbf{R}^2$. If we interpret each integral point of $\check{\sigma}$ as a (Laurent) monomial (here "Laurent" means that negative exponents are allowed) in variables (u_1, u_2), the algebra $\mathbf{C}[\check{\sigma} \cap \mathbf{Z}^2]$ is a polynomial algebra in two variables, say $\mathbf{C}[y_1, y_2]$.

Since σ is contained in the first quadrant, its dual $\check{\sigma}$ contains the dual of the first quadrant, which is the first quadrant of $\mathbf{R}^2$. If we remark that the integral points of the first quadrant correspond exactly to the polynomial algebra $\mathbf{C}[u_1, u_2]$, we see that there is therefore an inclusion

$$\mathbf{C}[u_1, u_2] \subset \mathbf{C}[y_1, y_2]$$

and it is an interesting exercise to check that it is given by

$$\begin{aligned} u_1 &\mapsto y_1^{a_1^1} y_2^{a_1^2} \\ u_2 &\mapsto y_1^{a_2^1} y_2^{a_2^2} \end{aligned}$$

where a_i^j is the i-th coordinate of the vector a^j.

The transform of a monomial $u^m = u_1^{m_1} u_2^{m_2}$ is, if we write $m = (m_1, m_2)$:

$$u_1^{m_1} u_2^{m_2} \mapsto y_1^{\langle a^1, m\rangle} y_2^{\langle a^2, m\rangle},$$

so that the transform of a binomial $u^m - \lambda_{mn} u^n$ is

$$u^m - \lambda_{mn} u^n \mapsto y_1^{\langle a^1, m\rangle} y_2^{\langle a^2, m\rangle} - \lambda_{mn} y_1^{\langle a^1, n\rangle} y_2^{\langle a^2, n\rangle}.$$

Now the key observation is that if $\langle a^1, m-n\rangle$ and $\langle a^2, m-n\rangle$ are both non zero, they have the same sign, which means that the two vectors a^1 and a^2 are in the same half space determined the hyperplane H_{m-n} dual to the vector $m-n$, or equivalently that the cone σ is *compatible* with H_{m-n} in the sense that $\sigma \cap H_{m-n}$ is a face of σ, then we can factor the transform of the binomial. Assume that $\langle a^i, m-n\rangle \geq 0$. We have non negative exponents in the identity

$$y_1^{\langle a^1, m\rangle} y_2^{\langle a^2, m\rangle} - \lambda_{mn} y_1^{\langle a^1, n\rangle} y_2^{\langle a^2, n\rangle} =$$

$$y_1^{\langle a^1, n\rangle} y_2^{\langle a^2, n\rangle} \big(y_1^{\langle a^1, m-n\rangle} y_2^{\langle a^2, m-n\rangle} - \lambda_{mn}\big).$$

Now we have an exceptional divisor defined by

$$y_1^{\langle a^1, n\rangle} y_2^{\langle a^2, n\rangle} = 0$$

and a strict transform defined by

$$y_1^{\langle a^1, m-n\rangle} y_2^{\langle a^2, m-n\rangle} - \lambda_{mn} = 0$$

The next observation is that the strict transform is non singular, and meets the exceptional divisor if and only if $\sigma \cap H_{m-n}$ is an edge of σ, i.e., is not $\{0\}$. Say that a^2 is in H_{m-n}; the strict transform is then

$$y_1^{\langle a^1, m-n\rangle} - \lambda_{mn} = 0.$$

Now if we assume that the binomial $u^m - \lambda_{mn} u^n$ is irreducible in $\mathbf{C}[u_1, u_2]$, which is equivalent since $\mathbf{C}$ is algebraically closed to the fact that the vector $m - n$ is *primitive*, in the sense that it is not an integral multiple of an integral vector, then it is not difficult to check (see [T1], Proposition 6.2) that $\langle a^1, m - n\rangle = 1$, so that finally our strict transform in this case is $y_1 - \lambda_{mn} = 0$, which is indeed non singular and transversal to the exceptional divisor.

Actually the same proof works if the binomial is reducible but there are then several points above the origin in the strict transform of the curve.

The next observation is that in two variables our binomial has to be of the form $u_1^m - \lambda u_2^n$ unless the curve contains a coordinate axis, which we exclude in the irreducible case. By a change of variable we may assume $\lambda = 1$ and by irreducibility, we have $(m, n) = 1$. Now to study the strict transform under one of our monomial maps $\pi(\sigma)$ we have seen that the only interesting case is when one of the generating vectors of σ, say a^1, is the vector (n, m). Let us assume that $n < m$. Set $a^2 = (a, b)$ and say that $am - bn = 1$ (we know it has to be ± 1). The transform of a monomial $u_1^i u_2^j$ is $y_1^{ni+mj} y_2^{ai+bj}$. From this follows that if we consider a curve with equation

$$(*) \qquad u_1^m - u_2^n + \sum_{ni+mj>mn} a_{ij} u_1^i u_2^j = 0$$

it transforms into

$$y_1^{mn} y_2^{am}\big(y_2 - 1 + \sum_{ni+mj>mn} a_{ij} y_1^{ni+mj-mn} y_2^{ai+bj-am}\big)$$

and one checks that all exponents are positive. The strict transform of our curve is still non singular in a neighborhood of the exceptional divisor, and transversal to the exceptional divisor at the point $y_1 = 0, y_2 = 1$.

If we consider the other cone σ' having the vector (n, m) as an edge, we find that the point where the strict transform meets the exceptional divisor lies in the open set of the corresponding chart $Z(\sigma')$ which is identified with an open set of $Z(\sigma)$; we are looking at the same object in two charts. This shows that the toric maps which provides an embedded resolution for the binomial $u_1^m - u_2^n = 0$ in fact also gives an embedded resolution for all the

curves of the type $(*)$, where one deforms the binomial by adding terms of higher weight, where the weight of u_1 is n and the weight of u_2 is m.
Now by a general combinatorial result (see [Ew]), for any integer $d \geq 2$, given a finite collection of hyperplanes whose equation has integral coefficients in the first quadrant $\mathbf{R}^d_{\geq 0}$ of $\mathbf{R}^d$, it is possible to find a *regular fan* with support $\mathbf{R}^d_{\geq 0}$, that is a finite collection Σ of regular rational cones such as our σ above (but now with d generating vectors of determinant ± 1) and its faces, whose union is $\mathbf{R}^d_{\geq 0}$, and such that if $\sigma \in \Sigma$ its faces are also in Σ, and for any $\sigma, \sigma' \in \Sigma$, the intersection $\sigma \cap \sigma'$ is a face of each. To each σ of dimension d corresponds a polynomial ring $\mathbf{C}[\check{\sigma} \cap \mathbf{Z}^d]$ and therefore an affine space $\mathbf{A}^d(\mathbf{C})$ with a birational map

$$\pi(\sigma)\colon \mathbf{A}^d(\mathbf{C}) \to \mathbf{A}^d(\mathbf{C})$$

generalizing the map

$$\begin{aligned} u_1 &\mapsto y_1^{a_1^1} y_2^{a_1^2} \\ u_2 &\mapsto y_1^{a_2^1} y_2^{a_2^2} \end{aligned}$$

which we have seen above in the case $d = 2$.
The sources of all these maps can be glued up together (see [Ew]) to form a nonsingular algebraic rational variety $Z(\Sigma)$ in such a way that the maps $\pi(\sigma)$ glue up into a proper and birational (hence surjective) map

$$\pi(\Sigma)\colon Z(\Sigma) \to \mathbf{A}^d(\mathbf{C}).$$

Coming back to the case $d = 2$ and a binomial, this gives us the existence of a regular fan (= a fan made of regular cones) with support $\mathbf{R}^2_{\geq 0}$, and compatible with the line H_{m-n}, which means that this line in $\mathbf{R}^2_{\geq 0}$ is the common edge of two cones of the fan.
In fact in this case there is a minimal such fan, obtained as follows:
Consider the set H_- of integral points of $\mathbf{R}^2_{\geq 0}$ which are below the line H_{m-n}, and the set H_+ of the integral points which are above. The boundaries of the convex hulls of H_- and H_+ contain parts of coordinates axes, and they meet at the extremity of the primitive integral vector contained in H_{m-n}. Drawing lines connecting the origin to all the integral points which are on these boundaries defines a fan which has the required properties and is the coarsest such fan. It is closely connected with the continued fraction expansion of the slope of the line H_{m-n}.
To this fan Σ is associated a proper birational map

$$\pi(\Sigma)\colon Z(\Sigma) \to \mathbf{A}^2(\mathbf{C})$$

which is an isomorphism outside of the origin and provides an embedded resolution of singularities for all plane branches which have an equation of the form

$$u_2^n - u_1^m + \sum_{\frac{i}{m}+\frac{j}{n}>1} a_{ij} u_1^i u_2^j = 0.$$

as one verifies by checking in each of the charts $\mathbf{Z}(\sigma) \simeq \mathbf{A}^2(\mathbf{C})$.
Since we saw that every plane branch is similarly a deformation of the monomial curve with the same semi-group, which is defined by g binomial equations in variables $u_0, \dots, u_g$, adding to each binomial only monomials of higher weight, one is ready to believe that similarly, a regular fan in $\mathbf{R}_{\geq 0}^{g+1}$ which is compatible with the g hyperplanes corresponding to the g binomials will provide a toric map

$$\pi(\Sigma)\colon Z(\Sigma) \to \mathbf{A}^{g+1}(\mathbf{C})$$

which is an embedded resolution not only for the monomial curve, but also for our original plane curve re-embedded in $\mathbf{A}^{g+1}(\mathbf{C})$ as was explained above. This is described in detail in [G-T] and generalized in [GP] to a much larger class of singularities.

This method of embedded resolution is quite different from the resolution by point blowing ups explained above, but it assumes that one knows the existence of a parametrization. The connection between the toric map and the sequence of point blowing ups is rather subtle (see [GP]); in the case $g = 1$ it is equivalent to the relation between finding approximations of a rational number by the reduced fractions of its continued fraction expansion and finding approximations by Farey series.
So the deformation to the monomial curve also explains to us how to resolve the singularities, and it is perhaps the best description. Can we generalize it to higher dimensions?

10. Relation with topology

I refer to the lectures of Lê and to [B-K] for the Burau-Zariski topological interpretation of the characteristic sequence

$$(\beta_0, \beta_1, \dots \beta_g)$$

as a characteristic of the iterated torus knot that one obtains upon intersecting the branch X with a sufficiently small sphere in $\mathbf{C}^2$ centered at the origin.

Given a germ of a reduced plane curve X, it has a decomposition $X = \bigcup_{i=1}^{r} X_i$ into branches; each branch has its characteristic sequence $B(X_i)$, and as numerical characters of X, we have also the intersection numbers $(X_i, X_j)_0$ of distinct branches at 0.

If we remember that these intersection numbers are equal to the linking numbers in $\mathbf{S}^3$ of the knots corresponding to X_i and X_j and are therefore topological characters of the link $X \cap \mathbf{S}^3_\epsilon$, since Milnor proved (see Lê's lectures) that the curve X is homeomorphic to the cone with vertex 0 drawn on this link, we expect that the collection of the characteristic sequences of the branches and their intersection numbers may be a topological invariant of the curve X.

Let us define the local topological type of a germ of subspace of $\mathbf{C}^N$ as follows:

Definition.- Two subspaces X_1 and X_2 of $\mathbf{C}^N$ are topologically equivalent at 0 if there exist neighbourhoods U and V of 0 in $\mathbf{C}^N$ and an homeomorphism $\psi\colon U \to V$ such that $\psi(X_1 \cap U) = X_2 \cap V$. Two germs at 0 of subspaces are topologically equivalent if they have representatives which are topologically equivalent at 0.

Theorem 10.1. *(Zariski, Lejeune-Jalabert). Two germs of plane curves $X = \cup_{i\in I} X_i$ and $X' = \cup_{i\in I'} X'_i$ are topologically equivalent if and only if there exists a bijection $\phi\colon I \to I'$ between their branches which preserves characteristics and intersection numbers, that is, satisfies*

$$B(X'_{\phi(i)}) = B(X_i) \ \text{ for } i \in I, \quad (X'_{\phi(i)}, X'_{\phi(j)})_0 = (X_i, X_j)_0 \ \text{ for } i \neq j.$$

Topological equivalence is less strict a relation than analytic (or even $\mathcal{C}^1$) equivalence.

Let X_1 and X_2 each consist of four lines through the origin in $\mathbf{C}^2$. According to the previous theorem, these two germs are topologically equivalent. However, if there was a germ et 0 of a $\mathcal{C}^1$ (and in particular analytic) isomorphism of $\mathbf{C}^2$ to itself, sending X_1 to X_2, its tangent linear map at 0 would have to send X_1 onto X_2. But two quadruplets of lines through 0 are linearly equivalent if and only if they have the same cross-ratio. If the slopes of the lines of X_1 are a_1, b_1, c_1, d_1, and similarly for X_2, the cross ratios are

$$(\frac{a_1 - a_3}{a_1 - a_4})(\frac{a_2 - a_4}{a_2 - a_3})$$

and the numbers obtained by permutation. It is therefore easy to find examples where X_1 and X_2 are not $\mathcal{C}^1$-equivalent.

In particular, in an analytic family of curves such as the surface in $\mathbf{C}^3$ with equation

$$(y-x)(y+x)(y-2x)(y+tx)=0$$

for small values of t, the fibers are all analytically inequivalent but topologically equivalent.

Theorem 10.2. *Given two reduced germs of plane curves $(X,0)\subset(\mathbf{C}^2,0)$ and $(X',0)\subset(\mathbf{C}^2,0)$ the following conditions are equivalent:*
1) X and X' are topologically equivalent,
2) There exists an integer d, a germ of curve $(C,0)\subset(\mathbf{C}^d,0)$ and two linear projections $p,p'\colon\mathbf{C}^d\to\mathbf{C}^2$, both general for C at 0, and such that $p(C)=X,\ \ p'(C)=X'$,
3) There exists a one-parameter family of germs of plane curves that is a germ along $\{0\}\times U$ of a surface in $\mathbf{C}^2\times U$, where U is a disk in $\mathbf{C}$, say with equation $f(x,y,u)=0$ and $v,v'\in U$ such that the germs of plane curve $f(x,y,v)=0,\ f(x,y,v')=0$ are isomorphic to $X,\ X'$ respectively and all the germs $f(x,y,t)=0$ have the same topological type for $t\in U$.
4) There exists a bijection from the set of branches of $(X,0)$ to the set of branches of $(X',0)$ which preserves characteristic (Puiseux) exponents and intersection numbers.
5) The minimal embeded resolution processes of $(X,0)$ and $(X',0)$ are "the same" in the sense that one blows up at each step points with the same multiplicity.

In fact, the theory of Lipschitz saturation, summarized in [T4], shows that, given the topological type of a germ of plane curve $(X,0)$, there exists a germ of a space curve $(X^s,0)\subset(\mathbf{C}^N,0)$, unique up to isomorphism, such that the germs of plane curves having the same topological type as $(X,0)$ are exactly, up to isomorphism, the images of $(X^s,0)$ by the linear projections $(\mathbf{C}^N,0)\to(\mathbf{C}^2,0)$ which are general for $(X^s,0)$.

11. Duality

A line in the projective space $\mathbf{P}^2$ is by definition a point in the dual projective space $\check{\mathbf{P}}^2$.

Poncelet saw that given a nondegenerate conic Q, to any point $P\in\mathbf{P}^2$, one can associate the *polar curve* of P with respect to Q, which is the line joining the points of contact with Q of the tangents to Q passing through P. We get in this way an isomorphism between $\mathbf{P}^2$ and its dual $\check{\mathbf{P}}^2$

We shall, however, refrain from identifying $\mathbf{P}^2$ and its dual in this way, as was done at that time.

The collection of points of $\check{\mathbf{P}}^2$ corresponding to the lines in $\mathbf{P}^2$ tangent to an algebraic curve C is an algebraic curve $\check{C} \subset \check{\mathbf{P}}^2$. Here we use the fact that one can say that a line is tangent to a curve C at a singular point x if its direction is a limit direction of tangents to C at non singular points tending to x.

A point x in $\mathbf{P}^2$ corresponds to a line $\check{x}$ in $\check{\mathbf{P}}^2$; each point of this line represents a line in $\mathbf{P}^2$ which contains x, and the lines through x tangent to C correspond to the intersection points in $\check{\mathbf{P}}^2$ of the curve $\check{C}$ and the line $\check{x}$. So the class of the curve C, defined as the number of lines tangent to C at non singular points and passing through a given general point of $\mathbf{P}^2$, is the degree $\check{m}$ of $\check{C}$. Let us compute it:

Poncelet considered, following Monge, the *polar curve* (the terminology is his): Let

$$f(X, Y, X) = 0,$$

where f is a homogeneous polynomial of degree m, be an equation for C. The points of C where the tangent goes though the point of $\mathbf{P}^2(\mathbf{C})$ with coordinates (ξ, η, ζ) are on C and on the curve of degree $m-1$ with equation

$$P_{(\xi,\eta,\zeta)}(f) = \xi\frac{\partial f}{\partial X} + \eta\frac{\partial f}{\partial Y} + \zeta\frac{\partial f}{\partial Z} = 0$$

obtained by *polarizing* the polynomial f with respect to the point (ξ, η, ζ). If C is non singular, the points we seek are all the intersection points of C and $P_{(\xi,\eta,\zeta)}(C)$. By Bézout's theorem, the number of these points counted with multiplicity is $m(m-1)$, for every point (ξ, η, ζ) it is equal to $m(m-1)$ if C has no singularities.

It is "geometrically obvious" that $\check{\check{C}} = C$; this is called biduality (it is completely wrong if we do geometry over a field of positive characteristic, so beware of what is "geometrically obvious"). If the curve $\check{C}$ had no singularities as well, the computation of degrees would give $m(m-1)(m^2-m-1) = m$, which holds only for $m = 2$. So if $m > 2$ the dual of a non singular curve has singularities ; for a general non singular curve, double points (a.k.a. nodes) of $\check{C}$ correspond to double tangents of C and cusps correspond to its inflexion points.

To understand biduality better, it becomes important to find the class of a projective plane curve with singularities, at least when these singularities are the simplest: nodes and cusps. This was done by Plücker and the formula for a curve with δ nodes and κ cusps is

$$\check{m} = m(m-1) - 2\delta - 3\kappa.$$

One said that" a node decreases the class by two, and a cusp by three" This is perhaps the first example of a search of numerical invariants of singularities.

One can compute the diminution of class provoked by an arbitrary plane curve singularity as follows: let $f(x,y) = 0$ be a local equation for the singular curve at a singular point x which we take as origin. The ideal $j(f) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})\mathbf{C}\{x,y\}$ defines set-theoretically the origin, which is the only singular point (locally). Therefore it contains a power of the maximal ideal and it is a vector space of finite codimension $\mu(C,x)$ in $\mathbf{C}\{x,y\}$. It is called the Milnor number of the singularity. Let $m(C,x)$ be the multiplicity of C at 0, the order of the equation $f(x,y)$. Then (see [T6]) the diminution of class due to the singularity is

$$\Delta_{C,x} = \mu(C,x) + m(C,x) - 1.$$

This means that for an arbitrary reduced projctive plane curve C of degree m we have the equality (generalized Plücker formula)

$$\check{m} = m(m-1) - \Sigma_{x\in C}\Delta_{C,x}.$$

This number is also the local intersection number at x of the curve C and one of its "general local polar curves", defined by $\alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y} = 0$ for geneal values of α, β. In this way, one can see that if we locally deform our singular curve to a non singular one, say by taking the equation $f(x,y) = \lambda$, the number of points where the tangent to this non singular curve has a fixed direction and which coalesce to 0 as $\lambda \to 0$ is $\Delta_{C,x}$; its is the number of possible tangents that are "absorbed" by the singular point.
It is a remarkable fact that the "diminution of class" $\Delta_{C,x}$ depends only on the topological type of the germ of plane curve (C,x). In fact there is a formula to express it in terms of the Puiseux exponents of the branches of (C,x) and their local intersection numbers. The fact that the degree of the dual of a projective variety depends only on global characters like the degree and the "topology" of its singularities extends to an arbitrary singular projective variety if one makes the notion of topological type somewhat more stringent.

In the theory of algebraic curves, an important formula states that given an algebraic map $f\colon C \to C'$ between non singular algebraic curves, which is of degree $\deg f = d$ (meaning that for a general point $c' \in C'$, $f^{-1}(c')$

consists of d points, and is ramified at the points $x_i \in C,\ 1 \leq i \leq r$, which means that near x_i, in suitable local coordinates on C and C', the map f is of the form $t \mapsto t^{e_i+1}$ with $e_i \in \mathbf{N},\ \ e_i \geq 1$. The integer e_i is the*ramification index* of f at x_i. Then we have the Riemann-Hurwitz formula relating the genus of C and the genus of C' via d and the ramification indices:

$$2g(C) - 2 = d(2g(C') - 2) + \sum_i e_i$$

If we have a finite map $f\colon C \to C'$ between possibly singular curves, it extends in a unique manner to a finite map of the same degree between their normalizations, to which we can apply the Riemann-Hurwitz formula to get a relation between the geometric genera of C and C'. If we apply this formula to the case where C is non singular and $C' = \mathbf{P}^1$, knowing that any compact algebraic curve is a finite ramified covering of $\mathbf{P}^1$, we find that we can calculate the genus of C from any linear system of points made of the fibers of a map $C \to \mathbf{P}^1$ if we know its degree and its singularities: we get

$$2g(C) = 2 - 2d + \sum e_i$$

The ramification points x_i can be computed as the so-called *jacobian divisor* of the linear system, which consists of the singular points, properly counted, of the singular members of the linear system. In particular if C is a plane curve and the linear system is the system of its plane sections by lines through a general point $x = (\xi : \eta : \zeta)$ of $\mathbf{P}^2$, the map f is the projection from C to $\mathbf{P}^1$ from x; its degree is the degree m of C and its ramification points are exactly the points where the line from x is tangent to C. Since x is general, these are simple tangency points, so the e_i are equal to 1, and their number is equal to the class $\check{m}$ of C; the formula gives

$$2g(C) - 2 = -2m + \check{m}\ ,$$

thus giving for the genus an expression which is *linear in the degree and the class*, whereas our expression in terms of the degree alone is quadratic.

This is the first example of the relation between the "characteristic classes" (in this case only the genus) and the polar classes; in this case the curve itself, of degree m and the degree of the polar locus, or apparent contour from x, i.e. in this case the class $\check{m}$. The extension to a non singular projective algebraic variety in characteristic zero is due to Todd.

12. The polar curve

The (general) polar curve plays a much more important role in the study of the plane curve singularities than just giving the diminution of class by its intersection number with the curve at the singular point.

Given an equation $f(x, y) = 0$ for a germ reduced plane curve in $(\mathbf{C}^2, 0)$, let us denote by $\ell(x, y)$ a homogeneous linear form, i.e., the equation of a line through the origin. The we define a map

$$F_\ell \colon (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$$

by $t_0 = f(x, y), t_1 = \ell(x, y)$.
The critical locus P_ℓ of this map is the local avatar of the polar curve which we saw in the previous section. More precisely if our germ of plane curve comes from a projective plane curve, then P_ℓ is the germ at 0 of the polar curve in the projective plane corresponding to the point at infinity in the direction of ℓ.
Remark now that the image of P_ℓ by the map F_ℓ is a plane curve whose equations is given by the Fitting ideal of the algebra of P_ℓ as a $\mathbf{C}\{t_0, t_1\}$-module, at least when the map $(P_\ell, 0) \to (\mathbf{C}^2, 0)$ induced by F_ℓ is finite, which is the case when ℓ is general.
This image $D_\ell = F_\ell(P_\ell)$ is the *discriminant* of F_ℓ, and it lives in a plane with given coordinates t_0, t_1. The Newton polygon of D_ℓ in the coordinates t_0, t_1 for a general choice of the linear form ℓ is independant of ℓ and is called the *jacobian Newton polygon* of f.
M. Merle proved that if $f(x, y)$ is irreducibule, the jacobian Newton polygon is a complete invariant of equisingularity of the curve $f(x, y) = 0$; it can be computed from the Puiseux exponents and determines them. The extension to the reducible case is due to E. Garcia Barroso (for all this, see [G]).
The jacobian Newton polygon encodes the essence of the dynamics as λ goes to zero of the points of the non-singular curve $f(x, y) = \lambda$ where the tangent is parallel to $\ell(x, y) = 0$. Those are the points counting for the class of a non singular curve degenerating to our singular curve which are "absorbed" by the singularity and so decrease the class.

Bibliography

B. N. Bourbaki, *Algèbre commutative, Chap. I-IX*, Masson.

B-K. E. Brieskorn, H. Knörrer, *Plane algebraic curves*, Birkhaüser, 1986.

Ca. A. Campillo, *Algebroid curves in positive characteristic*, Springer Lecture Notes in Mathematics, No. 813 (1980).

CH-L. H.-J. Chiang-Hsieh and J. Lipman, *A numerical condition for simultaneous normalization.* To appear in Duke Math. J.

Ei. D. Eisenbud, *Commutative algebra with a view towards Algebraic Geometry*, Graduate Texts in Math., Springer 1995.

Ew. G. Ewald, *Combinatorial Convexity and Algebraic Geometry*, Graduate Texts in Math. No. 168, Springer 1996.

G. E.R. Garcia Barroso, *Sur les courbes polaires d'une courbe plane réduite*, Proc. London Math. Soc., **81**, 3 (2000), 1-28.

GP. P. González Pérez, *Toric embedded resolution of quasi-ordinary singularities*, Annales de l'Institut Fourier, 2003.

G-T. R. Goldin and B. Teissier, *Resolving plane branch singularities with one toric morphism*, *in* "Resolution of Singularities, a research textbook in tribute to Oscar Zariski", Birkhaüser, Progress in Math. No. 181, 2000, 315-340.

GP-T. P. González Pérez and B. Teissier, *Embedded resolutions of non necessarily normal affine toric varieties*, Comptes Rendus Acad. Sci. Paris, Ser.1, **334**, 2002, 379-382.

H. H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero I, II*, Annals of Math., **79**, No.1 1964, 109-203 and **79**, No.2, 205-326.

K. B. Kaup, L. Kaup, and G. Barthel, *Holomorphic functions of several variables,* De Gruyter Studies in Math., 003, W. de Gruyter 1983.

Ke. K. Kedlaya *The algebraic closure of the power series field in positive characteristic.* Proc. Amer. Math. Soc. 129 (2001), no. 12, 3461–3470 (electronic).

L. S. Łojasiewicz, *Introduction to complex analytic Geometry,* , Birkhäuser Verlag, 1991.

M. M. Merle, *Invariants polaires des courbes planes,* , Inventiones Math., **41**, 103-111 (1977).

M-T. M. Merle and B. Teissier, *Conditions d'adjonction, d'après Du Val*, Séminaire sur les singularités des surfaces, Palaiseau 1976-77, M. Demazure, H. Pinkham, B. Teissier, Editeurs, Springer Lecture Notes in Math., No. 777, 229-245.

Ok. M. Oka, *Non-Degenerate complete intersection singularity*, Actualités mathématiques, Hermann, Paris 1997.

PP. P. Popescu-Pampu, *Approximate roots*, Proceedings of the Saskatoon Conference and Workshop on valuation theory (second volume), F-V. Kuhlmann, S. Kuhlmann, M. Marshall, editors, Fields Institute Communications, 2003, 285-321.

S. M. Saia, *The integral closure of ideals and the Newton filtration*, J. Algebraic Geometry, **5**, 1996, 1-11.

Sm. H.J.S. Smith, *On the higher singularities of plane curves*, Proc. London Math. Soc., **1**, t. VI, 1873, p. 153.

T1. B. Teissier, *Valuations, deformations, and toric geometry*, Proceedings of the Saskatoon Conference and Workshop on valuation theory (second volume), F-V. Kuhlmann, S. Kuhlmann, M. Marshall, editors, Fields Institute Communications, 2003.

T2. B. Teissier, *Monômes, volumes et multiplicités*, *in* Introduction à la théorie des singularités, II, p. 127-141, Travaux en Cours No. 37, Hermann, Paris.

T3. B. Teissier, *Cycles évanescents, sections planes, et conditions de Whitney.* Singularités à Cargèse, Astérisque No. 7-8, SMF Paris 1973.

T4. B. Teissier, *Multiplicités polaires, sections planes, et conditions de Whitney*, Algebraic Geometry, La Rábida 1981, J-M. Aroca, R. Buchweitz, M. Giusti and M. Merle, editors, Springer Lect. Notes in Math., No. 961, 1982.

T5. B. Teissier, *The hunting of invariants in the geometry of discriminants*, Real and complex singularities, Proc. Nordic Summer School Oslo 1976, Per Holm ed., Sijthoff and Noordhof 1977, 555-677.

T6. B. Teissier, *Résolution simultanée, I et II.* Séminaire sur les singularités des surfaces, Palaiseau 1976-77, Demazure, H. Pinkham, B. Teissier, Editeurs, Springer Lecture Notes in Math., No. 777,

TE. Kempf, Knudsen, Mumford, St. Donat, *Toroidal embeddings*, Springer Lect. Notes in Math., No. 339, 1973.

Z. O. Zariski, *Le problème des modules pour les branches planes*, (avec un appendice de B. Teissier), Hermann, Paris, 1986. Translation by the A.M.S. to appear.