Cours du vendredi premier décembre 2017

Lagrangien invariant par difféomorphisme:

$L : TM \to \mathbb{R}$, $\boxed{L(x, \lambda v) = \lambda L(x,v),}$ $\forall \lambda > 0.$

<u>Exemples</u> 1) Fermat: $L(x, \lambda v) = n(x)\, |v|$

$$|v| = \sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2}$$

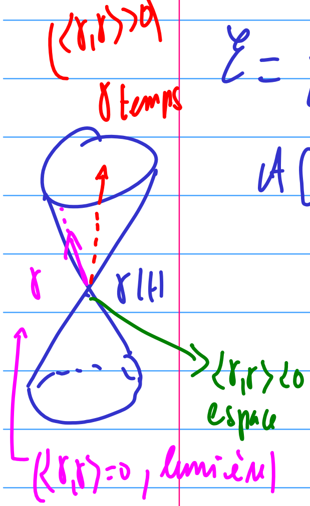
n: indice de réfraction.

2) (M, g): variété riemannienne $\left(\begin{array}{l} g_x : T_xM \times T_xM \to \mathbb{R} \\ \text{Produit scalaire sur } T_xM \end{array} \right)$

$$\mathcal{L}(\gamma) = \int_a^b \sqrt{g_{\mu\nu}(\gamma(t))\, \dot\gamma^\mu \dot\gamma^\nu}\, dt$$

3) $M \simeq (\mathbb{R}^4, \eta_{\mu\nu})$ $\eta_{\mu\nu}$: métrique pseudo-riemannienne

$$\langle v, v \rangle = \eta_{\mu\nu} v^\mu v^\nu = c^2 (v^0)^2 - (v^1)^2 - (v^2)^2 - (v^3)^2$$



$$\mathcal{E} = \{ \gamma \in \mathcal{C}^2([a,b], \mathbb{R}^4) ;\ \langle \dot\gamma(t), \dot\gamma(t) \rangle > 0 \}$$

$$A(\gamma) = K \int_a^b \sqrt{\eta_{\mu\nu}\, \dot\gamma^\mu \dot\gamma^\nu}\, dt$$: longueur de l'image de γ

= Lagrangien d'une particule <u>libre</u> en relativité restreinte.

$\mathbb{L} : TM \to T^*M$ <u>dégénère</u> $\boxed{\text{pour } L(t, \lambda v) = \lambda L(t, v)}$

pas le même qu'avant !

<u>Exemple</u> Partons de $L : I \times TM \to \mathbb{R}$
$(t, x, v) \mapsto L(t, x, v)$

Supposons $\mathbb{L} : I \times TM \to I \times T^*M$ est inversible
$(t, x^i, v^i) \mapsto \left(t, x^i, \frac{\partial L}{\partial v^i}(t, x^i, v^i)\right)$

Considérons : $\Lambda : T(I \times M) \to \mathbb{R}$

$$\underset{\substack{\uparrow \\ I}}{(x^0}, \underset{\substack{\uparrow \\ M}}{x^i}, w^0, w^i) \mapsto \boxed{L\left(x^0, x^i, \frac{w^i}{w^0}\right) w^0} = \Lambda$$

Alors si $\gamma : I \to M$, $\Gamma = \{(t, \gamma(t)) ; t \in I\} \subset I \times M$

Paramétrisons Γ par $s \mapsto (\eta^0(s), \eta^i(s))$

<u>Alors</u> $$\mathcal{L}[\gamma] = \int_{t_1}^{t_2} L(t, \gamma(t), \dot\gamma(t))\, dt = \int_a^b \boxed{L\left(\eta^0(s), \eta^i(s), \frac{\frac{d\eta^i}{ds}}{\frac{d\eta^0}{ds}}\right) \frac{d\eta^0}{ds}}\, ds$$

$$= \int_a^b \Lambda\left(\eta^\mu(s), \frac{d\eta^\mu}{ds}\right) ds \quad \text{où} \quad \left| \begin{array}{l} \eta^0(a) = t_1 \\ \eta^0(b) = t_2 . \end{array} \right.$$
$$= \Lambda[\eta]$$

La transformée $\hat{\mathbb{L}} : T(I \times M) \to T^*(I \times M)$
$(x^\mu, w^\mu) \mapsto \left(x^\mu, \frac{\partial \Lambda}{\partial x^\mu}\right)$] dégénérée

Car Λ est invariante par difféomorphisme de l'ensemble de départ $\ni s$.

<u>Retrouvons cela</u> : $\boxed{\pi_i = \frac{\partial \Lambda}{\partial w^i} = \frac{\partial L}{\partial v^i}\left(x^0, x^i, \frac{w^i}{w^0}\right)}$ $\Rightarrow \frac{w^i}{w^0} =$ Fonction (x^μ, π_i)

Puisque $\mathbb{L}$ est non dégénérée, je peut définir

$$\left| \begin{array}{l} H(x^0, x^i, p_i) = p_i v^i - L(x^0, x^i, v^i) \\ p_i = \frac{\partial L}{\partial v^i}(x^0, x^i, v^i) \end{array} \right.$$

Alors $\boxed{H(x^0, x^i, \pi_i) = \pi_i \frac{w^i}{w^0} - L\left(x^0, x^i, \frac{w^i}{w^0}\right)}$

$$\pi_0 = \frac{\partial \Lambda}{\partial w^0}(x^0, x^i, w^0, w^i)$$

$$= \frac{\partial}{\partial w^0}\left(L\left(x^0, x^i, \boxed{\frac{w^i}{w^0}}\right) \boxed{w^0} \right)$$

$$= \frac{\partial L}{\partial v^i}\left(x^\mu, \frac{w^i}{w^0}\right)\left(-\frac{w^i}{(w^0)^2}\right) w^0 + L\left(x^0, x^i, \frac{w^i}{w^0}\right)$$

$$= \boxed{L\left(x^\mu, \frac{w^i}{w^0}\right) - \frac{\partial L}{\partial v^i}\left(x^\mu, \frac{w^i}{w^0}\right) \frac{w^i}{w^0}}$$

Donc $\boxed{\pi_0 + H(x^0, x^i, \pi_i) = 0}$ Contrainte

Exemple $A[\eta] = K \int_a^b \sqrt{c^2 \left(\frac{d\eta^0}{ds}\right)^2 - \sum_{i=1}^{3} \left|\frac{d\eta^i}{ds}\right|^2} \, ds$

(particule relativiste). Choisissons (x^0, x^1, x^2, x^3) : coordonnées (implicite)

Choix du paramètre $s = t$ de $\boxed{\eta^0(t) = t}$ (choix de référentiel)

$$A[\eta] = K \int_a^b \sqrt{c^2 - \sum_i |\dot\eta^i|^2} \, dt$$

$$= K \int_a^b c \sqrt{1 - \frac{|v|^2}{c^2}} \, dt \simeq \int_a^b \left[Kc - \frac{Kc}{2} \frac{|v|^2}{c^2} + O\left(\frac{1}{c^3}\right) \right]$$

$$= \int_a^b \left[Kc - \underbrace{\frac{K}{2c} |v|^2}_{\frac{1}{2} m |v|^2} + O\left(\frac{1}{c^3}\right) \right] dt$$

$\frac{1}{2} m |v|^2$: Lagrangien Maupertuis

Donc $K = -mc \longrightarrow \underline{A}[\eta] = \int_a^b -mc\sqrt{c^2\left(\frac{d\eta^0}{ds}\right)^2 - \left|\frac{d\eta^i}{ds}\right|^2}$

$$\boxed{\Lambda = -mc\sqrt{c^2(w^0)^2 - \sum_i (w^i)^2}} = -mc\sqrt{\langle w, w\rangle}$$

$$\Pi_i = \frac{\partial \Lambda}{\partial w^i} = -mc\frac{-2w^i}{2\sqrt{\langle w,w\rangle}} = \frac{mc\,w^i}{\sqrt{\langle w,w\rangle}}$$

$$\langle w,w\rangle = c^2(w^0)^2 - \sum_i (w^i)^2$$

$$= c^2(w^0)^2\left(1 - \frac{|v|^2}{c^2}\right) \qquad |v|^2 = \sum\left(\frac{w^i}{w^0}\right)^2$$

$$\Rightarrow \Pi_i = \frac{mc\,w^i}{c\,w^0\sqrt{1-\frac{|v|^2}{c^2}}} = \frac{mc\,v^i}{\sqrt{1-\frac{|v|^2}{c^2}}} \quad \text{où } v^i = \frac{w^i}{w^0}$$

$$\Pi_0 = \frac{\partial\Lambda}{\partial w^0} = -mc\frac{2c^2 w^0}{2\sqrt{\langle w,w\rangle}} = -mc\frac{c}{\sqrt{1-\frac{|v|^2}{c^2}}} = \frac{-mc^2}{\sqrt{1-\frac{|v|^2}{c^2}}}$$

$$= -mc^2 - \frac{1}{2}m|v|^2 + O\left(\frac{1}{c^2}\right)$$

$$= -\text{Energie} = -E \text{ (dans le référentiel choisi)}$$

Contrainte $\qquad E^2 - |p|^2c^2 = \underbrace{(mc^2)^2}_{\text{indépendante du référentiel}}$ dans $T^\vee\mathbb{R}^4$.

Version hamiltonienne de l'action $\qquad L : I\times T\mathcal{M} \to \mathbb{R}$

$\to \Lambda : T(I\times\mathcal{M}) \to \mathbb{R}$, $\qquad \boxed{\Lambda(x^\mu, w^\mu) = L\left(x^\mu, \frac{w^i}{w^0}\right)w^0}$

$$(x^\mu) = (x^0, (x^i))$$

Soit $s \longmapsto \left(\eta^\mu(s)\right)_{0 \leq \mu \leq n} \in I \times \mathcal{M}$ (ou un espace-temps)

Notons $\boxed{\pi_i(s) = \frac{\partial L}{\partial v^i}\left(\eta^\mu(s), \frac{\frac{d\eta^i(s)}{ds}}{\frac{d\eta^0}{ds}}\right)}$ (inversible)

$$\pi_0(s) = -H\left(\eta^\mu(s), \pi^i(s)\right) = \frac{\partial \Lambda}{\partial w^0}\left(\eta^\mu, \frac{d\eta^\mu}{ds}\right)$$

$$A[\eta] = \int_a^b \underbrace{L\left(\eta^\mu(s), \frac{d\eta^\mu}{ds} \Big/ \frac{d\eta^0}{ds}\right)\frac{d\eta^0}{ds}}\, ds$$

$$= \int_a^b \left[\left(\underbrace{L} - \underbrace{\frac{\partial L}{\partial v^i}\frac{d\eta^i}{ds}\Big/\frac{d\eta^0}{ds}}\right)\underbrace{\frac{d\eta^0}{ds}} + \underbrace{\frac{\partial L}{\partial v^i}}\frac{d\eta^i}{ds}\right] ds$$

$$= \int_a^b \left(-H(\eta^\mu, \pi_i)\frac{d\eta^0}{ds} + \pi_i \frac{d\eta^i}{ds}\right) ds$$

$$= \int_a^b \pi_i\, d\eta^i - H\, d\eta^0 = \int_a^b u^*(p_\mu d\eta^\mu) = \int_a^b u^*\theta$$

où $u : [a,b] \longrightarrow T^*(I \times \mathcal{M})$ $\Big|$ $\pi_0 + H(\eta^\mu, \pi_i) = 0$

$\quad s \longmapsto \left(\eta^\mu(s), \pi_\mu(s)\right)$

Si $\underline{\pi_i = \frac{\partial L}{\partial v^i}\left(\eta^\mu, \frac{d\eta^i}{ds} \Big/ \frac{d\eta^0}{ds}\right)}$ $\quad \boxed{A[\eta] = \int_{u([a,b])} \theta}$

$\left(\int_a^b u^*\theta = \int_{u([a,b])} \theta \quad \text{si } u : [a,b] \to T^*(I \times \mathcal{M}) \text{ plongement}\right)$

Résultat général Soit $u : [a,b] \to (T^*Y, \theta)$

Soit $H : T^*Y$. u est point critique de

$$A[u] = \int_a^b u^*\theta - H\circ u \, dt$$

$$\iff \frac{du}{dt} = \xi_H(u) \quad \text{où} \quad \boxed{\xi_H \lrcorner \omega + dH = 0}$$

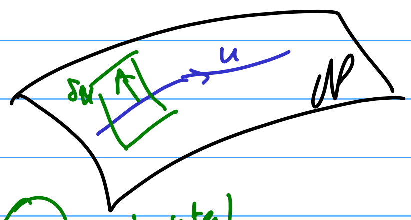
$$\omega = d\theta$$

Variation première de $A[u] = \int_0^1 u^*\theta$ **géométriquement**

où $\left| \begin{array}{l} u : [0,1] \longrightarrow \boxed{\mathcal{N} \subset T^*Y} \\ \quad \tau \longmapsto u(\tau) \end{array} \right.$ (hypersurface)

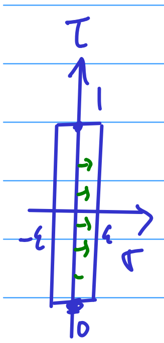
$$\mathcal{N} = \{ (x^\mu, p_\mu) ; \; p_0 = -H(x^\mu, p_i) \}$$

Soit $\delta u \in \Gamma(u^* T\mathcal{N})$



$\exists \, U :]-\varepsilon, \varepsilon[\times [0,1] \longrightarrow \mathcal{N}$ (contrainte) $\quad \left\{ \begin{array}{l} U(0,\tau) = u(\tau) \\ \frac{\partial U}{\partial \sigma}(0,\tau) = \delta u(\tau) \end{array} \right.$

$\qquad (\sigma, \tau) \longmapsto U(\sigma, \tau)$



$$u[\sigma](\tau) = U(\sigma, \tau)$$

$$\frac{d}{d\sigma} A[u[\sigma]] \Big|_{\sigma=0} \; ?$$

$\to \frac{\partial}{\partial \sigma}$

Introduisons $\Phi_s : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$

$\qquad (\sigma, \tau) \mapsto (\sigma + s, \tau)$

$$\Phi_s = e^{s \frac{\partial}{\partial \sigma}}$$

$$U \circ \Phi_s(\sigma, \tau) = U(\sigma + s, \tau)$$

$$\mathcal{A}[u[s]] = \int_0^1 u[s]^* \theta = \int_{\{s\}\times[0,1]} U^*\theta = \int_{\{0\}\times[0,1]} (U\circ\Phi_s)^*\theta$$

$$= \boxed{\int_{\{0\}\times[0,1]} \Phi_s^*(U^*\theta)} \qquad \left(\Phi_s = \exp s\frac{\partial}{\partial\sigma}\right)$$

Donc $\displaystyle\frac{d}{ds}\mathcal{A}[u[s]]\Big|_{s=0} = \int_{\{0\}\times[0,1]} \mathcal{L}_{\frac{\partial}{\partial\sigma}}(U^*\theta)$.

$$\overset{\text{Cartan}}{=} \int_{\{0\}\times[0,1]} \frac{\partial}{\partial\sigma}\lrcorner\, d(U^*\theta) + d\left(\frac{\partial}{\partial\sigma}\lrcorner\, U^*\theta\right)$$

$\downarrow\ d = U^* d\theta \qquad \downarrow$ Stokes $\quad \alpha = U^*\theta$

$d(U^*\theta) = U^* d\theta$

$$= \int_{\{0\}\times[0,1]} U^*\left(\frac{\partial U}{\partial\sigma}\lrcorner\, d\theta\right) + \left[U^*\left(\frac{\partial U}{\partial\sigma}\lrcorner\,\theta\right)\right]_{\{(0,0)\}}^{\{(0,1)\}}$$

On utilise $\boxed{\frac{\partial}{\partial\sigma}\lrcorner\, U^*\alpha = U^*\left(\frac{\partial U}{\partial\sigma}\lrcorner\,\alpha\right)}$

$$\boxed{= \int_{u([0,1])} \delta u\lrcorner\, d\theta + \left[\delta u\lrcorner\,\theta_u\right]_{t_1}^{t_2}}$$

a) Si δu est à support compact, $\displaystyle\frac{d}{ds}\mathcal{A}[u[s]] = \int_{u([0,1])} \delta u\lrcorner\,\omega$

C'est nul $\forall\, \delta u \in \Gamma(u^* T\mathcal{W})$

$\iff$ $\forall\, \tau\in[0,1]$, $\forall\, \xi \in T_{u(\tau)}\mathcal{W}$, $\omega_{u(\tau)}\left(\xi, \frac{du}{d\tau}\right) = 0$

$$\int_0^1 \omega_{u(\tau)}\left(\delta u(\tau), \frac{du}{d\tau}\right) d\tau$$

Autrement dit $\boxed{\frac{du}{d\tau}\lrcorner\,\omega_{u(\tau)}\Big|_{T_{u(\tau)}\mathcal{W}} = 0}$

(distribution $\ker\omega|_{\mathcal{W}}$: équations de Hamilton).

b) Si u est solution de $\boxed{\frac{du}{d\tau} \lrcorner \omega|_{\mathcal{W}} = 0}$ alors

$$\frac{d}{ds} \mathcal{A}[u[s]] = \left[\delta u \lrcorner \theta_u\right]_0^1 = \theta_{u(1)}(\delta u(1)) - \theta_{u(0)}(\delta u(0))$$

$\boxed{\text{Hypothèse}}$: $\forall$ $M_0, M_1 \in \mathcal{Y}$, $\exists !$ trajectoire $u : [0,1] \to \mathcal{W} \subset T^*\mathcal{Y}$

espace-temps

$$\left| \begin{array}{l} u(0) \in T^*_{M_0}\mathcal{Y} \\ u(1) \in T^*_{M_1}\mathcal{Y} \\ \boxed{\frac{du}{d\tau} \lrcorner \omega|_{\mathcal{W}} = 0} \end{array} \right.$$

$u = u[M_0, M_1]$

u Hamiltonienne

On définit $S(M_0, M_1) = \mathcal{A}[u[M_0, M_1]]$

Thm

$$\boxed{dS_{(M_0,M_1)}(\delta M_0, \delta M_1) = \theta_{(M_1,P_1)}(\delta M_1) - \theta_{(M_2,P_2)}(\delta M_0)}$$

où $(M_0, P_0) = u(0)$, $(M_1, P_1) = u(1)$ (fonctions de (M_1, M_2)

Dans $\mathcal{G} = \left\{ (z_0, z_1) \in \mathcal{W}^2 ; \exists u : [0,1] \to \mathcal{W}, \; u(0) = z_0, u(1) = z_1, \; u \text{ hamiltonienne} \right\} \subset \mathcal{W} \times \mathcal{W}$

$$\mathcal{W} \xleftarrow{\pi_1} \mathcal{G} \xrightarrow{\pi_2} \mathcal{W}$$

Hypothèse $\Longleftrightarrow$ $\mathcal{G}$ est paramétrisé par
$\Phi : \{(M_0, M_1) \in \mathcal{Y} \times \mathcal{Y}\} \to \mathcal{W}^2$
$(M_0, M_1) \mapsto (z_0, z_1)$

$\mathcal{G} \ni (z_0, z_1) \mapsto \underbrace{\mathcal{A}[u]}$ où u : trajectoire qui joint z_0 à z_1

$$=: \underbrace{\tilde{S}(z_0, z_1)}_{\mathcal{W}^2} = \tilde{S} \circ \Phi(M_0, M_1) = S(M_0, M_1)$$

$$\boxed{d\tilde{S} = \pi_1^* \theta - \pi_0^* \theta}$$

$M_0 = (x_0, t)$
$M_1 = (x_1, t)$

$$d\tilde{S} = \frac{\partial S}{\partial x_0^{\mu}} dx_0^{\mu} + \frac{\partial S}{\partial x_1^{\mu}} dx_1^{\mu} = p_{1\mu}(x_0, x_1)\, dx_1^{\mu} - p_{0\mu}(x_0, x_1)\, dx_0^{\mu}$$