REDUCIBLE DEFORMATIONS AND SMOOTHING OF PRIMITIVE MULTIPLE CURVES

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Resume. A *primitive multiple curve* is a Cohen-Macaulay irreducible projective curve Y that can be locally embedded in a smooth surface, and such that $C = Y_{red}$ is smooth. In this case, $L = \mathcal{I}_C/\mathcal{I}_C^2$ is a line bundle on C.

This paper continues the study of deformations of Y to curves with smooth irreducible components, when the number of components is maximal (it is then the multiplicity n of Y). If a primitive double curve Y can be deformed to reduced curves with smooth components intersecting transversally, then $h^0(L^{-1}) \neq 0$. We prove that conversely, if L is the ideal sheaf of a divisor with no multiple points, then Y can be deformed to reduced curves with smooth components intersecting transversally. We give also some properties of reducible deformations in the case of multiplicity $n > 2$.

Summary

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1. Introduction

A *primitive multiple curve* is an algebraic variety Y over $\mathbb{C}$ which is Cohen-Macaulay, such that the induced reduced variety $C = Y_{red}$ is a smooth projective irreducible curve, and that every closed point of Y has a neighborhood that can be embedded in a smooth surface. These curves have been defined and studied by C. Bănică and O. Forster in [2]. The simplest examples are infinitesimal neighborhoods of projective smooth curves embedded in a smooth surface (but most primitive multiple curves cannot be globally embedded in smooth surfaces). Primitive multiple curves of multiplicity 2 (called *ribbons*) have been parametrized in [3]. Primitive multiple curves of any multiplicity and the coherent sheaves on them have been studied in [5], [4], [6] and [7].

Let Y be a primitive multiple curve with associated reduced curve C, and suppose that $Y \neq C$. Let $\mathcal{I}_C$ be the ideal sheaf of C in Y. The *multiplicity* of Y is the smallest integer n such that $\mathcal{I}_C^n = 0$. We have then a filtration

$$C = C_1 \subset C_2 \subset \cdots \subset C_n = Y$$

where C_i is the subscheme corresponding to the ideal sheaf $\mathcal{I}_C^i$ and is a primitive multiple curve of multiplicity i. The sheaf $L = \mathcal{I}_C/\mathcal{I}_C^2$ is a line bundle on C, called the *line bundle on C associated to Y*.

1.1. Deformations to reduced reducible curves

Deformations of primitive multiple curves $Y = C_n$ of any multiplicity $n \geq 2$ to reduced curves having multiple components which are smooth, intersecting transversally, have been studied in [9]: n is the maximal number of components of such deformations of Y, and in this case we say that the deformation is *maximal*, and the number of intersection points of two components is exactly $-\deg(L)$ (these deformations are called *maximal reducible deformations*). In [9] the case $\deg(L) = 0$ has been completely treated: a primitive multiple curve of multiplicity n can be deformed in disjoint unions of n smooth curves if and only if $\mathcal{I}_C$ is isomorphic to the trivial bundle on C_{n-1}. In [8] it has been proved that this last condition is equivalent to the following: there exists a flat family of smooth curves $\mathcal{C} \to S$, parametrized by a smooth curve S, $s_0 \in S$ such that $\mathcal{C}_{s_0} = C$, such that Y is isomorphic to the n-th infinitesimal neighborhood of C in $\mathcal{C}$.

The problem of determining which primitive multiple curves of multiplicity n can be deformed to reduced curves having exactly n components, allowing intersections of the components, is more difficult. A necessary condition is $h^0(L^*) > 0$.

We treat mainly the case of primitive double curves. We introduce the notion of *local reducible deformation* (that can be obtained for example from a global deformation by restricting it to a suitable infinitesimal neighbourhood of C). We prove that if $Y = C_2$ is such that $h^0(L^*) > 0$, then it admits a local reducible deformation (theorem 3.2.6 (i)). We conjecture that any local reducible deformation can be extended to a global one. We prove this in the following case: there exist distinct points $P_1, \ldots, P_p \in C$ such that $L = \mathcal{O}_C(-P_1 - \cdots - P_p)$ (theorem 3.2.6 (ii)). So in this case Y admits a global maximal reducible deformation.

The simplest primitive double curve (with associated smooth curve C and associated line bundle L on C) is the *trivial curve* : to obtain it we embed C in L^* (a smooth surface) using the zero section, and take for Y the second infinitesimal neigbourhood of C in L^*. But there is no similar easy way to construct a general primitive double curve. Most of them cannot be embedded in a smooth surface: in the case $C = \mathbb{P}_1$, D. Bayer and D. Eisenbud have proved in [3], theorem 7.1, that the only primitive double curves that can be embedded in a smooth surface are the trivial ones, and the conic in $\mathbb{P}_2$. But the non trivial primitive double curves (corresponding to $\mathbb{P}_1$ and L) are parametrized by the projective space $\mathbb{P}(H^1(L(2))$, so there are many non embeddable primitive double curves when $\deg(L) \leq 5$. So to treat the general case we have to use a more abstract construction of primitive double curves. We use the parametrization of primitive double curves: it was obtained for the first time in [3], and in another way in [5], using Čech cohomology : primitive double curves are constructed by gluing smaller pieces of the form $U \times \mathrm{spec}(\mathbb{C}[t]/(t^2))$, where U is an open subset of C. This construction can also be used to construct local reducible deformations (at least when there exist distinct points $P_1, \ldots, P_p \in C$ such that $L = \mathcal{O}_C(-P_1 - \cdots - P_p)$). This is done in chapter 3.

In chapter 4, the properties of reducible deformations of double curves used in chapter 3 are extended to the case of multiplicity $n > 2$. This could be useful to determine which primitive multiple curves of multiplicity n can be deformed to reducible curves with n smooth components, using the parametrization of primitive multiple curves given in [5].

1.2. Smoothing of primitive multiple curves

Since curves with smooth components intersecting transversally are smoothable (cf. 2.5), any primitive multiple curve having a maximal reducible deformation is smoothable.

The deformations of primitive double (i.e. of multiplicity 2) curves (also called *ribbons*) to smooth projective curves have been studied by M. González in [13]: he proved that such a curve Y, with associated smooth curve C and associated line bundle L on C is smoothable if $h^0(L^{-2}) \neq 0$. Here we prove that Y can be deformed to curves with 2 components intersecting transversally only if $h^0(L^{-1}) \neq 0$. So there exist smoothable primitive double curves that cannot be deformed to curves with 2 components intersecting transversally.

1.3. Deformations of coherent sheaves

Another motivation for the study of the deformations of primitive multiple curves in reducible ones is the understanding of the moduli spaces of semi-stable coherent sheaves on primitive multiple curves. A coherent sheaf $\mathcal{E}$ on $Y = C_n$ is not in general locally free on some nonempty open subset of C. It has been proved in [4] that on some nonempty open subset of C, there exist uniquely determined integers $m_i \geq 0$, $1 \leq i \leq n$, such that $\mathcal{E}$ is locally isomorphic to a sheaf of the form

$$m_1\mathcal{O}_C \oplus m_2\mathcal{O}_{C_2} \oplus \cdots \oplus m_n\mathcal{O}_{C_n} \ .$$

In some cases, nonempty components of moduli spaces of semi-stable sheaves on Y contain no generically locally free sheaves (cf. [4], [6], [7]). I conjecture that if C_n has a maximal reducible deformation $\pi : \mathcal{C} \to S$, then this kind of sheaf can be deformed to sheaves $\mathcal{F}$ on the fibers $\mathcal{C}_s$ such that the ranks of the restrictions of $\mathcal{F}$ to the components of $\mathcal{C}_s$ are not the same, and determined by the integers m_i. Semi-stable vector bundles on curves with many components have already been studied in [20], [21], [22].

I am grateful to the anonymous referee for pointing some incomplete proofs in 3.2 and for example 3.2.8.

Notations : If X is an algebraic variety and $Y \subset X$ a subvariety, $\mathcal{I}_{Y,X}$ (or $\mathcal{I}_Y$ if there is no risk of confusion) denotes the ideal sheaf of Y in X.

2. Preliminaries

2.1. Primitive multiple curves

(cf. [2], [3], [4], [5], [6], [7], [8], [11]).

Let C be a smooth connected projective curve. A *multiple curve with support* C is a Cohen-Macaulay scheme Y such that $Y_{red} = C$.

Let n be the smallest integer such that $Y = C^{(n-1)}$, $C^{(k-1)}$ being the k-th infinitesimal neighborhood of C, i.e. $\mathcal{I}_{C^{(k-1)}} = \mathcal{I}_C^k$. We have a filtration $C = C_1 \subset C_2 \subset \cdots \subset C_n = Y$ where C_i is the biggest Cohen-Macaulay subscheme contained in $Y \cap C^{(i-1)}$. We call n the *multiplicity* of Y.

We say that Y is *primitive* if, for every closed point x of C, there exists a smooth surface S, containing a neighborhood of x in Y as a locally closed subvariety. In this case, $L = \mathcal{I}_C/\mathcal{I}_{C_2}$ is a line bundle on C and we have $\mathcal{I}_{C_j} = \mathcal{I}_C^j$, $\mathcal{I}_{C_j}/\mathcal{I}_{C_{j+1}} = L^j$ for $1 \leq j < n$. We call L the line bundle on C *associated* to Y. Let $P \in C$. Then there exist elements y, t of $m_{S,P}$ (the maximal ideal of $\mathcal{O}_{S,P}$) whose images in $m_{S,P}/m_{S,P}^2$ form a basis, and such that for $1 \leq i < n$ we have $\mathcal{I}_{C_i,P} = (y^i)$.

The simplest case is when Y is contained in a smooth surface S. Suppose that Y has multiplicity n. Let $P \in C$ and $f \in \mathcal{O}_{S,P}$ a local equation of C. Then we have $\mathcal{I}_{C_i,P} = (f^i)$ for $1 < j \leq n$, in particular $I_{Y,P} = (f^n)$, and $L = \mathcal{O}_C(-C)$.

For any $L \in \mathrm{Pic}(C)$, the *trivial primitive curve* of multiplicity n, with induced smooth curve C and associated line bundle L on C is the n-th infinitesimal neighborhood of C, embedded by the zero section in the dual bundle L^*, seen as a surface.

2.2. Construction of primitive multiple curves

(cf. [5]).

Let $(U_i)_{i\in I}$ be an affine open cover of C. Let n be a positive integer and $Z_n = \mathrm{spec}(\mathbb{C}[t]/(t^n))$. For $i, j \in I$, $i \neq j$, let

$$\Psi_{ij} : U_{ij} \times Z_n \longrightarrow U_{ij} \times Z_n$$

be an automorphism leaving U_{ij} invariant. Suppose that these automorphisms verify the cochain condition $\Psi_{ik} = \Psi_{jk} \circ \Psi_{ij}$. Then by gluing the $U_i \times Z_n$ using the Ψ_{ij} we obtain a primitive multiple curve C_n of multiplicity n whose associated reduced curve is C. Every primitive multiple curve can be obtained in this way.

Let $U \subset C$ be a proper open subset. Suppose that ω_C is trivial on U. Let $x \in \mathcal{O}_C(U)$ such that dx is a generator of $\omega_C(U)$. Then the automorphisms of $U \times Z_n$ leaving U invariant are the same as the automorphisms ϕ of the $\mathbb{C}$-algebra $\mathcal{O}_C(U)[t]/(t^n)$ such that for every $\gamma \in \mathcal{O}_C(U)[t]/(t^n)$, $\phi(\gamma)_{|C} = \gamma_{|C}$. Such automorphisms are of the form $\phi_{\mu,\nu}$, with $\mu, \nu \in \mathcal{O}_C(U)[t]/(t^{n-1})$, ν invertible, defined by: for every $\alpha \in \mathcal{O}_C(U)$

$$\phi_{\mu,\nu}(\alpha) \;=\; \sum_{i=0}^{n-1} (\mu t)^i \frac{\partial^i \alpha}{\partial^i x} \;,$$

and

$$\phi_{\mu,\nu}(t) \ = \ \nu t \ .$$

Suppose that ω_C is trivial on each U_{ij} and that we have fixed a generator $dx = dx_{ij} \in \omega_C(U_{ij})$. Then we can write $\Psi_{ij} = \phi_{\mu_{ij},\nu_{ij}}$, with $\mu_{ij}, \nu_{ij} \in \mathcal{O}_C(U_{ij})[t]/(t^{n-1})$, ν_{ij} invertible. The family $(\nu_{ij|C})$ is a cocycle defining the line bundle L associated to C_n. If the ideal sheaf $\mathcal{I}_{C,C_n}$ of C in C_n is isomorphic to the trivial line bundle on C_{n-1} then we can assume that $\nu_{ij} = 1$ for all i, j.

2.2.1. *The case of double curves* – Now we suppose that $n = 2$. In this case we have $\mu, \nu \in \mathcal{O}_C(U)$, ν is invertible, and $\phi_{\mu,\nu}$ can also been represented by a matrix $\begin{pmatrix} 1 & 0 \\ \mu\frac{\partial}{\partial x} & \nu \end{pmatrix}$ in such a way that the composition of morphisms is equivalent to the multiplication of matrices, i.e we have

$$\phi_{\mu',\nu'} \circ \phi_{\mu,\nu} \ = \ \phi_{\mu'',\nu''} \ ,$$

with $\mu'' = \mu' + \nu'\mu$, $\nu'' = \nu\nu'$.

We can see $\phi_{\mu_{ij},\nu_{ij}}$ as a matrix

$$\begin{pmatrix} 1 & 0 \\ \mu_{ij}\frac{\partial}{\partial x_{ij}} & \nu_{ij} \end{pmatrix} \ .$$

Here (ν_{ij}) is a cocycle representing L (the line bundle on C associated to C_2), and $(\mu_{ij}\frac{\partial}{\partial x_{ij}})$ is a cocycle representing an element η of $H^1(T_C \otimes L) = \mathrm{Ext}^1(\omega_C, L)$, associated to the canonical exact sequence

$$0 \longrightarrow L \longrightarrow \Omega_{Y|C} \longrightarrow \omega_C \longrightarrow 0 \ .$$

Moreover, C_2 is trivial if and only if $\eta = 0$.

According to [3] and [5], $\mathbb{C}\eta$ defines completely C_2. More precisely, we say two primitive double curves C_2, C_2', with the same induced smooth curve C, are *isomorphic* if there exists an isomorphism $C_2 \simeq C_2'$ inducing the identity on C. Of course in this case the associated line bundles on C are also the same. By associating $\mathbb{C}\eta$ to C_2, we define a bijection from the set of isomorphism classes of non trivial primitive double curves with induced smooth curve C and associated line bundle L and $\mathbb{P}(H^1(T_C \otimes L))$.

Let S be a smooth curve and $P \in C$. Let $\pi : \mathcal{C} \to S$ be a flat family of projective irreducible smooth curves parametrized by S, and suppose that $C = \pi^{-1}(P)$. Let C_2 be the second infinitesimal neighbourhood of C in $\mathcal{C}$, which is a primitive double curve with associated line bundle $\mathcal{O}_C$. To this double curve we have associated $\eta \in H^1(T_C)$. Then the image of the Kodaïra-Spencer map of the deformation $\mathcal{C}$ of C

$$T_P S \longrightarrow H^1(T_C)$$

is $\mathbb{C}\eta$.

2.3. Maximal reducible deformations

We recall here some definitions and results of [9].

2.3.1. Let C be a projective irreducible smooth curve, $n \geq 2$ an integer and C_n a primitive multiple curve of multiplicity n, with underlying smooth curve C. Let S be a smooth curve, $P \in S$ and $\pi : \mathcal{C} \to S$ a *maximal reducible deformation of* C_n (cf. [9]). This means that

(i) $\mathcal{C}$ is a reduced algebraic variety with n irreducible components $\mathcal{C}_1, \ldots, \mathcal{C}_n$.
(ii) We have $\pi^{-1}(P) = C_n$. So we can view C as a curve in $\mathcal{C}$.
(iii) For $i = 1, \ldots, n$, let $\pi_i : \mathcal{C}_i \to S$ be the restriction of π. Then $\pi_i^{-1}(P) = C$ and π_i is a flat family of smooth irreducible projective curves.
(iv) For every $z \in S\backslash\{P\}$, the components $\mathcal{C}_{1,z}, \ldots, \mathcal{C}_{n,z}$ of $\mathcal{C}_z$ meet transversally and any three components don't have a common point.

In this case we say also that $\mathcal{C}$ is a maximal reducible deformation of C_n.

Let L be the line bundle on C associated to C_n. Then for every $z \in S\backslash\{P\}$, any two components $\mathcal{C}_{i,z}$, $\mathcal{C}_{j,z}$ of $\mathcal{C}_z$ meet in exactly $-\deg(L)$ points.

2.3.2. Let $\mathcal{Z} \subset \mathcal{C}$ be the closure in $\mathcal{C}$ of the locus of the intersection points of the components of $\pi^{-1}(z)$, $z \neq P$. It is a curve in $\mathcal{C}$.

2.3.3. Let $I \subset \{1, \ldots, n\}$ be a proper subset with m elements, and $\mathcal{C}_I \subset \mathcal{C}$ the union of the components $\mathcal{C}_i$, $i \in I$. Then the restriction of π, $\pi_I : \mathcal{C}_I \to S$ is a maximal reducible deformation of C_m, the inclusion $\mathcal{C}_I \subset \mathcal{C}$ inducing the inclusion $C_m \subset C_n$.

2.3.4. Let $\pi : \mathcal{C} \to S$ be a morphism of algebraic varieties satisfying (i), (iii) and (iv) above, in such a way that the subvarieties $C \subset \mathcal{C}_i$ are identified in $\mathcal{C}$ and C is the underlying reduced subscheme of $\pi^{-1}(P)$. Then $C_n = \pi^{-1}(P)$ is a primitive multiple curve of multiplicity n if and only if for every closed point x of C there exists an open neighborhood of x in $\mathcal{C}$ that can be embedded in a smooth variety of dimension 3 (the proof is the same as that of proposition 4.1.6 of [9]). In this case of course π is a maximal reducible deformation of C_n.

2.3.5. *Gluings and fragmented deformations* – For $1 \leq i \leq n$, let

$\pi_i : \mathcal{C}_i \to S$

be a flat family of smooth projective irreducible curves, with a fixed isomorphism $\pi_i^{-1}(P) \simeq C$. A *gluing of* $\mathcal{C}_1, \cdots, \mathcal{C}_n$ *along* C is an algebraic variety $\mathcal{D}$ such that

- for $1 \leq i \leq n$, $\mathcal{C}_i$ is isomorphic to a closed subvariety of $\mathcal{D}$, also denoted by $\mathcal{C}_i$, and $\mathcal{D}$ is the union of these subvarieties.
- $\coprod_{1 \leq i \leq n}(\mathcal{C}_i \backslash C)$ is an open subset of $\mathcal{D}$.
- There exists a morphism $\pi : \mathcal{D} \to S$ inducing π_i on $\mathcal{C}_i$, for $1 \leq i \leq n$.
- The subvarieties $C = \pi_i^{-1}(P)$ of $\mathcal{C}_i$ coincide in $\mathcal{D}$.

For example, if $\deg(L) = 0$ in 2.3.1, i.e. if the irreducible components of the fibers $\pi^{-1}(z)$, $z \neq P$, are disjoint, then a maximal reducible deformation $\mathcal{C}$ of C_n (which is called a *fragmented deformation* in this case) is a gluing of $\mathcal{C}_1, \cdots, \mathcal{C}_n$ along C.

All the gluings of $\mathcal{C}_1, \cdots, \mathcal{C}_n$ along C have the same underlying Zariski topological space.

Let $\boldsymbol{\mathcal{A}}$ be the *initial gluing* of the $\mathcal{C}_i$ along C. It is an algebraic variety whose underlying Zarisky topological space is the same as that of any fragmented deformation with the same components, in particular the closed points are

$$(\coprod_{i=1}^{n} \mathcal{C}_i)/\sim \quad ,$$

where $\sim$ is the equivalence relation: if $x \in \mathcal{C}_i$ and $y \in \mathcal{C}_j$, $x \sim y$ if and only if $x = y$, or if $x \in \mathcal{C}_{i,P} \simeq C$, $y \in \mathcal{C}_{j,P} \simeq C$ and $x = y$ in C. The structural sheaf is defined by: for every open subset U of $\boldsymbol{\mathcal{A}}$

$$\mathcal{O}_{\boldsymbol{\mathcal{A}}}(U) \ = \ \{(\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_{\mathcal{C}_1}(U \cap \mathcal{C}_1) \times \cdots \times \mathcal{O}_{\mathcal{C}_n}(U \cap \mathcal{C}_{\eta \in H^1(T_C)} n); \alpha_{1|C} = \cdots = \alpha_{n|C}\}.$$

For every gluing $\mathcal{D}$ of $\mathcal{C}_1, \cdots, \mathcal{C}_n$, we have an obvious dominant morphism $\boldsymbol{\mathcal{A}} \to \mathcal{D}$. If follows that the sheaf of rings $\mathcal{O}_{\mathcal{D}}$ can be seen as a subsheaf of $\mathcal{O}_{\boldsymbol{\mathcal{A}}}$.

If we consider the maximal reducible deformation $\mathcal{C}$ of 2.3.1 the situation is slightly more complicated. We still have a dominant morphism $\boldsymbol{\mathcal{A}} \to \mathcal{C}$. For every $x \in C$, $\mathcal{O}_{\mathcal{C},x}$ can be seen a subalgebra of $\mathcal{O}_{\boldsymbol{\mathcal{A}},x}$. It contains elements $(\alpha_1, \ldots, \alpha_n)$ with the additional property that if $1 \leq i < j \leq n$ then α_i and α_j coincide on $\mathcal{C}_i \cap \mathcal{C}_j$, which is larger than C if $\deg(L) \neq 0$. We will see in 4.1.11 that it is not true that there always exists a fragmented deformation above a reducible one.

2.4. Gluings of families of curves

Let S be a smooth affine irreducible curve, and $P \in S$ a closed point. By a *family of curves* parametrized by S we mean a flat projective morphism $\pi : \mathcal{C} \to S$ such that the fibers of π are smooth projective irreducible curves. The scheme $\mathcal{C}$ is then a smooth irreducible surface.

2.4.1. Lemma: *For every finite subset Δ of $\mathcal{C}$ there exists an affine open subset $U \subset \mathcal{C}$ such that $\Delta \subset U$.*

Proof. Since π is a projective morphism, there exists a projective space $\mathbb{P}_N$ and a closed immersion $\mathcal{C} \subset \mathbb{P}_N \times S$ such that π is the restriction of the projection $p_S : \mathbb{P}_N \times S \to S$. Let $p_N : \mathbb{P}_N \times S \to \mathbb{P}_N$ be the other projection and $\Delta' = p_N(\Delta)$. There exists an integer $k > 0$ and $\sigma \in \mathcal{O}_{P_N}(k)$ such that s does not vanish at any point of Δ'. By [14], prop. 5.5.7 and corr. 1.3.4, the open subset $\mathcal{C}^\sigma = \{x \in \mathcal{C}; s(p_N(x)) \neq 0\}$ is affine, and it contains Δ. $\square$

2.4.2. *Existence of families of curves with prescribed sections* – Let C be a smooth projective irreducible curve, and $x_1, \ldots, x_p$ distinct points of C.

2.4.3. Lemma: *For any $\eta \in H^1(T_C)$, there exists a smooth curve S, $P \in S$, and a family of curves $\pi : \mathcal{C} \to S$ such that*

- *we have $\pi^{-1}(P) = C$, and the image of the Kodaïra-Spencer map $T_P S \to H^1(T_C)$ is $\mathbb{C}\eta$.*
- *for $1 \leq i \leq p$, there exists a section of π, $r_i : S \to \mathcal{C}$, such that $r_i(P) = x_i$.*

Proof. If $\eta = 0$ then we can take $\mathcal{C} = C \times S$ with the obvious sections.

Suppose that $\eta \neq 0$. There exists a flat family of smooth projective curves $\theta : \mathcal{B} \to B$ parametrized by a smooth variety B, such that θ is a projective morphism, and $b_0 \in B$ with

$B_{b_0} = \theta^{-1}(b_0) = C$, such that the Kodaïra-Spencer map $\omega_{b_0} : TB_{b_0} \to H^1(T_C)$ is surjective. To obtain θ one can use an embedding of C in a projective space $C \hookrightarrow \mathbb{P}_N$ such that the degree of C is sufficiently high, take for B a suitable open subset of the Hilbert scheme of curves in $\mathbb{P}_N$ with the same genus and degree as C, and for $\mathcal{B}$ the universal curve (cf. [8], prop. 4.3.1).

Let
$$\mathcal{B}_p \ = \ \{(\beta_1, \ldots, \beta_p) \in \mathcal{B}^p \ ; \ \theta(\beta_1) = \cdots = \theta(\beta_p)\} \ .$$
It is a smooth variety and we have
$$T(\mathcal{B}_p)_{b_0} \ = \ \{(u_1, \ldots, u_p) \in T\mathcal{B}_{b_0} \times \cdots T\mathcal{B}_{b_0} \ ; \ T\theta_{b_0}(u_1) = \cdots = T\theta_{b_0}(u_p)\} \ .$$
Moreover, the projection $\mathcal{B}_d \to B$ is a submersion. Let
$$\overline{x} \ = \ (x_1, \ldots, x_p) \ \in \ (\mathcal{B})_{b_0} \ .$$
Let $S \subset \mathcal{B}_p$ be a smooth curve such that $\overline{x} \in S$. Let $f : S \to \mathcal{B}$ be the inclusion $S \subset \mathcal{B}_p$, followed by the projection $\mathcal{B}_p \to B$. Let $\mathcal{C} = f^*(\mathcal{B})$. It is a family of curves parametrized by S. It is possible to choose S such that, if $u \in TS_{\overline{x}}$ is a generator, the image in $H^1(T_C)$ (by the Kodaïra-Spencer map of $\mathcal{C}$) of the projection of u on TB_{b_0} is η. The family of curves $\mathcal{C}$ has all the required properties. □

2.4.4. *Gluing of families of curves* – Let C be a smooth projective irreducible curve, and $\pi_1 : \mathcal{C}_1 \to S$, $\pi_2 : \mathcal{C}_1 \to S$, families of smooth curves such that $\pi_1^{-1}(P) = \pi_2^{-1}(P) = C$. Let p be a positive integer, and $x_1, \ldots, x_p$ distinct points of C. Let $r_1^i : S \to \mathcal{C}_1$ (resp. $r_2^i : S \to \mathcal{C}_2$), $1 \leq i \leq p$, be sections of π_1 (resp. π_2), such that $r_1^i(P) = r_2^i(P) = x_i$. Let
$$\Gamma \ = \ C \cup r_1^1(S) \cup \cdots \cup r_1^p(S) \ \subset \mathcal{C}_1 \ ,$$
which is a closed subvariety of $\mathcal{C}_1$. It is canonically isomorphic to the corresponding closed subvariety of $\mathcal{C}_2$:
$$C \cup r_2^1(S) \cup \cdots \cup r_2^p(S) \ .$$
It is the unique scheme obtained by gluing C and p copies of S by identifying x_i and P in each copy of S, in such a way that all the curves S and C are transverse at the intersection points.

According to [12], 4-, 5- théorème 5.4, and lemma 2.4.1, there exists a scheme $\mathcal{D}$, obtained by gluing $\mathcal{C}_1$ and $\mathcal{C}_2$ along Γ, in such a way that we have a cocartesian diagram

$$\begin{tikzcd} \Gamma \arrow[r, hook] \arrow[d, hook] & \mathcal{C}_1 \arrow[d, hook] \\ \mathcal{C}_2 \arrow[r, hook] & \mathcal{D} = \mathcal{C}_1 \sqcup_\Gamma \mathcal{C}_2 \end{tikzcd}$$

If $x \in \Gamma$, $\mathcal{O}_{\mathcal{D},x}$ is the ring of pairs $(\phi_1, \phi_2) \in \mathcal{O}_{\mathcal{C}_1,x} \times \mathcal{O}_{\mathcal{C}_2,x}$ such that $\phi_{1|\Gamma} = \phi_{2|\Gamma}$. Moreover the morphisms π_1, π_2 can be extended to a morphism $\pi : \mathcal{D} \to S$.

Let $\mathcal{O}_{\mathcal{C}_i}(1)$, $i = 1, 2$, be ample line bundles on $\mathcal{C}_i$, such that $\mathcal{O}_{\mathcal{C}_1}(1)_{|C} \simeq \mathcal{O}_{\mathcal{C}_2}(1)_{|C}$, and that $\mathcal{O}_{\mathcal{C}_i}(1)_{r_i^j(S)}$ is trivial for $1 \leq j \leq p$. Then we have $\mathcal{O}_{\mathcal{C}_1}(1)_{|\Gamma} \simeq \mathcal{O}_{\mathcal{C}_2}(1)_{|\Gamma}$. By the same argument as in [12], 6.3, we can define a line bundle $\mathcal{O}_{\mathcal{D}}(1)$ on $\mathcal{D}$ extending $\mathcal{O}_{\mathcal{C}_1}(1)$ and $\mathcal{O}_{\mathcal{C}_2}(1)$. By [15], 2.6.2, applied to $\pi : \mathcal{D} \to S$ and $(\pi_1, \pi_2) : \mathcal{C}_1 \sqcup \mathcal{C}_2 \to S$, $\mathcal{O}_{\mathcal{D}}(1)$ is ample. Hence $\mathcal{D}$ is a quasiprojective variety. According to [9] (cf. 2.3), $\pi^{-1}(P)$ is a primitive double curve with associated smooth curve S, and $\mathcal{D}$ is a maximal reducible deformation of $\pi^{-1}(P)$.

2.5. Smoothing of reducible deformations of primitive multiple curves

Let X be an algebraic variety. We say that X is *smoothable* if there exists a flat morphism $\psi : \mathcal{X} \to T$, where $\mathcal{X}$ and T are algebraic varieties, such that T is integral, there exists $t_0 \in T$ such that $\psi^{-1}(t_0) \simeq X$, and if $s \neq t \in T$, then $\psi^{-1}(s)$ is smooth.

2.5.1. Proposition : *Let D be a projective primitive multiple curve. If there exists a maximal reducible deformation of D, then D is smoothable.*

Proof. Let $\pi : \mathcal{C} \to S$ be a maximal reducible deformation of D, and $s_0 \in S$ such that $D = \mathcal{C}_{s_0}$. Let $i : \mathcal{C} \to \mathbb{P}_N$ be an embedding of $\mathcal{C}$ in a projective space. We may assume that for every $s \in S$ we have $h^1(\mathcal{O}_{\mathcal{C}_s}(1)) = 0$, and the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_N} \longrightarrow \mathcal{O}_{\mathbb{P}_N}(1) \otimes H^0(\mathcal{O}_{\mathbb{P}_N}(1))^* \longrightarrow T\mathbb{P}_N \longrightarrow 0$$

implies that we have also $h^1(T\mathbb{P}_{N|\mathcal{C}_s}) = 0$. Let $\mathcal{Z}$ be the component of the Hilbert scheme of curves in $\mathbb{P}_N$ containing D. Then by the local structure of D there is a canonical surjective morphism

$$T\mathbb{P}_N \longrightarrow\!\!\!\!\!\rightarrow \mathcal{H}om(\mathcal{I}_D, \mathcal{O}_D)$$

(where $\mathcal{I}_D$ denotes the ideal sheaf of D in $\mathbb{P}_N$). It follows that $H^1(\mathcal{H}om(\mathcal{I}_D, \mathcal{O}_D)) = \{0\}$. The same holds if we replace D with $\mathcal{C}_s$, for every $s \in S$. It follows that $\mathcal{Z}$ is smooth at all the fibers of π. Let $\mathcal{Z}^0 \subset \mathcal{Z}$ be the open subset of smooth points. We have thus $\mathcal{C}_s \in \mathcal{Z}^0$ for every $s \in S$.

It is clear that all the singularities of the fibers $\mathcal{C}_s$, $s \in S\backslash\{s_0\}$, are smoothable. It follows from proposition 29.9 of [17] that the fibers $\mathcal{C}_s$, $s \in S\backslash\{s_0\}$, are smoothable in $\mathbb{P}_N$. In particular some points in $\mathcal{Z}^0$ are smooth curves. It follows that $D = \mathcal{C}_{s_0}$ is smoothable. □

3. Maximal reducible deformations of primitive double curves

In this chapter S denotes a smooth curve, and $P \in S$. Let $t \in \mathcal{O}_{S,P}$ be a generator of the maximal ideal of P. We can suppose that t is defined on the whole of S, and that the ideal sheaf of P in S is generated by t.

3.1. Properties of maximal reducible deformations of primitive double curves

Let C_2 be a primitive double curve, with underlying projective smooth curve C and associated line bundle L on C. Let $\pi : \mathcal{C} \to S$ be a maximal reducible deformation of C_2, and $P \in S$ such that $\pi^{-1}(P) = C_2$. Then $\mathcal{C}$ has two irreducible components $\mathcal{C}_1$, $\mathcal{C}_2$ which are flat families of smooth irreducible curves parametrized by S. If $z \in S\backslash\{P\}$, the two components $\mathcal{C}_{1,z}$ and $\mathcal{C}_{2,z}$ of $\mathcal{C}_z$ intersect transversally in $-\deg(L)$ points. We suppose that $\deg(L) < 0$.

Let $\mathcal{Z} \subset \mathcal{C}$ be the closure in $\mathcal{C}$ of the locus of the intersection points of the components of $\pi^{-1}(z)$, $z \neq P$. Since S is a curve, $\mathcal{Z}$ is a curve of $\mathcal{C}_1$ and $\mathcal{C}_2$. It intersects C in a finite number of points. If x is one of these points, let r_x be the number of branches of $\mathcal{Z}$ at x and s_x the sum of the multiplicities of the intersections of these branches with C, so that we have $r_x \leq s_x$, with equality if and only if all the branches intersect transversally with C. Moreover, since for every $z\in S\backslash\{P\}$, $\mathcal{C}_{1z}\cap\mathcal{C}_{2z}$ consists of $-\deg(L)$ distinct points, we have $\sum_{x\in\mathcal{Z}\cap C} r_x = -\deg(L)$.

For $i=1,2$, let $\pi_i:\mathcal{C}_i\to S$ be the restriction of π. We will also denote π^*t by π, and π_i^*t by π_i. So we have $\pi=(\pi_1,\pi_2)\in\mathcal{O}_{\mathcal{C}}(-C)$.

3.1.1. Theorem: 1 − *Let $x\in C$. Then there exists an unique integer $p>0$ such that $\mathcal{I}_{C,x}/\langle(\pi_1,\pi_2)\rangle$ is generated by the image of $(\pi_1^p\lambda,0)$, for some $\lambda\in\mathcal{O}_{\mathcal{C}_1,x}$ not divisible by π_1. This integer does not depend on x.*

2 − *λ is unique up to multiplication by an invertible element of $\mathcal{O}_{\mathcal{C}_1,x}$, and $(\pi_1^p\lambda,0)$ is a generator of the ideal $\mathcal{I}_{\mathcal{C}_1,\mathcal{C},x}$ of $\mathcal{C}_1$ in $\mathcal{C}$.*

3 − *There are only a finite number of points $x\in C$ such that λ is not invertible.*

4 − *Let m_x be the multiplicity of $\lambda_{|C}\in\mathcal{O}_{C,x}$. Then we have $m_x>0$ if and only if $x\in\mathcal{Z}\cap C$, and in this case we have $m_x=r_x=s_x$, and the branches of $\mathcal{Z}$ at x intersect transversally with C. Moreover*

$$L \ \simeq\ \mathcal{O}_C(-\sum_{x\in\mathcal{Z}\cap C} r_x x)\ \simeq\ \mathcal{I}_{\mathcal{Z}\cap C,C}\ .$$

Proof. the proof of 1- is similar to the proof of proposition 4.2.1, 1- of [9]. Let $x\in C$ and $u=(\pi_1\alpha,\pi_2\beta)$ whose image is a generator of $\mathcal{I}_C/\langle(\pi_1,\pi_2)\rangle$ at x ($\mathcal{I}_C/\langle(\pi_1,\pi_2)\rangle$ is a locally free sheaf of rank 1 of $\mathcal{O}_C$-modules). Let $\beta_0\in\mathcal{O}_{\mathcal{C}_1,x}$ be such that $(\beta_0,\beta)\in\mathcal{O}_{\mathcal{C},x}$. Then the image of

$$u-(\pi_1,\pi_2)(\beta_0,\beta)=(\pi_1(\alpha-\beta_0),0)$$

is also a generator of $\mathcal{I}_C/\langle(\pi_1,\pi_2)\rangle$ at x. We can write it $(\pi_1^p\lambda,0)$, where λ is not a multiple of π_1.

As in proposition 4.2.1 of [9] it is easy to see that if $(\pi_1^q\mu,0)\in\mathcal{O}_{\mathcal{C},x}$, with μ not divisible by π_1 and $q>0$, and if k is a positive integer, then we can write

$$(\pi_1^q\mu,0)\ =\ \gamma.(\pi_1^p\lambda,0)+\delta.(\pi_1^k,\pi_2^k)\ ,$$

with $\gamma,\delta\in\mathcal{O}_{\mathcal{C},x}$. From this it follows that $q\geq p$, so p is unique. Since the image of $(\pi_1^p\lambda,0)$ is also a generator of $\mathcal{I}_C/\langle(\pi_1,\pi_2)\rangle$ in a neigborhood of x, p does not depend on x and 1- is proved, as well as 3-.

Suppose that the image of $(\pi_1^p\lambda',0)$ is also a generator of $\mathcal{I}_C/\langle(\pi_1,\pi_2)\rangle$ at x. Then for every positive integer k we can write

$$(\pi_1^p\lambda',0)\ =\ (\gamma_1,\gamma_2)(\pi_1^p\lambda,0)+(\delta_1,0).(\pi_1^{p+k},\pi_2^{p+k})\ ,$$

with $(\gamma_1,\gamma_2),(\delta_1,0)\in\mathcal{O}_{\mathcal{C},x}$, whence $\lambda'=\gamma_1\lambda+\pi_1^k\delta_1$. We have then

$$\lambda'\ \in\ \bigcap_{k>0}((\lambda)+(\pi_1)^{p+k}$$

in $\mathcal{O}_{\mathcal{C}_1,x}$, and the latter is equal to (λ) according to [19], vol. II, chap. VIII, theorem 9. So we can write $\lambda'=\beta\lambda$ with $\beta\in\mathcal{O}_{\mathcal{C}_1,x}$. In the same way, $\lambda=\beta'\lambda'$ with $\beta'\in\mathcal{O}_{\mathcal{C}_1,x}$. So $\beta\beta'=1$ and β is invertible, which proves the first assertion 2-. The proof of the second assertion is similar.

Suppose that $x\in\mathcal{Z}\cap C$. Since $(\pi_1^p\lambda,0)\in\mathcal{O}_{\mathcal{C},x}$, we have $\lambda_{|\mathcal{Z}\cap C}=0$, hence $m_x\geq s_x$.

It follows from 1-, 2-, 3- that $\mathcal{I}_C/\langle(\pi_1,\pi_2)\rangle$ (which is isomorphic to L) can be viewed as a subsheaf of $(\pi_1^p)/(\pi_1^{p+1})\simeq\mathcal{O}_C$, and this subsheaf is exactly $\mathcal{O}_C(-\sum_{x\in\mathcal{Z}\cap C}m_xx)$. Hence $-\deg(L)=\sum_{x\in C}m_x$. So

$$-\deg(L)=\sum_{x\in C}m_x\ \geq\sum_{x\in\mathcal{Z}\cap C}s_x\ \geq\sum_{x\in\mathcal{Z}\cap C}r_x=-\deg(L)\ ,$$

hence all the preceding inequalities are equalities and 4- follows. □

Similarly there exists an unique integer $q>0$ such that for every $x\in C$, $\mathcal{I}_{C,x}/\langle(\pi_1,\pi_2)\rangle$ is generated by the image of $(0,\pi_2^q\mu)$, for some $\mu\in\mathcal{O}_{\mathcal{C}_2,x}$ not divisible by π_2 and unique up to multiplication by an invertible element of $\mathcal{O}_{\mathcal{C}_2,x}$.

3.1.2. Corollary: *Let $x\in C$ and $(\pi_1^p\lambda,0)\in\mathcal{O}_{\mathcal{C},x}$ (resp. $(0,\pi_2^p\mu)\in\mathcal{O}_{\mathcal{C},x}$) be such that its image is a generator of $\mathcal{I}_{C,x}/\langle(\pi_1,\pi_2)\rangle$. Let $\theta\in\mathcal{O}_{\mathcal{C}_2,x}$ such that $(\lambda,\theta)\in\mathcal{O}_{\mathcal{C},x}$. Then*

1 – *There exists $\gamma\in\mathcal{O}_{\mathcal{C}_2,x}$ invertible such that $\theta=\gamma\mu$.*

2 – *The ideal of $\mathcal{Z}$ in $\mathcal{C}_{1,x}$ (resp. $\mathcal{C}_{2,x}$) id generated by λ (resp. μ).*

Proof. Since the image of $\pi^p(\lambda,\theta)-(\pi_1^p\lambda,0)=(0,-\pi_2^p\theta)$ is also a generator of $\mathcal{I}_{C,x}/\langle(\pi_1,\pi_2)\rangle$, 1- follows from theorem 3.1.1, 2- (for the other coordinate). The second assertion follows from theorem 3.1.1, 4-. □

3.1.3. Proposition: 1 – *There exists a canonical isomorphism*

$$\Phi:\mathcal{O}_{\mathcal{C}_1}/(\pi_1^p\mathcal{I}_{\mathcal{Z},\mathcal{C}_1})\longrightarrow\mathcal{O}_{\mathcal{C}_2}/(\pi_2^q\mathcal{I}_{\mathcal{Z},\mathcal{C}_2})$$

such that for every $x\in C$ and $(\alpha_1,\alpha_2)\in\mathcal{O}_{\mathcal{C}_1,x}\times\mathcal{O}_{\mathcal{C}_2,x}$, if $[\alpha_1]$, $[\alpha_2]$ denote the images of α_1, α_2 in $\mathcal{O}_{\mathcal{C}_1,x}/(\pi_1^p\mathcal{I}_{\mathcal{Z},\mathcal{C}_1,x})$, $\mathcal{O}_{\mathcal{C}_2,x}/(\pi_1^p\mathcal{I}_{\mathcal{Z},\mathcal{C}_2,x})$ respectively, we have $(\alpha_1,\alpha_2)\in\mathcal{O}_{\mathcal{C},x}$ if and only if $\Phi([\alpha_1])=[\alpha_2]$. For every $\alpha\in\mathcal{O}_{\mathcal{C},x}$, we have $\Phi([\alpha])_{|C}=[\alpha]_{|C}$, and $\Phi([\pi_1])=[\pi_2]$.

2 – *We have $q=p$.*

Proof. The first assertion is an easy consequence of the second statement of the preceding theorem. To prove 2-, let $x\in C\backslash\mathcal{Z}\cap C$. Then $(\pi_1^p\lambda,0)\in\mathcal{O}_{\mathcal{C},x}$ for some $\lambda\in\mathcal{O}_{\mathcal{C}_1,x}$ such that $\lambda_{|C}\neq0$. There exists $\mu\in\mathcal{O}_{\mathcal{C}_2,x}$ such that $(\lambda,\mu)\in\mathcal{O}_{\mathcal{C},x}$, and $\mu_{|C}=\lambda_{|C}\neq0$. We have $(0,\mu\pi_2^p)=\pi^p(\lambda,\mu)-(\pi_1^p\lambda,0)\in\mathcal{O}_{\mathcal{C},x}$, so $q\leq p$ by theorem 3.1.1, 2-. Similarly $p\leq q$, hence $p=q$. □

3.1.4. Proposition: *Let $J_p\subset\mathcal{O}_\mathcal{C}$ be the ideal sheaf consisting of pairs (u_1,u_2) such that u_1 is a multiple of π_1^{p+1} and u_2 a multiple of π_2^{p+1}. Then we have $J_p\subset(\pi)$.*

Proof. Let $a_1\in\mathcal{O}_{\mathcal{C}_1,x},a_2\in\mathcal{C}_{2,x}$ be such that $\Phi([a_1\pi_1^{p+1}])=[a_2\pi_2^{p+1}]$. We must prove that $\Phi([a_1\pi_1^p])=[a_2\pi_2^p]$. Let $b\in\mathcal{O}_{\mathcal{C}_2,x}$ be such that $\Phi([a_1])=[b]$. Then we have $\Phi([a_1\pi_1^{p+1}])=[b\pi_2^{p+1}]$, hence $a_2\pi_2^{p+1}-b\pi_2^{p+1}$ is a multiple of $\mu\pi_2^p$:

$$a_2\pi_2^{p+1}-b\pi_2^{p+1}\ =\beta\mu\pi_2^p$$

for some $\beta \in \mathcal{O}_{\mathcal{C}_2,x}$. It follows that β is a multiple of π_2 : $\beta = \alpha\pi_2$, and $a_2 - b = \alpha\mu$. Hence

$$\begin{aligned}\Phi([a_1\pi_1^p]) &= [b\pi_2^p] \\ &= [a_2\pi_2^p - \alpha\mu\pi_2^p] \\ &= [a_2\pi_2^p] \ .\end{aligned}$$

□

It follows that for the associated primitive multiple curve C_2 we have

$$\mathcal{O}_{C_2} \ = \ \mathcal{O}_{\mathcal{C}}/(\pi) \ = \ \big(\mathcal{O}_{\mathcal{C}}/J_p\big)/(\pi) \ .$$

3.1.5. *Localization* – Let

$$\overline{\Phi} : \mathcal{O}_{\mathcal{C}_1}/\big((\pi_1^p\mathcal{I}_{\mathcal{Z},\mathcal{C}_1}) + (\pi_1^{p+1})\big) \longrightarrow \mathcal{O}_{\mathcal{C}_2}/\big(\pi_2^q\mathcal{I}_{\mathcal{Z},\mathcal{C}_2}) + (\pi_1^{p+1})\big)$$

be the isomorphism induced by Φ. Let $x \in C$. For $i = 1$ or 2, and $\alpha_i \in \mathcal{O}_{\mathcal{C}_i,x}/(\pi_i^{p+1})$, let $\epsilon(\alpha_i)$ denote the image of α_i in $\big[\mathcal{O}_{\mathcal{C}_i}/\big((\pi_i^p\mathcal{I}_{\mathcal{Z},\mathcal{C}_i}) + (\pi_1^{p+1})\big)\big]_x$. Let

$$\mathcal{A}_x \ = \ \{(\alpha_1,\alpha_2) \in \mathcal{O}_{\mathcal{C}_1,x}/(\pi_1^{p+1}) \times \mathcal{O}_{\mathcal{C}_2,x}/(\pi_2^{p+1}); \overline{\Phi}(\epsilon(\alpha_1)) = \epsilon(\alpha_2)\} \ .$$

3.1.6. Proposition: *The natural projection* $\mathcal{O}_{\mathcal{C},x} \to \mathcal{A}_x$ *induces an isomorphism*

$$\mathcal{O}_{\mathcal{C},x}/(\pi) \ \simeq \ \mathcal{A}_x/(\pi) \ .$$

Proof. Let $\theta : \mathcal{O}_{\mathcal{C},x}/(\pi) \to \mathcal{A}_x/(\pi)$ be the natural morphism.

First we prove that θ is surjective. Let $(\alpha_1,\alpha_2) \in \mathcal{A}_x$. Let $a_i \in \mathcal{O}_{\mathcal{C}_i,x}$ be over α_i. Then, since $\overline{\Phi}(\epsilon(a_1)) = \epsilon(a_2)$, we can write

$$\Phi([a_1]) \ = \ [a_2] + [\beta_2]\pi_2^{p+1} \ ,$$

for some $\beta_2 \in \mathcal{O}_{\mathcal{C}_2,x}$. Let $a_2' = a_2 + \pi_2^{p+1}\beta_2$. Then the image of (a_1,a_2') in $\mathcal{A}_x$ is (α_1,α_2), and clearly $(a_1,a_2') \in \mathcal{O}_{\mathcal{C}_x}$ and its image in $\mathcal{A}_x$ is (α_1,α_2).

Now we prove that θ is injective. Let $u = (u_1,u_2) \in \mathcal{O}_{\mathcal{C},x}$, $\overline{u}$ its image in $\mathcal{O}_{\mathcal{C},x}/(\pi)$, and suppose that $\theta(\overline{u}) = 0$. Let v be the image of u in $\mathcal{A}_x$. Then we can write $v = \pi w$, for some $w \in \mathcal{A}_x$. Since θ is surjective, we can find $w' \in \mathcal{O}_{\mathcal{C},x}$ over w. Then $u' = u - \pi w'$ is of the form $(\pi_1^{p+1}\alpha_1, \pi_2^{p+1}\alpha_2)$, for some $\alpha_i \in \mathcal{O}_{\mathcal{C}_i,x}$. From proposition 3.1.4, u' is a multiple of π. Hence u is a multiple of π and $\overline{u} = 0$. □

3.2. Construction of maximal reducible deformations of primitive double curves

We keep the notations of 3.1 and we suppose that $p = 1$ (we don't need to study the case $p > 1$ to prove theorem 3.2.7, (ii)).

3.2.1. *Maximal reducible deformations of primitive double curves and gluings of families of curves* – We consider the maximal reducible deformation $\pi : \mathcal{C} \to S$ of 3.1 (with $p = 1$). It follows from the description of the local rings of the closed points of $\mathcal{C}$ given in 3.1 that $\mathcal{C}$

is a *gluing* of $\mathcal{C}_1$ and $\mathcal{C}_2$ along the closed subvariety $\Gamma = \mathcal{Z} \cup C$. This means that we have a cocartesian diagram

$$\begin{tikzcd} \Gamma \arrow[r, hook] \arrow[d, hook] & \mathcal{C}_1 \arrow[d, hook] \\ \mathcal{C}_2 \arrow[r, hook] & \mathcal{C} = \mathcal{C}_1 \sqcup_\Gamma \mathcal{C}_2 \end{tikzcd}$$

and that for every closed point $x \in \Gamma$, $\mathcal{O}_{\mathcal{C},x}$ is the ring of pairs $(\phi_1, \phi_2) \in \mathcal{O}_{\mathcal{C}_1,x} \times \mathcal{O}_{\mathcal{C}_2,x}$ such that $\phi_{1|Y} = \phi_{2|Y}$. This comes also from the fact that $\Gamma = \mathcal{C}_1 \cap \mathcal{C}_2$ as schemes (of course this is not true if $p > 1$).

3.2.2. *The cocycles defining* C_2 – For $i = 1, 2$, let $\mathcal{C}_i^{(2)}$ denote the infinitesimal neighborhood of order 2 of C in $\mathcal{C}_i$ (i.e. $\mathcal{O}_{\mathcal{C}_i^{(2)}} = \mathcal{O}_{\mathcal{C}_i}/(\pi_i^2)$). It is a primitive double curve. Let $(U_j)_{j\in J}$ be a finite open affine cover of C. Then for every $j \in J$, the restriction $U_j^{(2)}$ of $\mathcal{C}_i^{(2)}$ to U_j is trivial, i.e. there is an isomorphism $U_j^{(2)} \simeq U_j \times Z_2$ (where $Z_2 = \operatorname{spec}(\mathbb{C}[t])/(t^2))$ inducing the identity on U_j. We can suppose that U_j contains at most one point of $\mathcal{Z} \cap C$, and that each point in $\mathcal{Z} \cap C$ is contained in only one U_j. We suppose also that for every distinct $j, k \in J$, $\omega_{C|U_{jk}}$ is trivial, generated by dx_{jk}. Then $\mathcal{C}_i^{(2)}$ can be constructed by a cocycle $(\mu_{jk}^{(i)})_{j,k\in J}$, $\mu_{jk}^{(i)} \in \mathcal{O}_C(U_{jk})$, as in 2.2.1: $\mathcal{C}_i^{(2)}$ is obtained by gluing the $U_j^{(2)} \simeq U_j \times Z_2$ with the automorphisms of $U_{jk} \times Z_2$ defined by the matrices $\begin{pmatrix} 1 & 0 \\ \mu_{jk}^{(i)} \frac{\partial}{\partial x_{jk}} & 1 \end{pmatrix}$.

If U_j does not contain any point of $\mathcal{Z} \cap C$, let $r_j^{(i)} = 1$. If U_j contains a point of $\mathcal{Z} \cap C$, this point x is unique, and in this case let $r_j^{(i)} \in \mathcal{O}_C(U_j)[t]/(t^2)$ be an equation of $\mathcal{Z} \cap C$ in the open subset $U_j \times Z_2$ of $\mathcal{C}_2^{(i)}$. If $\tau \in \mathcal{O}_C(U_j)$ is an equation of x, we can write (by theorem 3.1.1)

$$r_j^{(i)} \ = \ \tau^{r_x} + t.h_j^{(i)} \ ,$$

where $h_j^{(i)}$ vanishes to order $\geq r_x - 1$ at x. It follows that we can suppose that $r_j^{(1)} = r_j^{(2)}$. More precisely, there exists an automorphism σ of $\mathcal{O}_C(U_j)[t]/(t^2)$ such that the induced automorphism of $\mathcal{O}_C(U_j)$ is the identity, and sending $r_j^{(1)}$ to $r_j^{(2)}$ (this comes from the description of these automorphisms in 2.2). Let $r_j = r_j^{(1)} = r_j^{(2)}$. Note that by theorem 3.1.1 we can assume that r_j is a product

$$r_j \ = \ \prod_{m=1}^{r_x} (\tau + \lambda_m t) \ ,$$

where $\lambda_m \in \mathbb{C}$ for $1 \leq m \leq r_x$, and the λ_m are distinct. Let $\rho_j = r_{j|C}$.

Let $\mathcal{C}^{(2)}$ be the scheme corresponding to the sheaf of algebras $\mathcal{A}$ (cf. 3.1.5). We have $(\mathcal{C}^{(2)})_{red} = C$. For every $j \in J$, we can view $\mathcal{A}(U_j)$ as the algebra of pairs $(a + bt, a + (b + \rho_j\beta)t)$, with $a, b, \beta \in \mathcal{O}_C(U_j)$ and the rule $t^2 = 0$. These sets must then be glued to build $\mathcal{A}$, using the automorphisms, for distinct $j,k\in J$

$$\begin{array}{rcl} \mathcal{A}(U_{jk}) & \longrightarrow & \mathcal{A}(U_{jk}) \\ (a+bt, a+(b+\rho_j\beta)t) & \longmapsto & \big(a+(\mu^{(1)}_{jk}\frac{\partial a}{\partial x_{jk}}+b)t, a+(\mu^{(2)}_{jk}\frac{\partial a}{\partial x_{jk}}+b+\rho_j\beta)t\big) \end{array} \tag{1}$$

(note that ρ_j and ρ_k are invertible on U_{jk}).

Now we can also describe the double primitive curve C_2 with a cocycle, unsing the fact that $\mathcal{O}_{C_2}=\mathcal{O}_{\mathcal{C}}/(\pi)=\mathcal{A}/(\pi)$ (cf. prop. 3.1.6). If $u\in\mathcal{A}(U_j)$, let $[u]$ denote its image in $\mathcal{A}(U_j)/(\pi)$. In $\mathcal{A}(U_j)$ we have $\pi=(t,t)$, hence we have an isomorphism

$$\begin{array}{rcl} (\mathcal{A}/(\pi))(U_j) & \longrightarrow & \mathcal{O}_C(U_j)[z]/(z^2) \\ (a+bt, a+(b+\rho_j\beta)t) & \longmapsto & a+\beta z\ . \end{array}$$

Hence the automorphism of $\mathcal{O}_C(U_{jk})[z]/(z^2)$ defining C_2 is given by

$$a+bz\longmapsto a+\big(\frac{\mu^{(2)}_{jk}-\mu^{(1)}_{jk}}{\rho_k}\frac{\partial a}{\partial x_{jk}}+\frac{\rho_j}{\rho_k}\big)z\ .$$

It follows that C_2 is defined by the cocycles $\left(\dfrac{\rho_j}{\rho_k}\right)$ and $\left(\dfrac{\mu^{(2)}_{jk}-\mu^{(1)}_{jk}}{\rho_k}\right)$. It is easy to verify that the first one defines $L=\mathcal{O}_C(-\sum_{x\in\mathcal{Z}\cap C}r_x x)$.

3.2.3. *Local reducible deformations* – Let $\mathcal{D}_1$, $\mathcal{D}_2$ be primitive double curves with associated smooth curve C and associated line bundle $\mathcal{O}_C$. Suppose that $\mathcal{D}_i$, $i=1,2$, is defined by the cocycle $(\rho^{(i)}_{jk}\frac{\partial}{\partial x_{jk}})$ (it is not necessary to use another cocycle since $L=\mathcal{O}_C$). For every $j\in J$ let $r_j\in\mathcal{O}_C(U_j)[t]/(t^2)$ such that

- r_j vanishes in at most one point in U_j, and not on $U_j\cap U_k$ if $k\in J\backslash\{j\}$.
- if r_j vanishes at $x_j\in U_j$ and τ is a generator of the ideal of x in $\mathcal{O}_C(U_j)$, then $r_j=\prod_{m=1}^{p_j}(\tau+\lambda_m t)$, for some integer $p_j\geq 1$, where the λ_m are distinct scalars.

Let $J'\subset J$ be the set of points j such that r_j vanishes at some point of U_j, and Z be the divisor $\sum_{j\in J'}p_jx_j$.

Then we can define from these data, using a sheaf of algebras $\mathcal{A}$ as in 3.2.2, a scheme $\mathcal{D}$ such that $\mathcal{D}_{red}=C$, and a global section π of $\mathcal{O}_{\mathcal{D}}$. We simply define $\mathcal{A}(U_j)$ as the algebra of pairs $(a+bt, a+(b+r_j\beta)t)$, with $a,b,\beta\in\mathcal{O}_C(U_j)$ and the rule $t^2=0$. We then glue the schemes $\operatorname{spec}(\mathcal{A}(U_j))$ using automorphisms similar to (1). Now the sheaf $\mathcal{A}/(\pi)$ if the structure sheaf of a double primitive curve C_2 with associated smooth curve C and associated line bundle $L=\mathcal{O}_C(-Z)$.

We call the scheme $\mathcal{D}$ a *local reducible deformation* of the primitive double curve C_2. The reducible deformations $\mathcal{C}$ defined previously can also be called *global* reducible deformations. It is clear from 3.2.2 that a global reducible deformation of C_2 induces a local one.

3.2.4. Remark: It is possible to define local reducible deformations in the case $p > 1$. But in this case we would need to use schemes of the form $U_i \times Z_{p+1}$ instead of $U_i \times Z_2$.

3.2.5. Proposition: *Let D be a primitive double curve with associated smooth curve C and associated line bundle L' on C. Suppose that $h^0(L'^*) \neq 0$. Then there exists a local reducible deformation of D.*

Proof. The hypothesis $h^0(L'^*) \neq 0$ means that L' is an ideal sheaf, so we can write $L' = \mathcal{O}_C(-\sum_{i=1}^{p} n_i x_i)$, where $x_1, \ldots, x_p$ are distinct points of C and $n_1, \ldots, n_p$ positive integers. We can choose an open affine cover $(U_i)_{i \in I}$ of C such that any U_i contains at most one of the points x_j, and that every point x_j is contained in only one U_i. We take now $r_j \in \mathcal{O}_C(U_j)[t]/(t^2)$ such that

- r_j is invertible if U_j contains no point x_i.
- If $x_i \in U_j$, and $\tau \in \mathcal{O}_C(U_j)$ a generator of the ideal of x_i, then r_j is of the form $r_j = \prod_{1 \leq m \leq n_i} (\tau + \lambda_m t)$, where $\lambda_1, \ldots, \lambda_m \in \mathbb{C}^*$.

For $1 \leq i \leq p$, let $\rho = r_{i|C}$. Then $(\dfrac{\rho_j}{\rho_k})$ is a cocycle of invertible functions (with respect to (U_{jk})) which defines L'. Suppose that D comes from $\sigma \in H^1(T_C \otimes L')$, defined by the cocycle (ν_{jk}). Now we define two primitive double curves with associated smooth curve C and associated line bundle $\mathcal{O}_C$:

- The first one in the trivial primitive double curve, defined by the cocycle (0).
- The second one is defined by the cocycle $(\rho_k \nu_{jk})$.

It is clear from 3.2.2 that these data define a local reducible deformation of D. □

We have seen that a reducible deformation of C_2 induces a local reducible deformation of it. The converse is true (for suitable L):

3.2.6. Proposition: *Let D be a primitive double curve, with associated smooth curve C and associated line bundle L. Suppose that there exist distinct points $P_1, \ldots, P_m$ of C, such that*

$$L \simeq \mathcal{O}_C(-P_1 - \cdots - P_m) .$$

Then there exists a global maximal reducible deformation of D.

Proof. We begin as in 3.2.5 with a local reducible deformation of D. We have to show that it can be extended to a global reducible deformation of D. This is an immediate consequence of lemma 2.4.3 (for the construction of adequate $\mathcal{C}_1$ and $\mathcal{C}_2$) and 2.4.4 (to glue them and make $\mathcal{C}$). □

The preceeding results can then be summarized in

3.2.7. Theorem: *Let D be a primitive double curve with associated smooth curve C and associated line bundle L on C. Then*

(i) *There exists a local maximal reducible deformation of D if and only if $h^0(L^*) \neq 0$.*

(ii) *If there exists a divisor $\Delta = P_1 + \cdots + P_m$ of C, with distinct points $P_1, \ldots, P_m$ of C, such that $L \simeq \mathcal{O}_C(-\Delta)$. Then there exists a global maximal reducible deformation of D.*

3.2.8. Example : Let Δ be a divisor as in theorem 3.2.7, and $L = \mathcal{O}_C(-\Delta)$. Let $\mathcal{C}_1 = \mathcal{C}_2 = C \times \mathbb{C}$ and

$$\Gamma \ = \ C \cup (\{P_1\} \times \mathbb{C}) \cup \cdots \cup (\{P_m\} \times \mathbb{C}) \ .$$

This curve is a closed subvariety of $\mathcal{C}_1$ and $\mathcal{C}_2$. We can glue $\mathcal{C}_1$ and $\mathcal{C}_2$ along Γ (using 2.4.4), and we obtain a maximal reducible deformation $\mathcal{C}$ of the trivial double curve associated to C and L.

Another way to obtain $\mathcal{C}$ is as follows: let s_0 be the zero section of L^*, and $s_1 \in H^0(L^*)$ vanishing at $P_1, \ldots, P_m$. We can view C as embedded in the surface L^* via s_0. Then we can realize $\mathcal{C}_1$ and $\mathcal{C}_2$ as closed subvarieties of $L^* \times \mathbb{C}$ in the following way : $\mathcal{C}_1 = C \times \mathbb{C}$, and $\mathcal{C}_2$ is over $t \in \mathbb{C}$ the image of the section $s_0 + ts_1$. Then we have $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$.

3.3. Smoothing of primitive double curves

The following result follows immediately from proposition 2.5.1 and theorem 3.2.7: let D be a primitive double curve with associated smooth curve C and associated line bundle L on C. If there exists a divisor $P_1 + \cdots + P_m$ of C, with distinct points $P_1, \ldots, P_m$ of C, such that $L \simeq \mathcal{O}_C(-P_1 - \cdots - P_m)$, then D is smoothable. Of course this is also a consequence of [13] (cf. the Introduction), since we have also $h^0(L^{-2}) \neq 0$.

4. Maximal reducible deformations in the general case

4.1. Properties of maximal reducible deformations of primitive multiple curves

Let C be a projective irreducible smooth curve, $n \geq 2$ an integer and C_n a primitive multiple curve of multiplicity n, with underlying smooth curve C and associated line bundle L on C. Let S be a smooth curve, $P \in C$ and $\pi : \mathcal{C} \to S$ a maximal reducible deformation of C_n, with $\pi^{-1}(P) = C_n$ (cf. 2.3). Let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be the irreducible components of $\mathcal{C}$ and $d = -\deg(L)$. We suppose that $d > 0$, i.e. π is not a fragmented deformation, so for every $s \in S \backslash \{P\}$ and i, j distinct integers in $\{1, \ldots, n\}$, the two components $\mathcal{C}_{i,s}$, $\mathcal{C}_{j,s}$ of $\mathcal{C}_s$ intersect transversally in d distinct points. As in 3, we denote also $t \circ \pi_i$ by π_i.

4.1.1. *Intersections of components (notations)* – Let $\mathcal{Z} \subset \mathcal{C}$ be the closure in $\mathcal{C}$ of the locus of the intersection points of the components of $\mathcal{C}_z = \pi^{-1}(z)$, $z \neq P$. Since S is a curve, $\mathcal{Z}$ is a curve. It intersects C in a finite number of points.

For every proper subset $I \subset \{1, \ldots, n\}$, let I^c denote the complement of I. If $J \subset \{1, \ldots, n\}$ is such that $I \subset J$, let $\mathcal{Z}_I^J$ denote the subvariety of $\mathcal{Z}$ which is the closure in $\mathcal{C}$ of the locus of intersection points of the components $\mathcal{C}_{i,z}$ and $\mathcal{C}_{j,z}$ of $\mathcal{C}_z$, where $i \in I$ and $j \in J \backslash I$. Of course $\mathcal{Z}_I^J$ is a union of components of $\mathcal{C}_I \cap \mathcal{C}_{J \backslash I}$ (the only other component being a curve with associated reduced curve C). If $i, j \in \{1, \ldots, n\}$ are distinct, we will note more simply $\mathcal{Z}_{\{i\}}^{\{i,j\}} = \mathcal{Z}_{ij}$. Similarly we will note $\mathcal{Z}_I = \mathcal{Z}_I^{\{1,\ldots,n\}}$.

If $\emptyset \neq I \subsetneq J \subset \{1, \ldots, n\}$, let $\mathcal{I}(I, J)$ denote the ideal sheaf of $\mathcal{Z}_I^J$ in $\mathcal{O}_{\mathcal{C}_I}$. If $J = \{1, \ldots, n\}$, we note $\mathcal{I}(I) = \mathcal{I}(I, \{1, \ldots, n\})$.

4.1.2. *Spectrum* – Let i, j be distinct integers such that $1 \leq i, j \leq n$, and $I = \{i, j\}$. According to 3 there exists a unique integer $p > 0$ such that $\mathcal{I}_{C,\mathcal{C}_I}/(\pi)$ is generated at any point $x \in C$ by the image of an element of the form $(\pi_i^p \alpha, 0)$, where $\alpha \in \mathcal{O}_{\mathcal{C}_i,x}$ is such that $\alpha_{|C} \neq 0$ (and also by the image of an element $(0, \pi_j^p \beta)$, where $\beta \in \mathcal{O}_{\mathcal{C}_j,x}$ is such that $\beta_{|C} \neq 0$). Recall that p is the smallest integer q such that $\mathcal{I}_{C,\mathcal{C}_I}$ contains a non zero element of the form $(\pi_i^q \lambda, 0)$ (or $(0, \pi_j^q \mu)$), with $\lambda_{|C} \neq 0$ (resp. $\mu_{|C} \neq 0$). Let

$$p_{ij} \;=\; p_{ji} \;=\; p,$$

and $p_{ii} = 0$ for $1 \leq i \leq n$. The symmetric matrix $(p_{ij})_{1 \leq i,j \leq n}$ is called the *spectrum* of $\mathcal{C}$.

4.1.3. *Generators of* $(\mathcal{I}_C^p + (\pi))/(\mathcal{I}_C^{p+1} + (\pi))$ - Let $i, j \in \{1, \ldots, n\}$ be such that $i \neq j$ and $x \in C$. Since $\mathcal{C}_{\{i,j\}} \subset \mathcal{C}$ there exists an element $\mathbf{u}_i = (v_m)_{1 \leq m \leq n}$ of $\mathcal{O}_{\mathcal{C},x}$ such that $v_i = 0$ and that the image of $(0, v_j)$ generates $\mathcal{I}_{C,\mathcal{C}_{\{i,j\}},x}/(\mathcal{I}^2_{C,\mathcal{C}_{\{i,j\}},x} + (\pi))$. According to 2.3.3, the image of $\mathbf{u}_i$ generates $\mathcal{I}_C/(\mathcal{I}_C^2 + (\pi))$ at x, and for any $k \in \{1, \ldots, n\}$ such that $k \neq i$, the image of $(0, v_k)$ generates $\mathcal{I}_{C,\mathcal{C}_{\{i,k\}},x}/(\mathcal{I}^2_{C,\mathcal{C}_{\{i,k\}},x} + (\pi))$. Hence from theorem 3.1.1 there exists $\alpha_{ik} \in \mathcal{O}_{\mathcal{C}_k,x}$ such that $v_k = \alpha_{ik} \pi_k^{p_{ik}}$, and

$$(\alpha_{ik}) \;=\; \mathcal{I}_{\mathcal{Z}_{ik},\mathcal{C}_k,x} \; .$$

Let I be a proper subset of $\{1, \ldots, n\}$, p its number of elements, and

$$\mathbf{u}_I \;=\; \prod_{i \in I} \mathbf{u}_i \; .$$

Then the image of $\mathbf{u}_I$ is a generator of $(\mathcal{I}^p_{C,x} + (\pi))/(\mathcal{I}^{p+1}_{C,x} + (\pi))$.

4.1.4. Proposition: 1 – *The ideal sheaf* $\mathcal{I}_{\mathcal{C}_I,\mathcal{C}}$ *of* $\mathcal{C}_I$ *in* $\mathcal{C}$ *at* x *is generated by* $\mathbf{u}_I$. *In particular the ideal sheaf of* $\mathcal{C}_i$ *in* $\mathcal{C}$ *at* x *is generated by* $\mathbf{u}_i$.

2 – $\mathcal{I}_{\mathcal{C}_I,\mathcal{C}}$ *is a line bundle on* $\mathcal{C}_{I^c}$.

Of course **2-** is a consequence of **1-**. The proof of **1-** is similar to that of proposition 4.3.3 in [9].

In particular, for every $i \in \{1, \ldots, n\}$, let $J_i = \{1, \ldots, n\} \backslash \{i\}$. Then the ideal sheaf of $\mathcal{C}_{J_i} \subset \mathcal{C}$ at x is generated by

$$\mathbf{u}_{J_i} \;=\; (0, \ldots, 0, A_i \pi_i^{q_i}, 0, \ldots, 0) \; ,$$

with

$$A_i \;=\; \prod_{1 \leq k \leq n, k \neq i} \alpha_{ki}, \qquad q_i \;=\; \sum_{1 \leq k \leq n} p_{ik} \; .$$

4.1.5. Proposition: *Let* $\mathbf{u}_I = (w_i)_{1\leq i\leq n}$. *For every* $j \in \{1,\ldots,n\}\backslash I$, *let* $w_j = v_j\pi_j^{m_j}$, *with* $v_{j|C} \neq 0$. *Then* v_j *is a generator of the ideal of* $\mathcal{Z}_{\{j\}}^{I\cup\{j\}}$ *at* x.

Proof. This follows from corollary 3.1.2, 2-. □

4.1.6. Proposition: **1 −** *Let* i,j,k *be distinct integers such that* $1 \leq i,j,k \leq n$. *Then if* $p_{ij} < p_{jk}$, *we have* $p_{ik} = p_{ij}$.

2 − *Let* $i \in \{1,\ldots,n\}$. *Then we have* $\mathcal{I}_{C,x} = (\mathbf{u}_i) + (\pi)$.

3 − *Let* $i \in \{1,\ldots,n\}$ *and* $v = (v_m)_{1\leq m\leq n} \in \mathcal{I}_{C,x}$ *such that* v_i *is a multiple of* π_i^p, *with* $p > 0$. *Then we have* $v \in (\mathbf{u}_i) + (\pi^p)$

The proof is similar to that of propositions 4.3.4 and 4.3.5 in [9].

4.1.7. *Construction by induction on* n – Let i be an integer such that $1 \leq i \leq n$. Let $\mathcal{B}_i$ be the image of $\mathcal{O}_{\mathcal{C}_{J_i}}$ in the sheaf of $\mathbb{C}$-algebras on C which at any point x is $\prod_{1\leq j\leq n, j\neq i}\mathcal{O}_{\mathcal{C}_j}/(A_j\pi_j^{q_j})$; it is also a sheaf of $\mathbb{C}$-algebras on C. For every point x of C and every $\alpha = (\alpha_m)_{1\leq m\leq n}$ in $\prod_{1\leq j\leq n}\mathcal{O}_{\mathcal{C}_j,x}$, we denote by $b_{i,x}(\alpha)$ its image in $\prod_{1\leq j\leq n, j\neq i}\mathcal{O}_{\mathcal{C}_j,x}/(A_j\pi_j^{q_j})$ (obtained by forgetting the i-th coordinate of α).

We denote by $\mathcal{A}_i$ the sheaf of $\mathbb{C}$-algebras on C which at any point x is $\mathcal{O}_{\mathcal{C}_i}/(A_i\pi_i^{q_i})$.

If $1 \leq k \leq n$, $x \in C$ and $\alpha \in \mathcal{O}_{\mathcal{C}_k,x}$, let $[\alpha]$ denote the image of α in $\mathcal{O}_{\mathcal{C}_k,x}/(A_k\pi_k^{q_k})$.

4.1.8. Proposition: *There exists a morphism of sheaves of* $\mathbb{C}$*-algebras on* C

$$\Phi_i : \mathcal{B}_i \longrightarrow \mathcal{A}_i$$

such that for every point x *of* C *and all* $(\alpha_m)_{1\leq m\leq n, m\neq i} \in \mathcal{O}_{\mathcal{C}_{J_i},x}$, $\alpha_i \in \mathcal{O}_{\mathcal{C}_i,x}$, *we have* $\alpha = (\alpha_m)_{1\leq m\leq n} \in \mathcal{O}_{\mathcal{C},x}$ *if and only if* $\Phi_{i,x}(b_{i,x}(\alpha)) = [\alpha_i]$.

The proof is similar to that of proposition 4.4.1 of [9].

For $1 < i \leq n$, let

$$A^{[i]} \ = \ \prod_{1\leq k<i}\alpha_{ki}\ .$$

The following result is a generalization of corollary 4.4.3 of [9]:

4.1.9. Corollary: *Let* N *be an integer such that* $N \geq \max_{1\leq i\leq n}(q_i)$. *Let* $x \in C$, $\beta \in \mathcal{O}_{\mathcal{C}_1,x}\times\cdots\mathcal{O}_{\mathcal{C}_n,x}$ *and* $u = (u_i)_{1\leq i\leq n} \in \mathcal{O}_{\mathcal{C},x}$ *such that* u_i *and* $A^{[i]}$ *are relatively prime in* $\mathcal{O}_{\mathcal{C}_i,x}$, *for* $1 < i \leq n$. *Suppose that* $[\beta u] \in \mathcal{O}_{\mathcal{C},x}/(\pi^N)$. *Then we have* $[\beta] \in \mathcal{O}_{\mathcal{C},x}/(\pi^N)$.

Proof. By induction on n. It is obvious if $n = 1$. Suppose that the lemma is true for $n-1$. Let $I = \{1,\ldots,n-1\}$. So we have $[\beta_{|\mathcal{C}_1\times\cdots\mathcal{C}_{n-1}}] \in \mathcal{O}_{\mathcal{C}_I,x}/(\pi_1,\ldots,\pi_{n-1})^N$ by the induction hypothesis. Let γ (resp. v) be the image of β (resp. u) in $\mathcal{B}_n$. To show that $[\beta] \in \mathcal{O}_{\mathcal{C},x}/(\pi^N)$ it is enough to verify that

$$\Phi_n(\gamma) \ = \ [\beta_n].$$

We have $\Phi_n(\gamma v) = [\beta_n u_n]$ because $[\beta u] \in \mathcal{O}_{\mathcal{C},x}/(\pi^N)$, and $\Phi_n(v) = [u_n]$ because $u \in \mathcal{O}_{\mathcal{C},x}$. So we have

$$\Phi_n(\gamma)[u_n] \;=\; \Phi_n(\gamma)\Phi_n(v) \;=\; \Phi_n(\gamma v) \;=\; [\beta_n u_n] \;=\; [\beta_n][u_n].$$

Since $u_{|C} \neq 0$, $[u_n]$ is not a zero divisor in $\mathcal{O}_{\mathcal{C}_n,x}/(A_n\pi_n^{q_n})$, so we have $\Phi_n(\gamma) = [\beta_n]$. □

The following result is a generalization of proposition 4.4.7 of [9]:

4.1.10. Proposition: *Let* $(\alpha_1\pi_1^{m_1}, \ldots, \alpha_n\pi_n^{m_n}) \in \mathcal{O}_{\mathcal{C},x}$ *be such that for* $1 < i \leq n$, $\alpha_{i|C} \neq 0$ *and that* α_i *and* $A^{[i]}$ *are relatively prime. Let* $\beta = (\beta_1, \ldots, \beta_n) \in \mathcal{O}_{\mathcal{C},x}$ *be such that for* $1 \leq i \leq n$, β_i *is a multiple of* α_i *in* $\mathcal{O}_{\mathcal{C}_i,x}$. *Let* $M = m_1 + \cdots m_n$. *Then*

$$\left(\frac{\beta_1}{\alpha_1}\pi_1^{M-m_1}, \ldots, \frac{\beta_n}{\alpha_n}\pi_n^{M-m_n}\right) \;\in\; \mathcal{O}_{\mathcal{C},x}\ .$$

Proof. By induction on n. It is obvious for $n = 1$. Suppose that it is true for $n-1 \geq 1$. Let $I = \{1, \ldots, n-1\}$. Then $(\alpha_1\pi_1^{m_1}, \ldots, \alpha_{n-1}\pi_{n-1}^{m_{n-1}}) \in \mathcal{O}_{\mathcal{C}_I,x}$. Hence, by the induction hypothesis, we have

$$\Big(\frac{\beta_1}{\alpha_1}\pi_1^{M-m_1-m_n}, \ldots, \frac{\beta_{n-1}}{\alpha_{n-1}}\pi_{n-1}^{M-m_{n-1}-m_n}\Big) \;\in\; \mathcal{O}_{\mathcal{C}_I,x}\ .$$

So there exists $\gamma \in \mathcal{O}_{\mathcal{C}_n,x}$ such that

$$u \;=\; \Big(\frac{\beta_1}{\alpha_1}\pi_1^{M-m_1-m_n}, \ldots, \frac{\beta_{n-1}}{\alpha_{n-1}}\pi_{n-1}^{M-m_{n-1}-m_n}, \gamma\Big) \;\in\; \mathcal{O}_{\mathcal{C},x}\ .$$

Multiplying by $(\alpha_1\pi_1^{m_1}, \ldots, \alpha_n\pi_n^{m_n})$ we see that $(\beta_1\pi_1^{M-m_n}, \ldots, \beta_{n-1}\pi_{n-1}^{M-m_n}, \gamma\alpha_n\pi_n^{m_n}) \in \mathcal{O}_{\mathcal{C},x}$. Subtracting $\beta\pi^{M-m_n}$, we find that $(0, \ldots, 0, \gamma\alpha_n\pi_n^{m_n} - \beta_n\pi_n^{M-m_n}) \in \mathcal{O}_{\mathcal{C},x}$. By proposition 4.1.4, the ideal of $\mathcal{C}_I \subset \mathcal{C}$ at x is generated by $(0, \ldots, 0, A_n\pi^{q_n})$. Hence we can write

$$\gamma\alpha_n\pi_n^{m_n} - \beta_n\pi_n^{M-m_n} \;=\; \lambda A_n\pi_n^{q_n}$$

for some $\lambda \in \mathcal{O}_{\mathcal{C}_n,x}$. Since α_n divides β_n, it divides also λA_n, and since α_n and A_n are relatively prime, α_n divides λ, and we have $\gamma\pi_n^{m_n} - \dfrac{\beta_n}{\alpha_n}\pi_n^{M-m_n} = \mu A_n$ for some $\mu \in \mathcal{O}_{\mathcal{C}_n,x}$. We have $v = (0, \ldots, 0, \mu A_n\pi_n^{q_n}) \in \mathcal{O}_{\mathcal{C},x}$. Now we have

$$\pi^{m_n}u - v \;=\; \left(\frac{\beta_1}{\alpha_1}\pi_1^{M-m_1}, \ldots, \frac{\beta_n}{\alpha_n}\pi_n^{M-m_n}\right) \;\in\; \mathcal{O}_{\mathcal{C},x}\ .$$

□

4.1.11. *Reducible deformations and fragmented deformations* – We show here that it is not true that there always exists a fragmented deformation above a reducible one (one could think that a reducible deformation could be obtained simply by gluing some curves together in a fragmented one).

Let C be a smooth curve of degree $d > 0$ in $\mathbb{P}_2$, and $f \in H^0(\mathcal{O}_{\mathbb{P}_2}(d))$ an equation of C. Let $h \in H^0(\mathcal{O}_{\mathbb{P}_2}(d))$,

$$\mathcal{C}(h) \;=\; \{(x,t) \in \mathbb{P}_2 \times \mathbb{C}; (f+th)(x) = 0\}$$

and $\pi_h : \mathcal{C}(h) \to \mathbb{C}$ the projection. Now let $h_1, \ldots, h_n \in H^0(\mathcal{O}_{\mathbb{P}_2}(d))$. Let $\mathcal{D} = \mathcal{C}(h_1) \cup \cdots \cup \mathcal{C}(h_n) \subset \mathbb{P}_2 \times \mathbb{C}$ and $\pi = (\pi_1, \ldots, \pi_n) : \mathcal{D} \to \mathbb{C}$. If $h_1, \ldots, h_n$ are sufficiently general, there exists a suitable neighborhood U of 0 in $\mathbb{C}$ such that $\mathcal{C} = \pi^{-1}(U)$ is a reducible deformation of C. Of course $\pi^{-1}(0)$ is the multiple curve in $\mathbb{P}_2$ defined by f^n. Now we have $p_{ij} = 1$ for $1 \leq i < j \leq n$, and

$$\mathbf{u}_1 \;=\; (0, (h_2 - h_1)t, \ldots, (h_n - h_1)t) \; .$$

Let $x \in C$ and suppose that $h_2 - h_1$ vanishes at x, but not $h_3 - h_1$. We will show that there is no fragmented deformation above $\mathcal{C}$, i.e. there is no fragmented deformation $\mathcal{C}'$ of C with components $\mathcal{C}(h_1)_{|U}, \ldots, \mathcal{C}(h_n)_{|U}$ such that $\mathcal{O}_{\mathcal{C}} \subset \mathcal{O}_{\mathcal{C}'}$. If such a fragmented deformation exists, its spectrum is the same as that of $\mathcal{C}$, and the ideal of $\mathcal{C}(h_1)_{|U}$ in $\mathcal{C}'$ is generated at x by an element

$$\mathbf{u}' \;=\; (0, t, a_2 t, \ldots, a_n t)$$

with a_i invertible in $\mathcal{O}_{\mathcal{C}(h_i),x}$ for $2 \leq i \leq n$. Then $\mathbf{u}'$ is a multiple of $\mathbf{u}$, i.e there exists $\alpha \in \mathcal{O}_{\mathcal{C},x}$ of the form

$$\alpha \;=\; (\alpha_1, h_2 - h_1, \frac{1}{a_3}(h_3 - h_1), \ldots, \frac{1}{a_n}(h_n - h_1)) \; .$$

But this is impossible because $(h_2 - h_1)_{|C} \neq \dfrac{1}{a_3}(h_3 - h_1)_{|C}$.

4.2. Localization and construction of maximal reducible deformations

We keep the notations of 4.1. The following result generalizes proposition 3.1.4:

4.2.1. Proposition: *Let* $u = (\alpha_1 \pi_1^{m_1}, \ldots, \alpha_n \pi_n^{m_n}) \in \mathcal{O}_{\mathcal{C},x}$ *, with* $m_i > q_i$ *for* $1 \leq i \leq n$*. Then* u *is a multiple of* π *in* $\mathcal{O}_{\mathcal{C},x}$ *, i.e.* $(\alpha_1 \pi_1^{m_1 - 1}, \ldots, \alpha_n \pi_n^{m_n - 1}) \in \mathcal{O}_{\mathcal{C},x}$.

Proof. By induction on n. The result is true for $n = 1$. Suppose that $n > 1$ and that it is true for $n - 1$. Let $I = \{0, \ldots, n-1\}$. Then $u' = (\alpha_1 \pi_1^{m_1 - 1}, \ldots, \alpha_{n-1} \pi_{n-1}^{m_{n-1} - 1}) \in \mathcal{O}_{\mathcal{C}_I,x}$ (by the induction hypothesis). Let $v \in \mathcal{O}_{C,x}$ be such that its image in $\mathcal{O}_{\mathcal{C}_I,x}$ is u', i.e $v = (v_i)_{1 \leq i \leq n}$ and $v_i = \alpha_i \pi_1^{m_i - 1}$ for $1 \leq i \leq n - 1$. Then $u - \pi v \in \mathcal{I}_{\mathcal{C}_I, \mathcal{C}, x}$, hence there exists $\lambda \in \mathcal{O}_{\mathcal{C}_n,x}$ such that

$$u - \pi v \;=\; (0, \ldots, 0, \lambda A_n \pi_n^{q_n}) \; .$$

We have only to show that λ is divisible by π_n. We have $v_n = \mu \pi_n^{q_n - 1}$, with $\mu = \pi_n^{m_n - q_n} a_n - \lambda A_n$, so we have to show that μ is divisible by π_n. Let $\tau = (\tau_i)_{1 \leq i \leq n} \in \mathcal{I}_{\mathcal{Z},x}$, such that $\tau_{|C} \neq 0$. Then we have

$$\tau v \;=\; (\pi_1^{m_1 - 1} a_1 \tau_1, \ldots, \pi_{n-1}^{m_{n-1} - 1} a_{n-1} \tau_{n-1}, \pi_n^{q_n - 1} \mu \tau_n) \; .$$

Since $m_i - 1 \geq q_i$ and A_i divides τ_i for $1 \leq i \leq n - 1$, we have

$$(\pi_1^{m_1 - 1} a_1 \tau_1, \ldots, \pi_{n-1}^{m_{n-1} - 1} a_{n-1} \tau_{n-1}, 0) \in \mathcal{O}_{\mathcal{C},x} \; .$$

Hence $w = (0, \ldots, 0, \pi_n^{q_n - 1} \mu \tau_n) \in \mathcal{O}_{\mathcal{C},x}$. Hence w is a multiple of $(0, \ldots, 0, \pi_n^{q_n} A_n)$. Since τ_n is not divisible by π_n, μ is divisible by π_n. $\square$

Let $I=\{1,\ldots,n-1\}$ and $x\in C$. Let $\boldsymbol{B_n}$ be the image of $\mathcal{O}_{\mathcal{C}_I,x}$ in $\prod_{i=1}^{n-1}\mathcal{O}_{\mathcal{C}_i,x}/\big((A_i\pi_i^{q_i})+(\pi_i^{q_i+1})\big)\times\ldots\times\mathcal{O}_{\mathcal{C}_{n-1},x}/\big((A_{n-1}\pi_{n-1}^{q_{n-1}})+(\pi_{n-1}^{q_{n-1}+1})\big)$. As in 3.1.5, we can deduce from the morphism Φ_{n-1} of proposition 4.1.8 the morphism

$$\overline{\Phi}_{n-1}:\boldsymbol{B_n}\longrightarrow\mathcal{O}_{\mathcal{C}_n,x}/\big((A_n\pi_n^{q_n})+(\pi_n^{q_n+1})\big)\ .$$

We can then define a new sheaf of algebras $\boldsymbol{\mathcal{A}}$ on C by

$$\boldsymbol{\mathcal{A}}_x\ =\ \{(u,v)\in\boldsymbol{B_n}\times\mathcal{O}_{\mathcal{C}_n,x}/\big((A_n\pi_n^{q_n})+(\pi_n^{q_n+1})\big)\ ;\ \overline{\Phi}_{n-1}(u)=v\}\ .$$

The proof of the following result is similar to that of proposition 3.1.6:

4.2.2. Proposition: *The natural projection* $\mathcal{O}_{\mathcal{C},x}\to\mathcal{A}_x$ *induces an isomorphism*

$$\mathcal{O}_{\mathcal{C},x}/(\pi)\ \simeq\ \boldsymbol{\mathcal{A}}_x/(\pi)\ .$$

It is then possible to define as for double primitive curves in 3.2.3 the notion of a *local reducible deformation* of a primitive multiple curve of multiplicity n. By proposition 4.2.2, a global reducible deformation induces a local one. And as for double curves, if a primitive multiple curve of multiplicity n has a local reducible deformation, it has also a global one.

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