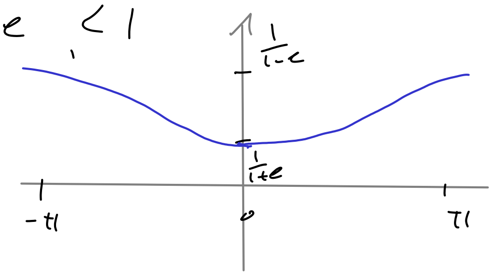
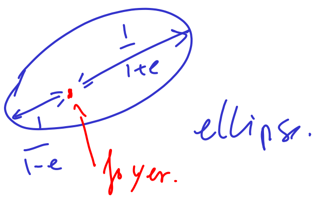
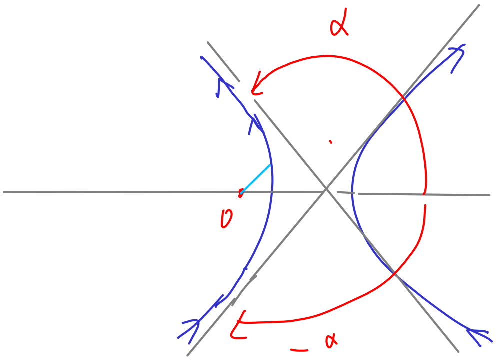
Devoir pour le 23 mars : à rendre jusqu'au 30 <u>mars</u>.

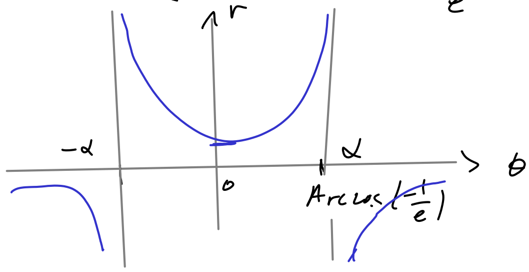
$$r = \frac{h}{1 + e\cos\theta} \qquad 0 < e < 1$$





$1 < e$: $\quad 1 + e\cos\theta = 0 \iff \cos\theta = -\frac{1}{e} \in \,]-1,0[$.





Equations de Hamilton.

Si $\boxed{\gamma : \mathbb{R} \to \underbrace{\mathbb{R}^3}}$ $\mathcal{C}^2$ décrit le mouvement d'un point dans l'espace physique.

ou Ω : ouvert $\subset \mathbb{R}^3$

Sur $\Omega \times \mathbb{R}^3$, on définit

(x, p)

$\uparrow$ position $\quad \uparrow$ (moments cinétiques | ou impulsion)

$$\boxed{\begin{aligned} H : \Omega \times \mathbb{R}^3 &\longrightarrow \mathbb{R} \quad \text{(Hamiltonien)} \\ (x, p) &\longmapsto H(x,p) \end{aligned}} \quad \mathcal{C}^1$$

Exemple : $\boxed{H(x,p) = \underbrace{V(x)}_{\text{"énergie potentielle"}} + \frac{\|p\|^2}{2m}}$

Rappel : $p = mv \implies \|p\|^2 = m^2 \|v\|^2$

$$\implies \frac{\|p\|^2}{2m} = \frac{1}{2} m \|v\|^2 \quad \text{(énergie cinétique)}$$

$$[\gamma : I \to \Omega] \longrightarrow M(t) = \begin{pmatrix} \gamma(t) \\ m\,\dot\gamma(t) \end{pmatrix} \in \Omega \times \mathbb{R}^3 \subset \mathbb{R}^3 \times \mathbb{R}^3 \simeq \mathbb{R}^6$$

$$= \begin{pmatrix} \gamma(t) \\ \pi(t) \end{pmatrix} \quad \text{où} \quad \pi(t) = m\,\dot\gamma(t)$$

Hamilton
$$\begin{cases} \dfrac{d\gamma^i}{dt} = \dfrac{\partial H}{\partial p_i}(\gamma, \pi) = \dfrac{\partial H}{\partial p_i} \circ M = \dfrac{\pi_i}{m} \\ \dfrac{d\pi_i}{dt} = -\dfrac{\partial H}{\partial x^i}(\gamma, \pi) = -\dfrac{\partial H}{\partial x^i} \circ M = -\dfrac{\partial V}{\partial x^i}(\gamma) \end{cases}$$

$$\begin{cases} \dfrac{\partial H}{\partial p_i}(x,p) = \dfrac{\partial}{\partial p_i}\left[V(x) + \dfrac{\|p\|^2}{2m}\right] = \dfrac{1}{2m} 2p_i = \dfrac{p_i}{m} \Rightarrow \dfrac{\partial H}{\partial p_i} \circ M = \dfrac{\partial H}{\partial p_i}(\gamma, \pi) = \dfrac{\pi_i}{m} \\ \dfrac{\partial H}{\partial x^i}(x,p) = \dfrac{\partial V}{\partial x^i}(x) \Rightarrow -\dfrac{\partial H}{\partial x^i} \circ M = -\dfrac{\partial H}{\partial x^i}(\gamma, \pi) = -\dfrac{\partial V}{\partial x^i}(\gamma) \end{cases}$$

$\left(V(x) = -\dfrac{G m_1 m_2}{\|x\|}\right.$ par exemple$\left.\right)$

Hamilton $\Leftrightarrow$
$$\begin{cases} \dfrac{d\gamma^i}{dt} = \dfrac{\pi_i}{m} \Leftrightarrow \pi_i = m\dfrac{d\gamma^i}{dt} \\ \boxed{m\dfrac{d^2\gamma^i}{dt^2}} = \dfrac{d}{dt}\left(m\dfrac{d\gamma^i}{dt}\right) = \dfrac{d\pi_i}{dt} = \boxed{-\dfrac{\partial V}{\partial x^i}(\gamma)} \end{cases}$$
(force dérivant d'un potentiel)

Marche toutes les fois que $F(x)$ (force) $= -\underbrace{\nabla V(x)}_{\text{gradient}}$ (gradient de $V : \mathbb{R}^3 \to \mathbb{R}$)

Cas de deux corps massifs en interaction (lié au devoir)

Deux corps A et B, masses m_A et m_B, positions $\gamma_A : \mathbb{R} \to \mathbb{R}^3$
$\gamma_B : \mathbb{R} \to \mathbb{R}^3$

Principe d'action & réaction : $F_{A/B} = -F_{B/A}$

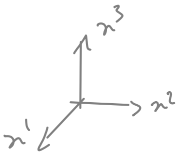
$$\Rightarrow \frac{d}{dt}\left(m_A \dot\gamma_A + m_B \dot\gamma_B\right) = 0$$

Cas
$$\begin{cases} m_A \ddot\gamma_A = -\dfrac{G m_A m_B}{\|\gamma_A - \gamma_B\|}(\gamma_A - \gamma_B) \\ m_B \ddot\gamma_B = -\dfrac{G m_A m_B}{\|\gamma_A - \gamma_B\|}(\gamma_B - \gamma_A) \end{cases}$$

$M : \mathbb{R} \longrightarrow \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ (position A) (position B) (impulsion A) (impulsion B)

$t \longmapsto (\gamma_A(t), \gamma_B(t), \pi_A(t), \pi_B(t))$

$$H : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$(x_A, x_B, p_A, p_B) \longmapsto H(x_A, x_B, p_A, p_B).$$



$$H(x_A, x_B, p_A, p_B) = \underbrace{-\frac{G m_A m_B}{\|x_A - x_B\|}}_{\text{énergie (potentielle) d'interaction gravitationnelle}} + \underbrace{\frac{\|p_A\|^2}{2m_A} + \frac{\|p_B\|^2}{2m_B}}_{\text{somme des énergies cinétiques}} \quad \text{(énergie totale du système)}$$

$i = 1, 2, 3$

$$\left\{ \begin{array}{l} \gamma(t) = (\gamma_A, \gamma_B)(t) \in \mathbb{R}^6 \\ \pi(t) = (\pi_A, \pi_B)(t) \in \mathbb{R}^6 \end{array} \right.$$

$$\boxed{\begin{array}{ll} \dfrac{d\gamma_A^i}{dt} = \dfrac{\partial H}{\partial p_{Ai}}(\gamma, \pi) \; ; \; \dfrac{d\gamma_B^i}{dt} = \dfrac{\partial H}{\partial p_{Bi}}(\gamma, \pi) \\ \dfrac{d\pi_{Ai}}{dt} = -\dfrac{\partial H}{\partial x_A^i}(\gamma, \pi) \; ; \; \dfrac{d\pi_{Bi}}{dt} = -\dfrac{\partial H}{\partial x_B^i}(\gamma, \pi) \end{array}}$$

← $\frac{d}{dt}$ (positions des particules)

← $\frac{d}{dt}$ (impulsions)

$$\frac{\partial H}{\partial p_{Ai}}(x, p) = \frac{p_{Ai}}{m_A} \; ; \; \frac{\partial H}{\partial p_{Bi}}(x, p) = \frac{p_{Bi}}{m_B}$$

$$\frac{\partial H}{\partial x_A^i}(x, p) = \frac{\partial}{\partial x_A^i}\left(-\frac{G m_A m_B}{\|x_A - x_B\|}\right) \underset{(x_B \text{ gelé})}{=} (-G m_A m_B) \frac{-1}{2\left(\|x_A - x_B\|^2\right)^{3/2}} \, 2x_A^i$$

$$\left(\begin{array}{ccccc} (x_A, x_B) & \longmapsto & \sum_{i=1}^{3} (x_A^i - x_B^i)^2 = \|x_A - x_B\|^2 & \longmapsto & -\dfrac{G m_A m_B}{\|x_A - x_B\|} \\ \mathbb{R}^6 \setminus \{(x, x)\} & \xrightarrow{G} & \mathbb{R} \setminus \{0\} & \xrightarrow{F} & \mathbb{R} \end{array} \right)$$

$$\frac{\partial (F \circ G)}{\partial x_A^i}(x_A, x_B) = F' \circ G(x_A, x_B) \, \frac{\partial G}{\partial x_A^i}(x_A, x_B)$$

$$= \frac{G m_A m_B}{\|x_A - x_B\|^3} \, x_A^i$$

De même $\dfrac{\partial H}{\partial x_B^i} = \dfrac{G m_A m_B}{\|x_A - x_B\|^3} \, x_B^i$

→ Hamilton
$$\left\{ \begin{array}{ll} \dot{\gamma}_A^i = \dfrac{\pi_{Ai}}{m_A} & \dot{\gamma}_B^i = \dfrac{\pi_{Bi}}{m_B} \iff \left\{ \begin{array}{l} \pi_{Ai} = m_A \dot{\gamma}_A^i \\ \pi_{Bi} = m_B \dot{\gamma}_B^i \end{array} \right. \\ \dot{\pi}_{Ai} = -\dfrac{G m_A m_B}{\|\gamma_A - \gamma_B\|^3} \gamma_A^i & \dot{\pi}_{Bi} = -\dfrac{G m_A m_B}{\|\gamma_A - \gamma_B\|^2} \gamma_B^i \end{array} \right.$$

$$\Rightarrow m_A \ddot{\gamma}_A^i = -\frac{G m_A m_B}{\|\gamma_A - \gamma_B\|^3} \gamma_A^i \quad \& \quad m_B \ddot{\gamma}_B^i = -\frac{G m_A m_B}{\|\gamma_A - \gamma_B\|^3} \gamma_B^i$$

Conservation de l'énergie : $\quad H(x,p) = V(x) + \frac{\|p\|^2}{2m}$

$$\mathbb{R} \xrightarrow{(\gamma,\pi)} \mathbb{R}^6 \xrightarrow{H} \mathbb{R} \qquad \pi = m\dot\gamma$$

$H \circ (\gamma,\pi) = E$: fonction qui donne l'énergie de la particule à chaque instant.

$$\left(H\circ(\gamma,\pi)(t) = V(\gamma) + \frac{\|\pi\|^2}{2m} = V(\gamma) + \frac{m\|\dot\gamma\|^2}{2} \right)$$: énergie à l'instant t.

Retrouvons cela :

$$\frac{dE}{dt} = \frac{d\, H\circ(\gamma,\pi)}{dt} = \sum_{i=1}^{3} \frac{\partial H}{\partial x^i}(\gamma,\pi)\frac{d\gamma^i}{dt} + \sum_{i=1}^{3} \frac{\partial H}{\partial p_i}(\gamma,\pi)\frac{d\pi_i}{dt}$$

$$\mathbb{R} \xrightarrow{u} \mathbb{R}^p \xrightarrow{F} \mathbb{R}$$

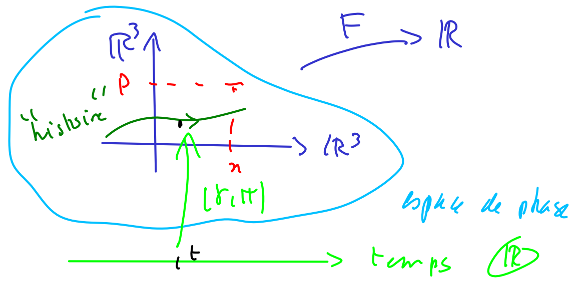
$$\frac{d}{dt}(F\circ u) = \sum_{\alpha=1}^{p} \left(\frac{\partial F}{\partial x^\alpha}\circ u\right)\frac{du^\alpha}{dt} = \left\langle (\nabla F)\circ u , \frac{du}{dt} \right\rangle$$

si (γ,π) est solution des équations de Hamilton

$$= \sum_{i=1}^{3} \frac{\partial H}{\partial x^i}(\gamma,\pi)\frac{\partial H}{\partial p_i}(\gamma,\pi) + \sum_{i=1}^{3} \frac{\partial H}{\partial p_i}(\gamma,\pi)\left(-\frac{\partial H}{\partial x^i}(\gamma,\pi)\right)$$

$= 0 \quad \longrightarrow$ conservation de $E(t)$ si (γ,π) est solution de Hamilton.

Que se passe-t-il pour n'importe quelle $F : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$?



Calculer $\frac{d\left(F\circ(\gamma,\pi)\right)}{dt}$

sachant que (γ,π) est solution de Hamilton

$$(H)\begin{cases} \frac{d\gamma^i}{dt} = \frac{\partial H}{\partial p_i}(\gamma,\pi) \\ \frac{d\pi_i}{dt} = -\frac{\partial H}{\partial x^i}(\gamma,\pi) \end{cases}$$

$$\frac{d}{dt}(F\circ u) = \sum_{\alpha=1}^{p} \left(\frac{\partial F}{\partial x^\alpha}\circ u\right)\frac{du^\alpha}{dt}$$

$$\mathbb{R} \xrightarrow{u} \mathbb{R}^p \xrightarrow{F} \mathbb{R}$$

$$\frac{d}{dt}\left[F\circ(\gamma,\pi)\right] = \sum_{i=1}^{3} \frac{\partial F}{\partial x^i}(\gamma,\pi)\frac{d\gamma^i}{dt} + \sum_{i=1}^{3} \frac{\partial F}{\partial p_i}(\gamma,\pi)\frac{d\pi_i}{dt}$$

$$= \sum_{i=1}^{3} \frac{\partial F}{\partial x^i}(\gamma,\pi)\frac{\partial H}{\partial p_i}(\gamma,\pi) + \frac{\partial F}{\partial p_i}(\gamma,\pi)\left(-\frac{\partial H}{\partial x^i}(\gamma,\pi)\right)$$

$$= \{H, F\}(\gamma,\pi)$$

où $\boxed{\{H,F\}(x,p) := \sum_{i=1}^{3} \frac{\partial H}{\partial p_i}(x,p)\frac{\partial F}{\partial x_i}(x,p) - \frac{\partial H}{\partial x_i}(x,p)\frac{\partial F}{\partial p_i}(x,p)}$

Crochet de Poisson de H et F.

Théorème Si (γ,π) est solution de $\frac{d\gamma^i}{dt} = \frac{\partial H}{\partial p_i}\circ(\gamma,\pi)$ & $\frac{d\pi_i}{dt} = -\frac{\partial H}{\partial x_i}\circ(\gamma,\pi)$

Alors $\boxed{\frac{d(F\circ(\gamma,\pi))}{dt} = \{H,F\}\circ(\gamma,\pi)}$

Ça généralise à N particules : $H, F : \underset{\text{(positions)}}{(\mathbb{R}^3)^N} \times \underset{\text{(impulsions)}}{(\mathbb{R}^3)^N} \to \mathbb{R}$

Exemple 2 particules A et B dans $\mathbb{R}^3$

$$H(x_A, x_B, p_A, p_B) = \frac{\|p_A\|^2}{2m_A} + \frac{\|p_B\|^2}{2m_B} + \frac{G m_A m_B}{\|x_A - x_B\|}$$

$$F = m_A x_A^1 + m_B x_B^1$$ (première composante de $m_A x_A + m_B x_B$)

$$\{H,F\} = \sum_{i=1}^{3} \frac{\partial H}{\partial p_{A^i}}\underbrace{\frac{\partial F}{\partial x_A^i}} + \frac{\partial H}{\partial p_{B^i}}\underbrace{\frac{\partial F}{\partial x_B^i}} - \frac{\partial H}{\partial x_A^i}\underbrace{\frac{\partial F}{\partial p_A^i}}_{0} - \frac{\partial H}{\partial x_B^i}\underbrace{\frac{\partial F}{\partial p_B^i}}_{0}$$

$$= \frac{\partial H}{\partial p_{A1}} m_A + \frac{\partial H}{\partial p_{B1}} m_B$$

$$= \frac{p_{A1}}{m_A} m_A + \frac{p_{B1}}{m_B} m_B = p_{A1} + p_{B1} =: P_1$$ (quantité de mouvement totale)

$G = p_{A1} + p_{B1} \Rightarrow \frac{\partial G}{\partial x_A^i} = \frac{\partial G}{\partial x_B^i} = 0$ et $\frac{\partial G}{\partial p_{A1}} = 1$, $\frac{\partial G}{\partial p_{A2}} = 0$, $\frac{\partial G}{\partial p_{B1}} = 1$

$$\{H,G\} = \left(\sum_{i=1}^{3} \frac{\partial H}{\partial p_{A^i}} \times 0 + \frac{\partial H}{\partial p_{B^i}} \times 0\right) - \frac{\partial H}{\partial x_A^1} \times 1 - \frac{\partial H}{\partial x_B^1} \times 1$$

$$= -\frac{\partial H}{\partial x_A^1} - \frac{\partial H}{\partial x_B^1}$$

Or $H = \frac{\|p_A\|^2}{2m_A} + \frac{\|p_B\|^2}{2m_B} - \frac{G m_A m_B}{\|x_A - x_B\|}$

$$\|x_A - x_B\|^2 = \sum_{i=1}^{3}(x_A^i - x_B^i)^2$$

$$\left.\begin{aligned} \frac{\partial}{\partial x_A^1}\left(\|x_A - x_B\|^2\right) &= 2(x_A^1 - x_B^1) \\ &\updownarrow \text{opposés} \\ \frac{\partial}{\partial x_B^1}\left(\|x_A - x_B\|^2\right) &= 2(x_B^1 - x_A^1) \end{aligned}\right\} \Rightarrow \forall f : \mathbb{R}_+ \to \mathbb{R}$$

$$\frac{\partial f(\|x_A - x_B\|^2)}{\partial x_A^1} + \frac{\partial f(\|x_A - x_B\|^2)}{\partial x_B} = 0$$

Donc $\{H, p_A^1 + p_B^1\} = 0.$ $\qquad G(x,p) = p_A^1 + p_B^1$

Conséquence
$$\frac{d(\pi_A^1 + \pi_B^1)}{dt} = \frac{d\, G\circ(\gamma,\pi)}{dt} = \underbrace{\{H, G\}}_{\text{toujours nul}}\circ(\gamma,\pi) = 0$$

Conclusion : $\pi_A^i + \pi_B^i$ est conservé, $\forall i = 1,2,3.$

(quantité de mouvement total)

Lien avec principe d'incertitude ("indétermination") de Heisenberg.

|| On ne peut pas déterminer position et impulsion simultanément avec une précision arbitraire.

$\Updownarrow$

$p_1 : \mathbb{R}^6 \to \mathbb{R}$, $\underbrace{x^1 : \mathbb{R}^6 \to \mathbb{R}}$

fonction "position" (observable)

$$\{p_1, x^1\} = \sum_{i=1}^{3} \frac{\partial p_1}{\partial p_i}\frac{\partial x^1}{\partial x^i} - \cancel{\frac{\partial p_1}{\partial x^i}}\,\cancel{\frac{\partial x^1}{\partial p_i}}$$
$$= 1\cdot 1 + 0\cdot 0 + 0\cdot 0 - 0\cdot = 1.$$

$$\boxed{\{p_1, x^1\} = 1}$$