REACHABLE SHEAVES ON RIBBONS AND DEFORMATIONS OF MODULI SPACES OF SHEAVES

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Resume. A *primitive multiple curve* is a Cohen-Macaulay irreducible projective curve Y that can be locally embedded in a smooth surface, and such that $C = Y_{red}$ is smooth. In this case, $L = \mathcal{I}_C/\mathcal{I}_C^2$ is a line bundle on C. If Y is of multiplicity 2, i.e. if $\mathcal{I}_C^2 = 0$, Y is called a *ribbon*. If Y is a ribbon and $h^0(L^{-2}) \neq 0$, then Y can be deformed to smooth curves, but in general a coherent sheaf on Y cannot be deformed in coherent sheaves on the smooth curves.

It has been proved in [11] that a ribbon with associated line bundle L such that $\deg(L) = -d < 0$ can be deformed to reduced curves having 2 irreducible components if L can be written as

$$L = \mathcal{O}_C(-P_1 - \cdots - P_d) ,$$

where $P_1, \cdots, P_d$ are distinct points of C. In this case we prove that quasi locally free sheaves on Y can be deformed to torsion free sheaves on the reducible curves with two components. This has some consequences on the structure and deformations of the moduli spaces of semi-stable sheaves on Y.

Summary

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1. Introduction

A *primitive multiple curve* is an algebraic variety Y over $\mathbb{C}$ which is Cohen-Macaulay, such that the induced reduced variety $C = Y_{red}$ is a smooth projective irreducible curve, and that every closed point of Y has a neighbourhood that can be embedded in a smooth surface. These curves have been defined and studied by C. Bănică and O. Forster in [1]. The simplest examples are

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infinitesimal neighbourhoods of projective smooth curves embedded in a smooth surface (but most primitive multiple curves cannot be globally embedded in smooth surfaces).

Let Y be a primitive multiple curve with associated reduced curve C, and suppose that $Y \neq C$. Let $\mathcal{I}_C$ be the ideal sheaf of C in Y. The *multiplicity* of Y is the smallest integer n such that $\mathcal{I}_C^n = 0$. The sheaf $L = \mathcal{I}_C/\mathcal{I}_C^2$ is a line bundle on C, called the *line bundle on C associated to Y*.

Primitive multiple curves of multiplicity 2 are called *ribbons*. They have been parametrised in [2]. Primitive multiple curves of any multiplicity and the coherent sheaves on them have been studied in [6], [5], [7] and [8].

The deformations of ribbons to smooth projective curves have been studied by M. González in [16]: he proved a ribbon Y, with associated smooth curve C and associated line bundle L on C is smoothable if $h^0(L^{-2}) \neq 0$.

1.1. Deformations to reduced reducible curves

Deformations of primitive multiple curves $Y = C_n$ of any multiplicity $n \geq 2$ to reduced curves having multiple components which are smooth, intersecting transversally, have been studied in [10]: we consider flat morphisms $\pi : \mathcal{C} \to S$, where S is a smooth connected curve, for some $P \in S$, $\mathcal{C}_P$ is isomorphic to Y and for $s \neq P$, $\mathcal{C}_s$ is a curve with multiple components which are smooth and intersect transversally. Then n is the maximal number of components of such deformations of Y. In this case we say that the deformation is *maximal*, the number of intersection points of two components is exactly $-\deg(L)$ and the genus of the components is the one of C (these deformations are called *maximal reducible deformations*). If $\deg(L) = 0$ we call π a *fragmented deformation*. This case $\deg(L) = 0$ has been completely treated in [10]: a primitive multiple curve of multiplicity n can be deformed in disjoint unions of n smooth curves if and only if $\mathcal{I}_C$ is isomorphic to the trivial bundle on C_{n-1}. In [9] it has been proved that this last condition is equivalent to the following: there exists a flat family of smooth curves $\mathcal{C} \to S$, parametrised by a smooth curve S, $s_0 \in S$ such that $\mathcal{C}_{s_0} = C$, such that Y is isomorphic to the n-th infinitesimal neighbourhood of C in $\mathcal{C}$.

The problem of determining which primitive multiple curves of multiplicity n can be deformed to reduced curves having exactly n components, allowing intersections of the components, is more difficult. A necessary condition is $h^0(L^*) > 0$. It has been proved in [11] that a ribbon with associated line bundle L such that $\deg(L) = -d < 0$ can be deformed to reduced curves having 2 irreducible components if L can be written as

$$L \ = \ \mathcal{O}_C(-P_1 - \cdots - P_d) \ ,$$

where $P_1, \cdots, P_d$ are distinct points of C.

1.2. Deformations of sheaves

Let Y be a ribbon, $C = Y_{\text{red}}$, and L the associated line bundle on C. We suppose that L can be written as

$$L \ = \ \mathcal{O}_C(-P_1 - \cdots - P_d) \ ,$$

where $P_1, \cdots, P_d$ are distinct points of C, or that $L = \mathcal{O}_C$. Let $\mathcal{E}$ be a coherent sheaf on Y. Let $p : \mathcal{E} \to \mathcal{E}_{|C}$ be the restriction morphism and $\mathcal{E}_1 = \ker(p)$, which is concentrated on C. We call $R(\mathcal{E}) = \mathrm{rk}(\mathcal{E}_{|C}) + \mathrm{rk}(\mathcal{E}_1)$ the *generalised rank* of $\mathcal{E}$, and $\mathrm{Deg}(\mathcal{E}) = \deg(\mathcal{E}_{|C}) + \deg(\mathcal{E}_1)$ the *generalised degree* of $\mathcal{E}$. It has been proved in [5] that there exist unique integers $a, b \geq 0$ and a nonempty open subset $U \subset C$ such that for every $x \in U$ we have $\mathcal{E}_x \simeq a\mathcal{O}_{Y,x} \oplus b\mathcal{O}_{C,x}$. If this is true for every $x \in C$, $\mathcal{E}$ is called a *quasi locally free sheaf*, and this is the case if and only if $\mathcal{E}_{|C}$ and $\mathcal{E}_1$ are vector bundles on C.

Let S be a smooth connected curve, $P \in S$ and $\rho : \mathcal{X} \to S$ a deformation of Y to smooth curves, i.e. ρ is flat, $\rho^{-1}(P) \simeq Y$, and for every $s \in S\backslash\{P\}$, $\mathcal{X}_s = \rho^{-1}(s)$ is a smooth projective irreducible curve. It is easy to see (using Hilbert polynomials with respect to an ample line bundle on $\mathcal{X}$) that if $\mathcal{F}$ is a coherent sheaf on $\mathcal{X}$ flat on S, then $R(\mathcal{F}_{|Y})$ must be even. This means that not all sheaves on Y can be deformed to sheaves on the smooth fibres. Moreover if $\mathcal{F}_{|Y}$ is locally free, $\deg(\mathcal{F}_s)$ is even.

Now let $\pi : \mathcal{C} \to S$ be a maximal reducible deformation of Y: we have $\pi^{-1}(P) \simeq Y$, and for every $s \in S\backslash\{P\}$, $\mathcal{C}_s = \pi^{-1}(s)$ is a reduced projective curve having 2 irreducible components, smooth and intersecting transversally in $-\deg(L)$ points (see 6.1 for the precise definition). Let E be a coherent sheaf on Y. We say that E *is reachable* (with respect to π) if there exists a smooth connected curve S', $P' \in S'$, a non constant morphism $f : S' \to S$ such that $f(P') = P$ and a coherent sheaf $\mathbb{E}$ on $f^*(\mathcal{C}) = \mathcal{C} \times_S S'$, flat on S', such that $\mathbb{E}_{P'} \simeq E$.

The main result of this paper is theorem 7.1.2

1.2.1. Theorem: *Every quasi locally free sheaf on Y is reachable.*

To prove this result we need to study the torsion free sheaves on reducible curves with several components, and the moduli spaces of such semi-stable sheaves. Some work has been done on this subject by M. Teixidor i Bigas in [26], [27], [28], [29], but only for vector bundles. We will need here to consider sheaves which have not the same rank on all the components. We will also use blowing-up of ribbons, which are also used in [3] in another context.

1.3. Motivation

1.3.1. *Study of the non reduced structure of some moduli spaces of stable sheaves on Y* – Some coherent sheaves on projective varieties have a non reduced versal deformation space. In particular, some moduli spaces of stable sheaves are non reduced.

We treat here the case of some sheaves on ribbons: the *quasi locally free sheaves of rigid type* (see. 1.5 and 1.6 for more details). Let Y be a ribbon and $\pi : \mathcal{C} \to S$ a deformation of Y to reduced curves with two components as in 1.1.

Let E be a stable quasi locally free sheaf of rigid type on Y. From [7], E belongs to an unique irreducible component of the corresponding moduli space of stable sheaves. Let $\mathbf{M}$ be the open subset of this component corresponding to quasi locally free sheaves of rigid type. It is proved in [7] that $\mathbf{M}_{reg}$ is smooth, and that the tangent sheaf $T\mathbf{M}$ of $\mathbf{M}$ is locally free. Hence we obtain a vector bundle $T\mathbf{M}/T(\mathbf{M}_{reg})$ on $\mathbf{M}_{reg}$, its rank is $h^0(L^*)$.

We find, using theorem 1.2.1, that E can be deformed in two distinct ways to sheaves on the reduced curves. In particular $\mathbf{M}$ deforms to two components of the moduli spaces of sheaves on the reduced curves, and $\mathbf{M}$ appears as the "limit" of varieties with two components, whence the non reduced structure of $\mathbf{M}$.

In [12] we show that this limit is responsible of a sub-line bundle $\mathcal{L}$ of $T\mathbf{M}/T(\mathbf{M}_{reg})$, and that $\mathcal{L}$ is closely related to the deformation π.

1.3.2. *Moduli spaces of vector bundles* – If instead of a deformation of Y to reducible curves on considers a deformation $\rho : \mathcal{D} \to S$ to *smooth* curves, among the quasi locally free sheaves on Y only some of them can be deformed to sheaves on the smooth fibers $\mathcal{D}_s$. For example if Y can be deformed to reduced curves having 2 irreducible components, then it can be deformed to smooth curves ([11], prop. 2.5.1), and it follows easily from theorem 1.2.1 that the vector bundles on Y can be deformed to vector bundles on the smooth fibers. The moduli spaces of stable vector bundles on Y (which are smooth) deform then to moduli spaces of vector bundles on the smooth fibers $\mathcal{D}_s$. It would then be possible to deduce properties of moduli spaces of vector bundles on the smooth fibers from the study of the moduli spaces of stable bundles on the special fiber Y. The study of the moduli spaces of stable bundles Y can also use that of moduli spaces of more complicated sheaves as those of quasi locally free sheaves of rigid type (if, as in [7] some proofs by induction are needed).

1.4. Moduli spaces of (semi-)stable sheaves on primitive reducible curves

1.4.1. *Primitive reducible curves* – (cf. 4-). A *primitive reducible curve* is an algebraic curve X such that

- X is connected.
- The irreducible components of X are smooth projective curves.
- Any three irreducible components of X have no common point, and any two components are transverse.

Suppose that X has two components D_1, D_2. Let $\mathcal{E}$ be a coherent sheaf on X. Then $\mathcal{E}$ is torsion free if and only if there exist vector bundles E_i on D_i, $i = 1, 2$, and for every $x \in D_1 \cap D_2$ a vector space W_x and surjective maps $f_{i,x} : E_{i,x} \to W_x$, such that $\mathcal{E}$ is isomorphic to the kernel of the surjective morphism of sheaves on X

$$E_1 \oplus E_2 \xrightarrow{\bigoplus(f_{1,x}, -f_{2,x})} \bigoplus_{x \in D_1 \cap D_2} \overline{W_x}$$

(where $\overline{W_x}$ is the skyscraper sheaf concentrated on x with fibre W_x at this point). For $i = 1, 2$, we have $E_i = \mathcal{E}_{|D_i}/T_i$, where T_i is the torsion subsheaf. We call

$$\tau(\mathcal{E}) \;=\; (\mathrm{rk}(E_1), \deg(E_1), \mathrm{rk}(E_2), \deg(E_2))$$

the *type* of $\mathcal{E}$. We have, for every $x \in D_1 \cap D_2$, $\dim(W_x) \leq \inf(\mathrm{rk}(E_1), \mathrm{rk}(E_2))$. If for every $x \in D_1 \cap D_2$, $\dim(W_x) = \inf(\mathrm{rk}(E_1), \mathrm{rk}(E_2))$, we say that $\mathcal{E}$ *is linked*. For example, if $\mathrm{rk}(E_1) = \mathrm{rk}(E_2)$, $\mathcal{E}$ is linked if and only if $\mathcal{E}$ is locally free. We prove in 4- that deformations of linked sheaves are linked, and that $\deg(E_1)$ and $\deg(E_2)$ (and hence also $\tau(\mathcal{E})$) are invariant by deformations. Moreover, any two linked sheaves with the same type can be put in a flat family of linked sheaves on X parametrised by an integral variety (theorem 4.3.3). We prove also that every torsion free sheaf on X can be deformed to linked sheaves (theorem 4.3.9).

1.4.2. *Moduli spaces of sheaves* – Let $\mathcal{O}_X(1)$ be an ample line bundle on X and H a polynomial in one variable with rational coefficients. Let $\mathcal{M}_X(H)$ be the moduli space of semi-stable torsion free sheaves on X with Hilbert polynomial H with respect to $\mathcal{O}_X(1)$. It follows from 1.4.1 that every irreducible component $\mathcal{N}$ of $\mathcal{M}_X(H)$ has a dense open subset corresponding to linked sheaves (having all the same type τ), and every semi-stable linked sheaf of type τ defines a point of $\mathcal{N}$. It follows that the components of $\mathcal{M}_X(H)$ are indexed by the types of the linked semi-stable sheaves of Hilbert polynomial H. We will denote by $\mathcal{N}_X(r_1, d_1, r_2, d_2)$ the component containing points corresponding to linked sheaves of type (r_1, d_1, r_2, d_2). We prove that $\mathcal{N}_X(r_1, d_1, r_2, d_2)$ is smooth at every point corresponding to a stable linked sheaf $\mathcal{E}$ such that E_1 and E_2 are simple vector bundles (theorem 4.3.7). But in general, we have $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \neq \{0\}$.

1.5. Moduli spaces of sheaves of rigid type on ribbons

Let Y be a ribbon, $C = Y_{\mathrm{red}}$ and L the associated line bundle on C. A quasi locally free sheaf $\mathcal{E}$ on Y is called *of rigid type* if it is locally free, or locally isomorphic to $a\mathcal{O}_Y \oplus \mathcal{O}_C$, for some integer $a \geq 0$. If $\mathcal{E}$ is of rigid type, then the deformations of $\mathcal{E}$ are also of rigid type, and $\deg(\mathcal{E}_{|C})$, $\deg(\mathcal{E}_1)$ are invariant by deformations. Moreover, any two quasi locally free sheaves of rigid type with the same $\mathrm{rk}(\mathcal{E}_{|C})$, $\deg(\mathcal{E}_{|C})$, $\deg(\mathcal{E}_1)$ can be put in a flat family of rigid sheaves with the same invariants parametrised by an integral variety (cf. [7]).

The semi-stability conditions on Y are the same for every choice of an ample line bundle on Y (cf. 3-). The Hilbert polynomial of a coherent sheaf $\mathcal{E}$ on Y depends only on the two invariants $R(\mathcal{E})$ and $\mathrm{Deg}(\mathcal{E})$. Let $\mathcal{M}_Y(R, D)$ be the moduli space of semi-stable torsion free sheaves on Y with generalised rank R and generalised degree D. Suppose that $R = 2a+1$ and d_0, d_1 are integers such that $D = d_0 + d_1 + a.\deg(L)$. Let $\mathcal{N}(R, d_0, d_1)$ denote the open irreducible subset of $\mathcal{M}_Y(R, D)$ corresponding to stable sheaves of rigid type $\mathcal{E}$ such that $\deg(\mathcal{E}_{|C}) = d_0$ and $\deg(\mathcal{E}_1 \otimes L^*) = d_1$ (it is denoted by $\mathcal{N}(a, 1, d_0, d_1)$ in [7] and [8]). We give in [8] sufficient conditions for the non-emptiness of $\mathcal{N}(R, d_0, d_1)$ (cf. 3.3.5). If it is non empty, then $\mathcal{N}(R, d_0, d_1)_{\mathrm{red}}$ is smooth, and at the point corresponding to the stable sheaf $\mathcal{E}$, its tangent space is canonically isomorphic to $H^1(\mathrm{End}(\mathcal{E}))$, whereas the tangent space of $\mathcal{N}(R, d_0, d_1)$ is isomorphic to $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$. The quotient

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})/H^1(\mathrm{End}(\mathcal{E})) \;\simeq\; H^0(\mathcal{E}xt^1(\mathcal{E}, \mathcal{E}))$$

is canonically isomorphic to $H^0(L^*)$ (cf. [5], cor. 6.2.2).

1.6. Relative moduli spaces of semi-stable sheaves

We use the notations of 1.2. Let $\mathcal{E}$ be a quasi locally free sheaf of rigid type on Y, locally isomorphic to $a\mathcal{O}_Y \oplus \mathcal{O}_C$. Let $\pi : \mathcal{C} \to S$ be a maximal reducible deformation of Y. Then by the main result, in 1.2, $\mathcal{E}$ can be deformed to torsion free sheaves on the primitive reducible curves $\mathcal{C}_s$, $s \neq P$.

Let $\mathcal{O}_{\mathcal{C}}(1)$ be an ample line bundle on $\mathcal{C}$. Then we have, for every $s \in S\backslash\{P\}$, $\deg(\mathcal{O}_{\mathcal{C}_{1,s}}(1)) = \deg(\mathcal{O}_{\mathcal{C}_{2,s}}(1))$, and so the semi-stability of sheaves on the curves $\mathcal{C}_s$ does not depend on the choice of $\mathcal{O}_{\mathcal{C}}(1)$. Let H be the Hilbert polynomial of $\mathcal{E}$. Let

$$\tau : \mathbf{M}(\mathcal{C}/S, H) \longrightarrow S$$

be the relative moduli space of sheaves on the fibres of π, so that for every closed point $s \in S$, $\tau^{-1}(s)$ is the moduli space $\mathcal{M}_{\mathcal{C}_s}(H)$ of semi-stable sheaves on $\mathcal{C}_s$ with Hilbert polynomial H.

We have $\mathrm{rk}(\mathcal{E}_{|C}) = a+1$ and $\mathrm{rk}(\mathcal{E}_1) = a$. Let $d_0 = \deg(\mathcal{E}_{|C})$, $d_1 = \deg(\mathcal{E}_1)$. Let S' be a smooth connected curve, $P' \in S'$, $f : S' \to S$ a non constant morphism such that $f(P') = P$ and $\mathbb{E}$ a coherent sheaf on $f^*(\mathcal{C}) = \mathcal{C} \times_S S'$, flat on S', such that $\mathbb{E}_{P'} \simeq \mathcal{E}$. We prove in proposition 6.4.5 that for $t \neq P'$ in a suitable neighbourhood of P', $\mathbb{E}_t$ is a linked sheaf such that, if $E_i = \mathbb{E}_{t|\mathcal{C}_{i,f(t)}}/T_i$ (where T_i is the torsion subsheaf), then either

$$\mathrm{rk}(E_1) = a\ , \quad \deg(E_1) = d_1\ , \quad \mathrm{rk}(E_2) = a+1\ , \quad \deg(E_2) = d_0\ ,$$

or

$$\mathrm{rk}(E_2) = a\ , \quad \deg(E_2) = d_1\ , \quad \mathrm{rk}(E_1) = a+1\ , \quad \deg(E_1) = d_0\ .$$

In other words, in the moduli space $\mathbf{M}(\mathcal{C}/S, H)$, $\mathcal{N}(2a+1, d_0, d_1) \subset \mathcal{M}_Y(H)$ is the "limit" of the only components $\mathcal{N}_{\mathcal{C}_s}(a+1, d_0, a, d_1)$, $\mathcal{N}_{\mathcal{C}_s}(a, d_1, a+1, d_0)$ of $\mathcal{M}_{\mathcal{C}_s}(H)$, $s \neq P$. In particular this explains (cf. 1.3) why $\mathcal{N}(2a+1, d_0, d_1)$ cannot be reduced.

1.7. The proof of the main theorem

We use the notations of 1.2. Let $\mathcal{Z} \subset \mathcal{C}$ be the closure in $\mathcal{C}$ of the locus of the intersection points of the components of $\pi^{-1}(s)$, $s \neq P$. It consists of $d = -\deg(L)$ smooth curves intersecting C in distinct points $P_1, \cdots, P_d$ such that $L = \mathcal{O}_C(-P_1 - \cdots - P_d)$. The blowing-up $\mathcal{D} \to \mathcal{C}$ of $\mathcal{Z}$ is a fragmented deformation of the blowing up of $P_1, \cdots, P_d$ in Y, which is a ribbon with associated smooth curve C and associated line bundle $\mathcal{O}_C$. We first prove the theorem for $\mathcal{D}$. By studying the relations between sheaves on $\mathcal{C}$ and sheaves on $\mathcal{D}$ we can prove the theorem for $\mathcal{C}$.

1.8. Outline of the paper

Section 2 contains reminders of definitions and results, and some technical lemmas used in the rest of the paper.

Section 3 contains mainly reminders of the main properties of coherent sheaves on ribbons, with some useful lemmas.

Section 4 is devoted to the study of torsion free sheaves on primitive reducible curves (cf. 1.4). The results obtained here could probably be useful if one wants to study more precisely the moduli spaces of sheaves on these curves, without restricting to the locally free ones.

In **Section 5** we consider the blowing-up of a finite number of points of a ribbon Y. It is again a ribbon $\widetilde{Y}$. We study the relations between torsion free sheaves on Y and torsion free sheaves on $\widetilde{Y}$.

In **Section 6** we study the coherent sheaves $\mathcal{E}$ on $\mathcal{C}$ such that $\mathcal{E}_{|Y}$ is a quasi locally free sheaf.

In **Section 7** we prove the main theorem, using the results of the preceding sections.

Notations – If X is a smooth variety and D a divisor of X, $\mathcal{O}_X(D)$ is the line bundle associated to D, i.e. the dual of the ideal sheaf $\mathcal{O}_X(-D)$ of D. If $\mathcal{E}$ is a coherent sheaf on X, let $\mathcal{E}(D) = \mathcal{E} \otimes \mathcal{O}_X(D)$.

– In this paper, an *algebraic variety* is a quasi-projective scheme over $\mathbb{C}$.

– If X is an algebraic variety and $Y \subset X$ a closed subvariety, let $\mathcal{I}_{Y,X}$ denote the ideal sheaf of Y in X. If there is no ambiguity, we will denote it also by $\mathcal{I}_Y$.

– if $x \in X$ is a closed point, $\mathbb{C}_x$ will denote the skyscraper sheaf concentrated on x with fibre $\mathbb{C}$ at x.

– if $f : \mathcal{X} \to S$ is a flat morphism of algebraic varieties, and $f : T \to S$ another morphism, we shall sometimes use the notation $f^*(\mathcal{X}) = \mathcal{X} \times_S T$.

2. Preliminaries

2.1. Relative moduli spaces of (semi-)stable sheaves

(cf. [21], [24])

Let X, S be algebraic varieties and $X \to S$ a flat projective morphism. Let $\sigma : S' \to S$ be an S-variety. Let $X' = X \times_S S'$. A *family of sheaves* on X'/S' is a coherent sheaf on $X \times_S S'$, flat on S'.

Let $\mathcal{O}_X(1)$ be an ample line bundle on X and H a polynomial in one variable with rational coefficients. A *family of semi-stable sheaves* with Hilbert polynomial H on X'/S' is a family of sheaves $\mathcal{E}$ on X'/S' such that for every closed point $s' \in S'$, the restriction $\mathcal{E}_{s'}$ of $\mathcal{E}$ to $X_{\sigma(s')} = X'_{s'}$ is a semi-stable sheaf with Hilbert polynomial H with respect to $\mathcal{O}_{X_{\sigma(s')}}(1) = \mathcal{O}_X(1)_{|X_{\sigma(s')}}$. According to [24], there exists a moduli space

$$\tau : \mathbf{M}(X/S, H) \longrightarrow S$$

such that for every closed point $s \in S$, $\tau^{-1}(s) = \mathbf{M}(X_s, H)$ is the moduli space of semi-stable sheaves on X_s with Hilbert polynomial H. To every family $\mathcal{E}$ of semi-stable sheaves with Hilbert polynomial H on X'/S', one associates a canonical morphism

$$f_{\mathcal{E}} : S' \longrightarrow \mathbf{M}(X/S, H)$$

such that for every $s' \in S'$, $f_{\mathcal{E}}(s')$ is the point of $\mathbf{M}(X_{\sigma(s')}, H)$ corresponding to $\mathcal{E}_{s'}$. The morphism τ needs not to be flat (cf. [19]).

2.2. Restrictions of sheaves on hypersurfaces

Let X be a smooth algebraic variety, and Y, Z hypersurfaces such that $Y \subset Z$. Let $Z = Y \cup T$, with T a minimal hypersurface. Then we have $\mathcal{O}_X(-Z) \subset \mathcal{O}_X(-Y)$. Let E be a vector bundle on X. Then we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r] \arrow[d, equal] & E(Y) \arrow[r] \arrow[d, hook] & E(Y)_{|Y} \arrow[d, "\phi_E"] \\ 0 \arrow[r] & E \arrow[r] & E(Z) \arrow[r] & E(Z)_{|Z} \end{tikzcd}$$

2.2.1. Lemma: *Let $z \in Z$. Then*

$$\mathrm{im}(\phi_E) \;=\; \mathcal{I}_{T,Z}\big[E(Z)_{|Z}\big]\ .$$

(recall that $\mathcal{I}_{T,Z}$ is the ideal sheaf of T in Z)

Proof. Let $f, g \in \mathcal{O}_{X,z}$ be equations of Y, T respectively. Then fg is an equation of Z. Then $\mathcal{O}_X(Y)_z$ is the dual of $\mathcal{O}_X(-Y)_z = (f) \subset \mathcal{O}_{X,z}$, and is generated by

$$\begin{aligned} (f) &\longrightarrow \mathcal{O}_{X,z} \\ \alpha.f &\longmapsto \alpha \end{aligned}$$

Similarly, $\mathcal{O}_X(Z)_z$ is the dual of $\mathcal{O}_X(-Z)_z = (fg) \subset \mathcal{O}_{X,z}$, and is generated by

$$\begin{aligned} (fg) &\longrightarrow \mathcal{O}_{X,z} \\ \alpha.fg &\longmapsto \alpha \end{aligned}$$

It is clear that $\mathrm{im}(\phi_{\mathcal{O}_Y,z}) = g.\mathcal{O}_X(Z)_z$. The lemma follows immediately. □

2.2.2. Lemma: *Let $\mathbb{E}$ be a vector bundle on X and F a vector bundle on Y. Then there is a canonical isomorphism*

$$\mathrm{Ext}^1_{\mathcal{O}_X}(F, \mathbb{E}) \;\simeq\; \mathrm{Hom}(F, \mathbb{E}(Y)_{|Y})\ .$$

Proof. We have $\mathcal{H}om(F, \mathbb{E}) = 0$, hence, from the Ext spectral sequence (cf. [15], 7.3) it suffices to prove that $\mathcal{E}xt^1_{\mathcal{O}_X}(F, \mathbb{E}) \simeq \mathcal{H}om(F, \mathbb{E}(Y)_{|Y})$.

Let $r = \mathrm{rk}(F)$. Locally we have a locally free resolution of F:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{E}_1 \arrow[r] \arrow[d, equal] & \mathbb{E}_0 \arrow[r] \arrow[d, equal] & F \arrow[r] & 0 \\ & \mathbb{E}_0(-Y) & r\mathcal{O}_X & & \end{tikzcd}$$

hence we have an exact sequence

$$0 \longrightarrow \mathcal{H}om(\mathbb{E}_0, \mathbb{E}) \longrightarrow \mathcal{H}om(\mathbb{E}_0(-Y), \mathbb{E}) \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_X}(F, \mathbb{E}) \longrightarrow 0\ .$$

The result follows immediately. □

2.3. A lemma on extensions

Let X be an algebraic variety, and A, B, C, D, $\mathcal{E}$, coherent sheaves, such that we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r] \arrow[d, "\alpha"] & \mathcal{E} \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & A \arrow[r] & \mathcal{E} \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

Suppose that α is injective, and let $N = \operatorname{coker}(\alpha)$. Then we have a canonical isomorphism $\ker(\beta) \simeq N$. Let $i : N \to D$ be the inclusion.

Let $q \geq 0$ be an integer and

$$\delta : H^q(N) \longrightarrow H^{q+1}(C) \ , \qquad \delta' : H^q(D) \to H^{q+1}(C)$$

be the mappings coming from the exact sequences

$$0 \longrightarrow C \longrightarrow A \longrightarrow N \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow C \longrightarrow \mathcal{E} \longrightarrow D \longrightarrow 0$$

respectively. The following lemma is easily proved (by using Čech cohomology):

2.3.1. Lemma: *The following triangle is commutative*

$$\begin{tikzcd} H^q(N) \arrow[r, "\delta"] \arrow[d, "H^q(i)"] & H^{q+1}(C) \\ H^q(D) \arrow[ur, "\delta'"'] & \end{tikzcd}$$

2.4. Generalised elementary modifications of vector bundles on curves

(cf. [22])

Let C be a smooth algebraic curve, $x \in C$, E a vector bundle on C and $N \subset E_x$ a linear subspace. If W is a finite dimensional vector space, let W_x denote the skyscraper sheaf concentrated on x with fibre W at x. Let E_N denote the kernel of the canonical morphism

$$E \longrightarrow (E_x/N)_x \ .$$

It is a vector bundle. We call E_N the *elementary modification of* E defined by N (elementary modifications are well known for rank 2 vector bundles).

We can also define elementary deformations from a finite set of distinct points of C: $x_1, \ldots, x_n$, and for $1 \leq i \leq n$, a linear subspace $N_i \subset E_{x_i}$. We obtain the vector bundle $E_{N_1,\ldots,N_n}$, which is also obtained by successive elementary transformations involving one point, i.e. we define $E_{N_1,\ldots,N_i}$ inductively by the relation $E_{N_1,\ldots,N_{i+1}} = (E_{N_1,\ldots,N_i})_{N_{i+1}}$.

2.4.1. *Periodicity* – We have $E(-x) \subset E_N$, and $E(-x)$ is itself an elementary modification of E_N. We have a canonical exact sequence

$$0 \longrightarrow E(-x) \longrightarrow E_N \longrightarrow N_x \longrightarrow 0 \ .$$

The kernel of the induced map $E_{N,x} \to N$ is canonically isomorphic to $E(-x)_x/(N \otimes \mathcal{O}_C(-x)_x))$. i.e we have

$$E(-x) \ = \ (E_N)_{E(-x)_x/(N\otimes\mathcal{O}_C(-x)_x)}$$

(this can be seen for example by taking a local trivialisation of E).

2.4.2. *Duality* – We can see $(E_x/N)^*$ as a linear subspace of E_x^*, and we have

$$(E^*)_{(E_x/N)^*} \ = \ (E_N)^*(-x) \ .$$

2.5. The Ext spectral sequence

(cf. [15])

Let $\mathcal{E}$, $\mathcal{F}$ be coherent sheaves on an algebraic variety X. Then there exists a spectral sequence such that

$$E_2^{pq} \ = \ H^p(X, \mathcal{E}xt^q(\mathcal{E},\mathcal{F})) \ \Longrightarrow \ \mathrm{Ext}^{p+q}(\mathcal{E},\mathcal{F}) \ .$$

It follows that we have an exact sequence

$$0 \longrightarrow H^1(X, \mathcal{H}om(\mathcal{E},\mathcal{F})) \xrightarrow{\tau} \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) \longrightarrow H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})) \longrightarrow H^2(X, \mathcal{H}om(\mathcal{E},\mathcal{F})).$$

In particular, if $H^2(X, \mathcal{H}om(\mathcal{E},\mathcal{F})) = \{0\}$, for example if $\dim(X) = 1$, then we have an exact sequence

$$0 \longrightarrow H^1(X, \mathcal{H}om(\mathcal{E},\mathcal{F})) \xrightarrow{\tau} \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) \longrightarrow H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})) \longrightarrow 0.$$

The morphism τ can be described using Čech cohomology: let (U_i) be an open cover of X. Let $\alpha \in H^1(X, \mathcal{H}om(\mathcal{E},\mathcal{F}))$ corresponding to a cocycle (α_{ij}), where $\alpha_{ij} \in \mathrm{Hom}(\mathcal{E}_{|U_{ij}}, \mathcal{F}_{|U_{ij}})$. Then $\tau(\alpha)$ corresponds to the extension $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{E} \to 0$, where $\mathcal{G}$ is obtained by gluing the sheaves $(\mathcal{F}\oplus\mathcal{E})_{|U_i}$ with the automorphisms of $(\mathcal{F}\oplus\mathcal{E})_{|U_{ij}}$ defined by the matrices $\begin{pmatrix} I_{\mathcal{F}} & \alpha_{ij} \\ 0 & I_{\mathcal{E}} \end{pmatrix}$.

Let $Y \subset X$ be a closed subvariety. Then if

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow 0 \tag{1}$$

is an extension coming from an element of $\mathrm{im}(\tau)$, the restriction

$$0 \longrightarrow \mathcal{F}_{|Y} \longrightarrow \mathcal{G}_{|Y} \longrightarrow \mathcal{E}_{|Y} \longrightarrow 0 \tag{2}$$

is exact, because locally on X, the equation (1) is split The map

$$H^1(X, \mathcal{E}nd(E)) \longrightarrow H^1(X, \mathcal{E}nd(\mathcal{E}_{|Y}))$$

induced by the canonical morphism $\mathcal{E}nd(E) \to \mathcal{E}nd(\mathcal{E}_{|Y})$ sends the element corresponding to (2) to the one corresponding to (1).

2.6. Infinitesimal deformations of coherent sheaves

2.6.1. *Deformations of sheaves* – Let X be a projective algebraic variety and E a coherent sheaf on X. A *deformation* of E is a quadruplet $\mathcal{D} = (S, s_0, \mathcal{E}, \alpha)$, where (S, s_0) is the germ of an analytic variety, $\mathcal{E}$ is a coherent sheaf on $S \times X$, flat on S, and α an isomorphism $\mathcal{E}_{s_0} \simeq E$. If there is no risk of confusion, we also say that $\mathcal{E}$ is an infinitesimal deformation of E. Let $Z_2 = \mathrm{spec}(\mathbb{C}[t]/(t^2))$. When $S = Z_2$ and s_0 is the closed point $*$ of Z_2, we say that $\mathcal{D}$ is an *infinitesimal deformation* of E. Isomorphisms of deformations of E are defined in an obvious way. If $f : (S', s'_0) \to (S, s_0)$ is a morphism of germs, the deformation $f^\#(\mathcal{D})$ is defined as well. A deformation $\mathcal{D} = (S, s_0, \mathcal{E}, \alpha)$ is called *semi-universal* if for every deformation $\mathcal{D}' = (S', s'_0, \mathcal{E}', \alpha')$ of E, there exists a morphism $f : (S', s'_0) \to (S, s_0)$ such that $f^\#(\mathcal{D}) \simeq \mathcal{D}'$, and if the tangent map $T_{s'_0}S' \to T_{s_0}S$ is uniquely determined. There always exists a semi-universal deformation of E (cf. [25], theorem I).

Let $\mathcal{E}$ be an infinitesimal deformation of E. Let p_X denote the projection $Z_2 \times X \to X$. Then there is a canonical exact sequence

$$0 \longrightarrow E \longrightarrow p_{X*}(\mathcal{E}) \longrightarrow E \longrightarrow 0 \ ,$$

i.e. an extension of E by itself. In fact, by associating this extension to $\mathcal{E}$ one defines a bijection between the set of isomorphism classes of infinitesimal deformations of E and the set of isomorphism classes of extensions of E by itself, i.e. $\mathrm{Ext}^1_{\mathcal{O}_X}(E, E)$.

2.6.2. *Kodaïra-Spencer morphism* – Let $\mathcal{D} = (S, s_0, \mathcal{E}, \alpha)$ be a deformation of E, and $X^{(2)}_{s_0}$ the infinitesimal neighbourhood of order 2 of $X_{s_0} = \{s_0\} \times X$ in $S \times X$. Then we have an exact sequence on $X^{(2)}_{s_0}$

$$0 \longrightarrow T_{s_0}S \otimes E \longrightarrow \mathcal{E}/m_s^2\mathcal{E} = \mathcal{E}_{|X^{(2)}_{s_0}} \longrightarrow E \longrightarrow 0 \ .$$

By taking the direct image by p_X we obtain the exact sequence on X

$$0 \longrightarrow T_{s_0}S \otimes E \longrightarrow p_{X*}(\mathcal{E}/m_s^2\mathcal{E}) \longrightarrow E \longrightarrow 0 \ ,$$

hence a linear map

$$\omega_{s_0} : T_{s_0}S \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(E, E) \ ,$$

which is called the *Kodaïra-Spencer morphism* of $\mathcal{E}$ at s_0.

We say that $\mathcal{E}$ is a *complete deformation* if ω_{s_0} is surjective. If $\mathcal{D}$ is a semi-universal deformation, ω_{s_0} is an isomorphism.

2.7. The Kodaïra-Spencer map for vector bundles on families of curves

2.7.1. *Vector bundles on families of smooth curves* – A *family of smooth curves* parametrised by an algebraic variety U is a flat morphism $\rho : \mathcal{X} \to U$ such that for every closed point $x \in U$, $\mathcal{X}_x = \rho^{-1}(x)$ is a smooth projective connected curve. If $f : Y \to U$ is a morphism of algebraic varieties, $f^*(\mathcal{X}) = \mathcal{X} \times_U Y$ is a family of smooth curves parametrised by Y. If $\mathbb{E}$ is a vector bundle on $\mathcal{X}$, let $f^\#(\mathbb{E}) = p^*_{\mathcal{X}}(\mathbb{E})$ (where $p_{\mathcal{X}}$ is the projection $\mathcal{X} \times_U Y \to \mathcal{X}$).

Let S be a smooth curve, $s_0 \in S$, $\pi : \mathcal{X} \to S$ a flat family of smooth projective curves and $C = \mathcal{X}_{s_0} = \pi^{-1}(s_0)$. When dealing with morphisms of smooth curves $\phi : T \to S$, we will always assume that there is exactly one closed point of T over s_0 (if ϕ is not constant and $s_0 \in \mathrm{im}(\phi)$, it is always possible to remove from T some points in the finite set $\phi^{-1}(s_0)$).

Let T, T' be smooth curves, $t_0 \in T$, $t'_0 \in T'$, $\alpha : T \to S$, $\alpha' : T' \to S$ morphisms such that $\alpha(t_0) = s_0$, $\alpha'(t'_0) = s_0$. Suppose that the tangent maps $T_{t_0}\alpha : T_{t_0}T \to T_{s_0}S$, $T_{t_0}\alpha' : T_{t'_0}T' \to T_{s_0}S$ are injective. Let $Z = T \times_S T'$, p_T, $p_{T'}$, β the projections $Z \to T$, $Z \to T'$, $Z \to S$ respectively. Let $\mathcal{Y} = \beta^*(\mathcal{X})$. Let $z_0 = (t_0, t'_0) \in Z$.

2.7.2. *Semi-universal families* – Let E be a vector bundle on C. There exists a smooth connected variety R, a flat morphism $\eta : R \to Z$, a vector bundle $\mathbb{E}$ on $\eta^*(\mathcal{Y})$ such that:

- There exists $r_0 \in R_{z_0}$ such that $\mathbb{E}_{r_0} \simeq E$.
- For every $z \in Z$, $\mathbb{E}_{|R_z \times \mathcal{X}_{\beta(z)}}$ is a complete deformation, i.e. for every $r \in R_z$, the Kodaïra-Spencer map
$$\omega_r : (TR_z)_r \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{X}_{\beta(z)}}}(\mathbb{E}_r, \mathbb{E}_r)$$
is surjective.
- R has the following local universal property: for every $z \in Z$, every $r \in R_z$ every neighbourhood U of z, every vector bundle $\mathcal{F}$ on $\beta^{-1}(U)$ such that $\mathcal{F}_z \simeq \mathbb{E}_r$, there exists a neighbourhood $U' \subset U$ of z and a morphism $\phi : U' \to R$ such that $\phi(z) = r$, $\eta \circ \phi = I$ and $\phi^{\#}(\mathcal{E}) \simeq \mathcal{F}$.

(for example R can be constructed by using relative Quot schemes). A useful consequence of the existence of R is that *there exists a smooth curve S', $s'_0 \in S'$, a morphism $\phi : S' \to S$ such that $\phi(s'_0) = s_0$, and a vector bundle $\mathcal{E}$ on $\phi^*(\mathcal{X})$ such that* $\mathcal{E}_{s'_0} \simeq E$ (one can take for S' an appropriate curve in R through r_0).

2.7.3. *The Kodaïra-Spencer map* – Let T, T' be smooth curves, $t_0 \in T$, $t'_0 \in T'$, $\alpha : T \to S$, $\alpha' : T' \to S$ morphisms such that $\alpha(t_0) = s_0$, $\alpha'(t'_0) = s_0$. Suppose that the tangent maps $T_{t_0}\alpha : T_{t_0}T \to T_{s_0}S$, $T_{t_0}\alpha' : T_{t'_0}T' \to T_{s_0}S$ are injective. Let $Z = T \times_S T'$, p_T, $p_{T'}$, β the projections $Z \to T$, $Z \to T'$, $Z \to S$ respectively. Let $\mathcal{Y} = \beta^*(\mathcal{X})$. Let $z_0 = (t_0, t'_0) \in Z$. Let E be a vector bundle on C, $\mathcal{E}$ a vector bundle on $\alpha^*(\mathcal{X})$, $\mathcal{E}'$ a vector bundle on $\alpha'^*(\mathcal{X})$, such that there are isomorphisms $\mathcal{E}_{s_0} \simeq E$, $\mathcal{E}'_{s_0} \simeq E$. We have two vector bundles $p_T^{\#}(\mathcal{E})$, $p_{T'}^{\#}(\mathcal{E}')$ on $\beta^*(\mathcal{X})$. There exist a neighbourhood U of z_0 and morphisms $f : U \to R$, $f' : U \to R$ such that $\eta \circ f = \eta \circ f' = I$ and $f^{\#}(\mathbb{E}) \simeq p_T^{\#}(\mathcal{E})$, $f'^{\#}(\mathbb{E}) \simeq p_{T'}^{\#}(\mathcal{E}')$. We have
$$T\eta_{r_0} \circ Tf_{z_0} \ = \ T\eta_{r_0} \circ Tf'_{z_0} \ = \ I_{TZ_{z_0}} \ ,$$
hence $\mathrm{im}(Tf_{z_0} - Tf'_{z_0}) \subset TR_{z_0, r_0}$. Now we define the linear map
$$\omega_{\mathcal{E},\mathcal{E}'} : TS_{s_0} \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_C}(E, E)$$
by:
$$\omega_{\mathcal{E},\mathcal{E}'} \ = \ \omega_{r_0} \circ (Tf_{z_0} - Tf'_{z_0}) \circ (T\beta_{z_0})^{-1}.$$
It is easy to see that this definition is independent of the choice of R satisfying the above properties.

If $\mathcal{X}$ is the trivial family, i.e. $\mathcal{X} = C \times S$ and $T = T' = S$, then there is a canonical choice for $\mathcal{E}'$: $\mathcal{E}_0 = p_C^*(E)$ (where $p_C : C \times S \to C$ is the projection). We can then define

$$\omega_{\mathcal{E}} \;=\; \omega_{\mathcal{E},\mathcal{E}_0} : TS_{s_0} \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_C}(E,E) \ ,$$

This is the usual Kodaïra-Spencer morphism.

2.7.4. *Other equivalent definition* – It uses the properties of coherent sheaves on primitive double curves (cf. 3). We assume for simplicity that $T = T' = Z$. Let $\mathfrak{m}_{z_0} \subset \mathcal{O}_{Z,z_0}$ be the maximal ideal, and

$$Y \;=\; C^{(2)} \;=\; \pi^{-1}(\mathrm{spec}(\mathcal{O}_{S,s_0}/\mathfrak{m}_{s_0}^2)) \;=\; \beta^{-1}(\mathrm{spec}(\mathcal{O}_{Z,z_0}/\mathfrak{m}_{z_0}^2))$$

the second infinitesimal neighbourhood of C in $\mathcal{X}$ or $\beta^*(\mathcal{X})$, which is a primitive double curve with associated smooth curve C and associated line bundle

$$\mathcal{I}_C/\mathcal{I}_C^2 \;\simeq TS^*_{s_0} \otimes \mathcal{O}_C \;\simeq\; \mathcal{O}_C \ .$$

(the isomorphism $\mathcal{I}_C/\mathcal{I}_C^2 \simeq \mathcal{O}_C$ depends on the choice of a generator of $T_{s_0}S$). We have an exact sequence

$$0 \longrightarrow E \otimes TZ^*_{z_0} \longrightarrow \mathcal{E}_{|Y} \longrightarrow E \longrightarrow 0$$

(given by the canonical filtration of $\mathcal{E}_{|Y}$). Let $\sigma_{\mathcal{E}} \in \mathrm{Ext}^1_{\mathcal{O}_Y}(E \otimes TZ^*_{z_0}, E)$ be the element associated to this exact sequence. The associated morphism $E \otimes TZ^*_{z_0} \to E \otimes TZ^*_{z_0}$ (cf. 3.4) is of course the identity. We define similarly $\sigma_{\mathcal{E}'} \in \mathrm{Ext}^1_{\mathcal{O}_Y}(E \otimes TZ^*_{z_0}, E)$. Then we have

$$\omega_{\mathcal{E},\mathcal{E}'} \;=\; (\sigma_{\mathcal{E}} - \sigma_{\mathcal{E}'}) \circ (T\beta_{z_0})^{-1} \ .$$

2.7.5. *Properties of the Kodaïra-Spencer map* – Let W, W' be smooth curves, $w_0 \in W$, $w_0' \in W'$, $\lambda : W \to T$, $\lambda' : W' \to T'$ morphisms such that $\lambda(w_0) = t_0$ and $\lambda'(w_0') = t_0'$, and such that the tangent maps $T\lambda_{w_0}$ and $T\lambda_{w_0'}$ are isomorphisms. Then we have

$$\omega_{\lambda^\#(\mathcal{E}),\lambda'^\#(\mathcal{E}')} \;=\; \omega_{\mathcal{E},\mathcal{E}'} \ .$$

2.7.6. Proposition: *Let u be a non zero element of TS_{s_0}. Let $\sigma \in \mathrm{Ext}^1_{\mathcal{O}_C}(E,E)$. Then there exists a smooth curve $\mathbf{T}$, $\mathbf{t}_0 \in T$, a morphism $\boldsymbol{\alpha} : \mathbf{T} \to S$ such that $\boldsymbol{\alpha}(\mathbf{t}_0) = s_0$, and a vector bundle $\boldsymbol{\mathcal{E}}$ on $\boldsymbol{\alpha}^*(\mathcal{X})$ such that $\boldsymbol{\mathcal{E}}_{\mathbf{t}_0} \simeq E$ and that*

$$\omega_{\boldsymbol{\mathcal{E}},\mathcal{E}}(u) \;=\; \sigma \ .$$

Proof. It suffices to take appropriate curves in R. □

3. Coherent sheaves on primitive double curves

3.1. Primitive double curves

(cf. [1], [2], [5], [6], [7], [8], [9], [13]).

Let C be a smooth connected projective curve. A *multiple curve with support* C is a Cohen-Macaulay scheme Y such that $Y_{red} = C$.

Let n be the smallest integer such that $Y \subset C^{(n-1)}$, $C^{(k-1)}$ being the k-th infinitesimal neighbourhood of C, i.e. $\mathcal{I}_{C^{(k-1)}} = \mathcal{I}_C^k$. We have a filtration $C = C_1 \subset C_2 \subset \cdots \subset C_n = Y$ where C_i is the biggest Cohen-Macaulay subscheme contained in $Y \cap C^{(i-1)}$. We call n the *multiplicity* of Y.

We say that Y is *primitive* if, for every closed point x of C, there exists a smooth surface S, containing a neighbourhood of x in Y as a locally closed subvariety. In this case, $L = \mathcal{I}_C/\mathcal{I}_{C_2}$ is a line bundle on C and we have $\mathcal{I}_{C_i} = \mathcal{I}_C^i$, $\mathcal{I}_{C_i}/\mathcal{I}_{C_{i+1}} = L^i$ for $1 \leq i < n$. We call L the line bundle on C *associated* to Y. Let $P \in C$. Then there exist elements z, x of $m_{S,P}$ (the maximal ideal of $\mathcal{O}_{S,P}$) whose images in $m_{S,P}/m_{S,P}^2$ form a basis, and such that for $1 \leq i < n$ we have $\mathcal{I}_{C_i,P} = (z^i)$. The image of x in $\mathcal{O}_{C,P}$ is then a generator of the maximal ideal.

The simplest case is when Y is contained in a smooth surface S. Suppose that Y has multiplicity n. Let $P \in C$ and $f \in \mathcal{O}_{S,P}$ a local equation of C. Then we have $\mathcal{I}_{C_i,P} = (f^i)$ for $1 < j \leq n$, in particular $I_{Y,P} = (f^n)$, and $L = \mathcal{O}_C(-C)$.

For any $L \in \mathrm{Pic}(C)$, the *trivial primitive curve* of multiplicity n, with induced smooth curve C and associated line bundle L on C is the n-th infinitesimal neighbourhood of C, embedded by the zero section in the dual bundle L^*, seen as a surface.

In this paper we are interested only in the case $n = 2$. Primitive multiple curves of multiplicity 2 are called *primitive double curves* or *ribbons*.

3.2. Canonical filtrations and invariants of coherent sheaves

Let Y be a ribbon, $C = Y_{\text{red}}$, and L the associated line bundle on C. Let $P \in C$ be a closed point, $z \in \mathcal{O}_{Y,P}$ an equation of C and M a $\mathcal{O}_{Y,P}$-module of finite type. Let $\mathcal{E}$ be a coherent sheaf on Y. We denote by $\tau_{\mathcal{E}} : \mathcal{E} \otimes L \to \mathcal{E}$ the canonical morphism.

3.2.1. *First canonical filtration* – The *first canonical filtration of* M is

$$\{0\} \subset G_1(M) \subset M$$

where $G_1(M)$ is the kernel of the canonical surjective morphism $M \to M \otimes_{\mathcal{O}_{Y,P}} \mathcal{O}_{C,P}$. So we have $G_1(M) = zM$. We will note $G_0(M) = M/G_1(M)$. The direct sum

$$\mathrm{Gr}(M) \ = \ G_1(M) \oplus G_0(M)$$

is a $\mathcal{O}_{C,P}$-module. The following properties are obvious:

– we have $G_1(M) = \{0\}$ if and only if M is a $\mathcal{O}_{C,P}$-module,

– a morphism of $\mathcal{O}_{Y,P}$-modules sends the first canonical filtration of the first module to the one of the second.

We define similarly the *first canonical filtration of* $\mathcal{E}$:

$$0 \subset G_1(\mathcal{E}) \subset \mathcal{E}$$

where $G_1(\mathcal{E})$ is the kernel of the canonical surjective morphisms $\mathcal{E} \to \mathcal{E}_{|C}$. We will note $G_0(\mathcal{E}) = \mathcal{E}/G_1(\mathcal{E}) = \mathcal{E}_{|C}$. We will also use the notation $\mathcal{E}_1 = G_1(\mathcal{E})$. Let

$$\mathrm{Gr}(\mathcal{E}) \ = \ G_1(\mathcal{E}) \oplus G_0(\mathcal{E}) \ .$$

It is a $\mathcal{O}_C$-module. The following properties are obvious:
- we have $G_1(\mathcal{E}) = \mathcal{I}_C\mathcal{E}$,
- we have $G_1(\mathcal{E}) = 0$ if and only if $\mathcal{E}$ is a sheaf on C,
- every morphism of coherent sheaves on Y sends the first canonical filtration of the first to that of the second.

3.2.2. *Second canonical filtration* – The *second canonical filtration of* M is

$$0 \subset G^{(1)}(M) \subset M \ ,$$

with $G^{(1)}(M) = \big\{u \in M; z^i u = 0\big\}$. We have $G_1(M) \subset G^{(1)}(M)$. We will note $G^{(2)}(M) = M/G^{(1)}(M)$. The direct sum

$$\mathrm{Gr}_2(M) \ = \ G^{(1)}(M) \oplus G^{(2)}(M)$$

is a $\mathcal{O}_{C,P}$-module. Every morphism of $\mathcal{O}_{Y,P}$-modules sends the second canonical filtration of the first module to that of the second.

We define similarly the *second canonical filtration of* $\mathcal{E}$:

$$0 \subset G^{(1)}(\mathcal{E}) \subset \ \mathcal{E} \ , \qquad G^{(2)}(\mathcal{E}) \ = \ \mathcal{E}/G^{(1)}(\mathcal{E}) \ .$$

We will also use the notation $\mathcal{E}^{(1)} = G^{(1)}(\mathcal{E})$. The direct sum

$$\mathrm{Gr}_2(\mathcal{E}) \ = \ G^{(1)}(\mathcal{E}) \oplus G^{(2)}(\mathcal{E})$$

is a $\mathcal{O}_C$-module. We have $G_1(\mathcal{E}) \subset G^{(1)}(\mathcal{E})$, and every morphism of coherent sheaves on Y sends the second canonical filtration of the first to that of the second.

3.2.3. *Relations between the two filtrations* - we have a canonical isomorphism

$$G_1(\mathcal{E}) \ \simeq \ G^{(2)}(\mathcal{E}) \otimes L \ .$$

The canonical morphism $\mathcal{E} \otimes \mathbb{L} \to \mathcal{E}$ induces two morphisms of sheaves on C:

$$\lambda_{\mathcal{E}} =: G^{(2)}(\mathcal{E}) \otimes L \longrightarrow G^{(1)}(\mathcal{E}) \ , \qquad \mu_{\mathcal{E}} =: G_0(\mathcal{E}) \otimes L \longrightarrow G_1(\mathcal{E}) \ .$$

The first morphism of sheaves is injective, the second is surjective, and we have

$$\ker(\mu_{\mathcal{E}}) \ \simeq \ \mathrm{coker}(\lambda_{\mathcal{E}}) \otimes L \ .$$

3.2.4. *Characterisation of the canonical filtrations* – (cf. lemma 3.4.3). Let

$$0 \subset E \subset \mathcal{E}$$

be a filtration such that E is a sheaf on C. Then we have $E \subset G^{(1)}(\mathcal{E})$, and $F = \mathcal{E}/E$ is a sheaf on C if and only if $G_1(\mathcal{E}) \subset E$. Suppose that $G_1(\mathcal{E}) \subset E$. Then the canonical morphism $\tau_{\mathcal{E}}$ induces $\phi : F \otimes L \to E$, and we have

– $E = G_1(\mathcal{E})$ if and only if ϕ is surjective.
– $E = G^{(1)}(\mathcal{E})$ if and only ϕ is injective.

3.2.5. *Invariants* – The integer $R(M) = \mathrm{rk}(\mathrm{Gr}(M))$ is called the *generalised rank* of M.

The integer $R(\mathcal{E}) = \mathrm{rk}(\mathrm{Gr}(\mathcal{E}))$ is called the *generalised rank* of $\mathcal{E}$. So we have $R(\mathcal{E}) = R(\mathcal{E}_P)$ for every P in a nonempty open subset of C.

The integer $\mathrm{Deg}(\mathcal{E}) = \deg(\mathrm{Gr}(\mathcal{E}))$ is called the *generalised degree* of $\mathcal{E}$. The generalised rank and degree of sheaves are deformation invariants.

If $R(\mathcal{E}) > 0$, let $\mu(\mathcal{E}) = \mathrm{Deg}(\mathcal{E})/R(\mathcal{E})$. We call this rational number the *slope* of $\mathcal{E}$.

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of $\mathcal{O}_{Y,P}$-modules of finite type. Then we have $R(M) = R(M') + R(M'')$.

Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be an exact sequence of coherent sheaves on Y. Then we have $R(\mathcal{F}) = R(\mathcal{E}) + R(\mathcal{G})$, $\mathrm{Deg}(\mathcal{F}) = \mathrm{Deg}(\mathcal{E}) + \mathrm{Deg}(\mathcal{G})$.

If $\mathcal{E}$ is a coherent sheaf on Y, we have

$$\chi(\mathcal{E}) \;=\; \mathrm{Deg}(\mathcal{E}) + R(\mathcal{E})(1-g) \;,$$

where g is the genus of C (Riemann-Roch theorem). It follows that if $\mathcal{O}_Y(1)$ is an ample line bundle on Y, $\mathcal{O}_C(1) = \mathcal{O}_Y(1)_{|C}$ and $\delta = \deg(\mathcal{O}_C(1))$, the Hilbert polynomial of $\mathcal{E}$ with respect to $\mathcal{O}_Y(1)$ is given by

$$P_{\mathcal{E}}(m) \;=\; R(\mathcal{E})\delta m + \mathrm{Deg}(\mathcal{E}) + R(\mathcal{E})(1-g) \;.$$

3.3. Quasi locally free sheaves – Reflexive sheaves

3.3.1. *Torsion* – Let M be a $\mathcal{O}_{Y,P}$-module of finite type. Le *torsion sub-module* $T(M)$ *of* M consists of the elements u of M such that there exists an integer $p > 0$ such that $x^p u = 0$ (recall that $x \in \mathcal{O}_{Y,P}$ is such that its image in $\mathcal{O}_{C,P}$ is a generator of the maximal ideal). We say that M is *torsion free* if $T(M) = \{0\}$.

Let $\mathcal{E}$ be a coherent sheaf on Y. The *torsion subsheaf* $T(\mathcal{E})$ *of* $\mathcal{E}$ is the maximal subsheaf of $\mathcal{E}$ with finite support. For every closed point P of C we have $T(\mathcal{E})_P = T(\mathcal{E}_P)$. We say that $\mathcal{E}$ is a *torsion sheaf* if $\mathcal{E} = T(\mathcal{E})$, or equivalently, if its support is finite. We say that $\mathcal{E}$ is *torsion free* if $T(\mathcal{E}) = 0$.

The following conditions are equivalent:

(i) $\mathcal{E}$ is torsion free.
(ii) $G^{(1)}(\mathcal{E})$ is locally free.

Moreover, if $\mathcal{E}$ is torsion free, $G^{(2)}(\mathcal{E})$ is also locally free.

3.3.2. *Quasi locally free sheaves* – Let M be a $\mathcal{O}_{Y,P}$-module of finite type. We say that M is *quasi free* if there exist integers $m_1, m_2 \geq 0$ such that $M \simeq m_1\mathcal{O}_{C,P} \oplus m_2\mathcal{O}_{Y,P}$. These integers are uniquely determined. In this case we say that M is *of type* (m_1, m_2). We have $R(M) = m_1 + 2m_2$.

Let $\mathcal{E}$ be a coherent sheaf Y. We say that $\mathcal{E}$ is *quasi locally free* at a point P of C if there exists a neighbourhood U of P and integers $m_1, m_2 \geq 0$ such that for every $Q \in U$, $\mathcal{E}_Q$ is quasi free of type m_1, m_2. The integers m_1, m_2 are uniquely determined and depend only of $\mathcal{E}$, and (m_1, m_2) is called the *type of* $\mathcal{E}$.

We say that $\mathcal{E}$ is *quasi locally free* if it is quasi locally free at every point of C.

The following conditions are equivalent:

(i) $\mathcal{E}_P$ is quasi locally free at P.
(ii) $G_0(\mathcal{E})$ and $G_1(\mathcal{E})$ are free $\mathcal{O}_{C,P}$-modules.
(iii) $\operatorname{coker}(\lambda_{\mathcal{E}})$ and $G_0(\mathcal{E})$ are free $\mathcal{O}_{C,P}$-modules.

The following conditions are equivalent:

(i) $\mathcal{E}$ is quasi locally free.
(ii) $G_0(\mathcal{E})$ and $G_1(\mathcal{E})$ are locally free on C.
(iii) $\operatorname{coker}(\lambda_{\mathcal{E}})$ and $G_0(\mathcal{E})$ are locally free on C.

Assume that $\mathcal{E}$ is torsion free. Then $\lambda_{\mathcal{E}}$ is a morphism of vector bundles, which is an injective morphism of sheaves, and $\mathcal{E}$ is quasi locally free if and only if $\lambda_{\mathcal{E}}$ is an injective morphism of vector bundles.

3.3.3. *Reflexive sheaves* – Let $P \in C$ and M be a $\mathcal{O}_{Y,P}$-module of finite type. Let $M^\vee$ be the *dual* of M: $M^\vee = \operatorname{Hom}(M, \mathcal{O}_{Y,P})$. If N is $\mathcal{O}_{C,P}$-module of finite type, we will denote N^* the dual of N: $N^* = \operatorname{Hom}(N, \mathcal{O}_{C,P})$.

Let $\mathcal{E}$ be a coherent sheaf on Y. Let $\mathcal{E}^\vee$ be the *dual* of $\mathcal{E}$: $\mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathcal{O}_Y)$. If E is a coherent sheaf on C, we will denote by E^* the dual of E: $E^* = \mathcal{H}om(E, \mathcal{O}_C)$. We have $E^\vee = E^* \otimes L$.

We have $G^{(1)}(\mathcal{E}^\vee) = G_0(\mathcal{E})^* \otimes L$. If $\mathcal{E}$ is quasi locally free we have

$$G_1(\mathcal{E}^\vee) \ = \ G_1(\mathcal{E})^* \otimes L^2 \ , \qquad G^{(2)}(\mathcal{E}^\vee) \ = \ G_1(\mathcal{E})^* \otimes L \ .$$

The following properties are equivalent (cf. [5], [7]):

(i) $\mathcal{E}$ is reflexive.
(ii) $\mathcal{E}$ is torsion free.
(iii) We have $\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{E}, \mathcal{O}_Y) = 0$.

Moreover we have in this case $\mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{E}, \mathcal{O}_Y) = 0$ for every $i \geq 1$.

It follows that if $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is an exact sequence of coherent torsion free sheaves on Y, the dual sequence $0 \to \mathcal{G}^\vee \to \mathcal{F}^\vee \to \mathcal{E}^\vee \to 0$ is also exact.

3.3.4. *Moduli spaces of semi-stable reflexive sheaves* – A reflexive sheaf $\mathcal{E}$ is semi-stable (resp. stable) (as per M. Maruyama [21] or C. Simpson [24]) if for every proper subsheaf $\mathcal{F} \subset \mathcal{E}$ we have

$$\mu(\mathcal{F}) \ \leq \ \mu(\mathcal{E}) \quad (\text{resp.} \quad <) \ .$$

As for smooth curves, the definition of semi-stability does not depend on the choice of an ample line bundle on Y. If $R > 0$ and D are integers, we will denote by $\mathcal{M}(R, D)$ the moduli space of semi-stable sheaves on Y of rank R and degree D, and by $\mathcal{M}_s(R, D)$ the open subset corresponding to stable sheaves.

3.3.5. *Quasi locally free sheaves of rigid type* – (cf. [7], [8]) A quasi locally free sheaf $\mathcal{E}$ on Y is called *of rigid type* if it is locally free, or locally isomorphic to $a\mathcal{O}_Y \oplus \mathcal{O}_C$, for some integer $a \geq 0$. In the latter case, we have $\mathrm{rk}(\mathcal{E}_{|C}) = a+1$ and $\mathrm{rk}(\mathcal{E}_1) = a$. The deformations of quasi locally free sheaves of rigid type are quasi locally free sheaves of rigid type, and a, $\deg(\mathcal{E}_1)$ and $\deg(\mathcal{E}_{|C})$ are invariant by deformation.

Let $R = 2a+1$ and d_0, d_1, D integers such that $D = d_0 + d_1$. The stable sheaves $\mathcal{E}$ of rigid type, of rank R and such that $\deg(\mathcal{E}_1 \otimes L^*) = d_1$ and $\deg(\mathcal{E}_{|C}) = d_0$ form an open subset $\mathcal{N}(R, d_0, d_1)$ of an irreducible component of $\mathcal{M}_s(R, D)$. If $\mathcal{N}(R, d_0, d_1)$ is non empty, we have

$$\dim(\mathcal{N}(R, d_0, d_1)) \; = \; 1 - a(a+1)\deg(L) + (g-1)(2a^2 + 2a + 1) \; .$$

In general $\mathcal{N}(R, d_0, d_1)$ is not reduced. The reduced subvariety $\mathcal{N}(R, d_0, d_1)_{\mathrm{red}}$ is smooth and its tangent sheaf at the point corresponding to a sheaf $\mathcal{F}$ is canonically isomorphic to $H^1(\mathcal{E}nd(\mathcal{F}))$. We have also canonical isomorphisms

$$T\mathcal{N}(R, d_0, d_1)_{\mathcal{F}} / T\mathcal{N}(R, d_0, d_1)_{\mathrm{red},\mathcal{F}} \; \simeq \; H^0(\mathcal{E}xt^1(\mathcal{F}, \mathcal{F})) \; \simeq \; H^0(L^*)$$

(for the last isomorphism see [5], cor. 6.2.2). It is known that $\mathcal{N}(R, d_0, d_1)$ is non empty if

$$\frac{\epsilon}{a+1} \; < \; \frac{\delta}{a} \; < \; \frac{\epsilon - \deg(L)}{a+1} \; .$$

3.4. Properties and construction of coherent sheaves

If E is a vector bundle on C, there exists a vector bundle $\mathbb{E}$ of Y such that $\mathbb{E}_{|Y} \simeq E$ ([5], th. 3.1.1). In particular there exists a line bundle $\mathbb{L}$ on Y such that $\mathbb{L}_{|C} \simeq L$. It follows that we have a locally free resolution of E :

$$\cdots \mathbb{E} \otimes \mathbb{L}^3 \longrightarrow \mathbb{E} \otimes \mathbb{L}^2 \longrightarrow \mathbb{E} \otimes \mathbb{L} \longrightarrow \mathbb{E} \longrightarrow E \longrightarrow 0 \; .$$

The following results follows easily:

3.4.1. Lemma: *Let F be a coherent sheaf on C and G a vector bundle on C. Then we have, for every integer $i \geq 0$*

$$\mathrm{Tor}^i_{\mathcal{O}_Y}(F, \mathcal{O}_C) \; \simeq \; F \otimes L^i \; , \qquad \mathcal{E}xt^1_{\mathcal{O}_Y}(G, F) \; \simeq \; \mathcal{H}om(G \otimes L^i, F) \; .$$

It follows that we have a canonical exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^1_{\mathcal{O}_C}(G, F) \arrow[r, "i"] \arrow[d, equal] & \mathrm{Ext}^1_{\mathcal{O}_Y}(G, F) \arrow[r] & \mathrm{Hom}(G \otimes L, F) \arrow[r] \arrow[d, equal] & 0 \; . \\ & H^1(\mathcal{H}om(G, F)) & & H^0(\mathcal{E}xt^1_{\mathcal{O}_Y}(G, F)) & \end{tikzcd}$$

3.4.2. We can give an explicit construction of the morphism i in the preceding exact sequence, using Čech cohomology: let (U_i) an open cover of C and $\alpha \in \mathrm{Ext}^1_{\mathcal{O}_C}(G, F)$, represented by a cocycle (α_{ij}), where $\alpha_{ij} \in \mathrm{Hom}(G_{|U_{ij}}, F_{|U_{ij}})$. Let $\sigma \in \mathrm{Ext}^1_{\mathcal{O}_Y}(G, F)$, corresponding to an extension

$$0 \longrightarrow F \longrightarrow \mathcal{E} \longrightarrow G \longrightarrow 0 \; .$$

We can view α_{ij} as an endomorphism of $\mathcal{E}_{|U_{ij}}$. Then $\sigma + i(\alpha)$ corresponds to the extension

$$0 \longrightarrow F \longrightarrow \mathcal{E}' \longrightarrow G \longrightarrow 0 \ ,$$

where $\mathcal{E}'$ is the sheaf obtained by gluing the sheaves $\mathcal{E}_{|U_{ij}}$ using the automorphisms $I + \alpha_{ij} \in \mathrm{Aut}(\mathcal{E}_{|U_{ij}})$.

3.4.3. Lemma: *Suppose that F is locally free. Let $\sigma \in \mathrm{Ext}^1_{\mathcal{O}_Y}(G, F)$, corresponding to an extension*

$$0 \longrightarrow F \longrightarrow \mathcal{E} \longrightarrow G \longrightarrow 0 \ .$$

Then

1 − *We have $G = \mathcal{E}_{|C}$ and $F = \mathcal{E}_1$ if and only if $\theta(\sigma)$ is surjective. In this case $\mathcal{E}$ is quasi locally free.*

2 − *We have $G = \mathcal{E}^{(2)}$ and $F = \mathcal{E}^{(1)}$ if and only $\theta(\sigma)$ is injective. In this case $\mathcal{E}$ is quasi locally free if and only if $\theta(\sigma)$ is injective as a morphism of vector bundles.*

Proof. **1-** follows from the factorisation of the canonical morphism $\mathcal{E}_{|C} \otimes L \to \mathcal{E}_1$

$$\begin{tikzcd} \mathcal{E}_{|C} \otimes L \arrow[r, two heads] & G \otimes L \arrow[r, two heads] \arrow[rr, bend left, "\theta(\sigma)"] & \mathcal{E}_1 \arrow[r, hook] & F, \end{tikzcd}$$

and **2-** from the factorisation of $\theta(\sigma)$

$$\begin{tikzcd} G \otimes L \arrow[r, two heads] & \mathcal{E}/\mathcal{E}^{(1)} \otimes L \arrow[r, equal] & \mathcal{E}_1 \arrow[r, hook] & F. \end{tikzcd}$$

□

3.4.4. Lemma: *Let $\mathbb{E}$ be a vector bundle on Y, E a coherent sheaf on C, $\phi : \mathbb{E} \to E$ a surjective morphism and $\mathcal{N} = \ker(\phi)$. Then we have an exact sequence*

$$0 \longrightarrow E \otimes L \longrightarrow \mathcal{N}_{|C} \longrightarrow \mathbb{E}_{|C} \longrightarrow E \longrightarrow 0$$

and a canonical isomorphism

$$\mathcal{N}_{|C}/(E \otimes L) \ \simeq \ \mathcal{N}^{(2)} \ .$$

Proof. The exact sequence follows from lemma 3.4.1. Consider the exact sequence

$$0 \longrightarrow \mathcal{N} \xrightarrow{\ i\ } \mathbb{E} \xrightarrow{\ p\ } E \longrightarrow 0 \ .$$

We have a canonical surjective morphism $\phi : \mathcal{N}_{|C} = \mathcal{N}/\mathcal{N}_1 \to \mathcal{N}^{(2)} = \mathcal{N}/\mathcal{N}^{(1)}$. Let $x \in C$ and $z \in \mathcal{O}_{Y,x}$ and equation of C. Let $u \in \mathcal{N}_{|C,x}$, and $\overline{u} \in \mathcal{N}_x$ over u. Then we have

$$\begin{aligned} i_{|C,x}(u) = 0 \quad & \Longleftrightarrow \quad i_x(\overline{u}) \text{ is a multiple of } z \\ & \Longleftrightarrow \quad z.i_x(\overline{u}) = i_x(z\overline{u}) = 0, \quad \text{because } \mathbb{E} \text{ is locally free} \\ & \Longleftrightarrow \quad z\overline{u} = 0 \\ & \Longleftrightarrow \quad \phi_x(u) = 0. \end{aligned}$$

This proves the isomorphism of lemma 3.4.4. □

3.4.5. *Punctual thickening of vector bundles on* C. Let $D = \sum_i m_i P_i$ be a divisor on C, where the P_i are distinct points and the m_i positive integers. Let $x_i \in \mathcal{O}_{C_2,P_i}$ be over a generator of the maximal ideal of $\mathcal{O}_{C,P_i}$, and $z_i \in \mathcal{O}_{C_2,P_i}$ a local equation of C. Let $\mathcal{I}[D]$ be the ideal sheaf on C_2 defined by: $\mathcal{I}[D]_P = L_P$ for P distinct of all the P_i, and $\mathcal{I}[D]_{P_i} = (x_i^{m_i} z_i) = x_i^{m_i}.L_{P_i}$. It depends only on D.

Let E be a vector bundle on C. According to [5], thm. 3.1.1, E can be extended to a vector bundle $\mathbb{E}$ on C_2. Let

$$E[D] \ = \ \mathbb{E}/(\mathbb{E} \otimes \mathcal{I}[D]) \ .$$

We have a canonical quotient $E[D] \to E$ which is an isomorphism ouside the P_i. It is easy to see that the sheaf $E[D]$ depends only of E and D, i.e. if $\mathbb{E}'$ is another vector bundle on C_2 extending E, then there exists an isomorphism (not unique)

$$\mathbb{E}/(\mathbb{E} \otimes \mathcal{I}[D]) \ \simeq \ \mathbb{E}'/(\mathbb{E}' \otimes \mathcal{I}[D])$$

inducing the identity on E. We have $T(E[D]) = \sum T_i$, where T_i is the skyscraper sheaf concentrated at P_i with fibre $E_{P_i}/x_i^{m_i} E_{P_i}$ at P_i (where E_{P_i} the fibre of the sheaf E at this point). We have $E[D]/T(E[D]) = E$.

4. Coherent sheaves on primitive reducible curves

A *primitive reducible curve* is an algebraic curve X such that

– X is connected.
– The irreducible components of X are smooth projective curves.
– Any three irreducible components of X have no common point, and any two components are transverse.

Let X be a primitive reducible curve, $\mathbf{C}$ the set of components of X, and I the set of intersection points of the components of X.

4.1. reflexive sheaves

The following result is an easy consequence of [23], VII, VIII:

4.1.1. Proposition: *Let $\mathcal{E}$ be a coherent sheaf on X. Then the following conditions are equivalent:*

(i) $\mathcal{E}$ *is pure of dimension 1.*
(ii) $\mathcal{E}$ *is of depth 1.*
(iii) $\mathcal{E}$ *is locally free at every point of X belonging to only one component, and if a point x belongs to two components D_1, D_2, then there exist integers a, a_1, $a_2 \geq 0$ and an isomorphism*

$$\mathcal{E}_x \ \simeq \ a\mathcal{O}_{X,x} \oplus a_1\mathcal{O}_{D_1,x} \oplus a_2\mathcal{O}_{D_2,x} \ .$$

(iv) *$\mathcal{E}$ is torsion free, i.e. for every $x \in X$, every element of $\mathcal{O}_{X,x}$ which is not a zero divisor in $\mathcal{O}_{X,x}$ is not a zero divisor in $\mathcal{E}_x$.*

(v) *$\mathcal{E}$ is reflexive.*

In particular, every torsion free coherent sheaf on X is reflexive.

4.1.2. Definition: *Let $\mathcal{E}$ be a torsion free sheaf on X. For every $D \in \mathbf{C}$, let $\mathcal{E}_D = \mathcal{E}_{|D}/T$ (where T is the torsion subsheaf of $\mathcal{E}_{|D}$). The sequence of pairs $\tau(\mathcal{E}) = (\mathrm{rk}(\mathcal{E}_D), \deg(\mathcal{E}_D))_{D\in\mathbf{C}}$ is called the* type *of $\mathcal{E}$.*

4.2. Structure and properties of torsion free sheaves

4.2.1. *Local structure at intersection points* – Let $x \in X$, belonging to two irreducible components D_1, D_2. The ring $\mathcal{O}_{X,x}$ can be identified with the ring of pairs $(\phi_1, \phi_2) \in \mathcal{O}_{D_1,x} \times \mathcal{O}_{D_2,x}$ such that $\phi_1(x) = \phi_2(x)$. For $i = 1, 2$, let $t_i \in \mathcal{O}_{D_i,x}$ be a generator of the maximal ideal. We will also denote by t_1 (resp. t_2) the element $(t_1, 0)$ (resp. $(0, t_2)$) of $\mathcal{O}_{X,x}$. Let M be the $\mathcal{O}_{X,x}$-module

$$M \ = \ (V \otimes \mathcal{O}_{X,x}) \oplus (W_1 \otimes \mathcal{O}_{D_1,x}) \oplus (W_2 \otimes \mathcal{O}_{D_2,x})$$

(where V, W_1, W_2 are finite dimensional $\mathbb{C}$-vector spaces). Then $\mathrm{End}(M)$ consists of matrices

$$\begin{pmatrix} \alpha & \lambda_1 t_1 & \lambda_2 t_2 \\ \beta_1 & \mu_1 & 0 \\ \beta_2 & 0 & \mu_2 \end{pmatrix}$$

with $\alpha \in \mathrm{End}(V)$, $\lambda_i \in L(W_i, V) \otimes \mathcal{O}_{D_i,x}$, $\beta_i \in L(V, W_i) \otimes \mathcal{O}_{D_i,x}$, $\mu_i \in \mathrm{End}(W_i) \otimes \mathcal{O}_{D_i,x}$ for $i = 1, 2$. Such an endomorphism is invertible if and only if α, μ_1 and μ_2 are invertible, if and only if their images in $\mathrm{End}(V)$, $\mathrm{End}(W_1)$, $\mathrm{End}(W_2)$ respectively are invertible. It is easy to see that the sub-module

$$N \ = \ (V \otimes \mathfrak{m}_x) \oplus (W_1 \otimes \mathcal{O}_{D_1,x}) \oplus (W_2 \otimes \mathcal{O}_{D_2,x})$$

is invariant by all the endomorphisms of M. The module M can be identified with the $\mathcal{O}_{X,x}$-module of

$$(u_1, u_2, v_1, v_2) \ \in \ (V \otimes \mathcal{O}_{D_1,x}) \oplus (V \otimes \mathcal{O}_{D_2,x}) \oplus (W_1 \otimes \mathcal{O}_{D_1,x}) \oplus (W_2 \otimes \mathcal{O}_{D_2,x})$$

such that the images of u_1, u_2 in V are the same.

It follows easily that if U is a neighbourhood of x which contains only one multiple point and meets only the components through x, then the reflexive sheaves $\mathcal{E}$ on U are obtained in the following way: take two vector bundles, E_1 on $D_1 \cap U$, E_2 on $D_2 \cap U$ respectively, a $\mathbb{C}$-vector space W and surjective linear maps

$$f_i : E_{i,x} \longrightarrow W \ , \qquad i = 1, 2 \ .$$

Then $\mathcal{E} = E_1$ on $D_1 \backslash \{x\}$, $\mathcal{E} = E_2$ on $D_2 \backslash \{x\}$, and

$$\mathcal{E}_x \ = \ \{(\phi_1, \phi_2) \in E_1(x) \times E_2(x) \ ; \ f_1(\phi_1(x)) = f_2(\phi_2(x))\} \ ,$$

(where $E_i(x)$ is the module of sections of E_i defined in a neighbourhood on x). We have

$$E_i \ = \ \mathcal{E}_{|D_i \cap U}/T_i \ ,$$

where T_i is the torsion subsheaf of $\mathcal{E}_{|D_i\cap U}$. We have

$$\mathcal{E}_x \ \simeq \ (W\otimes\mathcal{O}_{X,x})\oplus(\ker(f_1)\otimes\mathcal{O}_{D_1,x})\oplus(\ker(f_2)\otimes\mathcal{O}_{D_2,x}) \ .$$

The sheaf $\mathcal{E}$ is locally free if and only if f_1 and f_2 are isomorphisms.

We have an exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow E_1\oplus E_2 \xrightarrow{p} \overline{W} \longrightarrow 0,$$

where $\overline{W}$ denotes the skyscraper sheaf concentrated on x with fibre W at x, and p is given at x by

$$(f_1,-f_2) : E_{1,x}\oplus E_{2,x} \longrightarrow W \ .$$

4.2.2. *Another way of constructing sheaves* – We keep the notations of 4.2.1. Let $H\subset E_{1,x}\times E_{2,x}$ be a linear subspace. We define a reflexive sheaf $\mathcal{E}$ on U by:

– $\mathcal{E}=E_1$ on $D_1\backslash\{x\}$, and $\mathcal{E}=E_2$ on $D_2\backslash\{x\}$,
– $\mathcal{E}_x \ = \ \{(\phi_1,\phi_2)\in E_1(x)\times E_2(x) \ ; \ (\phi_1(x),\phi_2(x))\in H\}$.

Let $H_1\subset E_{1,x}$ and $H_2\subset E_{2,x}$ be the minimal linear subspaces such that $H\subset H_1\times H_2$, and $W=(H_1\times H_2)/H$. Let $\tau_1 : H_1=H_1\times\{0\}\to W$, $\tau_2 : H_2=\{0\}\times H_2\to W$ be the restrictions of the quotient map. Let $E'_1=(E_1)_{H_1}$, $E'_2=(E_2)_{H_2}$ (cf. 2.4). We have then canonical maps $\lambda_1 : E'_{1,x}\to H_1$, $\lambda_2 : E'_{2,x}\to H_2$. Let

$$f_1 : \tau_1\circ\lambda_1 : E'_{1,x}\to W \ , \quad f_2 : \tau_2\circ\lambda_2 : E'_{2,x}\to W \ .$$

Then the two linear maps f_1, f_2 are surjective, and $\mathcal{E}$ is the reflexive sheaf constructed from E'_1, E'_2, f_1, f_2 by the method of 4.2.1.

4.2.3. *Example: sheaves of homomorphisms* – Let F_i, G_i, $i=1,2$, be vector bundles on $D_i\cap U$. Let V, W be vector spaces, and

$$f_i : F_{i,x} \longrightarrow V \ , \quad g_i : G_{i,x} \longrightarrow W$$

surjective maps. Let $\mathcal{F}$ (resp. $\mathcal{G}$) be the reflexive sheaf on U defined by F_1, F_2, f_1, f_2 (resp. G_1, G_2, g_1, g_2). We will describe the sheaf $\mathcal{H}om(\mathcal{F},\mathcal{G})$. Of course we have $\mathcal{H}om(\mathcal{F},\mathcal{G})=\mathcal{H}om(F_i,G_i)$ on $D_i\backslash\{x\}$. From the definitions of $\mathcal{F}$, $\mathcal{G}$ it is easy to see that

$$\mathcal{H}om(\mathcal{F},\mathcal{G})_x \ = \ \{(\mu_1,\mu_2)\in\mathcal{H}om(F_1,G_1)(x)\times\mathcal{H}om(F_2,G_2)(x) \ ; \ (\mu_1(x),\mu_2(x))\in H\} \ ,$$

where $H\subset L(F_{1,x},G_{1,x})\times L(F_{2,x},G_{2,x})$ is the linear subspace consisting of (α_1,α_2) such that for every $(u_1,u_2)\in F_{1,x}\times F_{2,x}$ such that $f_1(u_1)=f_2(u_2)$, we have also $g_1(\alpha_1(u_1))=g_2(\alpha_2(u_2))$. If follows that $\mathcal{H}om(\mathcal{F},\mathcal{G})$ is reflexive.

We can suppose that we have decompositions into direct sums $F_i=(V\oplus A_i)\otimes\mathcal{O}_{D_i\cap U}$, $G_i=(W\oplus B_i)\otimes\mathcal{O}_{D_i\cap U}$, in such a way that f_i (resp. g_i) is the projection to V (resp. W). We can then write linear maps $\alpha_i : F_{i,x}\to G_{i,x}$ as matrices $\begin{pmatrix}\phi_i & M_i\\ N_i & P_i\end{pmatrix}$. It is then easy to see that H consists of pairs (α_1,α_2) such that $M_1=0$, $M_2=0$, $\phi_1=\phi_2$. The minimal subspaces $H_i\subset L(F_{i,x},G_{i,x})$ such that $H\subset H_1\times H_2$ are then clearly

$$H_i \ = \ \{\alpha_i\in L(F_{i,x},G_{i,x}) \ ; \ \alpha_i(\ker(f_i))\subset\ker(g_i)\} \ ,$$

and $(H_1 \times H_2)/H$ is canonically isomorphic to $L(V,W)$. The linear maps $\gamma_i : \mathcal{H}om(F_i, G_i)_{H_i,x} \longrightarrow L(V,W)$ are as follows:

$$\gamma_i : \mathcal{H}om(F_i, G_i)_{H_i,x} \twoheadrightarrow H_i \xrightarrow{\rho_i} L(V,W)$$

where ρ_i associates to $\alpha : F_{i,x} \to G_{i,x}$ the induced map $V = F_{i,x}/\ker(f_i) \to W = G_{i,x}/\ker(g_i)$. The reflexive sheaf $\mathcal{H}om(\mathcal{F},\mathcal{G})$ is then defined (using the method of 4.2.1) by the vector bundles $\mathcal{H}om(F_i, G_i)_{H_i}$ on D_i, and the linear maps γ_i.

4.2.4. Lemma: *We have, for $i = 1,2$ and integers $j \geq 0$, $k \geq 1$*

- $\mathrm{Ext}^{2j+1}_{\mathcal{O}_{X,x}}(\mathcal{O}_{D_1,x}, \mathcal{O}_{D_2,x}) \simeq \mathbb{C}$, $\mathrm{Ext}^{2k}_{\mathcal{O}_{X,x}}(\mathcal{O}_{D_1,x}, \mathcal{O}_{D_2,x}) = 0$.
- $\mathrm{Ext}^{2j+1}_{\mathcal{O}_{X,x}}(\mathcal{O}_{D_i,x}, \mathcal{O}_{D_i,x}) = 0$, $\mathrm{Ext}^{2k}_{\mathcal{O}_{X,x}}(\mathcal{O}_{D_i,x}, \mathcal{O}_{D_i,x}) \simeq \mathbb{C}$.
- $\mathrm{Tor}^{2j+1}_{\mathcal{O}_{X,x}}(\mathcal{O}_{D_i,x}, \mathcal{O}_{D_i,x}) \simeq \mathbb{C}$, $\mathrm{Tor}^{2k}_{\mathcal{O}_{X,x}}(\mathcal{O}_{D_i,x}, \mathcal{O}_{D_i,x}) = 0$.
- $\mathrm{Tor}^{2j+1}_{\mathcal{O}_{X,x}}(\mathcal{O}_{D_1,x}, \mathcal{O}_{D_2,x}) = 0$, $\mathrm{Tor}^{2k}_{\mathcal{O}_{X,x}}(\mathcal{O}_{D_1,x}, \mathcal{O}_{D_2,x}) \simeq \mathbb{C}$.

Proof. We use a locally free resolution of $\mathcal{O}_{D_1,x}$:

$$\cdots \mathcal{O}_{X,x} \xrightarrow{t_1} \mathcal{O}_{X,x} \xrightarrow{t_2} \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{D_1,x} \longrightarrow 0 \ .$$

Hence $\mathrm{Ext}^1_{\mathcal{O}_{X,x}}(\mathcal{O}_{D_1,x}, \mathcal{O}_{D_2,x})$ is isomorphic to the middle cohomology of the complex

$$\mathcal{O}_{D_2,x} \xrightarrow{t_2} \mathcal{O}_{D_2,x} \xrightarrow{t_1=0} \mathcal{O}_{D_2,x} \ .$$

The result follows immediately. The other equalities are proved in the same way. $\square$

4.2.5. *Relative version of lemma 4.2.4* – A similar proof shows that if S is an algebraic variety and $s \in S$, then we have, for $i = 1,2$ and integers $j \geq 0$, $k \geq 1$

$$\mathrm{Ext}^{2j+1}_{\mathcal{O}_{X\times S,(x,s)}}(\mathcal{O}_{D_1,x}\otimes\mathcal{O}_{S,s}, \mathcal{O}_{D_2,x}\otimes\mathcal{O}_{S,s}) \ \simeq \ \mathcal{O}_{S,s} \ , \quad \mathrm{Ext}^{2k}_{\mathcal{O}_{X\times S,(x,s)}}(\mathcal{O}_{D_1,x}\otimes\mathcal{O}_{S,s}, \mathcal{O}_{D_2,x}\otimes\mathcal{O}_{S,s}) \ = \ 0 \ ,$$

$$\mathrm{Ext}^{2j+1}_{\mathcal{O}_{X\times S,(x,s)}}(\mathcal{O}_{D_i,x}\otimes\mathcal{O}_{S,s}, \mathcal{O}_{D_i,x}\otimes\mathcal{O}_{S,s}) \ = \ 0 \ , \quad \mathrm{Ext}^{2k}_{\mathcal{O}_{X\times S,(x,s)}}(\mathcal{O}_{D_i,x}\otimes\mathcal{O}_{S,s}, \mathcal{O}_{D_i,x}\otimes\mathcal{O}_{S,s}) \ \simeq \ \mathcal{O}_{S,s} \ .$$

and the same as in lemma 4.2.4 for the Tor.

4.2.6. *Linked sheaves* – We keep the notations of 4.2.1. Let $r = \dim(W)$, $r_1 = \mathrm{rk}(E_1)$, $r_2 = \mathrm{rk}(E_2)$. Then we have $r \leq \inf(r_1, r_2)$, $\mathrm{rk}(\mathcal{E}_x) = r_1 + r_2 - r$. The sheaf $\mathcal{E}$ is called *linked at* x if $r = \inf(r_1, r_2)$, the maximal possible value (so one could say that the most unlinked sheaf is $E_1 \oplus E_2$). Suppose that $\mathcal{E}$ is linked at x. For $i = 1,2$, we have $E_i = \mathcal{E}_{|D_i}/T_i$ (where T_i is the torsion subsheaf of $\mathcal{E}_{|D_i}$). Suppose that $r_2 \geq r_1$. Then $T_2 = 0$ and $T_1 \simeq \mathbb{C}_x \otimes \mathbb{C}^{r_1 - r_2}$ (where $\mathbb{C}_x$ is the torsion sheaf concentrated on x with fibre $\mathbb{C}$ at that point).

A torsion free sheaf $\mathcal{F}$ on X is called *linked* if is linked at every intersection point of any two irreducible components of X.

4.2.7. *Extensions of vector bundles on two components* – Let V_1, V_2 be finite dimensional vector spaces. From lemma 4.2.4, we have a canonical isomorphism

$$\mathrm{Ext}^1_{\mathcal{O}_{X,x}}(\mathcal{O}_{D_1,x} \otimes V_1, \mathcal{O}_{D_2,x} \otimes V_2) \ \simeq \ L(V_1, V_2) \ . \tag{3}$$

Let $\phi \in L(V_1, V_2)$. We now give an explicit description of the extension

$$0 \longrightarrow \mathcal{O}_{D_2,x} \otimes V_2 \longrightarrow M \longrightarrow \mathcal{O}_{D_1,x} \otimes V_1 \longrightarrow 0$$

corresponding to ϕ. We have an exact sequence

$$0 \longrightarrow \mathcal{O}_{D_2,x} \otimes V_1 \xrightarrow{(t_2,\phi)} (\mathcal{O}_{X,x} \otimes V_1) \oplus (\mathcal{O}_{D_2,x} \otimes V_2) \longrightarrow M \longrightarrow 0$$

(cf. [4], 4–), and

$$M \ = \ (\mathrm{im}(\phi) \otimes \mathcal{O}_{X,x}) \oplus (\ker(\phi) \otimes \mathcal{O}_{D_1,x}) \oplus (\mathrm{coker}(\phi) \otimes \mathcal{O}_{D_2,x}) \ .$$

The module M can be the fibre of some linked torsion free sheaf if and only ϕ has maximal rank, i.e. is injective or surjective.

For vector bundles on $D_1 \cup D_2 \subset X$ we need more canonical isomorphisms. Let $D = D_1 \cap D_2$. We use the obvious locally free resolution of $\mathcal{O}_{D_1}$

$$\cdots \mathcal{L}_3 \longrightarrow \mathcal{L}_2 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{D_1} \longrightarrow 0 \ ,$$

where $\mathcal{L}_2$ is a line bundle on $D_1 \cup D_2$ with $\mathcal{L}_{2|D_2} = \mathcal{O}_{D_2}(-D)$, $\mathcal{L}_{2|D_1} = \mathcal{O}_{D_1}$, and $\mathcal{L}_3$ is a line bundle on $D_1 \cup D_2$ with $\mathcal{L}_{3|D_1} = \mathcal{O}_{D_1}(-D)$, $\mathcal{L}_{3|D_2} = \mathcal{O}_{D_2}$. In this way we obtain, if E_1, E_2 are vector bundles on D_1, D_2 respectively, a canonical isomorphism

$$\mathrm{Ext}^1_{\mathcal{O}_{D_1 \cup D_2}}(E_1, E_2) \ = \ \bigoplus_{x \in D} L(E_1(-x)_x, E_{2,x}) \ . \tag{4}$$

Now we make the link between this description of extensions and the construction of 4.2.1. Let E_1, E_2 be vector bundles on D_1, D_2 respectively. We want to describe torsion free sheaves $\mathcal{E}$ on $D_1 \cup D_2$ such that $\mathcal{E}_{|D_i}/T_i = E_i$ for $i = 1, 2$ (where T_i denotes the torsion subsheaf). As in in 4.2.1, $\mathcal{E}$ is defined by vector spaces W^x, $x \in D$, and surjective maps

$$f_1^x : E_{1,x} \longrightarrow W^x \ , \quad f_2^x : E_{2,x} \longrightarrow W^x \ . \tag{5}$$

Let $N_1^x = \ker(f_1^x)$, $N_2^x = \ker(f_2^x)$ for $x \in D$, $i = 1, 2$. We have an exact sequence

$$0 \longrightarrow \big(E_2\big)_{(N_2^x)_{x \in D_1 \cap D_2}} \longrightarrow \mathcal{E} \longrightarrow E_1 \longrightarrow 0 \tag{6}$$

(cf. 2.4 for the definition of $\big(E_2\big)_{(N_2^x)_{x \in D_1 \cap D_2}}$). The corresponding linear map (3) for every $x \in D$ is as follows:

$$(E_1(-x))_x \xrightarrow{f_1^x} W^x \otimes \mathcal{O}_X(-x)_x = E_2(-x)_x / N_2^x \otimes \mathcal{O}_X(-x)_x \hookrightarrow (E_2)_{(N_2^x),x}$$

(the last inclusion coming also from 2.4).

We have also a canonical exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow E_1 \oplus E_2 \xrightarrow{p} \bigoplus_{x \in D} \overline{W^x} \longrightarrow 0$$

where $\overline{W_x}$ denotes the skyscraper sheaf concentrated on x with fibre W_x at x, and p is given at x by

$$(f_1^x, -f_2^x) : E_{1,x} \oplus E_{2,x} \longrightarrow W^x \ .$$

This can of course be generalised to torsion free sheaves on the whole of X: recall that $\mathbf{C}$ denotes the set of components of X and I the set of intersection points. Let $\mathcal{F}$ be a torsion free sheaf on X, and for every $D\in\mathbf{C}$, $\mathcal{F}_D=\mathcal{F}_{|D}/T_D$ (where T_D is the torsion subsheaf). Then we have an exact sequence

$$0\longrightarrow\mathcal{F}\longrightarrow\bigoplus_{D\in\mathbf{C}}\mathcal{F}_D\longrightarrow\bigoplus_{P\in I}(\mathbb{C}_P\otimes W_P)\longrightarrow 0\ , \tag{7}$$

where for each $P\in I$, W_P is a vector space. If $\mathcal{F}$ is linked, such a P belongs to two components X_1, X_2, and $W_P=\mathcal{F}_{X_i,x}$, where i is such that $\mathrm{rk}(\mathcal{F}_i)=\inf(\mathrm{rk}(\mathcal{F}_1),\mathrm{rk}(\mathcal{F}_2))$.

4.2.8. *Duality* – To simplify the notations, we suppose in the rest of 4.2 that X has two components D_1, D_2 (it is easy to generalise the following results to the case where X has more than two components). Recall that $D=D_1\cap D_2$. If $\mathcal{E}$ is a coherent sheaf on X, let $\mathcal{E}^\vee$ denote its dual. If E_1 is a vector bundle on D_1 we have

$$E_1^\vee\ \simeq\ E_1^*(-D)\ .$$

We keep the notations of 4.2.7. For $i=1,2$, let $Q_i^x=E_{i,x}/N_i^x$. From equation (6) we deduce the exact sequence

$$0\longrightarrow E_1^*(-D)\longrightarrow\mathcal{E}^\vee\longrightarrow\Big[\big(E_2\big)_{(N_2^x)_{x\in D}}\Big]^*(-D)\longrightarrow 0$$

(cf. 2.4). According to 2.4.2, we have

$$\Big[\big(E_2\big)_{(N_2^x)_{x\in D}}\Big]^*(-D)\ =\ (E_2^*)_{(Q_2^{x*})_{x\in D}}\ .$$

It follows that if $G_i=\mathcal{E}^\vee_{|D_i}/T_i$, $i=1,2$, then we have

$$G_i\ =\ (E_i^*)_{(Q_i^{x*})_{x\in D}}\ .$$

The corresponding maps (5) are of course

$$f_i^{x*}:G_{i,x}=\big[(E_i^*)_{(Q_i^{x*})}\big]\longrightarrow(E_{i,x}/N_i^x)^*=W_x^*\ .$$

4.2.9. *Tensor products* – Let $\mathcal{E}$, $\mathcal{E}'$ be torsion free sheaves on $X=D_1\cup D_2$. Let $E_i=\mathcal{E}_{|D_i}/T_i$, $E'_i=\mathcal{E}'_{|D_i}/T'_i$ for $i=1,2$, where T_i, T'_i denote the torsion subsheaves, and $r_i=\mathrm{rk}(E_i)$, $r'_i=\mathrm{rk}(E'_i)$. For every $x\in D$, let

$$f_1^x:E_{1,x}\longrightarrow W^x\ ,\quad f_2^x:E_{2,x}\longrightarrow W^x\ ,\quad f_1'^x:E'_{1,x}\longrightarrow W'^x\ ,\quad f_2'^x:E'_{2,x}\longrightarrow W'^x$$

be surjective linear maps that define $\mathcal{E}$ and $\mathcal{E}'$ at x (cf. 4.2.7).

4.2.10. Proposition: *Suppose that $\mathcal{E}$ and $\mathcal{E}'$ are linked and that $(r_1-r_2)(r'_1-r'_2)\geq 0$. Then $\mathcal{E}\otimes\mathcal{E}'$ is torsion free and linked, and we have $(\mathcal{E}\otimes\mathcal{E}')_{|D_i}/\mathbb{T}_i\simeq E_i\otimes E'_i$ for $i=1,2$ (where $\mathbb{T}_i$ is the torsion subsheaf), and for every $x\in D$, $\mathcal{E}\otimes\mathcal{E}'$ is defined at x by the linear maps*

$$f_1^x\otimes f_1'^x:E_{1,x}\otimes E'_{1,x}\longrightarrow W_x\otimes W'_x\ ,\quad f_2^x\otimes f_2'^x:E_{2,x}\otimes E'_{2,x}\longrightarrow W_x\otimes W'_x\ .$$

Proof. The condition $(r_1-r_2)(r'_1-r'_2)\geq 0$ implies that for every $x\in D$

– if $r_1\geq r_2$, then there exist integers a, b, a', b' such that

$$\mathcal{E}_x\simeq a\mathcal{O}_{D_1,x}\oplus b\mathcal{O}_{X,x}\ ,\quad \mathcal{E}'_x\simeq a'\mathcal{O}_{D_1,x}\oplus b'\mathcal{O}_{X,x}\ ,$$

– if $r_1 < r_2$, then there exist integers a, b, a', b' such that

$$\mathcal{E}_x \simeq a\mathcal{O}_{D_2,x} \oplus b\mathcal{O}_{X,x} \ , \quad \mathcal{E}'_x \simeq a'\mathcal{O}_{D_2,x} \oplus b'\mathcal{O}_{X,x} \ .$$

This condition is necessary, otherwise $\mathcal{E}\otimes\mathcal{E}'$ would not be torsion free, because $\mathcal{O}_{D_1,x}\otimes\mathcal{O}_{D_2,x}$ is the torsion sheaf concentrated on x with fibre $\mathbb{C}$.

Let $\mathcal{F}$ be the linked sheaf defined by $E_1\otimes E'_1$, $E_2\otimes E'_2$ and the preceding maps. Then there is an obvious morphism $\mathcal{E}\otimes\mathcal{E}'\to\mathcal{F}$. Using local decompositions of $\mathcal{E}$, $\mathcal{E}'$ in direct sums (as indicated in the beginning of the proof) it is easy to see that it is an isomorphism. $\square$

4.2.11. *Computation of* $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})$ – Let $\mathcal{E}$ be a linked torsion free sheaf on $X = D_1\cup D_2$. According to 4.2.8 and 4.2.9, the sheaf $\mathcal{E}\otimes\mathcal{E}^\vee$ is defined by the vector bundles
$E_i\otimes(E_i^*)_{((E_{i,x}/N_i^x)^*)}$, and the maps $f_i^x\otimes f_i^{x*}$ (recall that $N_i^x = \ker(f_i^x)$).
According to 4.2.3, the sheaf $\mathcal{E}nd(\mathcal{E})$ is defined by the vector bundles $\mathcal{E}nd(E_i)_{(H_i^x)}$ and the maps $\gamma_i^x : \mathcal{E}nd(E_i)_{(H_i^x),x}\to\mathrm{End}(W^x)$, where for every $x\in D = D_1\cap D_2$,
$H_i^x = \{\alpha\in\mathrm{End}(E_{i,x}); \alpha(\ker(f_i^x))\subset\ker(f_i^x)\}$.

Since $\mathcal{E}$ is linked, we can assume that $r_1\leq r_2$. In this case we have $N_1^x = \{0\}$ for every $x\in D$, and

$$E_1\otimes(E_1^*)_{(Q_1^{x*})} \ = \ \mathcal{E}nd(E_1)_{(H_1^x)} \ = \ \mathcal{E}nd(E_1) \ ,$$

and only the vector bundles on D_2 can differ. We have two canonical commutative diagrams with exact rows and columns

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & \bigoplus_{x\in D}\overline{\mathrm{End}(N_2^x)} \arrow[d] & \\
0 \arrow[r] & E_2\otimes(E_2^*)_{(Q_2^{x*})} \arrow[r]\arrow[d] & \mathcal{E}nd(E_2) \arrow[r]\arrow[d, equal] & \bigoplus_{x\in D}\overline{L(N_{2x},E_{2,x})} \arrow[r]\arrow[d] & 0\\
0 \arrow[r] & \mathcal{E}nd(E_2)_{(H_2^x)} \arrow[r]\arrow[d] & \mathcal{E}nd(E_2) \arrow[r] & \bigoplus_{x\in D}\overline{L(N_2^x,Q_2^x)} \arrow[r]\arrow[d] & 0\\
 & \bigoplus_{x\in D}\overline{\mathrm{End}(N_2^x)} \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{E}^\vee\otimes\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E}nd(E_1)\oplus(E_2\otimes(E_2^*)_{(Q_2^{x*})}) \arrow[r] \arrow[d] & \bigoplus_{x\in D}\overline{\mathrm{End}(E_{1,x})} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{E}nd(\mathcal{E}) \arrow[r] \arrow[d] & \mathcal{E}nd(E_1)\oplus\mathcal{E}nd(E_2)_{(H_2^x)} \arrow[r] \arrow[d] & \bigoplus_{x\in D}\overline{\mathrm{End}(E_{1,x})} \arrow[r] & 0 \\
 & \bigoplus_{x\in D}\overline{\mathrm{End}(N_2^x)} \arrow[r, equal] \arrow[d] & \bigoplus_{x\in D}\overline{\mathrm{End}(N_2^x)} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

It follows that

4.2.12. Proposition: *Suppose that E_1 and E_2 are simple. Then there is a vector space $H_\mathcal{E}$ and canonical exact sequences*

$$0\longrightarrow\big(\bigoplus_{x\in D}\mathrm{End}(E_{1,x})\big)/\mathbb{C}\longrightarrow H_\mathcal{E}\longrightarrow\bigoplus_{x\in D}L(N_2^x,Q_2^x)\longrightarrow 0\ ,$$

$$0\longrightarrow H_\mathcal{E}\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{D_1}}(E_1,E_1)\oplus\mathrm{Ext}^1_{\mathcal{O}_{D_2}}(E_2,E_2)\longrightarrow 0\ .$$

In particular $\dim(\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ *depends only on* r_1, r_2, $\dim(\mathrm{Ext}^1_{\mathcal{O}_{D_1}}(E_1,E_1))$ *and* $\dim(\mathrm{Ext}^1_{\mathcal{O}_{D_2}}(E_2,E_2))$.

Proof. From lemma 4.2.4 we have $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})=0$, hence $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\simeq H^1(\mathcal{E}nd(\mathcal{E}))$.
The identity morphism $E_2\to E_2$ is also a section of $\mathcal{E}nd(E_2)_{(H_2^x)}$, hence it follows from the preceding diagram that we have an exact sequence

$$0\longrightarrow\big(\bigoplus_{x\in D}\mathrm{End}(E_{1,x})\big)/\mathbb{C}\longrightarrow H^1(\mathcal{E}nd(\mathcal{E}))\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{D_1}}(E_1,E_1)\oplus H^1(\mathcal{E}nd(E_2)_{(H_2^x)})\longrightarrow 0\ .$$

From the diagram before we deduce the exact sequence

$$0\longrightarrow\bigoplus_{x\in D}L(N_2^x,Q_2^x)\longrightarrow H^1(\mathcal{E}nd(E_2)_{(H_2^x)})\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{D_2}}(E_2,E_2)\longrightarrow 0\ .$$

The result follows immediately. □

4.3. Deformations of linked sheaves

4.3.1. *Local structure of families of linked sheaves* – We keep the notations of 4.2. Let Z be an algebraic variety and z_0 a closed point of Z. Let $\mathbb{F}$ be a coherent sheaf on $U \times Z$, flat on Z, and such that $\mathcal{F} = \mathbb{F}_{z_0}$ is a reflexive linked sheaf on U.

4.3.2. Lemma: *There exists a neighbourhood Z_0 of z_0 in Z such that for every closed point $z \in Z_0$, $\mathbb{F}_z$ is a linked torsion free sheaf on X.*

Proof. According to [20], prop. 2.1, the deformations of torsion free sheaves are torsion free. The fact that $\mathbb{F}_z$ is linked for z in a neighbourhood Z_0 of z_0 follows then easily from the semicontinuity of the rank of the fibres of coherent sheaves, since the rank of torsion free sheaves on X at the intersection points is minimal precisely when the sheaves are linked. □

From now on, we assume that $Z_0 = Z$. Let $r_i = \mathrm{rk}(\mathcal{F}_{|D_i})$ and suppose that $r_2 \geq r_1$. Then $\mathbb{F}_2 = \mathbb{F}_{|Z\times D_2}$ is a vector bundle on $Z \times D_2 \cap U$ (because it is locally free of rank r_2 on $D_2 \cap (U\backslash\{x\}) \times Z$ and also on $\{x\} \times Z$). Similarly the kernel $\mathbb{F}_1'$ of the restriction morphism $\mathbb{F} \to \mathbb{F}_{D_2\times Z}$ is a vector bundle on $Z \times D_1$. According to lemma 4.2.4 and 4.2.5, for every $z \in Z$ we have

$$\mathbb{F}_{z,x} \ \simeq \ r_1\mathcal{O}_{X\times Z,(x,z)} \oplus (r_2 - r_1)\mathcal{O}_{D_2\times Z,(x,z)} \ .$$

Hence if $\mathbb{T}_1$ is the torsion subsheaf of $\mathbb{F}_{|Z\times D_1}$, then $\mathbb{F}_1 = \mathbb{F}_{|Z\times D_1}/\mathbb{T}_1$ is a vector bundle on $Z \times D_1$, and we have a canonical isomorphism $\mathbb{F}_1' = \mathbb{F}_1 \otimes p_X^*(\mathcal{O}_X(-x))$ (where p_X is the projection $Z \times X \to X$).

4.3.3. Theorem: **1** – *Suppose that Z is connected. Let $\mathbb{U}$ be a coherent sheaf on $X \times Z$, flat on Z, and such that for every closed point $z \in Z$, $\mathbb{U}_z$ is a reflexive linked sheaf on Z. Let $D \in \mathbf{C}$. Let $\mathbb{T}_D$ be the torsion subsheaf of $\mathbb{U}_{|Z\times D}$ and $\mathbb{U}_D = \mathbb{U}_{|Z\times D}/\mathbb{T}_D$. Then $\mathbb{U}_D$ is a vector bundle on $Z \times D$. If $z \in Z$ is a closed point, then $\mathbb{U}_{D,z} = \mathbb{U}_{z|D}/T$ (where T is the torsion subsheaf of $\mathbb{U}_{z|D}$). In particular $\deg(\mathbb{U}_{D,z})$ is independent of z. So the type of $\mathbb{U}_z$ in independent of z.*

2 – *Conversely, let $\mathcal{E}$, $\mathcal{F}$ be linked torsion free sheaves on X, such that $\tau(\mathcal{E}) = \tau(\mathcal{F})$ (cf. 4.1.2). Then there exist an integral variety Y, a flat family $\mathbb{V}$ of linked torsion free sheaves on X parametrised by Y, and two closed points $s, t \in Y$ such that $\mathbb{V}_s \simeq \mathcal{E}$ and $\mathbb{V}_t \simeq \mathcal{F}$.*

Proof. **1** follows immediately from the discussion after lemma 4.3.2. To prove **2** we remark first that we have exact sequences

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{D\in\mathbf{C}} \mathcal{E}_D \longrightarrow \bigoplus_{P\in I}(\mathbb{C}_P \otimes W_P) \longrightarrow 0 \ ,$$

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_{D\in\mathbf{C}} \mathcal{F}_D \longrightarrow \bigoplus_{P\in I}(\mathbb{C}_P \otimes W_P) \longrightarrow 0 \ ,$$

(cf. (7)) with the same vector spaces W_P, where for every $D \in \mathbf{C}$, $\mathcal{E}_D = \mathcal{E}_{|D}/T_{\mathcal{E},D}$ and $\mathcal{F}_D = \mathcal{F}_{|D}/T_{\mathcal{F},D}$ ($T_{\mathcal{E},D}$, $T_{\mathcal{F},D}$ are the torsion subsheaves). For every $D \in \mathbf{C}$, since $\mathcal{E}_D$ and $\mathcal{F}_D$ have the same rank and degree, there exist an integral variety Y_D, a vector bundle $\mathbb{V}_D$ on $D \times Y_D$

and two closed points $s_S, t_D \in Y_D$ such that $\mathbb{V}_{D,s_D} \simeq \mathcal{E}_D$ and $\mathbb{V}_{D,t_D} \simeq \mathcal{F}_D$. Let $\mathbf{Y} = \prod_{D\in\mathbf{C}} \mathbb{Y}_D$, and

$$\mathbb{B} \ = \ p_{X*}(\mathcal{H}om(\bigoplus_{D\in\mathbf{C}} p_D^{\#}(\mathbb{V}_D), \bigoplus_{P\in I} p_P^*(\mathbb{C}_P \otimes W_P)) \ ,$$

(where p_X, p_D, p_P are the appropriate projections), which is a vector bundle on $\mathbf{Y}$. For every $y = (y_D)_{D\in\mathbf{C}} \in \mathbf{Y}$, $\mathbb{B}_y = \mathrm{Hom}(\bigoplus_{D\in\mathbf{C}} \mathbb{V}_{D,y_D}, \bigoplus_{P\in I} \mathbb{C}_P \otimes W_P)$. Now it suffices to take for Y the open subset $\mathbb{B}_0$ of $\mathbb{B}$ of surjective morphisms and for $\mathbb{V}$ the kernel of the obvious universal surjective morphism on $X \times \mathbb{B}_0$

$$\pi^{\#}(\bigoplus_{D\in\mathbf{C}} p_D^{\#}(\mathbb{V}_D)) \longrightarrow \pi^{\#}(\bigoplus_{P\in I} p_P^*(\mathbb{C}_P \otimes W_P))$$

(where π is the projection $\mathbb{B}_0 \to \mathbf{Y}$). $\square$

4.3.4. The description of the local structure of $\mathbb{U}$ implies also that the exact sequence (7) can be globalised:

$$0 \longrightarrow \mathbb{U} \longrightarrow \bigoplus_{D\in\mathbf{C}} \mathbb{U}_D \longrightarrow \bigoplus_{x\in I} p_{x*}(W_x) \longrightarrow 0 \ ,$$

where $\mathbf{C}$ is the set of components of X, I is the set of intersection points, and for each $x \in I$, W_x is a vector bundle on Z and p_x is the inclusion $\{x\} \times Z \to X \times Z$. More precisely, x belongs to two components D_1, D_2, and $W_x = \mathbb{U}_{D_i|\{x\}\times Z}$, where i is such that $\mathrm{rk}(\mathbb{U}_i) = \inf(\mathrm{rk}(\mathbb{U}_1), \mathrm{rk}(\mathbb{U}_2))$.

4.3.5. *Construction of complete families* – Let $\mathcal{E}$ be a linked torsion free sheaf on X, $E_D = \mathcal{E}_{|D}/T_D$ for $D \in \mathbf{C}$ (where T_D is the torsion subsheaf), $r_D = \mathrm{rk}(E_D)$. Then there exists a *complete deformation* $\mathbb{E}_D$ of E_D parametrised by a smooth irreducible variety Z_D: it is defined by a vector bundle of $D \times Z_D$ and a closed point $z_D \in Z_D$ such that $E_D \simeq \mathbb{E}_{D,z_D}$, with the following local universal property: for every flat family $\mathcal{F}$ of sheaves on D parametrised by an algebraic variety Y, and $y \in Y$ such that $\mathcal{F}_y \simeq E_D$, there exists a neighbourhood Y_0 of y and a morphism $f : Y_0 \to Z_D$ such that $f(y) = z_D$ and that there is an isomorphism $f^{\#}(\mathbb{E}_D) \simeq \mathcal{F}_{|D\times Y_0}$ compatible with the isomorphisms $\mathcal{F}_y \simeq E_D$, $E_D \simeq \mathbb{E}_{D,z_D}$. This deformation can be constructed for example using Quot schemes. Now let $\boldsymbol{Z} = \prod_{D\in\mathbb{C}} Z_D$ and p_D the projections $\boldsymbol{Z} \to Z_D$. For every intersection point x, let D_1^x, D_2^x be the components through x, such that $\mathrm{rk}(E_{D_1^x}) \leq \mathrm{rk}(E_{D_2^x})$. Let $\mathbb{F}_x$ be the vector bundle on $\boldsymbol{Z}$ with fibre at (z_D) equal to

$$\mathbb{F}_x \ = \ L(\mathbb{E}_{D_2^x,(x,z_{D_2^x})}, \mathbb{E}_{D_1^x,(x,z_{D_1^x})}) \ ,$$

and $\mathbb{F} = \bigoplus_{x\in I} \mathbb{F}_x$. Let $\mathbb{F}_0 \subset \mathbb{F}$ be the open subset corresponding to surjective morphisms, and $q : \mathbb{F}_0 \to \boldsymbol{Z}$ the projection. Let $\mathbb{G}$ be the torsion sheaf on $X \times \boldsymbol{Z}$

$$\mathbb{G} \ = \ \bigoplus_{x\in I} p_{D_1^x}^{\#}(\mathbb{E}_{D_1^x|\{x\}\times Z_{D_1^x}}) \ ,$$

i.e. for every $(z_D) \in \boldsymbol{Z}$, $\mathbb{G}_{(z_D)}$ is the torsion sheaf on X

$$\mathbb{G}_{(z_D)} \ = \ \bigoplus_{x\in I} \mathbb{E}_{D_1^x,(x,z_D)} \otimes \mathbb{C}_x \ .$$

Then we have a canonical surjective morphism of sheaves on $X \times \mathbb{F}_0$

$$\Phi : \bigoplus_{D \in \mathbf{C}} q^{\#}(p_D^{\#}(\mathbb{E}_D)) \longrightarrow q^{\#}(\mathbb{G}) \ .$$

Let $z = (z_D)_{D \in \mathbf{C}} \in \boldsymbol{Z}$, $\phi = (\phi_x)_{x \in I} \in \mathbb{F}_0$ over z, so ϕ_x is a surjective map $\mathbb{E}_{D_2^x,(x,z_{D_2^x})} \to \mathbb{E}_{D_1^x,(x,z_{D_1^x})}$. For $x \in I$, Φ at (x, ϕ) is the linear map which is

$$(-I, \phi_x) : \mathbb{E}_{D_1^x,(x,z_{D_1^x})} \oplus \mathbb{E}_{D_2^x,(x,z_{D_2^x})} \longrightarrow \mathbb{E}_{D_1^x,(x,z_{D_1^x})}$$

on the two summands $\mathbb{E}_{D_1^x,(x,z_{D_1^x})}$, $\mathbb{E}_{D_2^x,(x,z_{D_2^x})}$, and zero on the others. Let $\boldsymbol{\mathcal{E}} = \ker(\Phi)$.

It follows then easily from 4.3.4 that $\boldsymbol{\mathcal{E}}$ has the following *local universal property*: under the hypotheses of theorem 4.3.3, if $z \in Z$ is a closed point such that $\mathbb{U}_z \simeq \mathcal{E}$, there exists a neighbourhood Z_0 of z and a morphism $f : Z_0 \to \mathbb{F}_0$ such that $\mathbb{U}_{X \times Z_0} \simeq f^{\#}(\boldsymbol{\mathcal{E}})$.

4.3.6. *Properties of stable linked sheaves* – Let $\mathcal{O}_X(1)$ be an ample line bundle on X. For every polynomial P in one variable and rational coefficients, let $\mathcal{M}_X(P)$ denote the moduli space of sheaves on X, stable with respect to $\mathcal{O}_X(1)$ and with Hilbert polynomial P.

4.3.7. Theorem: *Let $\mathcal{E}$ be a linked reflexive sheaf on X and P its Hilbert polynomial. Suppose that $\mathcal{E}$ is stable with respect to $\mathcal{O}_X(1)$, and that the restrictions of $\mathcal{E}$ to the irreducible components of X modulo torsion are simple vector bundles. Then $\mathcal{M}_X(P)$ is smooth at the point corresponding to $\mathcal{E}$.*

Proof. We use the construction of $\mathcal{M}_X(P)$ as a good quotient of an open subset $\mathbf{R}$ of a *Quot* scheme (cd [21], [24]) by a reductive group G. The quotient morphism $\pi : \mathbf{R} \to \mathcal{M}_X(P)$ is associated to the universal sheaf $\mathcal{E}$ on $X \times \mathbf{R}$. Let $\mathbf{R}_\ell$ be the G-invariant open subset of points y of $\mathbf{R}$ such that $\mathcal{E}_y$ is linked. According to 4.3.5, π can be locally factorised, at any point y of $\mathbf{R}_\ell$, through a smooth variety: $\pi : \mathbf{R} \to \mathbb{F}_0 \to \mathcal{M}_X(P)$. It follows that $\pi(\mathbf{R}_\ell)$ is contained in $\mathcal{M}_X(P)_{\text{red}}$, hence $\pi(\mathbf{R}_\ell)$ is integral.

At any point z of $\mathcal{M}_X(P)$ corresponding to a stable sheaf $\mathcal{E}$, the tangent space of $\mathcal{M}_X(P)$ at z is canonically isomorphic to $\text{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$. According to proposition 4.2.12 and theorem 4.3.3, if $\mathcal{E}$ is linked, then the dimension of this space depends only on P, hence is constant. It follows that $\pi(\mathbf{R}_\ell)$ is smooth. $\square$

4.3.8. Remark: Let $\mathcal{E}$ be a linked torsion free sheaf on X. Then we have $H^0(\mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) \subset \text{Ext}^2_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$. It follows from lemma 4.2.4 that if $\mathcal{E}$ is not locally free then $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \neq \{0\}$.

4.3.9. Theorem: *Every torsion free sheaf on X can be deformed to linked torsion free sheaves.*

Proof. Let $\mathcal{F}$ be a torsion free sheaf. We will use the exact sequence (7):

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_{D \in \mathbf{C}} \mathcal{F}_D \xrightarrow{\phi} \bigoplus_{P \in I} (\mathbb{C}_P \otimes W_P) \longrightarrow 0 \ .$$

For each intersection point $x \in I$, belonging to the components $D_1, D_2 \in \mathbf{C}$, ϕ_x is non zero only on $\mathcal{F}_{D_1} \oplus \mathcal{F}_{D_2}$, and comes from two surjective maps $\mathcal{F}_{D_1,x} \to W_x$, $\mathcal{F}_{D_2,x} \to W_x$. And $\mathcal{F}$ is linked at x if and only $\dim(W_x)=\inf(\mathrm{rk}(\mathcal{F}_{D_1}),\mathrm{rk}(\mathcal{F}_{D_2}))$. Let $N_1\subset\mathcal{F}_{D_1,x}$ be a linear subspace, and $\mathcal{E}=(\mathcal{F}_{D_1})_{N_1}$ (cf. 2.4). Let $A_1=N_1\otimes\mathcal{O}_X(x)_x$. Then we have an exact sequence

$$0\longrightarrow\mathcal{F}_{D_1}\longrightarrow\mathcal{E}(x)\longrightarrow\mathbb{C}_x\otimes A_1\longrightarrow 0\ ,$$

associated to $\sigma\in\mathrm{Ext}^1_{\mathcal{O}_X}(\mathbb{C}_x\otimes A_1,\mathcal{F}_{D_1})$. Let $\sigma'\in\mathrm{Ext}^1_{\mathcal{O}_X}(\mathbb{C}_x\otimes A_1,W_x)$ be the image of σ by ϕ, and $0\to\mathbb{C}_x\otimes W_x\to\mathcal{M}\longrightarrow\mathbb{C}_x\otimes A_1\to 0$ the associated extension, so that we have a commutative diagram with exact rows

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d, equal] & \mathcal{F}_{D_1}\oplus\mathcal{F}_{D_2} \arrow[r, "\phi"] \arrow[d] & \mathbb{C}_x\otimes W_x \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{E}(x)\oplus\mathcal{F}_{D_2} \arrow[r] \arrow[d] & \mathcal{M} \arrow[r] \arrow[d] & 0 \\
 & & \mathbb{C}_x\otimes A_1 \arrow[r, equal] \arrow[d] & \mathbb{C}_x\otimes A_1 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where the sheaf $\mathcal{G}$ is identical to $\mathcal{F}$ in a neighbourhood of x. The sheaf $\mathcal{M}$ is concentrated on x if and only if $\mathcal{O}_X(-x)\mathcal{M}=0$, if and only if $\mathcal{O}_X(-x)\mathcal{E}(x)\subset\mathcal{G}$. We have $\mathcal{O}_X(-x)\mathcal{E}(x)\subset\mathcal{F}_{D_1}$ and the image of $\mathcal{O}_X(-x)\mathcal{E}(x)$ in $\mathcal{F}_{D_1,x}$ is precisely N_1. Hence $\mathcal{M}$ is concentrated on x if and only if $N_1\subset\ker(\phi_{x|\mathcal{F}_{D_1,x}})$. So it is possible to choose N_1 such that $\mathcal{M}$ is concentrated on x and of dimension $\inf(\mathrm{rk}(\mathcal{F}_{D_1}),\mathrm{rk}(\mathcal{F}_{D_2}))$. If we make such a modification at every $x\in I$, we obtain an exact sequence

$$0\longrightarrow\mathcal{F}\longrightarrow\bigoplus_{D\in\mathbf{C}}E_D\longrightarrow\bigoplus_{P\in I}(\mathbb{C}_P\otimes V_P)\longrightarrow 0\ ,$$

where for every $D\in\mathbf{C}$, E_D is a vector bundle on D, and for every $x\in I$, V_x is a vector space of dimension $\inf(\mathrm{rk}(E_{D_1}),\mathrm{rk}(E_{D_2}))$. Now we consider the flat family of kernels of surjective morphisms

$$\bigoplus_{D\in\mathbf{C}}\mathbb{E}_D\longrightarrow\bigoplus_{P\in I}(\mathbb{C}_P\otimes V_P)\ .$$

This family contains $\mathcal{F}$ and its generic sheaf is linked. $\square$

4.3.10. Remark: In the preceding proof it would be possible to use more generally pairs of subspaces N_1,N_2, with $N_i\subset\mathcal{F}_{D_i,x}$. In this case we would obtain vector bundles $\mathbb{E}_D$ with the same ranks as the ones of the theorem but not necessarily the same degrees.

4.4. Moduli spaces of semi-stable sheaves

4.4.1. *Polarisations* – An ample line bundle $\mathcal{O}_X(1)$ on X is defined by

- for every $D \in \mathbf{C}$, a line bundle $\mathcal{O}_D(1)$ of positive degree δ_D on D,
- for every intersection point $x \in I$, belonging to the components D_1 and D_2, an isomorphism $\mathcal{O}_{D_1}(1)_x \simeq \mathcal{O}_{D_2}(1)_x$.

Let $\mathcal{F}$ be a torsion free sheaf. Consider the exact sequence (7):

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_{D\in\mathbf{C}} \mathcal{F}_D \xrightarrow{\phi} \bigoplus_{P\in I}(\mathbb{C}_P \otimes W_P) \longrightarrow 0 ,$$

(recall that for every component $D \in \mathbf{C}$, $\mathcal{F}_D = \mathcal{F}_{|D}/T_D$, where T_D is the torsion subsheaf). For every $D \in \mathbf{C}$ and $P \in I$, let g_D be the genus of D and

$$r_D \ = \ \mathrm{rk}(\mathcal{F}_D) \ , \quad d_D \ = \ \deg(\mathcal{F}_D) \ , \quad h_P \ = \ \dim(W_P) \ .$$

Then the Hilbert polynomial of $\mathcal{F}$ corresponding to $\mathcal{O}_X(1)$ is:

$$P_{\mathcal{F}}(m) \ = \ \left(\sum_{D\in\mathbf{C}} r_D\delta_D\right).m + \sum_{D\in\mathbf{C}}(r_D(1-g_D)+d_D) - \sum_{P\in I} h_P \ . \tag{8}$$

4.4.2. We will be mainly interested in the following case: X has only two components D_1, D_2, of the same genus g, intersecting in d points. We will also suppose that $\delta_{D_1} = \delta_{D_2} = \delta$. We have then

$$P_{\mathcal{F}}(m) \ = \ (r_{D_1}+r_{D_2})\delta m + (r_{D_1}+r_{D_2})(1-g) + d_{D_1} + d_{D_2} - \sum_{P\in I} h_P \ .$$

If $\mathcal{F}$ is a linked sheaf, we have $h_P = \inf(r_{D_1}, r_{D_2})$ for every $P \in I$, hence

$$P_{\mathcal{F}}(m) \ = \ (r_{D_1}+r_{D_2})\delta m + (r_{D_1}+r_{D_2})(1-g) + d_{D_1} + d_{D_2} - d.\inf(r_{D_1}, r_{D_2}) \ .$$

It follows that $P_{\mathcal{F}}$ depends only on $r_{D_1}+r_{D_2}$ and $d_{D_1}+d_{D_2} - d.\inf(r_{D_1}, r_{D_2})$.

4.4.3. *Components of the moduli spaces of sheaves* – Recall that $\mathcal{M}_X(\boldsymbol{P})$ denotes the moduli space of sheaves on X, stable with respect to $\mathcal{O}_X(1)$ and with Hilbert polynomial $\boldsymbol{P}$. According to theorems 4.3.3 and 4.3.9

- Every component of $\mathcal{M}_X(\boldsymbol{P})$ contains points corresponding to linked sheaves.
- If $\mathcal{E}$ is a linked sheaf corresponding to a point in a component $\mathcal{N}$ of $\mathcal{M}_X(\boldsymbol{P})$, for every linked sheaf $\mathcal{D}$ corresponding to a point of $\mathcal{N}$, we have $\tau(\mathcal{F}) = \tau(\mathcal{E})$.
- If $\mathcal{E}$, $\mathcal{F}$ are stable linked sheaves of Hilbert polynomial $\boldsymbol{P}$ such that $\tau(\mathcal{F}) = \tau(\mathcal{E})$, then their corresponding points in $\mathcal{M}_X(\boldsymbol{P})$ belong to the same irreducible component.

It follows that the components of $\mathcal{M}_X(\boldsymbol{P})$ are indexed by the types of the linked sheaves that they contain.

Using theorem 4.3.7 and the same method as in the proof of theorem 4.3.9, it is easy to prove that every component of $\mathcal{M}_X(\boldsymbol{P})$ containing points corresponding to stable sheaves is generically smooth.

In the particular case of 4.4.2, $\boldsymbol{P}$ depends only on two parameters R and D: if $\mathcal{E}$ is a linked semi-stable sheaf of Hilbert polynomial $\boldsymbol{P}$, let $r_i = \mathrm{rk}(\mathcal{E}_{|D_i}/T_i)$ (where T_i is the torsion subsheaf).

Then $R = r_{D_1} + r_{D_2}$, $D = d_{D_1} + d_{D_2} - d.\inf(r_{D_1}, r_{D_2})$. We have seen that if $\mathcal{M}_X(\boldsymbol{P})$ is non empty, then there exist linked semi-stable sheaves of Hilbert polynomial $\boldsymbol{P}$, so we can use the following notation: $\mathcal{M}_X(R,D) = \mathcal{M}_X(\boldsymbol{P})$.

If R and D are fixed, the components of $\mathcal{M}_X(R,D)$ depend also on two parameters: r and ϵ. If it contains the point corresponding to $\mathcal{E}$, take $r = r_{D_1}$ and $\epsilon = d_{D_1}$. We will denote this component by $\mathcal{N}_X(R,D,r,\epsilon)$.

5. Coherent sheaves on blowing-ups of ribbons

5.1. Blowing-ups of ribbons

(cf. [2], 1-, [6], 5.4).

Let Y a primitive double curve with associated smooth curve C and associated line bundle L. Let D be a divisor on C and

$$\phi : \widetilde{Y} \longrightarrow Y$$

the blowing-up of Y along D. Then $\widetilde{Y}$ is a primitive double curve with associated smooth curve C and associated line bundle $L(D)$.

If $\mathbb{E}$ is a vector bundle on Y, then $\phi^*(\mathbb{E})$ is a vector bundle on $\widetilde{Y}$, and we have $\phi^*(\mathbb{E})_{|C} = \mathbb{E}_{|C}$. On the other hand, we have

$$\phi^*(\mathbb{E})_1 \ \simeq \ \mathbb{E}_1(D) \ = \ \mathbb{E}_{|C} \otimes L(D) \ .$$

Let P be a point of D, with multiplicity m. Let $x \in \mathcal{O}_{Y,P}$ be over a generator of the maximal ideal of $\mathcal{O}_{C,P}$ and $z \in \mathcal{O}_{Y,P}$ an equation of C. Then there is a canonical isomorphism

$$\mathcal{O}_{\widetilde{Y},P} \ \simeq \ \mathcal{O}_{Y,P}\left[\frac{z}{x^m}\right] \ ,$$

where $\mathcal{O}_{Y,P}\left[\frac{z}{x^m}\right]$ is the subring generated by $\mathcal{O}_{Y,P}$ and $\frac{z}{x^m}$ in the localisation $\mathcal{O}_{Y,P}\left[\frac{1}{x}\right]$. It follows easily that we have

5.1.1. Lemma: *Let E be a vector bundle on C. Then we have*

$$\phi^*(E) \ \simeq \ E[D] \ .$$

(cf. 3.4.5). Let $\mathbb{F}$ be a vector bundle on Y, and $F = \mathbb{F}_{|C}$. Let $f : E \to \mathbb{F}$ be a morphism, equivalent to a morphism $E \to F \otimes L$. Then the morphism $E \to F \otimes L(D)$ induced by $\phi^*(f) : \phi^*(E) \to \phi^*(\mathbb{F})$ is the composition

$$E \xrightarrow{f} F \otimes L \longrightarrow F \otimes L(D) \ ,$$

where the morphism on the right comes from the canonical section of $\mathcal{O}_C(D)$.

5.2. Inverse images of quasi-locally free sheaves

Let $\mathcal{E}$ be a quasi locally free sheaf on Y, and
$$0 \longrightarrow K \longrightarrow \mathcal{E} \xrightarrow{p} E \longrightarrow 0$$
an exact sequence, where K and E are vector bundles on C. Then we have a canonical exact sequence
$$0 \longrightarrow \mathcal{E}_1 \longrightarrow K \longrightarrow N \longrightarrow 0 \ ,$$
where N is the kernel of the canonical surjective morphism $\mathcal{E}_{|C} \mapsto E$. Let σ be the element of $\mathrm{Ext}^1_{\mathcal{O}_C}(N, \mathcal{E}_1)$ associated to this exact sequence. Let
$$\widetilde{\mathcal{E}} \ = \ \phi^*(\mathcal{E})/T(\phi^*(\mathcal{E})) \ .$$

5.2.1. Theorem: *We have* $(\widetilde{\mathcal{E}})_{|C} \simeq \mathcal{E}_{|C}$, $(\widetilde{\mathcal{E}})_1 \simeq \mathcal{E}_1(D)$, *and an exact sequence*
$$0 \longrightarrow \widetilde{K} \longrightarrow \widetilde{\mathcal{E}} \xrightarrow{\widetilde{p}} E \longrightarrow 0 \ ,$$
where $\widetilde{K}$ *is the vector bundle on* C *defined by the exact sequence*
$$0 \longrightarrow \mathcal{E}_1(D) \longrightarrow \widetilde{K} \longrightarrow N \longrightarrow 0$$
associated to the element of $\mathrm{Ext}^1_{\mathcal{O}_C}(N, \mathcal{E}_1(D))$ *image of* σ *by the morphism*
$$\mathrm{Ext}^1_{\mathcal{O}_C}(N, \mathcal{E}_1) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_C}(N, \mathcal{E}_1(D))$$
induced by the canonical section of $\mathcal{O}_C(D)$.

Proof. Let $U \subset C$ be a non-empty open subset, U_Y (resp $U_{\widetilde{Y}}$) the corresponding open subset of Y (resp. $\widetilde{Y}$), such that $\mathcal{E}$ is split as
$$\mathcal{E}_{|U_Y} \ \simeq \ \mathbb{E} \oplus F \ ,$$
where $\mathbb{E}$ is locally free on U_Y, and E is locally free on U. Then we have
$$\phi^*(\mathcal{E})_{|U_{\widetilde{Y}}} \ \simeq \ \phi^*(\mathbb{E}) \oplus F[D \cap U] \quad \text{and} \quad \widetilde{\mathcal{E}}_{|U_{\widetilde{Y}}} \ \simeq \ \phi^*(\mathbb{E}) \oplus F \ .$$
It follows that $(\widetilde{\mathcal{E}}_{|U_{\widetilde{Y}}})_{|U} \simeq \mathcal{E}_{|U} \simeq \mathbb{E}_{|U} \oplus F$, the first isomorphism being independent of the splitting $\mathcal{E}_{|U_Y} \simeq \mathbb{E} \oplus F$. It follows that $(\widetilde{\mathcal{E}})_{|C} \simeq \mathcal{E}_{|C}$. We have also $(\widetilde{\mathcal{E}})_{1|U_{\widetilde{Y}}} \simeq \mathcal{E}_{1|U_Y}(D) = \mathbb{E}_1(D)$, the first isomorphism being independent of the splitting $\mathcal{E}_{|U_Y} \simeq \mathbb{E} \oplus F$. It follows that $(\widetilde{\mathcal{E}})_1 \simeq \mathcal{E}_1(D)$.

Let $p_C : \mathcal{E}_{|C} \to E$ be the morphism induced by p. The fact that K is concentrated on C is equivalent to $\ker(p_C) \subset p(\mathcal{E}^{(1)})$. On U this means that $N_{|U} = \ker(p_{C|U}) \subset F$. The morphism $\widetilde{p}_C : \widetilde{\mathcal{E}}_{|C} \to E$ induced by $\widetilde{p}$ being the same as p_C, we see that $\ker(\widetilde{p})$ is concentrated on C. We have in fact
$$K_{|U} \ = \ (\mathbb{E}_{|C} \otimes L_{|U}) \oplus N_{|U} \ , \quad \widetilde{K}_{|U} \ = \ (\mathbb{E}_{|C} \otimes L(D)_{|U}) \oplus N_{|U} \ .$$

It is easy to check that the obvious commutative diagram

$$\begin{tikzcd}
 & \mathcal{E}_{1|U} \arrow[d, equal] & K_{|U} \arrow[d, equal] & & \\
0 \arrow[r] & \mathbb{E}_{|C}\otimes L_{|U} \arrow[r] \arrow[d, hook] & (\mathbb{E}_{|C}\otimes L_{|U})\oplus N_{|U} \arrow[r] \arrow[d, hook] & N_{|U} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{E}_{|C}\otimes L(D)_{|U} \arrow[r] \arrow[d, equal] & (\mathbb{E}_{|C}\otimes L(D)_{|U})\oplus N_{|U} \arrow[r] \arrow[d, equal] & N_{|U} \arrow[r] & 0 \\
 & \mathcal{E}_1(D)_{|U} & \widetilde{K}_{|U} & &
\end{tikzcd}$$

is independent of the splitting $\mathcal{E}_{|U_Y}\simeq\mathbb{E}\oplus F$. Hence all these diagrams can be glued to define this one on C

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}_1 \arrow[r] \arrow[d, hook] & K \arrow[r] \arrow[d, hook] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{E}_1(D) \arrow[r] & \widetilde{K} \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

This implies the last statement of the theorem by [4], prop. 4.3.2. $\square$

Recall that we have an exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_C}(E,K) \xrightarrow{\ i\ } \operatorname{Ext}^1_{\mathcal{O}_Y}(E,K) \xrightarrow{\ \theta\ } \operatorname{Hom}(E\otimes L,K) \longrightarrow 0$$

(cf. 3.4). On $\widetilde{Y}$ we have an exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_C}(E,\widetilde{K}) \xrightarrow{\ \widetilde{i}\ } \operatorname{Ext}^1_{\mathcal{O}_{\widetilde{Y}}}(E,\widetilde{K}) \xrightarrow{\ \widetilde{\theta}\ } \operatorname{Hom}(E\otimes L(D),\widetilde{K}) \longrightarrow 0.$$

Let $0\to K\to\mathcal{E}\to E\to 0$ be an exact sequence on Y, and $\eta\in\operatorname{Ext}^1_{\mathcal{O}_Y}(E,K)$ corresponding to it. The morphism $\theta(\eta)$ can be written as a composition

$$E\otimes L \xrightarrow{\ \tau\ } \mathcal{E}_1 \hookrightarrow K.$$

Let $\widetilde{\eta}\in\operatorname{Ext}^1_{\mathcal{O}_{\widetilde{Y}}}(E,\widetilde{K})$ corresponding to the first exact sequence of theorem 5.2.1. Then $\widetilde{\theta}(\widetilde{\eta})$ is the composition

$$E\otimes L(D) \xrightarrow{\ \tau\otimes I_{\mathcal{O}_C(D)}\ } \mathcal{E}_1(D) \hookrightarrow \widetilde{K}.$$

5.2.2. *The case of the first canonical filtration* – We consider the exact sequence on Y

$$0\longrightarrow\mathcal{E}_1\longrightarrow\mathcal{E}\longrightarrow\mathcal{E}_{|C}\longrightarrow 0\ ,$$

(i.e. we suppose that $K=\mathcal{E}_1$ and $E=\mathcal{E}_{|C}$) and $\eta\in\operatorname{Ext}^1_{\mathcal{O}_Y}(E,K)$ corresponding to it. In this case we have $N=0$, hence $\widetilde{K}=\mathcal{E}_1(D)=K(D)$, we have an exact sequence on $\widetilde{Y}$

$$0\longrightarrow K(D)\longrightarrow\widetilde{\mathcal{E}}\longrightarrow\widetilde{\mathcal{E}}_{|C}=\mathcal{E}_{|C}\longrightarrow 0$$

and $\widetilde{\theta}(\widetilde{\eta})=\theta(\eta)\otimes I_{\mathcal{O}_C(D)}$.

Now consider extensions
$$0 \longrightarrow K \longrightarrow \mathcal{E} \xrightarrow{p} E \longrightarrow 0$$
corresponding to $\eta \in \mathrm{Ext}^1_{\mathcal{O}_Y}(E, K)$ such that $\theta(\eta) : E \otimes L \to K$ is surjective. According to lemma 3.4.3 this condition is equivalent to the fact that $K = \mathcal{E}_1$ and $E = \mathcal{E}_{|C}$. Let $\mathrm{Surj}(E \otimes L, K)$ denote the set of surjective morphisms $E \otimes L \to K$.

Let $0 \to K \to \mathcal{E} \to E \to 0$ be an extension, corresponding to $\eta \in \mathrm{Ext}^1_{\mathcal{O}_Y}(E, K)$ such that $\theta(\eta)$ is surjective. Let $E' \subset E$ be a subbundle such that the restriction $E' \otimes L \to K$ of $\theta(\eta)$ is surjective. Let $\eta' \in \mathrm{Ext}^1_{\mathcal{O}_Y}(E', K)$ be induced by η, and $0 \to K \to \mathcal{E}' \to E' \to 0$ the corresponding exact sequence. Then the associated morphism $E' \otimes L \to K$ is the restriction of $\theta(\eta)$, and we have a commutative diagram on Y

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & \mathcal{E}' \arrow[r] \arrow[d, hook] & E' \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & K \arrow[r] & \mathcal{E} \arrow[r] & E \arrow[r] & 0
\end{tikzcd}$$

Similarly let $K \twoheadrightarrow K''$ be a quotient bundle of K. Let $\eta'' \in \mathrm{Ext}^1_{\mathcal{O}_Y}(E, K'')$ be induced by η, and $0 \to K'' \to \mathcal{E}'' \to E \to 0$ the corresponding exact sequence. The associated morphism $E \otimes L \to K''$ is the composition

$$E \otimes L \xrightarrow{\theta(\eta)} K \twoheadrightarrow K'',$$

hence it is surjective, and we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, two heads] & \mathcal{E} \arrow[r] \arrow[d, two heads] & E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K'' \arrow[r] & \mathcal{E}'' \arrow[r] & E \arrow[r] & 0
\end{tikzcd}$$

5.2.3. Lemma: **1 –** *We have a commutative diagram on* $\widetilde{Y}$

$$\begin{tikzcd}
0 \arrow[r] & K(D) \arrow[r] \arrow[d, equal] & \widetilde{\mathcal{E}'} \arrow[r] \arrow[d, hook] & E' \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & K(D) \arrow[r] & \widetilde{\mathcal{E}} \arrow[r] & E \arrow[r] & 0
\end{tikzcd}$$

(where the two horizontal sequences are implied by theorem 5.2.1).

2 – *We have a commutative diagram on* $\widetilde{Y}$

$$\begin{tikzcd}
0 \arrow[r] & K(D) \arrow[r] \arrow[d, two heads] & \widetilde{\mathcal{E}} \arrow[r] \arrow[d, two heads] & E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K''(D) \arrow[r] & \widetilde{\mathcal{E}''} \arrow[r] & E \arrow[r] & 0
\end{tikzcd}$$

(where the two horizontal sequences are implied by theorem 5.2.1).

Proof. We will prove only **1-** (**2-** is analogous). The exact sequences on Y are coming from the first canonical filtrations, and the exact sequences of lemma 5.2.3 too. Hence the existence of the diagram comes from the functoriality of the canonical filtrations. The fact that the vertical morphisms are equality on the left and inclusions on the middle and the right can be seen by taking local descriptions of $\widetilde{\mathcal{E}}'$ and $\widetilde{\mathcal{E}}$ as in theorem 5.2.1. $\square$

5.2.4. Theorem: *Suppose that* $\mathrm{Surj}(E\otimes L,K)\neq\emptyset$. *Then there exists a linear map*

$$\nu : \mathrm{Ext}^1_{\mathcal{O}_Y}(E,K) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{\widetilde{Y}}}(E,K(D))$$

such that

1 − *For every* $\eta\in\mathrm{Ext}^1_{\mathcal{O}_Y}(E,K)$ *such that* $\theta(\eta)$ *is surjective, if* $0\to K\to\mathcal{E}\to E\to 0$ *is the exact sequence corresponding to* η, *then* $0\to K(D)\to\widetilde{\mathcal{E}}\to E\to 0$ *is the exact sequence corresponding to* $\nu(\eta)$.

2 − *We have a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_{\mathcal{O}_C}(E,K) \arrow[r, "i"] \arrow[d, "\alpha"] & \mathrm{Ext}^1_{\mathcal{O}_Y}(E,K) \arrow[r, "\theta"] \arrow[d, "\nu"] & \mathrm{Hom}(E\otimes L,K) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1_{\mathcal{O}_C}(E,K(D)) \arrow[r, "\widetilde{i}"] & \mathrm{Ext}^1_{\mathcal{O}_{\widetilde{Y}}}(E,K(D)) \arrow[r, "\widetilde{\theta}"] & \mathrm{Hom}(E\otimes L(D),K(D)) \arrow[r] & 0
\end{tikzcd}$$

(where α *is induced by the canonical section of* $\mathcal{O}_C(D)$*).*

Proof. We can already define ν on the open subset of $\mathrm{Ext}^1_{\mathcal{O}_Y}(E,K)$ of elements η such that $\theta(\eta)$ is surjective. We will prove that it can be extended to a linear map.

Let $n\geq 2$ be an integer and $\eta_1,\ldots,\eta_n\in\mathrm{Ext}^1_{\mathcal{O}_Y}(E,K)$ such that $\theta(\eta_1),\ldots,\theta(\eta_n)$ are surjective, as well as $\theta(\eta_1+\cdots+\eta_n)$. For $1\leq i\leq n$ let

$$0\longrightarrow K\longrightarrow\mathcal{E}_i\longrightarrow E\longrightarrow 0$$

be the exact sequence corresponding to η_i. We have a commutative diagram on Y with exact horizontal sequences

$$\begin{tikzcd}
0 \arrow[r] & K\oplus\cdots\oplus K \arrow[r] \arrow[d, "f"] & \mathcal{E}_1\oplus\cdots\oplus\mathcal{E}_n \arrow[r] \arrow[d] & E\oplus\cdots\oplus E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r] & \mathcal{F} \arrow[r] & E\oplus\cdots\oplus E \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[u, equal] & \mathcal{E} \arrow[r] \arrow[u, hook] & E \arrow[r] \arrow[u, hook, "j"'] & 0
\end{tikzcd}$$

We can view $\mathrm{Ext}^1_{\mathcal{O}_Y}(E\oplus\cdots\oplus E,K\oplus\cdots\oplus K)$ as the space of $n\times n$-matrices with coefficients in $\mathrm{Ext}^1_{\mathcal{O}_Y}(E,K)$. The exact sequence at the top is associated to the matrix with zero terms outside the diagonal, and with i-th diagonal term η_i. It is also the direct sum of all the exact sequences $0\to K\to\mathcal{E}_i\to E\to 0$. The middle exact sequence is associated to $(\eta_1,\ldots,\eta_n)\in\mathrm{Ext}^1_{\mathcal{O}_Y}(E\oplus\cdots\oplus E,K)$. The exact sequence down corresponds to $\eta_1+\cdots+\eta_n$. The morphisms f and j are the identity on each factor.

We will how see how this diagram is transformed on $\widetilde{Y}$ (using theorem 5.2.1). The first transformed exact sequence will obviously be the direct sum of the exact sequences

$0 \to K(D) \to \widetilde{\mathcal{E}}_i \to E \to 0$. According to lemma 5.2.3 the whole transformed diagram is as follows

$$\begin{tikzcd}
0 \arrow[r] & K(D)\oplus\cdots\oplus K(D) \arrow[r] \arrow[d, "\widetilde{f}"] & \widetilde{\mathcal{E}_1}\oplus\cdots\oplus\widetilde{\mathcal{E}_n} \arrow[r] \arrow[d] & E\oplus\cdots\oplus E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K(D) \arrow[r] \arrow[d, equal] & \widetilde{\mathcal{F}} \arrow[r] & E\oplus\cdots\oplus E \arrow[r] & 0 \\
0 \arrow[r] & K(D) \arrow[r] & \widetilde{\mathcal{E}} \arrow[r] \arrow[u, hook] & E \arrow[r] \arrow[u, hook, "\widetilde{j}"'] & 0
\end{tikzcd}$$

The morphisms $\widetilde{f}$ and $\widetilde{j}$ are the identity on each factor. It follows from this diagram that the exact sequence $0 \to K(D) \to \widetilde{\mathcal{E}} \to F \to 0$ corresponds to $\nu(\eta_1) + \cdots \nu(\eta_n)$. Hence we have

$$\nu(\eta_1 + \cdots + \eta_n) \ = \ \nu(\eta_1) + \cdots + \nu(\eta_n) \ .$$

Let $\eta \in \mathrm{Ext}^1_{\mathcal{O}_Y}(E, K)$ such that $\theta(\eta)$ is surjective, and

$$0 \longrightarrow K \longrightarrow \mathcal{E} \xrightarrow{p} E \longrightarrow 0$$

the associated exact sequence. Let $\lambda \in \mathbb{C}^*$. Then $\lambda\eta$ corresponds to the exact sequence

$$0 \longrightarrow K \longrightarrow \mathcal{E} \xrightarrow{\frac{1}{\lambda}p} E \longrightarrow 0 \ .$$

It follows easily that $\nu(\lambda\eta) = \lambda\nu(\eta)$.

Now let $(\eta_1, \ldots, \eta_n)$ be a basis of $\mathrm{Ext}^1_{\mathcal{O}_Y}(E, K)$ such that for $1 \leq i \leq n$, $\theta(\eta_i)$ is surjective. We can now define coherently ν on the whole of $\mathrm{Ext}^1_{\mathcal{O}_Y}(E, K)$ by

$$\nu(\lambda_1\eta_1 + \cdots + \lambda_n\eta_n) \ = \ \lambda_1\nu(\eta_1) + \cdots + \lambda_n\nu(\eta_n) \ .$$

This proves **1-**.

It is clear that the right square of the diagram of **2-** is commutative, and we have to show that the left square is commutative. We will use 3.4.2. Let $\eta \in \mathrm{Ext}^1_{\mathcal{O}_Y}(E, K)$ such that $\theta(\eta)$ is surjective, and $0 \to K \to \mathcal{E} \to E \to 0$ the corresponding extension.

The sheaf $\mathcal{E}$ can be constructed using trivialisations as in the proof of 5.2.1: for every $P \in C$ there exists a neighbourhood U of P such that on the corresponding open subset U_Y of Y we have $\mathcal{E}_{|U_Y} \simeq \mathbb{E} \oplus F$, where $\mathbb{E}$ is a vector bundle on Y and F a vector bundle on C. So $\mathcal{E}$ can be constructed by gluing sheaves of type $\mathbb{E} \oplus F$. The gluing are isomorphisms

$$f : \mathbb{E} \oplus F \xrightarrow{\simeq} \mathbb{E}' \oplus F' \ ,$$

where $\mathbb{E}$, $\mathbb{E}'$ are vector bundles on U_Y, F, F' are vector bundles on U. Note that we have $\mathbb{E}_1 = \mathbb{E}'_1 = K_U$, hence $\mathbb{E}_{|U} = \mathbb{E}'_{|U} = (K \otimes L^*)_{|U}$, and $\mathcal{E}_U = \mathbb{E}_{|U} \oplus F = \mathbb{E}'_{|U} \oplus F'$. Suppose that on U_Y, U is defined by an equation $z \in \mathcal{O}_Y(U_Y)$. The isomorphism f is defined by a matrix $\begin{pmatrix} 1+za & zb \\ c & d \end{pmatrix}$ where $a \in \mathrm{End}(K_{|U})$, $b \in \mathrm{Hom}(F, K_{|U})$, $c \in \mathrm{Hom}(K_{|U}, F')$, $d \in \mathrm{Hom}(F, F')$. Let $u \in \mathrm{Ext}^1_{\mathcal{O}_C}(E, K)$. According to 3.4.2, the sheaf $\mathcal{E}'$ defined by $\eta + i(u)$ can be constructed by the same trivialisations as $\mathcal{E}$, but with gluings defined by matrices of type $\begin{pmatrix} 1+z(a+a') & z(b+b') \\ c & d \end{pmatrix}$ with $a' \in \mathrm{End}(K_{|U})$, $b' \in \mathrm{Hom}(F, K_{|U})$, (a', b') representing an element of $\mathrm{Hom}(E_{|U}, K_{|U})$. The pairs (a', b') (for various open subsets U covering C) make a cocycle representing u.

Now, according to lemma 5.1.1, the sheaf $\widetilde{\mathcal{E}}$ is obtained by gluing the sheaves $\phi^*(\mathbb{E})\oplus F$ on $U_{\widetilde{Y}}$ (the open subset of $\widetilde{Y}$ corresponding to U), using the isomorphisms $\widetilde{f}$ defined the matrices $\begin{pmatrix}1+zsa & zsb\\ c & d\end{pmatrix}$, where s is the canonical section of $\mathcal{O}_C(D)$. Similarly $\widetilde{\mathcal{E}}'$ is defined by the matrices $\begin{pmatrix}1+zs(a+a') & zs(b+b')\\ c & d\end{pmatrix}$. It follows that $\nu(\eta+i(u))=\nu(\eta)+u'$, where u' is defined by a cocycle of pairs of type (sa',sb'), i.e. we have $u'=\alpha(u)$. □

5.3. The dual construction

If instead of 5.2.2 we use the second canonical filtration of a quasi locally free sheaf $\mathcal{E}$:

$$0\longrightarrow\mathcal{E}^{(1)}\longrightarrow\mathcal{E}\longrightarrow\mathcal{E}/\mathcal{E}^{(1)}=\mathcal{E}_1\otimes L^*\longrightarrow 0\ ,$$

to describe the sheaf $\widetilde{\mathcal{E}}$ using theorem 5.2.1, we find that the sheaf $\widetilde{K}$ does not depend only on $\mathcal{E}_1\otimes L^*$ and $\mathcal{E}^{(1)}$. We have an exact sequence $0\to\mathcal{E}_1(D)\to\widetilde{K}\to N\to 0$, where N and the extension may vary even if $\mathcal{E}_1\otimes L^*$ and $\mathcal{E}^{(1)}$ remain constant.

To use the second canonical filtration, one can consider the following sheaf on $\widetilde{Y}$ associated to $\mathcal{E}$:

$$\widehat{\mathcal{E}}\ =\ \left(\widetilde{\mathcal{E}^\vee}\right)^\vee$$

(cf. 3.3.3). We have the exact sequence

$$0\longrightarrow(\mathcal{E}_1\otimes L^*)^\vee=\mathcal{E}_1^*\otimes L^2\longrightarrow\mathcal{E}^\vee\longrightarrow\mathcal{E}^{(1)\vee}=\mathcal{E}^{(1)*}\otimes L\longrightarrow 0\ ,$$

corresponding to the first canonical filtration of $\mathcal{E}^\vee$. We have then the exact sequences on $\widetilde{Y}$:

$$0\longrightarrow\mathcal{E}_1^*\otimes L^2(D)\longrightarrow\widetilde{\mathcal{E}^\vee}\longrightarrow\mathcal{E}^{(1)*}\otimes L\longrightarrow 0\ ,$$

corresponding to the first canonical filtration of $\widetilde{\mathcal{E}^\vee}$, and

$$0\longrightarrow\mathcal{E}^{(1)}(D)\longrightarrow\widehat{\mathcal{E}}\longrightarrow\mathcal{E}_1\otimes L^*\longrightarrow 0\ ,$$

corresponding to the second canonical filtration of $\widehat{\mathcal{E}}$.

Now consider extensions

$$0\longrightarrow K\longrightarrow\mathcal{E}\xrightarrow{p}E\longrightarrow 0$$

corresponding to $\eta\in\mathrm{Ext}^1_{\mathcal{O}_Y}(E,K)$ such that $\theta(\eta):E\otimes L\to K$ is injective (as a morphism of vector bundles). According to lemma 3.4.3 this condition is equivalent to the fact that $\mathcal{E}$ is quasi locally free, $K=\mathcal{E}^{(1)}$ and $E=\mathcal{E}_1\otimes L^*$. Let $\mathrm{Inj}(E\otimes L,K)$ denote the set of injective morphisms of vector bundles $E\otimes L\to K$. The following result is similar to theorem 5.2.4 and can be proved in the same way:

5.3.1. Theorem: *Suppose that* $\mathrm{Inj}(E\otimes L,K)\neq\emptyset$. *Then there exists a linear map*

$$\nu':\mathrm{Ext}^1_{\mathcal{O}_Y}(E,K)\longrightarrow\mathrm{Ext}^1_{\mathcal{O}_{\widetilde{Y}}}(E,K(D))$$

such that

1 − *For every $\eta \in \mathrm{Ext}^1_{\mathcal{O}_Y}(E,K)$ such that the morphism of vector bundles $\theta(\eta)$ is injective, if $0 \to K \to \mathcal{E} \to E \to 0$ is the exact sequence corresponding to η, then $0 \to K(D) \to \widehat{\mathcal{E}} \to E \to 0$ is the exact sequence corresponding to $\nu(\eta)$.*

2 − *We have a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_{\mathcal{O}_C}(E,K) \arrow[r, "i"] \arrow[d, "\alpha"] & \mathrm{Ext}^1_{\mathcal{O}_Y}(E,K) \arrow[r, "\theta"] \arrow[d, "\nu'"] & \mathrm{Hom}(E\otimes L, K) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1_{\mathcal{O}_C}(E,K(D)) \arrow[r, "\widetilde{i}"] & \mathrm{Ext}^1_{\mathcal{O}_{\widetilde{Y}}}(E,K(D)) \arrow[r, "\widetilde{\theta}"] & \mathrm{Hom}(E\otimes L(D), K(D)) \arrow[r] & 0
\end{tikzcd}$$

(where α is induced by the canonical section of $\mathcal{O}_C(D)$).

If $K = E\otimes L$, then of course $\nu = \nu'$, and in this case the injective (or surjective) morphisms correspond to locally free $\mathcal{E}$.

6. Coherent sheaves on reducible deformations of primitive double curves

6.1. Preliminaries

(cf. [10], [11])

Let C be a projective irreducible smooth curve and $Y = C_2$ a primitive double curve, with underlying smooth curve C, and associated line bundle L on C. Let S be a smooth curve, $P \in S$ and $\pi : \mathcal{C} \to S$ a *maximal reducible deformation of* Y (cf. [10]). This means that

(i) $\mathcal{C}$ is a reduced algebraic variety with two irreducible components $\mathcal{C}_1, \mathcal{C}_2$.
(ii) We have $\pi^{-1}(P) = Y$. So we can view C as a curve in $\mathcal{C}$.
(iii) For $i = 1,2$, let $\pi_i : \mathcal{C}_i \to S$ be the restriction of π. Then $\pi_i^{-1}(P) = C$ and π_i is a flat family of smooth irreducible projective curves.
(iv) For every $z \in S\backslash\{P\}$, the components $\mathcal{C}_{1,z}, \mathcal{C}_{2,z}$ of $\mathcal{C}_z$ meet transversally.

For every $z \in S\backslash\{P\}$, $\mathcal{C}_{1,z}$ and $\mathcal{C}_{2,z}$ meet in exactly $-\deg(L)$ points. If $\deg(L) = 0$, then π (or $\mathcal{C}$) is called a *fragmented deformation*.

Let $\mathcal{Z} \subset \mathcal{C}$ be the closure in $\mathcal{C}$ of the locus of the intersection points of the components of $\pi^{-1}(z)$, $z \neq P$. Since S is a curve, $\mathcal{Z}$ is a curve of $\mathcal{C}_1$ and $\mathcal{C}_2$. It intersects C in a finite number of points. If $x \in C$, let r_x be the number of branches of $\mathcal{Z}$ at x and s_x the sum of the multiplicities of the intersections of these branches with C.

Let $t \in \mathcal{O}_{S,P}$ be a generator of the maximal ideal. We will also denote π^*t by π and π_i^*t by π_i. So we have $\pi = (\pi_1,\pi_2) \in \mathcal{O}_{\mathcal{C}}(-C)$.

6.1.1. Theorem: *(cf.* [11], *th. 3.1.1) Let $x \in C$. Then*

1 − *There exists an unique integer $p > 0$ such that $\mathcal{I}_{C,x}/\langle(\pi_1,\pi_2)\rangle$ is generated by the image of $(\pi_1^p\lambda_1, 0)$, for some $\lambda_1 \in \mathcal{O}_{\mathcal{C}_1,x}$ not divisible by π_1. This integer does not depend on x.*

2 – *λ_1 is unique up to multiplication by an invertible element of $\mathcal{O}_{\mathcal{C}_1,x}$, and $(\pi_1^p\lambda_1, 0)$ is a generator of the ideal $\mathcal{I}_{\mathcal{C}_1,\mathcal{C},x}$ of $\mathcal{C}_1$ in $\mathcal{C}$.*

3 – *Let m_x be the multiplicity of $\lambda_{1|C} \in \mathcal{O}_{C,x}$. Then we have $m_x > 0$ if and only if $x \in \mathcal{Z} \cap C$, and in this case we have $m_x = r_x = s_x$, and the branches of $\mathcal{Z}$ at x intersect transversally with C. Moreover*

$$L \;\simeq\; \mathcal{O}_C(-\sum_{x\in\mathcal{Z}\cap C} r_x x) \;\simeq\; \mathcal{I}_{\mathcal{Z}\cap C,C} \ .$$

4 – *There exists $\lambda_2 \in \mathcal{O}_{\mathcal{C}_2,x}$ not divisible by π_2, such that $\mathcal{I}_{C,x}/\langle(\pi_1,\pi_2)\rangle$ is generated by the image of $(0, \pi_2^p\lambda_2)$, and λ_2 unique up to multiplication by an invertible element of $\mathcal{O}_{\mathcal{C}_2,x}$.*

In the preceding theorem, it is even possible to choose λ_1, λ_2 such that $(\lambda_1,\lambda_2) \in \mathcal{O}_{\mathcal{C},x}$ ([11], corollary 3.1.2). The ideal sheaf of $\mathcal{Z}$ in $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) at x is generated by λ_1 (resp. λ_2).

Let

$$\mathcal{Z}_0 \;=\; \mathcal{C}_1 \cap \mathcal{C}_2 \;\subset \mathcal{C} \ .$$

We have then $(\mathcal{Z}_0)_{red} = \mathcal{Z} \cup C$. The ideal sheaf $\mathbb{L}_1 = \mathcal{I}_{Z_0,\mathcal{C}_1}$ (resp. $\mathbb{L}_2 = \mathcal{I}_{Z_0,\mathcal{C}_2}$) of $\mathcal{Z}_0$ in $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) at x is generated by $\lambda_1\pi_1^p$ (resp. $\lambda_2\pi_2^p$). Hence $\mathbb{L}_1$ (resp. $\mathbb{L}_2$) is a line bundle on $\mathcal{C}_1$ (resp. $\mathcal{C}_2$). The ideal sheaf $\mathcal{I}_{Z,\mathcal{C}_1}$ (resp. $\mathcal{I}_{Z,\mathcal{C}_2}$) of $\mathcal{Z}$ in $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) is canonically isomorphic to $\mathbb{L}_1$ (resp. $\mathbb{L}_2$). The p-th infinitesimal neighbourhoods of C in $\mathcal{C}_1$, $\mathcal{C}_2$ (generated respectively by π_1^p and π_2^p) are canonically isomorphic, we will denote them by $C^{(p)}$. We have also a canonical isomorphism $\mathbb{L}_{1|C^{(p)}} \simeq \mathbb{L}_{2|C^{(p)}}$, and $\mathbb{L}_{1|C} \simeq \mathbb{L}_{2|C} \simeq L$. It is also possible, by replacing S with a smaller neighbourhood of P, to assume that $\mathbb{L}_{1|\mathcal{Z}_0} \simeq \mathbb{L}_{2|\mathcal{Z}_0}$. Let $\mathbb{L} = \mathbb{L}_{1|\mathcal{Z}_0} = \mathbb{L}_{2|\mathcal{Z}_0}$.

We have $\mathcal{I}_{\mathcal{C}_1,\mathcal{C}} = \mathbb{L}_2$ and $\mathcal{I}_{\mathcal{C}_2,\mathcal{C}} = \mathbb{L}_1$.

Let $\alpha \in \mathcal{O}_{\mathcal{C}_1,x}$. Then there exists $\beta \in \mathcal{O}_{\mathcal{C}_2,x}$ such that $(\alpha,\beta) \in \mathcal{O}_{\mathcal{C},x}$. By associating the class of β to that of α we obtain an isomorphism of rings

$$\Phi_x : \mathcal{O}_{\mathcal{C}_1,x}/(\lambda_1\pi_1^p) \longrightarrow \mathcal{O}_{\mathcal{C}_2,x}/(\lambda_1\pi_2^p)$$

(the two are of course isomorphic to $\mathcal{O}_{Z_0,x}$), and we have $\Phi_x(\pi_1) = \pi_2$.

6.1.2. *The associated fragmented deformation* – The preceding isomorphism Φ_x induces an isomorphism

$$\Psi_x : \mathcal{O}_{\mathcal{C}_1,x}/(\pi_1^p) \longrightarrow \mathcal{O}_{\mathcal{C}_2,x}/(\pi_2^p) \ .$$

These isomorphisms define a fragmented deformation $\rho : \mathcal{D} \to S$ with the same components $\mathcal{C}_1$, $\mathcal{C}_2$ (which meet only along C in $\mathcal{D}$), and we have a canonical surjective morphism $\eta : \mathcal{D} \to \mathcal{C}$, induced by the obvious inclusion $\mathcal{O}_\mathcal{C} \subset \mathcal{O}_\mathcal{D}$. So we have

$$\mathcal{O}_{\mathcal{C},x} \;=\; \{(\alpha,\beta) \in \mathcal{O}_{\mathcal{C}_1,x} \times \mathcal{O}_{\mathcal{C}_2,x} \ ; \ \alpha \equiv \beta \bmod \lambda_1\pi_1^p\}$$

and

$$\mathcal{O}_{\mathcal{D},x} \;=\; \{(\alpha,\beta) \in \mathcal{O}_{\mathcal{C}_1,x} \times \mathcal{O}_{\mathcal{C}_2,x} \ ; \ \alpha \equiv \beta \bmod \pi_1^p\}$$

6.1.3. Lemma: *The primitive double curve $\rho^{-1}(P)$ is the blowing up of $\mathcal{Z} \cap C = \sum_{x\in\mathcal{Z}\cap C} r_x x \subset Y$.*

Proof. Let $C_2' = \rho^{-1}(P)$. It is clear that the morphism induced by ρ, $C_2' \to Y$, is an isomorphism between the open subsets corresponding to $C\backslash C \cap \mathcal{Z}$. Let $x \in C\backslash C \cap \mathcal{Z}$. Let $\lambda_1 \in \mathcal{O}_{\mathcal{C}_1,x}$ be a generator of the ideal of $\mathcal{Z}$ in $\mathcal{C}_1$. Then the maximal ideal of $\mathcal{O}_{C_2',x}$ (resp. $\mathcal{O}_{Y,x}$) is generated by the image z' of $(\pi_1^p\lambda_1, 0)$ (resp. by the image z of $(\pi_1^p, 0)$). If $t \in \mathcal{O}_{\mathcal{C}_1,x}$ is over a generator of the maximal ideal of $\mathcal{O}_{C,x}$, then we have $\lambda_1 = \alpha t^{r_x}$, for some invertible $\alpha \in \mathcal{O}_{\mathcal{C}_1,x}$. It follows easily that there is a canonical isomorphism

$$\mathcal{O}_{C_2',x} \;\simeq\; \mathcal{O}_{Y,x}\left[\frac{z}{t^m}\right] \ ,$$

where $\mathcal{O}_{Y,x}\left[\dfrac{z}{z^m}\right]$ is the subring generated by $\mathcal{O}_{Y,z}$ and $\dfrac{z}{t^m}$ in the localisation $\mathcal{O}_{Y,x}\left[\dfrac{1}{t}\right]$. This proves the lemma (cf. 5.1). □

6.1.4. Proposition: *The morphism $\eta : \mathcal{D} \to \mathcal{C}$ is the blowing-up of $\mathcal{Z}$.*

Proof. It suffices to prove that η satisfies the universal property of blowing-ups: if X is an algebraic variety and $f : X \to \mathcal{C}$ is a morphism such that the ideal sheaf $f^{-1}(\mathcal{I}_{\mathcal{Z},\mathcal{C}})\mathcal{O}_X$ is invertible, then there is a unique morphism $g : X \to \mathcal{D}$ such that $f = \eta \circ g$. The problem is local, we need only to prove that for every $x \in X$, there exists a neighbourhood U of x such that $f_{|U}$ can be factorised through $\mathcal{D}$, and this factorisation is unique. This is obvious if $f(x) \in \mathcal{C}\backslash\mathcal{Z}$, and well known if $f(x) \in \mathcal{Z}\backslash C$. Suppose that $f(x) \in \mathcal{Z} \cap C$. For $i = 1, 2$, let $\lambda_i \in \mathcal{O}_{\mathcal{C}_i,x}$ be a generator of the ideal of $\mathcal{Z}$ in $\mathcal{C}_i$, such that $(\lambda_1, \lambda_2) \in \mathcal{O}_{\mathcal{C},x}$. Then $\mathcal{I}_{\mathcal{Z},\mathcal{C},x}$ is generated by (λ_1, λ_2) and $(\pi_1^p\lambda_1, 0)$. Let $\gamma \in (f^{-1}(\mathcal{I}_{\mathcal{Z},\mathcal{C}})\mathcal{O}_X)_x$ be a generator, it is not a zero divisor. Then there exist $a, b \in \mathcal{O}_{X,x}$ such that

$$\gamma \;=\; a.f^*(\lambda_1, \lambda_2) + b.f^*(\pi_1^p\lambda_1, 0) \ .$$

There exist $\beta_1, \beta_2 \in \mathcal{O}_{X,x}$ such that

$$f^*(\pi_1^p\lambda_1, 0) \;= \beta_1\gamma \ , \quad f^*(0, \pi_2^p\lambda_2) \;=\; \beta_2\gamma \ .$$

Since $(\pi_1^p\lambda_1, 0)(0, \pi_2^p\lambda_2) = 0$, we have $\beta_1\beta_2\gamma^2 = 0$, hence $\beta_1\beta_2 = 0$. Hence $\beta_1(x) = 0$ or $\beta_2(x) = 0$. We can suppose that $\beta_1(x) = 0$. We have

$$(1 - \beta_1 b)f^*(\pi_1^p\lambda_1, 0) \;=\; \beta_1 a.f^*(\lambda_1, \lambda_2) \ .$$

But $1 - \beta_1 b$ is invertible, let $\rho = \dfrac{\beta_1 a}{1 - \beta_1 b}$. We have then

$$f^*(\pi_1^p\lambda_1, 0) \;=\; \rho.f^*(\lambda_1, \lambda_2) \tag{9}$$

and we can suppose that $\gamma = f^*(\lambda_1, \lambda_2)$, i.e. $a = 1$ and $b = 0$. Now we describe the factorisation of f^* at x. According to 6.1.2, $\mathcal{O}_{\mathcal{D},f(x)}$ is generated by $\mathcal{O}_{\mathcal{C},f(x)}$ and $(\pi_1^p, 0)$. Since $(\pi_1^p\lambda_1, 0) = (\pi_1^p, 0)(\lambda_1, \lambda_2)$, the equation (9) suggests that we must take $f^*(\pi_1^p, 0) = \rho$. This will define uniquely the extension of f^* provided that we show that there is no ambiguity in this definition, that is if Q is a polynomial in one variable with coefficients in $\mathcal{O}_{\mathcal{C},f(x)}$ such that $Q((\pi_1^p, 0)) = 0$ in $\mathcal{O}_{\mathcal{D},f(x)}$, then $Q(\rho) = 0$ in $\mathcal{O}_{X,x}$. Let m be the degree of Q. Then $(\lambda_1, \lambda_2)^m Q((\pi_1^p, 0)) \in \mathcal{O}_{\mathcal{C},x}$. We have

$$f^*\left((\lambda_1, \lambda_2)^m Q((\pi_1^p, 0))\right) \;= 0 \;=\; f^*(\lambda_1, \lambda_2)^m Q(\rho) \ ,$$

and since $f^*(\lambda_1, \lambda_2)$ is not a zero divisor, we have $Q(\rho) = 0$. □

6.1.5. *The relative case* – Let S' be a smooth connected curve and $P' \in S'$. Let $f : S' \to S$ be a non constant morphism such that $f(P') = P$, and suppose that $f^{-1}(P) = \{P'\}$ (which can be done by shrinking S'). Then $f^*(\mathcal{C}) = \mathcal{C} \times_S S' \to S'$ is again a maximal deformation of Y. The integer p remains the same if and only if the tangent map of f at P is not zero.

6.2. Coherent sheaves on reducible deformations

We use the notations of 6.1, and we suppose that L can be written as

$$L \ = \ \mathcal{O}_C(-P_1 - \cdots - P_d) \ ,$$

where $P_1, \cdots, P_d$ are distinct points of C, or that $L = \mathcal{O}_C$.

6.2.1. *Locally free resolutions* – For every $x \in \mathcal{Z} \cap C$, let $(\lambda_{1x}, \lambda_{2x}) \in \mathcal{O}_{\mathcal{C},x}$ such that $\mathcal{I}_{C,x}/\langle(\pi_1, \pi_2)\rangle$ is generated by the image of $(\pi_1^p\lambda_1, 0)$, and also by the image of $(0, \pi_2^p\lambda_2)$. If $x \in C\backslash(\mathcal{Z} \cap C)$, let $(\lambda_{1x}, \lambda_{2x}) = (1, 1)$.

Let $\mathbb{D}$ the ideal sheaf on $\mathcal{C}$ such that

– $\mathbb{D} = \mathcal{O}_\mathcal{C}$ on $\mathcal{C}\backslash(\mathcal{Z} \cap C)$,
– For every $x \in \mathcal{Z} \cap C$, $\mathbb{D}_x = ((\lambda_{1x}, \lambda_{2x}))$.

The sheaf $\mathbb{D}$ is well defined (because $\mathcal{Z} \cap C$ is finite) and locally free. But it depends of the choice of $(\lambda_{1x}, \lambda_{2x})$ $(x \in \mathcal{Z} \cap C)$.

We have a canonical morphism

$$r_1 : \mathbb{D} \longrightarrow \mathcal{O}_\mathcal{C}$$

such that $\mathrm{im}(r_1) = \mathcal{I}_{\mathcal{C}_2}$, which, for every $x \in C$, sends $(\alpha, \beta) \in \mathbb{D}_x$ to $(\pi_1^p\alpha, 0)$. And similarly, we have a canonical morphism

$$r_2 : \mathbb{D} \longrightarrow \mathcal{O}_\mathcal{C} \ ,$$

with image $\mathcal{I}_{\mathcal{C}_1}$.

It is easy to see that we have exact sequences

$$\ldots \mathbb{D}^3 \xrightarrow{r_1 \otimes I_{\mathbb{D}^2}} \mathbb{D}^2 \xrightarrow{r_2 \otimes I_\mathbb{D}} \mathbb{D} \xrightarrow{r_1} \mathcal{O}_\mathcal{C} \longrightarrow \mathcal{O}_{\mathcal{C}_2} \longrightarrow 0 \ , \tag{10}$$

$$\ldots \mathbb{D}^3 \xrightarrow{r_2 \otimes I_{\mathbb{D}^2}} \mathbb{D}^2 \xrightarrow{r_1 \otimes I_\mathbb{D}} \mathbb{D} \xrightarrow{r_2} \mathcal{O}_\mathcal{C} \longrightarrow \mathcal{O}_{\mathcal{C}_1} \longrightarrow 0 \ , \tag{11}$$

i.e. locally free resolutions of $\mathcal{O}_{\mathcal{C}_1}$ and $\mathcal{O}_{\mathcal{C}_2}$. It follows immediately that

6.2.2. Corollary: *For $i = 1, 2$, we have* $\mathcal{E}xt^1_{\mathcal{O}_\mathcal{C}}(\mathcal{O}_{\mathcal{C}_i}, \mathcal{O}_\mathcal{C}) = 0$ *and* $\mathrm{Tor}^1_{\mathcal{O}_\mathcal{C}}(\mathcal{O}_{\mathcal{C}_1}, \mathcal{O}_{\mathcal{C}_2}) = 0$.

6.2.3. Lemma: *Let $x \in C$, and M a torsion free $\mathcal{O}_{\mathcal{C}_i,x}$-module ($i = 1$ or 2). Then for $j \geq 1$, we have* $\mathrm{Tor}^j_{\mathcal{O}_{\mathcal{C},x}}(\mathcal{O}_{Y,x}, M) = 0$.

Proof. This follows immediately from the free resolution of $\mathcal{O}_Y$

$$0 \longrightarrow \mathcal{O}_\mathcal{C} \xrightarrow{\pi} \mathcal{O}_\mathcal{C} \longrightarrow \mathcal{O}_Y \longrightarrow 0 \ .$$

□

6.2.4. *Extensions of vector bundles* – Let E_i, $i = 1, 2$, be a vector bundle on $\mathcal{C}_i$. We are interested in extensions on $\mathcal{C}$

$$0 \longrightarrow E_1 \longrightarrow \mathcal{E} \longrightarrow E_2 \longrightarrow 0 \tag{12}$$

(of course the case of extensions $0 \to E_2 \to \mathcal{E} \to E_1 \to 0$ is analogous). The sheaf $\mathcal{E}$ is then flat on S.

6.2.5. Proposition: *There are canonical isomorphisms*

$$\mathcal{E}xt^1_{\mathcal{O}_\mathcal{C}}(E_2, E_1) \simeq \mathcal{H}om(E_{2|\mathcal{Z}_0}, \mathbb{L}^*_{1|\mathcal{Z}_0} \otimes E_{1|\mathcal{Z}_0}) \ , \quad \mathrm{Ext}^1_{\mathcal{O}_\mathcal{C}}(E_2, E_1) \simeq \mathrm{Hom}(E_{2|\mathcal{Z}_0}, \mathbb{L}^* \otimes E_{1|\mathcal{Z}_0}) \ .$$

Proof. The first equality is an easy consequence of (10). The second follows from the Ext spectral sequence ([15], 7.3) and the fact that $\mathcal{H}om(E_2, E_1) = 0$. □

Consider an extension (12), corresponding to $\sigma \in \mathrm{Hom}(E_{2|\mathcal{Z}_0}, \mathbb{L}^*_{1|\mathcal{Z}_0} \otimes E_{1|\mathcal{Z}_0})$. Then according to lemma 6.2.3 we have an exact sequence

$$0 \longrightarrow E_{1|C} \longrightarrow \mathcal{E}_{|Y} \longrightarrow E_{2|C} \longrightarrow 0$$

Recall that we have a canonical exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_C}(E_{2|C}, E_{1|C}) \xrightarrow{\ i\ } \mathrm{Ext}^1_{\mathcal{O}_Y}(E_{2|C}, E_{1|C}) \xrightarrow{\ \theta\ } \mathrm{Hom}(E_{2|C} \otimes L, E_{1|C}) \longrightarrow 0.$$

6.2.6. Lemma: *There are canonical morphisms*

$$\tau : \mathrm{Ext}^1_{\mathcal{O}_\mathcal{C}}(E_2, E_1) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_Y}(E_{2|C}, E_{1|C}) \ , \quad \overline{\tau} : \mathcal{E}xt^1_{\mathcal{O}_\mathcal{C}}(E_2, E_1) \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_Y}(E_{2|C}, E_{1|C}) \ ,$$

τ *associating to the extension* $0 \to E_1 \to \mathcal{E} \to E_2 \to 0$ *the extension*
$0 \to E_{1|C} \to \mathcal{E}_{|Y} \to E_{2|C} \to 0$.

Proof. Let $\mathcal{O}_\mathcal{C}(1)$ be an ample line bundle on $\mathcal{C}$. There exists a vector bundle $\mathbb{F}_0$ on $\mathcal{C}$ such that $\mathrm{Ext}^1_{\mathcal{O}_\mathcal{C}}(\mathbb{F}_0, E_1) = \{0\}$, $\mathrm{Ext}^1_{\mathcal{O}_Y}(\mathbb{F}_{0|Y}, E_{1|C}) = \{0\}$ and such that there exists a surjective morphism $f_0 : \mathbb{F}_0 \to E_2$ (we can take $\mathbb{F}_0$ of the form $\mathbb{F}_0 = \mathcal{O}_\mathcal{C}(-n) \otimes \mathbb{C}^k$, for suitable k and n). Let $N_0 = \ker(f_0)$, so we have an exact sequence

$$0 \longrightarrow N_0 \longrightarrow \mathbb{F}_0 \longrightarrow E_2 \longrightarrow 0 \ .$$

It follows from lemma 6.2.3 that this sequence restricted to Y is exact:

$$0 \longrightarrow N_{0|Y} \longrightarrow \mathbb{F}_{0|Y} \longrightarrow E_{2|C} \longrightarrow 0 \ .$$

Hence we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}(\mathbb{F}_0, E_1) & \longrightarrow & \mathrm{Hom}(N_0, E_1) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}_\mathcal{C}}(E_2, E_1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
\mathrm{Hom}(\mathbb{F}_{0|Y}, E_{1|C}) & \longrightarrow & \mathrm{Hom}(N_{0|Y}, E_{1|C}) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}_Y}(E_{2|C}, E_{1|C}) & \longrightarrow & 0
\end{array}$$

where the vertical arrows are the canonical morphisms. The morphism τ is then deduced immediately, and it is easy to see that it does not depend on f_0. The existence of $\overline{\tau}$ can be proved in the same way. □

We have a commutative diagram

$$
\begin{tikzcd}
\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(E_2,E_1) \arrow[r, equal] \arrow[d, "\tau"] & H^0(\mathcal{E}xt^1_{\mathcal{O}_Y}(E_{2|C},E_{1|C})) = \mathrm{Hom}(E_{2|\mathcal{Z}_0},\mathbb{L}^*_{1|\mathcal{Z}_0}\otimes E_{1|\mathcal{Z}_0}) \arrow[d, "r"] \\
\mathrm{Ext}^1_{\mathcal{O}_Y}(E_{2|C},E_{1|C}) \arrow[r, "\theta"] & H^0(\mathcal{E}xt^1_{\mathcal{O}_Y}(E_{2|C},E_{1|C})) = \mathrm{Hom}(E_{2|C}\otimes L,E_{1|C})
\end{tikzcd}
\tag{13}
$$

where $r : \mathrm{Hom}(E_{2|\mathcal{Z}_0},\mathbb{L}^*_{1|\mathcal{Z}_0}\otimes E_{1|\mathcal{Z}_0}) \to \mathrm{Hom}(E_{2|C}\otimes L,E_{1|C})$ is the restriction to C.

6.2.7. *Duality* – If $\mathcal{F}$ is a coherent sheaf on $\mathcal{C}$, we will denote its dual $\mathrm{Hom}(\mathcal{F},\mathcal{O}_{\mathcal{C}})$ by $\mathcal{F}^\vee$, and if F is a coherent sheaf on $\mathcal{C}_i$, its dual on $\mathcal{C}_i$ will be denoted by F^*. Note that, when we consider F as a sheaf on $\mathcal{C}$, we have

$$F^\vee \;=\; F^*\otimes\mathbb{L}_i\ .$$

According to corollary 6.2.2, we have an exact sequence on $\mathcal{C}$

$$0\longrightarrow E_2^*\otimes\mathbb{L}_2\longrightarrow\mathcal{E}^\vee\longrightarrow E_1^*\otimes\mathbb{L}_1\longrightarrow 0\ .$$

It follows from proposition 6.2.5 that we have a canonical isomorphism

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(E_1^*\otimes\mathbb{L}_1,E_2^*\otimes\mathbb{L}_2)\;\simeq\;\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(E_2,E_1)\ .$$

Of course, the elements of $\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(E_2,E_1)$ corresponding to the exact sequence (12) and its dual are the same.

6.2.8. *Polarisations* – Let $\mathcal{O}_{\mathcal{C}}(1)$ be an ample line bundle on $\mathcal{C}$. As usually, if $Z\subset\mathcal{C}$ is a subvariety, we will denote by $\mathcal{O}_Z(1)$ the restriction of $\mathcal{O}_{\mathcal{C}}(1)$ to Z. For every $s\in S\backslash\{P\}$, let D_1, D_2 be the irreducible components of $\mathcal{C}_s$. Let $\mathbf{d}=\deg(\mathcal{O}_C(1))$. Then since C belongs to the two families of curves $\mathcal{C}_1$, $\mathcal{C}_2$, we have

$$\deg(\mathcal{O}_{D_1}(1))\;=\;\mathbf{d}\;=\;\deg(\mathcal{O}_{D_2}(1))\ .$$

Let $\mathcal{E}$ be a coherent sheaf on $\mathcal{C}$, flat on S. Suppose that for every $s\in S$, $\mathcal{E}_s$ is torsion free. For $i=1,2$, let $\mathcal{E}_{s,i}=\mathcal{E}_{s|D_i}/T_i$, where T_i is the torsion subsheaf. Then $\mathcal{E}_{s,i}$ is a vector bundle on D_i. Let $r_i=\mathrm{rk}(\mathcal{E}_{s,i})$ and $d_i=\deg(\mathcal{E}_{s,i})$. From 4.2 we have an exact sequence

$$0\longrightarrow\mathcal{E}_s\longrightarrow\mathcal{E}_{s,1}\oplus\mathcal{E}_{s,2}\longrightarrow\bigoplus_{x\in D_1\cap D_2}\mathcal{O}_{\mathcal{C}_s,x}\otimes W_x\longrightarrow 0\ ,$$

where for every $x\in D_1\cap D_2$, W_x is a finite dimensional vector space. Then we have (cf. 4.4.2)

$$P_{\mathcal{E}_s}(m)\;=\;(r_1+r_2)\mathbf{d}m+(r_1+r_2)(1-g)+d_1+d_2-\sum_{x\in D_1\cap D_2}\dim(W_x)\ .$$

On the other hand, we have (cf. 3.2.5)

$$P_{\mathcal{E}_{|Y}}(m)\;=\;R(\mathcal{E}_{|Y})\mathbf{d}m+R(\mathcal{E}_{|Y})(1-g)+\mathrm{Deg}(\mathcal{E}_{|Y})\ .$$

Since $P_{\mathcal{E}_s}=P_{\mathcal{E}_{|Y}}$, we have

$$R(\mathcal{E}_{|Y})\;=\;r_1+r_2\ ,\quad \mathrm{Deg}(\mathcal{E}_{|Y})\;=\;d_1+d_2-\sum_{x\in D_1\cap D_2}\dim(W_x)\ .$$

6.3. Coherent sheaves on fragmented deformations

We keep the notations of 6.2. If $\deg(L)=0$, we say that $\pi : \mathcal{C}\to C$ is a *fragmented deformation* of Y. Hence for every $s\in S\backslash\{P\}$, $\mathcal{C}_s=\pi^{-1}(s)$ is the disjoint union on $\mathcal{C}_{1,s}$ and $\mathcal{C}_{2,s}$. In this case we have $L=\mathcal{O}_C$.

Suppose that π is a fragmented deformation and that $p=1$. In this case $\mathcal{C}$ is simple, it is the gluing of $\mathcal{C}_1$ and $\mathcal{C}_2$ along C, i.e. for every $x\in C$ we have

$$\mathcal{O}_{\mathcal{C},x} \ = \ \big\{(\phi_1,\phi_2)\in \mathcal{O}_{\mathcal{C}_1,x}\times\mathcal{O}_{\mathcal{C}_2,x}\ ;\ \phi_{1|C}=\phi_{2|C}\big\}\ .$$

We have $\mathcal{Z}_0=C$, hence the diagram (13)is

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(E_2,E_1) \arrow[r, equal] \arrow[d, "\tau"] & \mathrm{Hom}(E_{2|C},E_{1|C}) \arrow[d, equal] \\
\mathrm{Ext}^1_{\mathcal{O}_Y}(E_{2|C},E_{1|C}) \arrow[r, "\theta"] & \mathrm{Hom}(E_{2|C},E_{1|C})
\end{tikzcd}$$

It follows that

6.3.1. Proposition: *Given $f:E_{2|C}\to E_{1|C}$, there exists only one extension on Y*

$$0\longrightarrow E_{1|C}\longrightarrow \mathcal{F}\longrightarrow E_{2|C}\longrightarrow 0$$

corresponding to $\sigma\in\mathrm{Ext}^1_{\mathcal{O}_Y}(E_{2|C},E_{1|C})$ such that $\theta(\sigma)=f$, and which is the restriction to Y on an extension (12) *on $\mathcal{C}$.*

6.3.2. *The case of vector bundles* – The sheaf $\mathcal{E}$ is locally free if and only f is an isomorphism. In this case $\mathcal{E}$ is obtained by gluing E_1 and E_2 along $E_{1|C}\simeq E_{2|C}$, so it is unique.

6.4. Regular sheaves on reducible deformations

6.4.1. Definition: *A coherent sheaf $\mathcal{E}$ on $\mathcal{C}$ is called* regular *if it is locally free on $\mathcal{C}\backslash\mathcal{Z}_0$, and if for every $x\in\mathcal{Z}_0$ there exists a neighbourhood of x in $\mathcal{C}$, a vector bundle $\mathbb{E}$ on U, $i\in\{1,2\}$, and a vector bundle F on $U\cap\mathcal{C}_i$, such that $\mathcal{E}_{|U}\simeq\mathbb{E}\oplus F$.*

6.4.2. Lemma: *In the preceding definition, i, and the ranks of $\mathbb{E}$ and F are unique.*

Proof. Suppose that $\mathcal{E}_{|U}\simeq\mathbb{E}'\oplus F'$, with $\mathbb{E}'$ locally free on U and F' a vector bundle on $U\cap\mathcal{C}_j$. The induced isomorphism $\mathbb{E}\oplus F\to\mathbb{E}'\oplus F'$ is defined by a matrix $\begin{pmatrix}A & B\\ C & D\end{pmatrix}$. We have $B_{|\mathcal{Z}_0}=0$, hence $A_{|\mathcal{Z}_0}$ and $D_{|\mathcal{Z}_0}$ are isomorphisms. The result follows immediately. □

6.4.3. Proposition: *Let $\mathcal{E}$ be a coherent sheaf on $\mathcal{C}$. Then the following assertions are equivalent:*

(i) *$\mathcal{E}$ is regular (with $i=1$ in definition 6.4.1).*

(ii) *There exists an exact sequence* $0 \to E_2 \to \mathcal{E} \to E_1 \to 0$, *where for* $j = 1, 2$, E_j *is a vector bundle on* $\mathcal{C}_j$, *such that the associated morphism* $E_{1|\mathcal{Z}_0} \to \mathbb{L}^* \otimes E_{2|\mathcal{Z}_0}$ *is surjective on a neighbourhood of* C.

(iii) *There exists an exact sequence* $0 \to E_1 \to \mathcal{E} \to E_2 \to 0$, *where for* $j = 1, 2$, E_j *is a vector bundle on* $\mathcal{C}_j$, *such that the associated morphism* $E_{2|\mathcal{Z}_0} \to \mathbb{L}^* \otimes E_{1|\mathcal{Z}_0}$ *is injective (as a morphism of vector bundles) on a neighbourhood of* C.

We have a similar result by taking $i = 2$ *in definition 6.4.1.*

Proof. Suppose that (i) is true. Let $x \in \mathcal{Z}_0$ and U a neighbourhood of x such that $\mathcal{E}_{|U} \simeq \mathbb{E} \oplus F$, with $\mathbb{E}$ locally free on U and F a vector bundle on $U \cap \mathcal{C}_1$. Then we have $\mathcal{E}_{|U\cap\mathcal{C}_1} \simeq \mathbb{E}_{|U\cap\mathcal{C}_1} \oplus F$. Hence $E_1 = \mathcal{E}_{|\mathcal{C}_1}$ is a vector bundle on $\mathcal{C}_1$. The kernel of the restriction morphism $\mathcal{E} \to \mathcal{E}_{|\mathcal{C}_1}$ is $\mathbb{E} \otimes \mathbb{L}_2$, which is a vector bundle on $U \cap \mathcal{C}_2$. Hence the kernel E_2 of the restriction morphism $\mathcal{E} \to E_1$ is a vector bundle on $\mathcal{C}_2$. On U, the associated morphism $E_{1|\mathcal{Z}_0} \to \mathbb{L}^* \otimes E_{2|\mathcal{Z}_0}$ is the projection

$$\mathbb{E}_{|U\cap\mathcal{Z}_0} \oplus F_{|U\cap\mathcal{Z}_0} \longrightarrow \mathbb{E}_{|U\cap\mathcal{Z}_0} \ ,$$

hence it is surjective. This proves (ii). The proof of (iii) is similar, in this case E_2 is $\mathcal{E}_{\mathcal{C}_2}$ quotiented by its torsion subsheaf.

Suppose that (ii) is true. Let Let $x \in \mathcal{Z}_0$ and U a neighbourhood of x such that $E_{1|U\cap\mathcal{C}_1}$, $E_{2|U\cap\mathcal{C}_2}$ and $\mathbb{L}_{|\mathcal{Z}_0\cap U}$ are trivial, say $E_{1|U\cap\mathcal{C}_1} \simeq \mathcal{O}_{U\cap\mathcal{C}_1} \otimes \mathbb{C}^{r_1}$, $E_{2|U\cap\mathcal{C}_2} \simeq \mathcal{O}_{U\cap\mathcal{C}_2} \otimes \mathbb{C}^{r_2}$. We can also assume that the associated surjective morphism $E_{1|\mathcal{Z}_0\cap U} \to \mathbb{L}^* \otimes E_{2|\mathcal{Z}_0\cap U}$ is the projection

$$\mathcal{O}_{U\cap\mathcal{Z}_0} \otimes \mathbb{C}^{r_1} = (\mathcal{O}_{U\cap\mathcal{Z}_0} \otimes \mathbb{C}^{r_1-r_2}) \oplus (\mathcal{O}_{U\cap\mathcal{Z}_0} \otimes \mathbb{C}^{r_2}) \longrightarrow \mathcal{O}_{U\cap\mathcal{Z}_0} \otimes \mathbb{C}^{r_2} \ .$$

The extension corresponding to this morphism (cf. prop. 6.2.5) is the direct sum of

$$0 \longrightarrow \mathcal{O}_{U\cap\mathcal{C}_2} \otimes \mathbb{C}^{r_2} \longrightarrow \mathcal{O}_U \otimes \mathbb{C}^{r_2} \longrightarrow \mathcal{O}_{U\cap\mathcal{C}_1} \otimes \mathbb{C}^{r_2} \longrightarrow 0$$

and

$$0 \longrightarrow 0 \longrightarrow \mathcal{O}_{U\cap\mathcal{C}_1} \otimes \mathbb{C}^{r_1-r_2} \longrightarrow \mathcal{O}_{U\cap\mathcal{C}_1} \otimes \mathbb{C}^{r_1-r_2} \longrightarrow 0 \ ,$$

hence $\mathcal{E}_{|U} \simeq (\mathcal{O}_U \otimes \mathbb{C}^{r_2}) \oplus (\mathcal{O}_{U\cap\mathcal{C}_1} \otimes \mathbb{C}^{r_1-r_2})$, and (i) is proved. The proof that (iii) implies (i) is similar. $\square$

6.4.4. *Geometric construction of regular sheaves on fragmented deformations* – We suppose now that $\mathcal{C}$ is a fragmented deformation as in 6.3, with $p = 1$. Let E_i, $i = 1, 2$, a vector bundle on $\mathcal{C}_i$, and $\phi : E^*_{1|C} \to E^*_{2|C}$ an injective morphism of vector bundles. Viewing E^*_1 and E^*_2 as algebraic varieties, let Y be the variety obtained by gluing $E^*_{1|C} \subset E^*_1$ and the corresponding subbundle of $E^*_{2|C} \subset E^*_2$. The existence of Y is easily seen for example by using théorème 5.4 of [14]. We have a canonical projection $\rho : Y \to \mathcal{C}$, whose fibres are vector spaces. Let $\mathcal{E}$ be the sheaf of $\mathcal{O}_\mathcal{C}$-modules defined by: for every open subset $U \subset \mathcal{C}$, $\mathcal{E}(U)$ is the subset of $\mathcal{O}_Y(\rho^{-1}(U))$ of regular maps which are linear on the fibres of ρ. By considering open subsets on which E_1 and E_2 are trivial, it is easy to see that $\mathcal{E}$ is a regular sheaf, and that we have an exact sequence

$$0 \longrightarrow E_1 \longrightarrow \mathcal{E} \longrightarrow E_2 \longrightarrow 0$$

associated with the surjective morphism ${}^t\phi : E_{2|C} \to E_{1|C}$.

6.4.5. Proposition: *Let* $\mathcal{E}$ *be a coherent sheaf on* $\mathcal{C}$, *flat on* S. *Suppose that for every* $s \in S$, $\mathcal{E}_s$ *is torsion free, and that* $\mathcal{E}_{|Y}$ *is quasi locally free of rigid type (cf. 3.3.5). Then* $\mathcal{E}$ *is regular.*

Proof. We will suppose that $\mathcal{E}_{|Y}$ is locally isomorphic to $a\mathcal{O}_Y \oplus \mathcal{O}_C$ for some integer $a \geq 0$ (the case where $\mathcal{E}_{|Y}$ is locally free is similar and easier). Let $s \in S\backslash\{P\}$, D_1, D_2 the irreducible components of $\mathcal{C}_s$, and for $i = 1, 2$, $F_i = \mathcal{E}_{s|D_i}/T_i$ (where T_i is the torsion subsheaf), it is a vector bundle on D_i. Let $r_i = \mathrm{rk}(F_i)$. Then from 6.2.8 we have

$$R(\mathcal{E}_{|Y}) \ = \ 2a+1 \ = \ r_1 + r_2 \ .$$

For every $x \in C$, we have $\mathrm{rk}(\mathcal{E}_{|Y,x}) = a+1$. By semi-continuity of the rank, we have $r_1 \leq a+1$ and $r_2 \leq a+1$. If follows that $r_1 = a$, $r_2 = a+1$ or $r_1 = a+1$, $r_2 = a$. We can suppose that we are in the first case.

Let $Z_1 \subset \mathcal{Z}$ be an irreducible component. It is a smooth irreducible curve meeting C in one point z. We have $\mathrm{rk}(\mathcal{E}_{Y,z}) = a+1$, hence for a general point $z' \in Z_1$, we have $\mathrm{rk}(\mathcal{E}_{|Z_1,z'}) \leq a+1$. Suppose that $\pi(z') = s \neq P$. Then from 4-, there exist integers $a_1 \geq 0$, $a_2 \geq 0$, $b \geq 0$ such that

$$\mathcal{E}_{s,z'} \ \simeq \ a_1\mathcal{O}_{D_1,z'} \oplus a_2\mathcal{O}_{D_2,z'} \oplus b\mathcal{O}_{\mathcal{C}_s,z'} \ .$$

Hence we have $a_1 + a_2 + b = \mathrm{rk}(\mathcal{E}_{|Z_1,z'}) \leq a+1$, $a_1 + b = r_1 = a$, $a_2 + b = r_2 = a+1$. It follows that $a_1 = 0$. Hence $\mathcal{E}_s$ is linked at z'. So we can suppose that there exists a neighbourhood U of P in S such that for every $s \in S\backslash\{P\}$, $\mathcal{E}_s$ is linked. From now on, we suppose that $U = S$.

The sheaf $E_2 = \mathcal{E}_{\mathcal{C}_2}$ is of rank $a+1$ at every point of $\mathcal{C}_2$, hence it is a vector bundle on $\mathcal{C}_2$. Let E_1 be the kernel of the restriction morphism $\mathcal{E} \to E_2$. It is a torsion free sheaf concentrated on $\mathcal{C}_1$. From lemma 6.2.3, we have an exact sequence

$$0 \longrightarrow E_{1|C} \longrightarrow \mathcal{E}_{|Y} \longrightarrow E_{2|C} \longrightarrow 0 \ .$$

Hence we have $E_{2|C} = \mathcal{E}_C$ and $E_{1|C} = (\mathcal{E}_{|Y})_1$. In particular $E_{1|C}$ is of rank a on C, as on $\mathcal{C}_1\backslash C$. It follows that E_1 is a vector bundle on $\mathcal{C}_1$. The morphism $E_{2|\mathcal{Z}_0} \to \mathbb{L}^* \otimes E_{1|\mathcal{Z}_0}$ corresponding to the exact sequence $0 \to E_1 \to \mathcal{E} \to E_2 \to 0$ is surjective, hence $\mathcal{E}$ is regular by proposition 6.4.3. $\square$

6.5. Regular sheaves and blowing-ups

We keep the notations of 6.2, and we suppose that $p = 1$.

We now consider the associated fragmented deformation $\rho : \mathcal{D} \to S$ (cf. 6.1.2). Recall that the canonical morphism $\eta : \mathcal{D} \to \mathcal{C}$ is the blowing-up of $\mathcal{Z}$. If $C'_2 = \rho^{-1}(P)$, the induced morphism $C'_2 \to Y$ is the blowing-up of $\mathcal{Z} \cap C$.

Let $\mathcal{E}$ be a regular sheaf on $\mathcal{C}$, $T(\eta^*(\mathcal{E}))$ the torsion subsheaf of $\eta^*(\mathcal{E})$, and

$$\widetilde{\mathcal{E}} \ = \ \eta^*(\mathcal{E})/T(\eta^*(\mathcal{E})) \ .$$

6.5.1. Proposition: *Let $\mathcal{E}$ be a regular sheaf on $\mathcal{C}$, and $0 \to E_2 \to \mathcal{E} \to E_1 \to 0$ an exact sequence, where E_i is a vector bundle on $\mathcal{C}_i$, for $i = 1, 2$, such that the associated morphism $\lambda : E_{1|\mathcal{Z}_0} \to \mathbb{L}^* \otimes E_{2|\mathcal{Z}_0}$ is surjective. Then we have an exact sequence on $\mathcal{D}$*

$$0 \longrightarrow E_2 \otimes \mathbb{L}_2^* \longrightarrow \widetilde{\mathcal{E}} \longrightarrow E_1 \longrightarrow 0$$

such that the associated morphism $E_{1|C} \to E_{2|C} \otimes L^*$ is the restriction of λ.

Proof. The proof is easy, using the description of ρ in 6.1.2 and the local description of $\mathcal{E}$ given in the proof of proposition 6.4.3. □

It is clear that the dual sheaf $\mathcal{E}^\vee$ is also regular. Let

$$\widehat{\mathcal{E}} \; = \; \left(\widetilde{\mathcal{E}^\vee}\right)^\vee \; .$$

The proof of the following result is similar to that of proposition 6.5.1:

6.5.2. Proposition: *Let $\mathcal{E}$ be a regular sheaf on $\mathcal{C}$, and $0 \to E_2 \to \mathcal{E} \to E_1 \to 0$ an exact sequence, where E_i is a vector bundle on $\mathcal{C}_i$, for $i = 1, 2$, such that the associated morphism of vector bundles $\lambda : E_{1|\mathcal{Z}_0} \to \mathbb{L}^* \otimes E_{2|\mathcal{Z}_0}$ is injective. Then we have an exact sequence on $\mathcal{D}$*

$$0 \longrightarrow E_2 \otimes \mathbb{L}_2^* \longrightarrow \widehat{\mathcal{E}} \longrightarrow E_1 \longrightarrow 0$$

such that the associated morphism $E_{1|C} \to E_{2|C} \otimes L^*$ is the restriction of λ.

6.6. Deformations of regular sheaves on fragmented deformations

We keep the notations of 6.5. Let $\mathcal{E}$, $\mathcal{E}'$ be regular sheaves on $\mathcal{D}$. We suppose that there are exact sequences

$$0 \longrightarrow F \longrightarrow \mathcal{E} \longrightarrow E \longrightarrow 0 \; , \quad 0 \longrightarrow F \longrightarrow \mathcal{E}' \longrightarrow E' \longrightarrow 0 \; ,$$

where F is a vector bundle on $\mathcal{E}_2$, E, E' are vector bundles on $\mathcal{C}_1$ such that $E_{|C} = E'_{|C} = E_0$. We can then define the Kodaïra-Spencer morphism

$$\omega_{E,E'} : TS_P \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_C}(E_0, E_0)$$

(cf. 2.7). We suppose also that the two morphisms $E_{|C} \to F_{|C}$, $E'_{|C} \to F_{|C}$ associated to these exact sequences are the same and surjective, there are denoted by $\phi : E_0 \to F_{|C}$.

Let

$$0 \longrightarrow F_{|C} \longrightarrow \mathcal{E}_{|C'_2} \longrightarrow E_0 \longrightarrow 0 \; , \quad 0 \longrightarrow F_{|C} \longrightarrow \mathcal{E}'_{|C'_2} \longrightarrow E_0 \longrightarrow 0$$

be the restrictions of the preceding exact sequences, and $\sigma, \sigma' \in \mathrm{Ext}^1_{\mathcal{O}_{C'_2}}(E_0, F_{|C})$ the corresponding elements. Let $\tau \in TS_P$ be the element corresponding to the isomorphism $\mathcal{I}_{C,C'_2} \simeq \mathcal{O}_C$. Let

$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_C}(E_0, F_{|C}) \xrightarrow{\iota} \mathrm{Ext}^1_{\mathcal{O}_{C'_2}}(E_0, F_{|C}) \xrightarrow{\theta} \mathrm{Hom}(E_0, F_{|C}) \longrightarrow 0$$

be the canonical exact sequence (cf. 3.4). Let

$$\overline{\phi} : \mathrm{Ext}^1_{\mathcal{O}_C}(E_0, E_0) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_C}(E_0, F_{|C})$$

be the map induced by ϕ.

6.6.1. Proposition: *We have $\sigma' - \sigma = \iota\left(\overline{\phi}(\omega_{E,E'}(\tau))\right)$.*

Proof. We will use Čech cohomology. According to 6.4, there exists an open cover $(U_i)_{i\in I}$ of a neighbourhood of C in $\mathcal{D}$ such that for every $i\in I$ we have isomorphisms

$$\begin{tikzcd}
\mathcal{E}_{|U_i} \arrow[dr, "\lambda_i"] & \\
& \mathbb{V}_i\oplus W_i \\
\mathcal{E}'_{|U_i} \arrow[ur, "\lambda'_i"] &
\end{tikzcd}$$

where $\mathbb{V}_i$, W_i are trivial vector bundles on U_i, $U_i\cap\mathcal{C}_1$ respectively. We can suppose that the induced isomorphisms $(\mathbb{V}_i\oplus W_i)_{|C}\simeq E_C$ and $(\mathbb{V}_i\oplus W_i)_{|C}\simeq E'_C$ are the same, as well as the two $F_{i|U_i}\simeq\mathbb{V}_i\otimes\mathcal{I}_{\mathcal{C}_1}$. Now let

$$\theta_{ij}=\lambda_j\circ\lambda_i^{-1},\ \theta'_{ij}=\lambda'_j\circ\lambda_i^{\prime -1}:(\mathbb{V}_i\oplus W_i)_{|U_{ij}}\to(\mathbb{V}_j\oplus W_j)_{|U_{ij}}\ ,$$

represented respectively by matrices

$$M_{ij}\ =\ \begin{pmatrix}A_{ij} & B_{ij}\\ C_{ij} & D_{ij}\end{pmatrix}\quad\text{and}\quad M'_{ij}\ =\ \begin{pmatrix}A'_{ij} & B'_{ij}\\ C'_{ij} & D'_{ij}\end{pmatrix}\ .$$

We can write

$$A'_{ij}\ =\ A_{ij}+\alpha_{ij}\ ,$$

where

$$\alpha_{ij}:\mathbb{V}_{i|U_{ij}\cap\mathcal{C}_1}\to(\pi_1,0)\mathbb{V}_{j|U_{ij}}=(\pi_1)\mathbb{V}_{j|U_{ij}\cap\mathcal{C}_1}\ ,$$

and the two matrices restricted to C are the same. Now we have induced trivialisations of E and E': $\mu_i:E_{U_i}\to\mathbb{V}_{i|\mathcal{C}_1}\oplus F_i$, $\mu'_i:\mathcal{E}'_{U_i}\to\mathbb{V}_{i|\mathcal{C}_1}\oplus F_i$, such that

$$\nu_{ij}=\mu_j\circ\mu_i^{-1},\ \nu'_{ij}=\mu'_j\circ\mu_i^{\prime -1}:\mathbb{V}_{i|U_{ij}\cap\mathcal{C}_1}\oplus W_{i|U_{ij}}\to\mathbb{V}_{j|U_{ij}\cap\mathcal{C}_1}\oplus W_{j|U_{ij}}$$

are represented by the matrices $M_{ij|\mathcal{C}_1}$ and $M'_{ij|\mathcal{C}_1}$ respectively. It follows that C_{ij}, C'_{ij} are multiples of π_1, and that D_{ij}, D'_{ij} are of the form $D_{ij}=I+\pi_1\delta_{ij}$, $D'_{ij}=I+\pi_1\delta'_{ij}$. We have then on C'_2

$$(M_{ij}-M'_{ij})_{|C'_2}\ =\ \begin{pmatrix}\alpha_{ij|C'_2} & (B_{ij}-B'_{ij})_{|C'_2}\\ 0 & 0\end{pmatrix}$$

and a similar formula for $(M_{ij}-M'_{ij})_{|C}$.

The result follows then easily from the interpretation in 2.7.4 of the Kodaïra-Spencer morphism and from 3.4.2. $\square$

We suppose that there are exact sequences

$$0\longrightarrow F\longrightarrow\mathcal{E}\longrightarrow E\longrightarrow 0\ ,\quad 0\longrightarrow F'\longrightarrow\mathcal{E}'\longrightarrow E\longrightarrow 0\ ,$$

where E is a vector bundle on $\mathcal{E}_1$, F, F' are vector bundles on $\mathcal{C}_2$ such that $F_{|C}=F'_{|C}=F_0$. We can then define the Kodaïra-Spencer morphism

$$\omega_{E,E'}:TS_P\longrightarrow\operatorname{Ext}^1_{\mathcal{O}_C}(E,E)\ .$$

We suppose also that the two morphisms $E_{|C} \to F_{|C}$, $E_{|C} \to F'_{|C}$ associated to these exact sequences are the same and surjective, there are denoted by $\psi : E_{|C} \to F_0$. Let

$$0 \longrightarrow F_0 \longrightarrow \mathcal{E}_{|C'_2} \longrightarrow E_{|C} \longrightarrow 0\ , \quad 0 \longrightarrow F_0 \longrightarrow \mathcal{E}'_{|C'_2} \longrightarrow E_{|C} \longrightarrow 0$$

be the restrictions of the preceding exact sequences, and $\rho, \rho' \in \operatorname{Ext}^1_{\mathcal{O}_{C'_2}}(E_{|C}, F_0)$ the corresponding elements. Let

$$0 \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_C}(E_{|C}, F_0) \xrightarrow{\kappa} \operatorname{Ext}^1_{\mathcal{O}_{C'_2}}(E_{|C}, F_0) \xrightarrow{\zeta} \operatorname{Hom}(E_{|C}, F_0) \longrightarrow 0$$

be the canonical exact sequence (cf. 3.4). Let

$$\widetilde{\psi} : \operatorname{Ext}^1_{\mathcal{O}_C}(F_0, F_0) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_C}(E_{|C}, F_0)$$

be the map induced by ψ. The following result is similar to proposition 6.6.1

6.6.2. Proposition: *We have* $\rho' - \rho = \kappa\left(\widetilde{\psi}(\omega_{F,F'}(\tau))\right)$.

6.6.3. Corollary: *Let $\mathbb{E}$ be a quasi locally free sheaf on Y. Then there exist a smooth curve T, $t_0 \in T$, a non constant morphism $\alpha : T \to S$ such that $\alpha(t_0) = P$, and a regular sheaf $\mathcal{E}$ on $\alpha^*(\mathcal{D})$ such that $\mathcal{E}_{|Y} \simeq \mathbb{E}$. It is possible to choose $\mathcal{E}$ satisfying conditions (ii) or (iii) of proposition 6.4.3.*

Proof. We will construct $\mathcal{E}$ satisfying (ii) (the case of (iii) is similar). We consider the first canonical filtration of $\mathcal{E}$ (cf. 3.2):

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_0 \longrightarrow 0\ ,$$

and the associated $\sigma \in \operatorname{Ext}^1_{\mathcal{O}_Y}(\mathcal{E}_0, \mathcal{E}_1)$ and surjective morphism $f : \mathcal{E}_0 \to \mathcal{E}_1$. According to 2.7.2 there exist smooth curves T^0, T^1, and for $i = 0, 1$, $t_0^i \in T^i$, morphisms $\phi_i : T^i \to S$ such that $\phi_i(t_0^i) = P$, and vector bundles $\mathcal{F}_i$ on $\phi_i^*(\mathcal{C}_{i+1})$ such that $\mathcal{F}_{i,t_0^i} \simeq \mathcal{E}_i$. Let $Z = T^0 \times_S T^1$, p_0, p_1, q the projections $Z \to T^0$, $Z \to T^1$, $Z \to S$ respectively. According to proposition 6.3.1, there exists a unique extension on $q^*(\mathcal{D})$

$$0 \longrightarrow p_1^*(\mathcal{F}_1) \longrightarrow \mathcal{F} \longrightarrow p_0^*(\mathcal{F}_0) \longrightarrow 0$$

corresponding to f. We have an exact sequence

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{F}_{|Y} \longrightarrow \mathcal{E}_0 \longrightarrow 0\ ,$$

associated to $\sigma' \in \operatorname{Ext}^1_{\mathcal{O}_Y}(\mathcal{E}_0, \mathcal{E}_1)$ (here Y is the fibre of $q^*(\mathcal{D}) \to Z$ over (t_0^0, t_0^1)). Recall the exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{E}_0, \mathcal{E}_1) \xrightarrow{\iota} \operatorname{Ext}^1_{\mathcal{O}_Y}(\mathcal{E}_0, \mathcal{E}_1) \xrightarrow{\theta} \operatorname{Hom}(\mathcal{E}_0, \mathcal{E}_1) \longrightarrow 0$$

(cf. 3.4). We have $\theta(\sigma) = \theta(\sigma')$, hence $\sigma'' = \sigma - \sigma' \in \operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{E}_0, \mathcal{E}_1)$.

Since f is surjective, the induced map

$$\rho : \operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{E}_0, \mathcal{E}_0) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{E}_0, \mathcal{E}_1)$$

is surjective. Let $\mu \in \operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{E}_0, \mathcal{E}_0)$ such that $\rho(\mu) = \sigma''$.

According to proposition 2.7.6, there exists a smooth curve T, $t_0 \in T$, a morphism $\beta : T \to Z$ such that $\beta(t_0) = (t_0^0, t_0^1)$, and a vector bundle $\mathcal{U}$ on $\beta^*(q^*(\mathcal{C}_2))$ such that $\mathcal{U}_{|C} \simeq \mathcal{E}_0$ and $\omega_{\beta^\#(p_0^*(\mathcal{F}_0)),\mathcal{U}} = \mu$. Let $\alpha = q \circ \beta : T \to S$. We now consider the extension on $\alpha^*(\mathcal{D})$

$$0 \longrightarrow \beta^*(p_1^*(\mathcal{F}_1)) \longrightarrow \mathcal{V} \longrightarrow \mathcal{U} \longrightarrow 0 .$$

It follows from proposition 6.6.1 that its restriction to Y

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{U}_{|Y} \longrightarrow \mathcal{E}_0 \longrightarrow 0$$

is associated to σ. □

7. Maximal reducible deformations and limit sheaves

Let C be a projective irreducible smooth curve and $Y = C_2$ a primitive double curve, with underlying smooth curve C, and associated line bundle L on C. Let S be a smooth affine curve, $P \in S$ and $\pi : \mathcal{C} \to S$ a maximal reducible deformation of Y. We keep the notations of 6-.

7.1. Reachable sheaves

7.1.1. Definition: *A coherent sheaf $\mathcal{E}$ on Y is called* reachable *(with respect to π) if there exists a smooth curve S', $P' \in S'$, a non constant morphism $f : S' \to S$ such that $f(P') = P$ and a coherent sheaf $\mathbb{E}$ on $f^*(\mathcal{C})$, flat on S', such that $\mathbb{E}_{P'} \simeq \mathcal{E}$.*

The rest of section 7 is devoted to the proof of the

7.1.2. Theorem: *Every quasi locally free sheaf on Y is reachable.*

Note that by corollary 6.6.3 the theorem is true on $\mathcal{D}$.

7.2. Sheaves concentrated on C

Let E_i, $i = 1, 2$, be a vector bundle on $\mathcal{C}_i$. If $0 \to E_1 \to \mathcal{E} \to E_2 \to 0$ is an exact sequence, the restriction to Y

$$0 \longrightarrow E_{1|Y} = E_{1|C} \longrightarrow \mathcal{E}_{|Y} \longrightarrow E_{2|Y} = E_{2|C} \longrightarrow 0 \qquad (14)$$

is also exact.

Recall that L^* has a canonical section $\mathbf{s}$ (defined by theorem 6.1.1, **3**). Let

$$\lambda_L : \mathrm{Ext}^1_{\mathcal{O}_C}(E_{2|C}, E_{1|C}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_C}(E_{2|C} \otimes L, E_{1|C})$$

be the induced linear map. The rest of 7.2 is dedicated to the proof of

7.2.1. Proposition: *Let*

$$0 \longrightarrow E_{1|C} \longrightarrow E \longrightarrow E_{2|C} \longrightarrow 0 \tag{15}$$

be an extension on C, associated to $\sigma \in \mathrm{Ext}^1_{\mathcal{O}_C}(E_{2|C}, E_{1|C})$. Then there exists an extension $0 \to E_1 \to \mathcal{E} \to E_2 \to 0$ on $\mathcal{C}$ such that $\mathcal{E}_{|Y}$ is concentrated on C and that (14) is isomorphic to (15) if and only if $\lambda_L(\sigma) = 0$.

Let $\mathcal{O}_{\mathcal{C}}(1)$ be an ample line bundle on $\mathcal{C}$. It $X \subset \mathcal{C}$ is a closed subvariety, let $\mathcal{O}_X(1) = \mathcal{O}_{\mathcal{C}}(1)_{|X}$. Let n_0 be an integer such that for every $n \geq n_0$, $E_2(n)$ is generated by its global sections and $H^1(E_1(n)) = H^1(E_{1|C}(n)) = H^1(E_1(n)(-C)) = \{0\}$. Suppose that $n \geq n_0$. Let $k = h^0(E_2(n))$. Then the morphism (of sheaves on $\mathcal{C}$)

$$\tau : k\mathcal{O}_{\mathcal{C}}(-n) = \mathcal{O}_{\mathcal{C}}(-n) \otimes H^0(E_2(n)) \longrightarrow E_2$$

induced by the evaluation morphism is surjective. Let $\mathcal{N} = \ker(\tau)$. Using the fact that

$$\mathrm{Hom}(E_2, E_1) \ = \ \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(k\mathcal{O}_{\mathcal{C}}(-n), E_1) \ = \ \{0\} \ ,$$

we deduce from the exact sequence

$$0 \longrightarrow \mathcal{N} \longrightarrow k\mathcal{O}_{\mathcal{C}}(-n) \longrightarrow E_2 \longrightarrow 0 \tag{16}$$

this one

$$0 \longrightarrow \mathrm{Hom}(k\mathcal{O}_{\mathcal{C}}(-n), E_1) \longrightarrow \mathrm{Hom}(\mathcal{N}, E_1) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(E_2, E_1) \longrightarrow 0 \ .$$

According to lemma 6.2.3, the restriction of (16) to Y

$$0 \longrightarrow \mathcal{N}_{|Y} \xrightarrow{\ \epsilon\ } k\mathcal{O}_Y(-n) \longrightarrow E_{2|C} \longrightarrow 0$$

is also exact, and induces a surjective map

$$\gamma : \mathrm{Hom}(\mathcal{N}_{|Y}, E_{1|C}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_Y}(E_{2|C}, E_{1|C}) \ .$$

Let $\sigma \in \mathrm{Ext}^1_{\mathcal{O}_Y}(E_{2|C}, E_{1|C})$, $\phi \in \mathrm{Hom}(\mathcal{N}_{|Y}, E_{1|C})$ such that $\gamma(\phi) = \sigma$, and

$$0 \longrightarrow E_{1|C} \longrightarrow \mathcal{E} \longrightarrow E_{2|C} \longrightarrow 0$$

the extension (on Y) corresponding to σ. Then $\mathcal{E}$ is isomorphic to the cokernel of the injective morphism

$$(\epsilon, \phi) : \mathcal{N}_{|Y} \longrightarrow k\mathcal{O}_Y(-n) \oplus E_{1|C} \ .$$

7.2.2. Lemma: *The sheaf $\mathcal{E}$ is concentrated on C if and only if ϕ vanishes on $(\mathcal{N}_{|Y})^{(1)}$.*

Proof. Let $x \in C$, $z \in \mathcal{O}_{Y,x}$ an equation of C. Then we have $z.k\mathcal{O}_Y(-n)_x \subset \mathcal{N}_{|Y,x}$, and $\mathcal{E}_x$ is a $\mathcal{O}_{C,x}$-module if and only if $z\mathcal{E}_x = 0$, if and only if for every $(u, e) \in k\mathcal{O}_Y(-n)_x \times E_{1|C,x}$, we have $z(u, e) = (zu, 0) \in \mathrm{im}(\epsilon_x, \phi_x)$, i.e. if and only if there exists $\nu \in \mathcal{N}_{|Y,x}$ such that $\epsilon_x(\nu) = zu$ and $\phi_x(\nu) = 0$. We have

$$\begin{aligned}
\epsilon_x(\nu) \text{ is a multiple of } z \ &\Longleftrightarrow\ z\epsilon_x(\nu) = 0 \\
&\Longleftrightarrow\ \epsilon_x(z\nu) = 0 \\
&\Longleftrightarrow\ z\nu = 0 \\
&\Longleftrightarrow\ \nu \in (\mathcal{N}_{|Y})^{(1)}_x \ .
\end{aligned}$$

Lemma 7.2.2 follows immediately. □

According to lemma 3.4.4, we have an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{N}_{|Y}/(\mathcal{N}_{|Y})^{(1)} \arrow[r] \arrow[d, equal] & k\mathcal{O}_C(-n) \arrow[r, "\rho"] & E_{2|C} \arrow[r] & 0. \\
& \mathcal{N}_{|C}/\big((\mathcal{N}_{|Y})^{(1)}/(\mathcal{N}_{|Y})_1\big) & & &
\end{tikzcd}$$

Let

$$U \ = \ \mathcal{N}_{|Y}/(\mathcal{N}_{|Y})^{(1)} = \mathcal{N}_{|C}/\big((\mathcal{N}_{|Y})^{(1)}/(\mathcal{N}_{|Y})_1\big) \ = \ \ker(\rho) \ .$$

Then it follows from lemma 7.2.2 that:

7.2.3. $\mathcal{E}$ *is concentrated on* C *if and only if* ϕ *can be factorised through* U.
We have a canonical surjection $\beta : \mathrm{Hom}(U, E_{1|C}) \to \mathrm{Ext}^1_{\mathcal{O}_C}(E_{1|C}, E_{2|C})$, and for every $\phi \in \mathrm{Hom}(U, E_{1|C})$, if $0 \to E_{1|C} \to E \to E_{2|C} \to 0$ is the extension on C associated to $\beta(\phi)$, then E is isomorphic to the cokernel of the injective morphism

$$(i, \phi) : U \longrightarrow n\mathcal{O}_C(-n) \oplus E_{1|C}$$

(where i is the inclusion).

Let $\mathcal{N}_1$ the kernel of the surjective composite morphism

$$k\mathcal{O}_{\mathcal{C}_1}(-n) \xrightarrow{\tau_1} E_{2|\mathcal{C}_1} = E_{2|\mathcal{Z}_0} \longrightarrow E_{2|\mathcal{Z}}$$

(cf. 6.1), with $\tau_1 = \tau_{|\mathcal{C}_1}$. Since $\mathrm{Tor}^1_{\mathcal{O}_{\mathcal{C}_1}}(\mathcal{O}_C, E_{2|\mathcal{Z}}) = \{0\}$, we have an exact sequence

$$0 \longrightarrow \mathcal{N}_{1|C} = \mathcal{N}_{|C} \longrightarrow k\mathcal{O}_C(-n) \longrightarrow E_{2|\mathcal{Z}\cap C} \longrightarrow 0 \ ,$$

and a commutative diagram with exact rows and columns

(17)
$$\begin{tikzcd}
& & & 0 \arrow[d] & \\
& 0 \arrow[d] & & E_{2|C} \otimes L \arrow[d] & \\
0 \arrow[r] & U \arrow[r] \arrow[d] & k\mathcal{O}_C(-n) \arrow[r] \arrow[d, equal] & E_{2|C} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{N}_{1|C} \arrow[r] \arrow[d] & k\mathcal{O}_C(-n) \arrow[r] & E_{2|\mathcal{Z}\cap C} \arrow[r] \arrow[d] & 0 \\
& E_{2|C} \otimes L \arrow[d] & & 0 & \\
& 0 & & &
\end{tikzcd}$$

Let

(18) $$0 \longrightarrow E_1 \longrightarrow \mathcal{E} \longrightarrow E_2 \longrightarrow 0$$

be an extension on $\mathcal{C}$, associated to $\sigma \in \mathrm{Ext}^1_{\mathcal{O}_\mathcal{C}}(E_2, E_1)$. Recall that

$$\mathrm{Ext}^1_{\mathcal{O}_\mathcal{C}}(E_2, E_1) \ \simeq \ \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{C}_1}}(E_{2|\mathcal{Z}_0}, E_1) \ \simeq \ \mathrm{Hom}(E_{2|\mathcal{Z}_0}, E_1(\mathcal{Z}_0)_{|\mathcal{Z}_0})$$

(cf. 6.1). The exact sequence (18) restricts to an exact sequence on Y

$$0 \longrightarrow E_{1|Y} \longrightarrow \mathcal{E}_{|Y} \longrightarrow E_{2|Y} \longrightarrow 0 \ .$$

Let $\sigma_0 \in \mathrm{Ext}^1(E_{2|C}, E_{1|C})$ be the associated element. Recall that we have a canonical exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_C}(E_{2|C}, E_{1|C}) \xrightarrow{i} \mathrm{Ext}^1(E_{2|C}, E_{1|C}) \xrightarrow{\theta} \mathrm{Hom}(E_{2|C} \otimes L, E_{1|C}) \longrightarrow 0$$

(cf. 3.4). We have a canonical obvious map

$$\theta_1 : \mathrm{Hom}(E_{2|\mathcal{Z}_0}, E_1(\mathcal{Z}_0)_{|kz_0}) \longrightarrow \mathrm{Hom}(E_{2|C} \otimes L, E_{1|C}) \ ,$$

and $\theta_1(\sigma) = \theta(\sigma_0)$.

We have a commutative diagram with exact rows

(19)
$$\begin{tikzcd}
0 \arrow[r] & \mathcal{N}_{|\mathcal{C}_1} \arrow[r] \arrow[d, hook] & k\mathcal{O}_{\mathcal{C}_1}(-n) \arrow[r] \arrow[d, equal] & E_{2|\mathcal{Z}_0} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \mathcal{N}_1 \arrow[r] & k\mathcal{O}_{\mathcal{C}_1}(-n) \arrow[r] & E_{2|\mathcal{Z}} \arrow[r] & 0
\end{tikzcd}$$

(the exactness of the first row comes from corollary 6.2.2),

From (19) and lemma 2.2.2 we deduce the commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{N}_1, E_1) \arrow[r] \arrow[d, hook] & \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{C}_1}}(E_{2|\mathcal{Z}}, E_1) = \mathrm{Hom}(E_{2|\mathcal{Z}}, E_1(\mathcal{Z})_{|\mathcal{Z}}) \arrow[r] \arrow[d, hook, "\Psi"'] & 0 \\
\mathrm{Hom}(\mathcal{N}_{|\mathcal{C}_1}, E_1) \arrow[r] & \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{C}_1}}(E_{2|\mathcal{Z}_0}, E_1) = \mathrm{Hom}(E_{2|\mathcal{Z}_0}, E_1(\mathcal{Z}_0)_{|\mathcal{Z}_0}) \arrow[r] & 0
\end{tikzcd}$$

From lemma 2.2.1, the image of Ψ is exactly the space of morphisms of vector bundles vanishing on C. It follows that:

7.2.4. *The extensions* (7) *such that* $\mathcal{E}_Y$ *is concentrated on* C *arise from the morphisms* $\mathcal{N} \to E_1$ *that can be factorised through* $\mathcal{N}_1$*.*
Let $\phi : \mathcal{N} \to \mathcal{N}_1 \xrightarrow{\alpha} E_1$ be such a morphism. Then the morphism $U \to E_{1|C}$ restriction of α can be used to define the extension $0 \to E_{1|C} \to \mathcal{E}_Y \to E_{2|C} \to 0$ (cf. 7.2.3).
We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(k\mathcal{O}_C(-n), E_{1|C}) \arrow[r] \arrow[d, hook] & \mathrm{Hom}(U, E_{1|C}) \arrow[r, two heads] \arrow[d, equal] & \mathrm{Ext}^1_{\mathcal{O}_C}(E_{2|C}, E_{1|C}) \arrow[d, "\lambda_L"'] \\
\mathrm{Hom}(\mathcal{N}_{1|C}, E_{1|C}) \arrow[r] & \mathrm{Hom}(U, E_{1|C}) \arrow[r, "\mu"] & \mathrm{Ext}^1_{\mathcal{O}_C}(E_{2|C} \otimes L, E_{1|C}),
\end{tikzcd}$$

where the two rows come from (17) and are exact. The commutativity of the right square follows from lemma 2.3.1. It follows that: *a morphism* $\alpha : U \to E_{1|C}$ *can be extended to* $\mathcal{N}_{1|C}$ *if and only if* $\gamma(\mu(\alpha)) = 0$.

7.2.5. Lemma: *The restriction morphism*

$$\lambda : \mathrm{Hom}(\mathcal{N}_1, E_1) \longrightarrow \mathrm{Hom}(\mathcal{N}_{1|C}, E_{1|C})$$

is surjective.

Proof. Let $T = \mathcal{Z} \cap C$. From the exact sequences

$$0 \to \mathcal{N}_1 \to k\mathcal{O}_{\mathcal{C}_1}(-n) \to E_{2|\mathcal{Z}} \to 0 \ , \quad 0 \to \mathcal{N}_{1|C} \to k\mathcal{O}_C(-n) \to E_{2|T} \to 0$$

We deduce the commutative diagram with exact rows

$$\begin{tikzcd}
 & & & \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{C}_1}}(E_{2|\mathcal{Z}}, E_1) \arrow[d, equal] & \\
0 \arrow[r] & \mathrm{Hom}(k\mathcal{O}_{\mathcal{C}_1}(-n), E_1) \arrow[r] \arrow[d, "f"] & \mathrm{Hom}(\mathcal{N}_1, E_1) \arrow[r] \arrow[d, "g"] & \mathrm{Hom}(E_{2|\mathcal{Z}}, E_1(\mathcal{Z})_{|\mathcal{Z}}) \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & \mathrm{Hom}(k\mathcal{O}_C(-n), E_{1|C}) \arrow[r] & \mathrm{Hom}(\mathcal{N}_{1|C}, E_{1|C}) \arrow[r] & \mathrm{Hom}(E_{2|T}, (E_{1|C})(T)_{|T}) \arrow[r] \arrow[d, equal] & 0 \\
 & & & \mathrm{Ext}^1_{\mathcal{O}_C}(E_{2|\mathcal{Z}\cap C}, E_{1|C}) &
\end{tikzcd}$$

The map f is surjective because $H^1(E_1(n)(-C)) = \{0\}$, and h is also surjective because $\mathcal{Z}$ is made of disjoint affine curves. Hence g is surjective. $\Box$

Proof of proposition 7.2.1. Suppose that the extension $0 \to E_1 \to \mathcal{E} \to E_2 \to 0$ on $\mathcal{C}$ exists. Then according to 7.2.4 we have $\lambda_L(\sigma) = 0$. This follows also from proposition 6.3.1.

Conversely, suppose that $\lambda_L(\sigma) = 0$. Let $\phi \in \mathrm{Hom}(U, E_{1|C})$ over σ (cf. 7.2.3). Then according to 7.2.4, ϕ can be extended to $\overline{\phi} : \mathcal{N}_{1|C} \to U$. From lemma 7.2.5, there exists $\Phi \in \mathrm{Hom}(\mathcal{N}_1, E_1)$ restriction to $\overline{\phi}$ on C. This $\overline{\phi}$ defines the extension $0 \to E_1 \to \mathcal{E} \to E_2 \to 0$ on $\mathcal{C}$. $\Box$

7.3. Proof of theorem 7.1.2

Let $\mathcal{E}$ be a quasi locally free sheaf on Y and

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_0 \longrightarrow 0 \tag{20}$$

the exact sequence coming from its first canonical filtration, corresponding to $\sigma \in \mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{E}_0, \mathcal{E}_1)$. Let $f : \mathcal{E}_0 \otimes L \to \mathcal{E}_1$ be the associated surjective morphism. Let $\rho : \mathcal{D} \to \mathcal{C}$ be the blowing-up of $\mathcal{Z}$, inducing $\phi : \widehat{Y} = C_2' \to Y$, the blowing-up of $\mathcal{Z} \cap C$ in Y (cf. 6.5). We have then an exact sequence on $\widehat{Y}$

$$0 \longrightarrow \mathcal{E}_1 \otimes L^* \longrightarrow \widetilde{\mathcal{E}} \longrightarrow \mathcal{E}_0 \longrightarrow 0 \tag{21}$$

corresponding to the first canonical filtration of $\widetilde{\mathcal{E}}$ (cf. 5). Using corollary 6.6.3 it is possible to find a smooth curve T, $t_0 \in T$, a morphism $\alpha : T \to C$ such that $\alpha(t_0) = P$, and vector bundles F_i on $\alpha^*(\mathcal{C}_i)$, $i = 1, 2$, such that

$$\mathbb{F}_{2|C} \ \simeq \ \mathcal{E}_1 \otimes L^* \ , \quad \mathbb{F}_{1|C} \ \simeq \ \mathcal{E}_0 \ ,$$

and an exact sequence

$$0 \longrightarrow F_2 \longrightarrow \mathbb{E} \longrightarrow F_1 \longrightarrow 0 \tag{22}$$

on $\alpha^*(\mathcal{D})$ restricting to (21) on $\widehat{Y}$. It is possible to extend f to a surjective morphism $\overline{f} : F_{1|\mathcal{Z}_0} \to F_{2|\mathcal{Z}_0}$, corresponding to an extension

$$0 \longrightarrow F_2 \otimes \mathbb{L} \longrightarrow \mathcal{U} \longrightarrow F_1 \longrightarrow 0 \tag{23}$$

on $\alpha^*(\mathcal{C})$ (cf. proposition 6.2.5). Then by proposition 6.5.1, (22) is the exact sequence

$$0 \longrightarrow F_2 \longrightarrow \widetilde{\mathcal{U}} \longrightarrow F_1 \longrightarrow 0$$

arising from (23) by blowing-up (cf. 6.5). The restriction of (23) to Y is an exact sequence

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E}_0 \longrightarrow 0$$

corresponding to $\sigma' \in \operatorname{Ext}^1_{\mathcal{O}_Y}(\mathcal{E}_0, \mathcal{E}_1)$, and such that the associated morphism $\mathcal{E}_0 \otimes L \to \mathcal{E}_1$ is f. We have

$$\sigma - \sigma' \ \in \ \operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{E}_0, \mathcal{E}_1) \ .$$

Let

$$\lambda : \operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{E}_0, \mathcal{E}_1) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{E}_0, \mathcal{E}_1 \otimes L^*)$$

be the map induced by the inclusion $L \subset \mathcal{O}_C$. Then we have $\sigma - \sigma' \in \ker(\lambda)$, from theorem 5.2.4. From proposition 7.2.1, it is possible to find an extension on $\alpha^*(\mathcal{C})$

$$0 \longrightarrow F_2 \longrightarrow \widetilde{\mathcal{V}} \longrightarrow F_1 \longrightarrow 0$$

such that its restriction to Y

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{V}_{|Y} \longrightarrow \mathcal{E}_0 \longrightarrow 0$$

is in fact concentrated on C and corresponds to $\sigma - \sigma'$. Let $g : F_{1|\mathcal{Z}_0} \to F_{2|\mathcal{Z}_0}$ corresponding to g. Then the extension

$$0 \longrightarrow F_2 \longrightarrow \widetilde{\mathcal{V}} \longrightarrow F_1 \longrightarrow 0$$

restricts to (20) on Y.

References

[1] Bănică, C., Forster, O. *Multiple structures on space curves.* In: Sundararaman , D. (Ed.) Proc. of Lefschetz Centennial Conf. (10-14 Dec. Mexico), Contemporary Mathematics 58, AMS, 1986, 47-64.

[2] Bayer, D., Eisenbud, D. *Ribbons and their canonical embeddings.* Trans. of the Amer. Math. Soc., 1995, 347-3, 719-756.

[3] Chen, D., Kass, J.L. *Moduli of generalized line bundles on a ribbon.* J. Pure Appl. Algebra 220 no. 2 (2016), 822-844.

[4] Drézet, J.-M. *Déformations des extensions larges de faisceaux.* Pacific Journ. of Math. 220, 2 (2005), 201-297.

[5] Drézet, J.-M. *Faisceaux cohérents sur les courbes multiples* . Collect. Math. 2006, 57-2, 121-171.

[6] Drézet, J.-M. *Paramétrisation des courbes multiples primitives* Adv. in Geom. 2007, 7, 559-612.

[7] Drézet, J.-M. *Faisceaux sans torsion et faisceaux quasi localement libres sur les courbes multiples primitives.* Mathematische Nachrichten, 2009, 282-7, 919-952.

[8] Drézet, J.-M. *Sur les conditions d'existence des faisceaux semi-stables sur les courbes multiples primitives.* Pacific Journ. of Math. 2011, 249-2, 291-319.

[9] Drézet, J.-M. *Courbes multiples primitives et déformations de courbes lisses.* Annales de la Faculté des Sciences de Toulouse 22, 1 (2013), 133-154.

[10] Drézet, J.-M. *Fragmented deformations of primitive multiple curves.* Central European Journal of Mathematics 11, n^o 12 (2013), 2106-2137.

[11] Drézet, J.-M. *Reducible deformations and smoothing of primitive multiple curves.* Manuscripta Mathematica 148 (2015), 447-469.
[12] Drézet, J.-M. *Non-reduced moduli spaces of sheaves on multiple curves.* preprint (2017), arXiv:1705.10634.
[13] Eisenbud, D., Green, M. *Clifford indices of ribbons.* Trans. of the Amer. Math. Soc., 1995, 347-3, 757-765.
[14] Ferrand, D. *Conducteur, descente et pincement.* Bull. Soc. math. France 131,4 (2003), 553-585.
[15] Godement, R. *Topologie algébrique et théorie des faisceaux.* Actualités scientifiques et industrielles 1252, Hermann, Paris (1964).
[16] González, M. *Smoothing of ribbons over curves.* Journ. für die reine und angew. Math., 2006, 591, 201-235.
[17] Hartshorne, R. *Algebraic geometry.* Grad. Texts in Math., Vol. 52, Springer (1977).
[18] Huybrechts, D., Lehn, M. *The Geometry of Moduli Spaces of Sheaves.* Aspect of Math. E31, Vieweg (1997).
[19] Inaba, M.-A. *On the moduli of stable sheaves on a reducible projective scheme and examples on a reducible quadric surface.* Nagoya Math. J. 2002, 166, 135-181.
[20] Maruyama, M. *Openness of a family of torsion free sheaves.* J. Math. Kyoto Univ. 16-3 (1976), 627-637.
[21] Maruyama, M. *Moduli of stable sheaves II.* J. Math. Kyoto Univ. 18 (1978), 577-614.
[22] Maruyama, M. *On a generalization of elementary transformations of algebraic vector bundles.* Conference on algebraic varieties of small dimension (Turin, 1985). Rend. Sem. Mat. Univ. Politec. Torino 44, Special Issue (1986), 1-13.
[23] Seshadri, C.S. *Fibrés vectoriels sur les courbes algébriques.* Astérisque 96 (1982).
[24] Simpson, C.T. *Moduli of representations of the fundamental group of a smooth projective variety I.* Publ. Math. IHES 79 (1994), 47-129.
[25] Siu Y., Trautmann, G. *Deformations of coherent analytic sheaves with compact supports.* Memoirs of the Amer. Math. Soc., Vol. 29, N. 238 (1981).
[26] Teixidor i Bigas, M. *Moduli spaces of (semi-)stable vector bundles on tree-like curves.* Math. Ann. 290 (1991), 341-348.
[27] Teixidor i Bigas, M. *Moduli spaces of vector bundles on reducible curves.* Amer. J. of Math. 117 (1995), 125-139.
[28] Teixidor i Bigas, M. *Compactifications of moduli spaces of (semi)stable bundles on singular curves: two points of view. Dedicated to the memory of Fernando Serrano.* Collect. Math. 49 (1998), 527-548.
[29] Teixidor i Bigas, M. *Vector bundles on reducible curves and applications.* Grassmannians, moduli spaces and vector bundles. Clay Math. Proc., 14, Amer. Math. Soc., Providence, RI, (2011), 169-180.

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