PRIMITIVE MULTIPLE SCHEMES

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RESUME. A primitive multiple scheme is a Cohen-Macaulay scheme Y such that the associated reduced scheme $X = Y_{red}$ is smooth, irreducible, and that Y can be locally embedded in a smooth variety of dimension $\dim(X)+1$. If n is the multiplicity of Y, there is a canonical filtration $X = X_1 \subset X_2 \subset \cdots \subset X_n = Y$, such that X_i is a primitive multiple scheme of multiplicity i. The simplest example is the trivial primitive multiple scheme of multiplicity n associated to a line bundle L on X: it is the n-th infinitesimal neighborhood of X, embedded in the line bundle L^* by the zero section.

Let $\mathbf{Z}_n = \mathrm{spec}(\mathbb{C}[t]/(t^n))$. The primitive multiple schemes of multiplicity n are obtained by taking an open cover (U_i) of a smooth variety X and by gluing the schemes $U_i \times \mathbf{Z}_n$ using automorphisms of $U_{ij} \times \mathbf{Z}_n$ that leave U_{ij} invariant. This leads to the study of the sheaf of nonabelian groups $\mathcal{G}_n$ of automorphisms of $X \times \mathbf{Z}_n$ that leave the X invariant, and to the study of its first cohomology set. If $n \geq 2$ there is an obstruction to the extension of X_n to a primitive multiple scheme of multiplicity $n+1$, which lies in the second cohomology group $H^2(X,E)$ of a suitable vector bundle E on X.

In this paper we study these obstructions and the parametrization of primitive multiple schemes. As an example we show that if $X = \mathbb{P}_m$ with $m >= 3$ all the primitive multiple schemes are trivial. If $X = \mathbb{P}_2$, there are only two non trivial primitive multiple schemes, of multiplicities 2 and 4, which are not quasi-projective. On the other hand, if X is a projective bundle over a curve, we show that there are infinite sequences $X = X_1 \subset X_2 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots$ of non trivial primitive multiple schemes.

SUMMARY

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1. Introduction

A *multiple scheme* is a Cohen-Macaulay scheme Y over $\mathbb{C}$ such that $Y_{red} = X$ is a smooth connected variety. We call X the *support* of Y. If Y is quasi-projective we say that it is a *multiple variety with support* X. In this case Y is projective if X is.

Let $\mathcal{I}_X$ be the ideal sheaf of X in Y. Let n be the smallest integer such that $Y = X^{(n-1)}$, $X^{(k-1)}$ being the k-th infinitesimal neighborhood of X, i.e. $\mathcal{I}_{X^{(k-1)}} = \mathcal{I}_X^k$. We have a filtration $X = X_1 \subset X_2 \subset \cdots \subset X_n = Y$ where X_i is the biggest Cohen-Macaulay subscheme contained in $Y \cap X^{(i-1)}$. We call n the *multiplicity* of Y.

Let $d = \dim(X)$. We say that Y is *primitive* if, for every closed point x of X, there exists a smooth variety S of dimension $d+1$, containing a neighborhood of x in Y as a locally closed subvariety. In this case, $L = \mathcal{I}_X/\mathcal{I}_{X_2}$ is a line bundle on X, X_j is a primitive multiple scheme of multiplicity j and we have $\mathcal{I}_{X_j} = \mathcal{I}_X^j$, $\mathcal{I}_{X_j}/\mathcal{I}_{X_{j+1}} = L^j$ for $1 \leq j < n$. We call L the line bundle on X *associated* to Y. The ideal sheaf $\mathcal{I}_X$ can be viewed as a line bundle on X_{n-1}. If $n = 2$, Y is called a *primitive double scheme*.

The simplest case is when Y is contained in a smooth variety S of dimension $d+1$. Suppose that Y has multiplicity n. Let $P \in X$ and $f \in \mathcal{O}_{S,P}$ a local equation of X. Then we have $\mathcal{I}_{X_i,P} = (f^i)$ for $1 < j \leq n$ in S, in particular $\mathcal{I}_{Y,P} = (f^n)$, and $L = \mathcal{O}_X(-X)$.

For any $L \in \mathrm{Pic}(X)$, the *trivial primitive variety* of multiplicity n, with induced smooth variety X and associated line bundle L on X is the n-th infinitesimal neighborhood of X, embedded by the zero section in the dual bundle L^*, seen as a smooth variety.

The case $\dim(X) = 1$ is well known. The primitive multiple curves have been defined and studied in [3]. Double curves had previously been used in [17]. The primitive double schemes have then been constructed and parametrized in [4]. Further work on primitive multiple curves has been done in [5], [6], [7], [8], [9], [10], [11], [12], [13] and [14], where higher multiplicity is treated, as well as coherent sheaves on primitive curves and their deformations. See also [23], [19], [27], [28] and [29]. Primitive double schemes of greater dimension are also studied in[1], [2], [20] and [21].

If $\dim(X) \geq 2$, the main difference with the case of curves is the presence of *obstructions* : given X_n they are

– Obstruction to the extension of a vector bundle on X_{n-1} to a vector bundle on X_n.
– Obstruction to the extension of X_n to a primitive multiple scheme of multiplicity $n+1$.

We will see that these obstructions depend on the vanishing of elements in cohomology groups $H^2(X, E)$, where E is a suitable vector bundle on X. Hence if $\dim(X) = 1$ the obstructions disappear, and it is always possible to extend vector bundles or primitive multiple schemes. This is why one can obtain many primitive multiple curves.

On the other hand in some cases, when $\dim(X) \geq 2$, it may happen that non trivial primitive multiple schemes are very rare. For example, we will see that if $X = \mathbb{P}_m$, $m \geq 2$, then there are exactly two non trivial primitive multiple schemes (for $m = 2$, in multiplicities 2 and 4), and these two schemes are not quasi-projective (the scheme of multiplicity 2 appears already in [2], see also [21], 2-). But in some other cases (for example if X is of dimension 2 and is a projective bundle over a curve) there are big families and infinite sequences

$$X = X_1 \subset X_2 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots$$

of non trivial primitive multiple schemes.

If $\dim(X) \geq 2$, it may also happen that primitive multiple schemes are not quasi-projective (this has also been observed in [2], [21]). In fact, X_n is quasi-projective (and even projective) if and only if it is possible to extend an ample line bundle on X to a line bundle on X_n. For example the two non trivial primitive multiple schemes for $X = \mathbb{P}_2$ are not quasi-projective.

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Notations: – In this paper, an *algebraic variety* is a quasi-projective scheme over $\mathbb{C}$. A *vector bundle* on a scheme is an algebraic vector bundle.

– If X is a scheme and $P \in X$ is a closed point, we denote by $m_{X,P}$ (or m_P) the maximal ideal of P in $\mathcal{O}_{X,P}$.

– If X is an scheme and $Z \subset X$ is a closed subscheme, $\mathcal{I}_Z$ (or $\mathcal{I}_{Z,X}$) denotes the ideal sheaf of Z in X.

– If V is a finite dimensional complex vector space, $\mathbb{P}(V)$ denotes the projective space of lines in V, and we use a similar notation for projective bundles.

Let X_n be a primitive multiple scheme of multiplicity n, $X = (X_n)_{red}$, and $L \in \text{Pic}(X)$ the associated line bundle.

1.1. Construction and parametrization of primitive multiple schemes

1.1.1. *The sheaves of groups associated to primitive multiple schemes* – For every open subset $U \subset X$, let $U^{(n)}$ denote the corresponding open subset of X_n. Let $\mathbf{Z}_n = \text{spec}(\mathbb{C}[t]/(t^n))$. It is proved in [8], th. 5.2.1, that X_n is locally trivial, i.e. there exists an open cover $(U_i)_{i\in I}$ of X such that for evey $i \in I$ there is an isomorphism

$$\delta_i : U_i^{(n)} \longrightarrow U_i \times \mathbf{Z}_n$$

inducing the identity on U_i. For every open subset U of X, we have $\mathcal{O}_{X\times\mathbf{Z}_n}(U) = \mathcal{O}_X(U)[t]/(t^n)$. It is then natural to introduce the sheaf of (non abelian) groups $\mathcal{G}_n$ on X, where $\mathcal{G}_n(U)$ is the group of automorphisms θ of the $\mathbb{C}$-algebra $\mathcal{O}_X(U)[t]/(t^n)$ such that for every $\alpha \in \mathcal{O}_X(U)[t]/(t^n)$, $\theta(\alpha)_{|U} = \alpha_{|U}$. We have $\mathcal{G}_1 = \mathcal{O}_X^*$.

Let

$$\delta_{ij} = \delta_j\delta_i^{-1} : U_{ij} \times \mathbf{Z}_n \xrightarrow{\simeq} U_{ij} \times \mathbf{Z}_n,$$

and $\delta_{ij}^* = \delta_i^{*-1}\delta_j^* \in \mathcal{O}_X(U_{ij})[t]/(t^n)$. Then (δ_{ij}^*) is a cocycle of $\mathcal{G}_n$, which describes completely X_n.

In this way we see that there is a canonical bijection between the cohomology set $H^1(X, \mathcal{G}_n)$ ans the set of isomorphism classes of primitive multiple schemes X_n such that $X = (X_n)_{red}$. Most results of this paper come from computations in *Čech cohomology*.

There is an obvious morphism $\rho_{n+1} : \mathcal{G}_{n+1} \to \mathcal{G}_n$, such that

$$H^1(\rho_{n+1}) : H^1(X, \mathcal{G}_{n+1}) \longrightarrow H^1(X, \mathcal{G}_n)$$

sends X_{n+1} to X_n if $n \geq 2$, whereas

$$H^1(\rho_2) : H^1(X, \mathcal{G}_2) \longrightarrow H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X)$$

sends X_2 to L.

We prove that $\ker(\rho_2) \simeq T_X$ (proposition 4.3.2) and $\ker(\rho_{n+1}) \simeq T_X \oplus \mathcal{O}_X$ if $n \geq 2$ (proposition 4.4.1), where T_X is the tangent bundle of X. The fact that they are sheaves of *abelian* groups allows to compute *obstructions*. Let $g_n \in H^1(X, \mathcal{G}_n)$, corresponding to the primitive multiple scheme X_n, and for $1 \leq i < n$, g_i the image of g_n in $H^1(X, \mathcal{G}_i)$. Let $\ker(\rho_{n+1})^{g_n}$ be the associated sheaf of groups (cf. 2.2). We prove that

$$\ker(\rho_2)^{g_1} \ \simeq \ T_X \otimes L$$

(proposition 4.3.1). By cohomology theory, we find that there is a canonical surjective map

$$H^1(X, T_X \otimes L) \longrightarrow H^1(\rho_2)^{-1}(L)$$

sending 0 to the trivial primitive scheme, and whose fibers are the orbits of the action of $\mathbb{C}^*$ on $H^1(X, T_X \otimes L)$ by multiplication. Hence there is a bijection between the set of non trivial double schemes with associated line bundle L, and $\mathbb{P}(H^1(X, T_X \otimes L))$. For a non trivial double scheme X_2, we have an exact sequence

$$0 \longrightarrow L \longrightarrow \Omega_{X_2|X} \longrightarrow \Omega_X \longrightarrow 0 \ ,$$

corresponding to $\sigma \in \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X, L) = H^1(T_X \otimes L)$, and $\mathbb{C}\sigma$ is the element of $\mathbb{P}(H^1(T_X \otimes L))$ corresponding to X_2. This result has already been proved in [4] using another method, and in [8] when X is a curve, in the same way as here.

If $n > 2$ we have

$$\ker(\rho_{n+1})^{g_n} \ \simeq \ (\Omega_{X_2|X})^* \otimes L^n$$

(proposition 4.3.2). We have then (by the theory of cohomology of sheaves of groups) an *obstruction map*

$$\Delta_n : H^1(X, \mathcal{G}_n) \longrightarrow H^2((\Omega_{X_2|X})^* \otimes L^n)$$

such that $g_n \in \mathrm{im}(H^1(\rho_{n+1}))$ if and only $\Delta_n(g_n) = 0$.

Suppose that $n > 2$ and $\Delta_n(g_n) = 0$, and let $g_{n+1} \in H^1(X, \mathcal{G}_{n+1})$ be such that $H^1(\rho_{n+1})(g_{n+1}) = g_n$. Then $H^1(\rho_{n+1})^{-1}(g_n)$ is the set of extensions of X_n to a primitive multiple scheme of multiplicity $n+1$. Let $\mathrm{Aut}(X_n)$ be the set of automorphisms of X_n inducing the identity on X. There is a canonical surjective map

$$\lambda_{g_{n+1}} : H^1(X, (\Omega_{X_2|X})^* \otimes L^n) \longrightarrow H^1(\rho_{n+1})^{-1}(g_n)$$

which sends 0 to g_n, and whose fibers are the orbits of an action of $\mathrm{Aut}(X_n)$ on $H^1(X, (\Omega_{X_2|X})^* \otimes L^n)$.

1.1.2. *Extensions of the ideal sheaf of X* – The ideal sheaf $\mathcal{I}_{X,X_n}$ is a line bundle on X_{n-1}. A necessary condition to extend X_n to a primitive multiple scheme X_{n+1} of multiplicity $n+1$ is that $\mathcal{I}_{X,X_n}$ can be extended to a line bundle on X_n (namely $\mathcal{I}_{X,X_{n+1}}$). This is why we can consider pairs $(X_n, \mathbb{L})$, where $\mathbb{L}$ is a line bundle on X_n such that $\mathbb{L}_{|X_{n-1}} \simeq \mathcal{I}_{X,X_n}$.

The corresponding sheaf of groups on X is defined as follows: for every open subset $U \subset X$, $\mathcal{H}_n(U)$ is the set of pairs (ϕ, u), where $\phi \in \mathcal{G}_n(U)$, and $u \in \mathcal{O}_X(U)[t]/(t^n)$ is such that $\phi(t) = ut$ (with an adequate definition of the group law, cf. 4.5). The set of isomorphism classes of the above pairs $(X_n, \mathbb{L})$ can then be identified with the cohomology set $H^1(X, \mathcal{H}_n)$.

There is an obvious morphism $\tau_n : \mathcal{G}_{n+1} \to \mathcal{H}_n$, such that

$$H^1(\tau_n) : H^1(X, \mathcal{G}_{n+1}) \longrightarrow H^1(X, \mathcal{H}_n)$$

sends X_{n+1} to $(X_n, \mathcal{I}_{X,X_{n+1}})$. Let $g \in H^1(X, \mathcal{H}_n)$. We prove in proposition 4.5.2 that $\ker(\tau_n) \simeq T_X$ and

$$\ker(\tau_n)^g \ \simeq \ T_X \otimes L^n \ .$$

Consequently there is again, by cohomology theory, an *obstruction map*

$$\Delta''_n : H^1(X, \mathcal{H}_n) \longrightarrow H^2(T_X \otimes L^n)$$

such that, if $(X_n, \mathbb{L})$ corresponds to g, there is an extension of X_n to a primitive multiple scheme X_{n+1} of multiplicity $n+1$ with $\mathcal{I}_{X,X_{n+1}} \simeq \mathbb{L}$ if and only if $\Delta''_n(g) = 0$.

Suppose $\Delta''_n(g) = 0$, and let $g_{n+1} \in H^1(X, \mathcal{G}_{n+1})$ be such that $H^1(\tau_n)(g_{n+1}) = g$. Then $H^1(\tau_n)^{-1}(g)$ is the set of extensions of X_n to a primitive multiple scheme X_{n+1} of multiplicity $n+1$, such that $\mathcal{I}_{X,X_{n+1}} \simeq \mathbb{L}$. Let $\mathrm{Aut}^{\mathbb{L}}(X_n)$ be the set of automorphisms σ of X_n inducing the identity on X, and such that $\sigma^*(\mathbb{L}) \simeq \mathbb{L}$. There is a canonical surjective map

$$\lambda''_{g_{n+1}} : H^1(X, T_X \otimes L^n) \longrightarrow H^1(\rho_{n+1})^{-1}(g_n)$$

which sends 0 to g_{n+1}, and whose fibers are the orbits of an action of $\mathrm{Aut}^{\mathbb{L}}(X_n)$ on $H^1(X, T_X \otimes L^n)$.

1.2. Automorphisms of primitive multiple schemes

The group $\mathrm{Aut}(X_n)$ appears in the parametrization of the extensions of X_n to multiplicity $n+1$. Let $\mathrm{Aut}_0(X_n)$ be group of automorphisms of X_n such that the induced automorphism of L is the identity. Let

$$\mathcal{T}_n \ = \ (\Omega_{X_n})^* \ .$$

We prove that there is a natural bijection

$$\mathrm{Aut}_0(X_n) \longrightarrow H^0(X_n, \mathcal{I}_X \mathcal{T}_n)$$

(theorem 6.3.2, unless $n = 2$ this is not a morphism of groups). Locally this corresponds to the following result: let $U \subset X$ be an open subset such that Ω_U is trivial. For every derivation D_0 of $\mathcal{O}_X(U)[t]/(t^n)$, if $D = tD_0$, then

$$\chi_D \ = \ \sum_{k \geq 0} \frac{1}{k!} D^k : \mathcal{O}_X(U)[t]/(t^n) \longrightarrow \mathcal{O}_X(U)[t]/(t^n)$$

is an element of $\mathcal{G}_n(U)$ such that $\chi_D(t)$ is of the form $\chi_D(t) = (1 + \beta t)t$ (for some $\beta \in \mathcal{O}_X(U)[t]/(t^n)$). Moreover for every $\chi \in \mathcal{G}_n(U)$ with this property, there exists a unique derivation D multiple of t such that $\chi = \chi_D$ (theorem 6.1.5).

We show that if X_n is not trivial, and $\mathrm{Aut}_0(X_n)$ is trivial, then $\mathrm{Aut}(X_n)$ is finite (theorem 6.3.6 and corollary 6.3.7).

1.3. Extensions and obstructions

1.3.1. *Extension of vector bundles* – Let r be a positive integer. If Y is a primitive multiple scheme, let $\mathrm{GL}(r,\mathcal{O}_Y)$ denote the sheaf of groups on Y of invertible $r\times r$-matrices with coefficients in Y (the group law being the multiplication of matrices). If $\mathcal{E}$ is a coherent sheaf on Y, let $M(r,\mathcal{E})$ denote the sheaf of groups on Y of matrices with coefficients in $\mathcal{E}$ (the group law being the addition of matrices). There is a canonical bijection between $H^1(Y,\mathrm{GL}(r,\mathcal{O}_Y))$ and the set of isomorphism classes of rank r vector bundles on Y.

Suppose that X_n can be extended to a primitive multiple scheme X_{n+1} of multiplicity $n+1$. The kernel of the canonical morphism $p_n:\mathrm{GL}(r,\mathcal{O}_{X_{n+1}})\to\mathrm{GL}(r,\mathcal{O}_{X_n})$ is isomorphic to $M(r,L^n)$. Let $g\in H^1(X_n,\mathrm{GL}(r,\mathcal{O}_{X_n}))$, corresponding to the vector bundle $\mathbb{E}$ on X_n. Let $E=\mathbb{E}_{|X}$. We prove that the associated sheaf $M(r,L^n)^g$ is isomorphic to $E^*\otimes E\otimes L^n$. It follows that we have a canonical obstruction map

$$\Delta: H^1(X_n,\mathrm{GL}(r,\mathcal{O}_{X_n}))\longrightarrow H^2(X,E^*\otimes E\otimes L^n)$$

such that $g\in\mathrm{im}(H^1(p_n))$ if and only $\Delta(g)=0$. Hence $\Delta(g)$ is the obstruction to the extension of $\mathbb{E}$ to X_{n+1}. If $\mathbb{E}$ is a line bundle, this obstruction lies in $H^2(X,L^n)$.

We have a canonical exact sequence of sheaves on X_n

$$0\longrightarrow L^n\longrightarrow \Omega_{X_{n+1}|X_n}\longrightarrow\Omega_{X_n}\longrightarrow 0\ ,$$

corresponding to $\sigma_n\in\mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\Omega_{X_n},L^n)$, inducing $\overline{\sigma}_n\in\mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\mathbb{E}\otimes\Omega_{X_n},\mathbb{E}\otimes L^n)$. Let $\nabla_0(E)\in\mathrm{Ext}^1(\mathbb{E},\mathbb{E}\otimes\Omega_{X_n})$ be the canonical class of $\mathbb{E}$ (cf. 3). We have a canonical product

$$\mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\mathbb{E}\otimes\Omega_{X_n},\mathbb{E}\otimes L^n)\times\mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\mathbb{E},\mathbb{E}\otimes\Omega_{X_n})\longrightarrow\mathrm{Ext}^2_{\mathcal{O}_{X_n}}(\mathbb{E},\mathbb{E}\otimes L^n)=H^2(X,E^*\otimes E\otimes L^n)\ .$$

We prove that $\Delta(g)=\overline{\sigma}_n\nabla_0(\mathbb{E})$ (theorem 7.1.2).

1.3.2. *Extension of primitive multiple schemes* – Suppose that $n=2$, and that $\mathcal{I}_{X,X_2}$ can be extended to a line bundle $\mathbb{L}$ on X_2. Let $g_2\in H^1(X,\mathcal{H}_2)$ be associated to X_2. We give in theorem 7.3.1 a description of the obstruction map

$$\Delta_2'': H^1(X,\mathcal{H}_2)\longrightarrow H^2(X,T_X\otimes L^2)$$

corresponding to $(X,\mathbb{L})$, used to decide if X_2 can be extended to multiplicity 3. This description is made with Čech cohomology.

We could not find a simpler expression for $\Delta_2''(g_2)$, except in the case $L=\mathcal{O}_X$. The parametrization of extensions of $\mathcal{O}_X$ to a line bundle on X_2 exhibits an element $\theta_{\mathbb{L}}\in H^1(\mathcal{O}_X)$ associated to $\mathbb{L}$. On the other hand, we have $\sigma\in H^1(T_X)$ associated to X_2 (from the construction of double primitive schemes, or by the exact sequence $0\to L\to\Omega_{X_2|X}\to\Omega_X\to 0$). We find that

$$\Delta_2''(g_2)\ =\ \theta_{\mathbb{L}}\sigma+\frac{1}{2}[\sigma,\sigma]\ .$$

(where $[-,-]$ is the Poisson bracket).

Suppose that X_2 is trivial. We give in 6.5 a complete description of the action of $\mathrm{Aut}(X_2)$ on $H^1(X,(\Omega_{X_2|X})^*\otimes L^2)$ (cf .1.1).

For any $n \geq 2$, if X_n is trivial, we describe in 6.6 the action of $\mathbb{C}^* \subset \mathrm{Aut}(X_n)$ on $H^0(X, (\Omega_{X_2|X})^* \otimes L^n)$. It follows that if $h^0(X_n, \mathcal{I}_X \mathcal{T}_n) = 0$, then $\mathrm{Aut}(X_n) \simeq \mathbb{C}^*$, and the extensions of X_n to multiplicity $n+1$ are parametrized by a *weighted projective space* (cf. 7.2.3).

1.4. Examples

1.4.1. *The case of surfaces and canonical associated line bundle* – Suppose that X is a surface and $L = \omega_X$. Let X_2 be a non trivial primitive double scheme with associated line bundle ω_X. As indicated in 1.1, such schemes are parametrized by $\mathbb{P}(H^1(\Omega_X))$. Let $\sigma \in H^1(\Omega_X)$ such that $\mathbb{C}\sigma$ corresponds to X_2. We prove in proposition 7.4.1 that if $H^0(T_X) = \{0\}$, then X_2 can be extended to a primitive multiple scheme of multiplicity 3 if and only if $\sigma.\nabla_0(\omega_X) = 0$ in $H^2(\omega_X)$.

1.4.2. *The case of projective spaces* – We suppose that $X = \mathbb{P}_m$, $m \geq 2$. We prove that if $m > 2$, then there are only trivial primitive multiple schemes, and if $m = 2$, there are exactly two non trivial primitive multiple schemes (theorem 8.0.1). The first, $\mathbf{X}_2$, is of multiplicity 2, with $L = \mathcal{O}_{\mathbb{P}_2}(-3)$. The second, $\mathbf{X_4}$, is of multiplicity 4, with $L = \mathcal{O}_{\mathbb{P}_2}(-1)$.

We prove that the only line bundles on $\mathbf{X}_2$ and $\mathbf{X}_4$ are the trivial line bundles. Consequently, $\mathbf{X}_2$ and $\mathbf{X}_4$ are not quasi-projective (theorems 8.0.2 and 8.0.3). Since a primitive multiple scheme is a locally complete intersection, it has a dualizing sheaf which is a line bundle. We must have $\omega_{\mathbf{X}_2} \simeq \mathcal{O}_{\mathbf{X}_2}$ and $\omega_{\mathbf{X}_4} \simeq \mathcal{O}_{\mathbf{X}_4}$, i.e. $\mathbf{X}_2$ $\mathbf{X}_4$ are *K3 carpets* according to definition 1.2 of [21].

1.4.3. *The case of projective bundles over curves* – Let C be a smooth irreducible projective curve and E a rank 2 vector bundle on C. Let $X = \mathbb{P}(E)$ be the associated projective bundle and $\pi : \mathbb{P}(E) \to C$ the canonical projection. We suppose that

$$L \ = \ \pi^*(D) \otimes \mathcal{O}_{\mathbb{P}(E)}(-k) \ ,$$

with $D \in \mathrm{Pic}(C)$ and $k \geq 3$. We prove in 9 that if $\deg(D)$ is sufficiently big, then there exist infinite sequences

$$X_1 = \mathbb{P}(E) \subset X_2 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots$$

of projective non trivial primitive multiple schemes (X_n of multiplicity n).

For example, if $C = \mathbb{P}_1$, $E = \mathcal{O}_{\mathbb{P}_1} \otimes \mathbb{C}^2$, we have $X = \mathbb{P}_1 \times \mathbb{P}_1$. Let $L = \mathcal{O}(d, -k)$, with $d > 0$, $k \geq 3$. Then such infinite sequences exist, and for every $n > 1$, the extensions of X_n to primitive multiple schemes of multiplicity $n+1$ form a family of dimension $d(3kn^2 - 5n - 2kn + k + 1) + 5kn - 7 - k$.

1.5. Outline of the paper

In chapter 2, we give several ways to use Čech cohomology, and some basic results on cohomology of sheaves of groups are recalled. We give also some technical results about polynomials that are used later.

In chapter 3 the definition and some properties of the canonical class of a vector bundle are given.

In chapter 4 we give the definition and the properties of the sheaves of groups that are used to define primitive multiple schemes, as described in 1.1.

The chapter 5 contains a detailed study of the groups $\mathcal{G}_n(U)$ of 1.1, and of the canonical sheaves of primitive multiple schemes. These results are used is to perform computations in Čech cohomology.

In chapter 6 we give some descriptions and properties of the automorphism group of a primitive multiple scheme. Some demonstrations are omitted when they are similar to the case of primitive multiple curves, treated in [8].

In chapter 7 we study the two extensions problems: extension of vector bundles and extension of primitive multiple schemes to higher multiplicity. It contains also a comparison of the present work and of the other possible construction of primitive multiple schemes given in [4].

In Chapter 8 we apply the previous results to the case $X = \mathbb{P}_m$, $m \geq 2$.

In chapter 9 we apply the previous results to the case $X = \mathbb{P}(E)$, where E is a rank 2 vector bundle on a smooth projective curve.

2. Preliminaries

2.1. Čech cohomology and trivializations

Let X be a scheme over $\mathbb{C}$ and E, F vector bundles on X of ranks r, s respectively. Let $(U_i)_{i\in I}$ be an open cover of X such that we have trivializations:

$$\alpha_i : E_{|U_i} \xrightarrow{\simeq} \mathcal{O}_{U_i} \otimes \mathbb{C}^r .$$

2.1.1. Let

$$\alpha_{ij} = \alpha_i \circ \alpha_j^{-1} : \mathcal{O}_{U_{ij}} \otimes \mathbb{C}^r \xrightarrow{\simeq} \mathcal{O}_{U_{ij}} \otimes \mathbb{C}^r ,$$

so that we have the relation $\alpha_{ij}\alpha_{jk} = \alpha_{ik}$. Let n be a positive integer, and for every sequence $(i_0, \ldots, i_n)$ of distinct elements of I, $\sigma_{i_0\cdots i_n} \in H^0(U_{i_0\cdots i_n}, \mathcal{O}_X \otimes \mathbb{C}^r)$. Let

$$\theta_{i_0\cdots i_n} \ = \ \alpha_{i_0}^{-1}\sigma_{i_0\cdots i_n} \ \in \ H^0(U_{i_0\cdots i_n}, E) \ .$$

The family $(\theta_{i_0\cdots i_n})$ represents an element of $H^n(X,E)$ if the cocycle relations are satisfied: for every sequence $(i_0, \ldots, i_{n+1})$ of distinct elements of I,

$$\sum_{k=0}^{n} (-1)^k \theta_{i_0\cdots\widehat{i_k}\cdots i_{n+1}} \ = \ 0 \ ,$$

which is equivalent to

$$\alpha_{i_0i_1}\sigma_{i_1\cdots i_{n+1}} + \sum_{k=1}^{n+1} (-1)^k \sigma_{i_0\cdots\widehat{i_k}\cdots i_{n+1}} \ = \ 0 \ .$$

For $n=1$, this gives that elements of $H^1(X,E)$ are represented by families (σ_{ij}), $\sigma_{ij}\in H^0(U_{ij},\mathcal{O}_X\otimes\mathbb{C}^r)$, such that

$$\alpha_{ij}\sigma_{jk}+\sigma_{ij}-\sigma_{ik} \;=\; 0\ .$$

In Čech cohomology, it is generally assumed that $\theta_{ji}=-\theta_{ij}$. This implies that $\sigma_{ji}=-\alpha_{ji}\sigma_{ij}$.

2.1.2. Similarly, an element of $H^n(X,E\otimes F)$ is represented by a family $(\mu_{i_0\cdots i_n})$, with $\mu_{i_0\cdots i_n}\in H^0(U_{i_0\cdots i_n},\mathbb{C}^r\otimes F)$, satisfying the relations

$$(\alpha_{i_0i_1}\otimes I_F)(\mu_{i_1\cdots i_{n+1}})+\sum_{k=1}^{n+1}(-1)^k\mu_{i_0\cdots\widehat{i_k}\cdots i_{n+1}} \;=\; 0\ .$$

The corresponding element of $H^0(U_{i_0\cdots i_n},E\otimes F)$ is $\theta_{i_0\cdots i_n}=(\alpha_{i_0}\otimes I_F)^{-1}(\mu_{i_0\cdots i_n})$.

Suppose that $n=1$. Let $\nu\in H^1(E\otimes F)$ be the element defined by (θ_{ij}). For every $i\in I$, let β_i be an automorphism of $\mathcal{O}_{U_i}\otimes\mathbb{C}^r$, and $\theta_i\in H^0(U_i,E\otimes F)$. The cocycle $(\theta_{ij}+\theta_i-\theta_j)$ represents also ν. Let $\rho_i=(\beta_i\alpha_i\otimes I_F)(\theta_i)\in H^0(U_i,\mathbb{C}^r\otimes F)$. If we replace α_i with $\beta_i\alpha_i$, we see that ν is represented by the family (μ'_{ij}), $\mu'_{ij}\in H^0(U_{ij},\mathbb{C}^r\otimes F)$, with

$$\mu'_{ij} \;=\; (\beta_i\otimes I_F)(\mu_{ij})+\rho_i-(\beta_i\alpha_{ij}\beta_j^{-1}\otimes I_F)(\rho_j)\ .$$

2.1.3. *Representation of morphisms* – Suppose that we have also local trivializations of F:

$$\beta_i : F_{|U_i}\xrightarrow{\ \simeq\ }\mathcal{O}_{U_i}\otimes\mathbb{C}^s.$$

We have then local trivializations of $\mathcal{H}om(E,F)$

$$\begin{aligned}\Delta_i : \mathcal{H}om(E,F)_{|U_i} &\xrightarrow{\ \simeq\ } \mathcal{O}_{U_i}\otimes L(\mathbb{C}^r,\mathbb{C}^s)\\ \phi &\longmapsto \beta_i\circ\phi\circ\alpha_i^{-1}\end{aligned}$$

such that, for every $\lambda\in H^0(\mathcal{O}_{U_{ij}}\otimes L(\mathbb{C}^r,\mathbb{C}^s))$, we have

$$\Delta_{ij}(\lambda) \;=\; \Delta_i\Delta_j^{-1}(\lambda) \;=\; \beta_{ij}\lambda\alpha_{ij}^{-1}\ .$$

2.1.4. *Construction of vector bundles via local isomorphisms* – Le Z be a scheme over $\mathbb{C}$, $(Z_i)_{i\in I}$ an open cover of Z, and for every $i\in I$, a scheme U_i, with an isomorphism $\delta_i:Z_i\to U_i$. Let $\delta_{ij}=\delta_j\delta_i^{-1}:\delta_i(Z_{ij})\to\delta_j(Z_{ij})$, which is an isomorphism. Then a vector bundle of X can be constructed (in an obvious way) using vector bundles E_i on U_i and isomorphisms

$$\Theta_{ij} : \delta_{ij}^*(E_{j|\delta_j(Z_{ij})})\longrightarrow E_{i|\delta_i(Z_{ij})}$$

such that $\Theta_{ij}\circ\delta_{ij}^*(\Theta_{jk})=\Theta_{ik}$ on $\delta_i(Z_{ijk})$.

2.1.5. *Representation of extensions* – Let E, F be vector bundles on X, and $\sigma\in\mathrm{Ext}^1(F,E)\simeq H^1(\mathrm{Hom}(F,E))$. Let

$$0\longrightarrow E\longrightarrow\mathcal{E}\longrightarrow F\longrightarrow 0$$

be the corresponding exact sequence. Suppose that σ is represented by a cocycle (σ_{ij}), $\sigma_{ij} : F_{|U_{ij}} \to E_{|U_{ij}}$. Then $\mathcal{E}$ can be constructed by gluing the vector bundles $(E \oplus F)_{|U_i}$ using the automorphisms of $(E \oplus F)_{|U_{ij}}$ defined by the matrices $\begin{pmatrix} I_E & \sigma_{ij} \\ 0 & I_F \end{pmatrix}$.

2.2. Cohomology of sheaves of non abelian groups

(cf. [18])

Let X be a paracompact topological space and G a sheaf of groups on X. We denote by $H^1(X, G)$ the first Čech cohomology group of G. If G is not a sheaf of *commutative* groups there is no natural structure of group on $H^1(X, G)$. Let $e \in H^1(X, G)$ be defined by the trivial section of G (on the trivial open cover (X) of X), we call e the *identity element* of $H^1(X, G)$.

Given an open cover $(U_i)_{i\in I}$ of X, a 1-cocycle of G is a family $(g_{ij})_{i,j\in I}$ such that $g_{ij} \in G(U_{ij})$ and that the relations $g_{ij}g_{jk} = g_{ik}$ are satisfied.

2.2.1. *Actions of sheaves of groups* – Let H be another sheaf of group on X, acting on G (the action is compatible with the group actions). Let $z \in H^1(X, H)$, represented by a cocycle (z_{ij}) with respect to an open cover $(U_i)_{i\in I}$ of X. Recall that the associated sheaf of groups G^z is defined as follows: we glue the sheaves $G_{|U_i}$ using the automorphisms (z_{ij}) of $G_{|U_{ij}}$. The sections of G^z are represented by families $(\gamma_i)_{i\in I}$, $\gamma_i \in G(U_i)$, such that $\gamma_i = z_{ij}\gamma_j$ on U_{ij}. The elements of $H^1(X, G^z)$ are represented by families (ρ_{ij}), $\rho_{ij} \in G(U_{ij})$, such that $\rho_{ij}^{-1}\rho_{ik} = z_{ij}\rho_{jk}$ on U_{ijk}. If G is a sheaf of *commutative* groups, the elements of the higher cohomology groups $H^k(X, G^z)$, $k \geq 2$ can be represented in the same way by appropriate families.

2.2.2. *Exact cohomology sequence* – Let $\Gamma \subset G$ be a subsheaf of normal subgroups of G, and G/Γ the quotient sheaf of groups. Hence we have an exact sequence of sheaves of groups

$$0 \longrightarrow \Gamma \xrightarrow{i} G \xrightarrow{p} G/\Gamma \longrightarrow 0.$$

Let $A_1, \ldots, A_m$ be sets, having a particular element $e_i \in A_i$ for $1 \leq i \leq m$. A sequence of maps

$$A_1 \xrightarrow{f_1} A_2 \longrightarrow \cdots \longrightarrow A_{m-1} \xrightarrow{f_{m-1}} A_m$$

is called *exact* if for $1 < i < m$ we have $f_i^{-1}(e_i) = f_{i-1}(A_{i-1})$. Then we have: *the canonical sequence*

$$0 \longrightarrow H^0(X,\Gamma) \xrightarrow{H^0(i)} H^0(X,G) \xrightarrow{H^0(p)} H^0(X,G/\Gamma) \xrightarrow{\delta}$$
$$H^1(X,\Gamma) \xrightarrow{H^1(i)} H^1(X,G) \xrightarrow{H^1(p)} H^1(X,G/\Gamma) \quad .$$

is exact. The subsequence

$$0 \longrightarrow H^0(X,\Gamma) \xrightarrow{H^0(i)} H^0(X,G) \xrightarrow{H^0(p)} H^0(X,G/\Gamma)$$

is an exact sequence of groups. Moreover, if $c, c' \in H^0(X, G/\Gamma)$, we have $\delta(c) = \delta(c')$ if and only if $cc'^{-1} \in \operatorname{im}(H^0(p))$.

The group $H^0(X,G/\Gamma)$ acts on $H^1(X,\Gamma)$ as follows: let $a\in H^1(X,\Gamma)$, represented by a cocycle (γ_{ij}) (with respect to an open cover (U_i) of X), and $c\in H^0(X,G/\Gamma)$. Suppose also that the cover (U_i) is such that $c_{|U_i}$ is the image of an element c_i of $G(U_i)$. Then $(c_i\gamma_{ij}c_j^{-1})$ is a cocycle of elements of Γ, and the corresponding element of $H^1(X,\Gamma)$ depends only on c and a; we denote it by $c.a$. We have then $\delta(c)=c^{-1}.e$. Moreover, let $\gamma,\gamma'\in H^1(X,\Gamma)$. Then we have: $H^1(i)(\gamma)=H^1(i)(\gamma')$ *if and only if* γ,γ' *are in the same* $H^0(X,G/\Gamma)$*-orbit.*

2.2.3. *The fibers of* $H^1(p)$ – The sheaf of groups G acts (by conjugation) on itself and Γ, G/Γ. Let $\omega\in H^1(X,G/\Gamma)$, $g\in H^1(X,G)$ be such that $H^1(p)(g)=\omega$. Suppose that g is represented by a cocycle (g_{ij}). Let $g'\in H^1(p)^{-1}(\omega)$. Then g' is represented by a cocycle of the form $(\gamma_{ij}g_{ij})$, with $\gamma_{ij}\in\Gamma(U_{ij})$. Conversely, a cochain $(\gamma_{ij}g_{ij})$ is a cocycle if and only if, for every i,j,k we have $\gamma_{ij}^{-1}\gamma_{ik}=g_{ij}\gamma_{jk}g_{ij}^{-1}$, i.e. if and only if (γ_{ij}) induces a cocycle in Γ^g (cf. 2.2.1). In this way we define a surjective map

$$\lambda_g : H^1(X,\Gamma^g)\longrightarrow H^1(p)^{-1}(\omega)$$

sending the identity element of $H^1(X,\Gamma^g)$ to g.

Note that we have an exact sequence $0\to\Gamma^g\to G^g\to G^g/\Gamma^g\to 0$.

The group $H^0(X,G^g/\Gamma^g)$ acts on $H^1(X,\Gamma^g)$ and the fibers of λ_g are the fibers of this action.

2.2.4. *The case where* Γ *is commutative* – The sheaf of groups G/Γ acts then by conjugation on Γ. Let $\omega\in H^1(X,G/\Gamma)$. We define $\Delta(\omega)\in H^2(X,\Gamma^\omega)$ as follows: we can represent ω by a cocycle (ω_{ij}) (for a suitable open cover (U_i) of X), such that for every i,j there exists $g_{ij}\in G(U_{ij})$ over ω_{ij}, and that $g_{ji}=g_{ij}^{-1}$. For every indices i,j,k let $\gamma_{ijk}=g_{ij}g_{jk}g_{ki}$. Then (γ_{ijk}) is a family representing $\Delta(\omega)$ (cf. 2.2.1). We have then: $\Delta(\omega)=0$ *if and only if* ω *belongs to the image of* $H^1(p)$.

The commutativity of Γ brings also a supplementary property of the map λ_g of 2.2.3. The sheaf of groups Γ^g can be seen naturally as a subsheaf of commutative groups of of G^g, and we have a canonical isomorphism $(G/\Gamma)^g\simeq G^g/\Gamma^g$. Hence there is a natural action of $H^0(X,G^g/\Gamma^g)$ on $H^1(X,\Gamma^g)$, and we have: *the fibers of* λ_g *are the orbits of the action of* $H^0(X,G^g/\Gamma^g)$. This action is as follows: let $h\in H^0(G^g/\Gamma^g)$, $\gamma\in H^1(X,\Gamma^g)$, represented by (γ_{ij}). For a suitable open cover (U_i), each $h_{|U_i}$ can be lifted to $h_i\in G(U_i)$, and we have $h_ig_{ij}h_jg_{ij}^{-1}\in\Gamma(U_{ij})$. Then $h.\gamma$ is represented by $(h_i.\gamma_{ij}.g_{ij}h_j^{-1}g_{ij}^{-1})$.

2.3. Extension of polynomials

The following results are used in 3. Let R be a commutative unitary ring and n a positive integer. Let $R_n=R[t]/(t^n)$, which can be seen as the set of polynomials of degree $<n$ in the variable t with coefficients in R. In this way we have an inclusion $R_n\subset R_{n+1}$. For every $\alpha\in R_n$ and every integer p such that $0\leq p<n$, let α_p denote the coefficient of t^p in α.

Let r be a positive integer, $M_{n,r}=M(r,R_n)$ the R-algebra of $r\times r$-matrices with coefficients if R_n, and $\mathrm{GL}_{n,r}=\mathrm{GL}(r,R_n)\subset M_{n,r}$ the group of invertible $r\times r$ matrices. We have inclusions $M_{n,r}\subset M_{n+1,r}$, $\mathrm{GL}_{n,r}\subset\mathrm{GL}_{n+1,r}$.

Let $A \in M_{n+1,r}$, and p an integer such that $0 \leq p \leq n$. We will denote by $A_p \in M(r,R)$ the coefficient of the term of degree p of A, so that $A = \sum\limits_{0 \leq p \leq n} A_p t^p$. Let

$$[A]_n \ = \ \sum_{0 \leq p < n} A_p t^p \ ,$$

so that $A = [A]_n + A_n t^n$.

Let $A \in \mathrm{GL}_{n,r}$. An element B of $\mathrm{GL}_{n+1,r}$ is called an *extension* of A if $[B]_n = A$. Here we will define canonical extensions compatible with some actions of automorphisms of R_{n+1}.

Let λ be an automorphism of R_{n+1} such that $\lambda(\alpha)_0 = \alpha$ for every $\alpha \in R$. We have $\lambda(t) = \mu_\lambda t$, with $\mu_\lambda \in R_n$, invertible, and $\mu_{\lambda^{-1}} = \mu_\lambda^{-1}$. We will also denote by λ the automorphism of R_n and the bijection $M_{n+1,r} \to M_{n+1,r}$ induced by λ.

Let $A \in \mathrm{GL}_{n+1,r}$ (or $\mathrm{GL}_{n,r}$). Let $\mathrm{Inv}_\lambda(A) = \lambda^{-1}(A^{-1})$. In this way we define a bijection $\mathrm{GL}_{n+1,r} \to \mathrm{GL}_{n+1,r}$ whose inverse is $A \mapsto \mathrm{Inv}_{\lambda^{-1}}(A)$. Let

$$\Gamma_\lambda(A) \ = \ \frac{1}{2}\big(\mu_\lambda^n A_0 \mathrm{Inv}_\lambda(A)_n A_0 - A_n\big) \ ,$$

so that we have also

$$\Gamma_{\lambda^{-1}}(\mathrm{Inv}_\lambda(A)) \ = \ \frac{1}{2}\big(\mu_{\lambda^{-1}}^n A_0^{-1} A_n A_0^{-1} - \mathrm{Inv}_\lambda(A)_n\big) \ .$$

Let

$$A_{ext,\lambda} \ = \ A + \Gamma_\lambda(A) t^n \ ,$$

which is an extension of A if $A \in \mathrm{GL}_{n,r}$.

2.3.1. Lemma: *We have* $\mathrm{Inv}_\lambda(A_{ext,\lambda}) \ = \ \mathrm{Inv}_\lambda(A)_{ext,\lambda^{-1}}$.

Proof. We have

$$\begin{aligned}
\mathrm{Inv}_\lambda(A_{ext,\lambda}) \ &= \ \lambda^{-1}\big([A + \Gamma_\lambda(A)t^n]^{-1}\big) \\
&= \ \lambda^{-1}\big([A(I + A_0^{-1}\Gamma_\lambda(A)t^n)]^{-1}\big) \\
&= \ \lambda^{-1}\big((I - A_0^{-1}\Gamma_\lambda(A)t^n))A^{-1}\big) \\
&= \ (I - \mu_\lambda^{-n} A_0^{-1}\Gamma_\lambda(A)t^n)\mathrm{Inv}_\lambda(A) \\
&= \ \mathrm{Inv}_\lambda(A) - \mu_\lambda^{-n} A_0^{-1}\Gamma_\lambda(A)A_0^{-1}t^n \ .
\end{aligned}$$

The result follows from the equality $\Gamma_{\lambda^{-1}}(\mathrm{Inv}_\lambda(A)) = -\mu_\lambda^{-n} A_0^{-1}\Gamma_\lambda(A)A_0^{-1}$, which is easily verified. □

2.3.2. Lemma: *Let* $A \in \mathrm{GL}(n,r)$, *viewed as an element of* $\mathrm{GL}(n+1,r)$. *Then we have*

$$\mathrm{Inv}_\lambda(A_{ext,\lambda}) \ = \ ([\mathrm{Inv}_\lambda(A)]_n)_{ext,\lambda^{-1}} \ .$$

Proof. We have from lemma 2.3.1

$$\begin{aligned}
\mathrm{Inv}_\lambda(A_{ext,\lambda}) &= \mathrm{Inv}_\lambda(A)_{ext,\lambda^{-1}} \\
&= \mathrm{Inv}_\lambda(A) - \frac{1}{2}\mathrm{Inv}_\lambda(A)_n t^n \\
&= [\mathrm{Inv}_\lambda(A)]_n + \frac{1}{2}\mathrm{Inv}_\lambda(A)_n t^n \ .
\end{aligned}$$

Let $B = \mathrm{Inv}_{\lambda^{-1}}([\mathrm{Inv}_\lambda(A)]_n)$, so that $[\mathrm{Inv}_\lambda(A)]_n = B_\lambda$. We have

$$([\mathrm{Inv}_\lambda(A)]_n)_{ext,\lambda^{-1}} = [\mathrm{Inv}_\lambda(A)]_n + \frac{1}{2}\mu_\lambda^{-n}A_0^{-1}B_nA_0^{-1}t^n \ ,$$

hence we have to show that $\mathrm{Inv}_\lambda(A)_n = \mu_\lambda^{-n}A_0^{-1}B_nA_0^{-1}$. We have

$$\begin{aligned}
B &= \lambda\big(([\mathrm{Inv}_\lambda(A)]_n)^{-1}\big) \\
&= \lambda\big((\mathrm{Inv}_\lambda(A) - \mathrm{Inv}_\lambda(A)_n t^n)^{-1}\big) \\
&= \lambda\big([\mathrm{Inv}_\lambda(A)(1 - A_0\mathrm{Inv}_\lambda(A)_n t^n)]^{-1}\big) \\
&= \lambda\big((1 + A_0\mathrm{Inv}_\lambda(A)_n t^n)\mathrm{Inv}_\lambda(A)^{-1}\big) \\
&= A + \mu_\lambda^n A_0\mathrm{Inv}_\lambda(A)_n A_0 t^n \ ,
\end{aligned}$$

and the result follows immediately since $A_n = 0$. □

2.3.3. *Example* – If $n = 2$, $\lambda = I$ and $A \in \mathrm{GL}_{2,r}$, we have

$$\begin{aligned}
A_{ext,I} &= A_0 + tA_1 + \frac{1}{2}t^2A_1A_0^{-1}A_1 \ , \\
(A^{-1})_{ext,I} &= A_0^{-1} - tA_0^{-1}A_1A_0^{-1} + \frac{1}{2}t^2A_0^{-1}A_1A_0^{-1}A_1A_0^{-1} \ .
\end{aligned}$$

3. Canonical class associated to a vector bundle

3.1. Definition

Let X be a scheme over $\mathbb{C}$ and E a vector bundle on X of rank r. Let $(U_i)_{i\in I}$ be an open cover of X such that we have trivializations:

$$\theta_i : E_{|U_i} \xrightarrow{\simeq} \mathcal{O}_{U_i} \otimes \mathbb{C}^r.$$

then

$$\theta_{ij} = \theta_i\theta_j^{-1} : \mathcal{O}_{U_{ij}} \otimes \mathbb{C}^r \xrightarrow{\simeq} \mathcal{O}_{U_{ij}} \otimes \mathbb{C}^r$$

can be viewed as a $r \times r$ matrix with coefficients in $\mathcal{O}(U_{ij})$. We can then define $d\theta_{ij}$, which is a $r \times r$ matrix with coefficients in $\Omega_X(U_{ij})$. Let

$$B_{ij} = (d\theta_{ij})\theta_{ij}^{-1} \ .$$

We have

$$\begin{aligned} B_{ik} &= (d\theta_{ik})\theta_{ik}^{-1} \\ &= \big(\theta_{ij}d\theta_{jk} + d\theta_{ij}\theta_{jk}\big)\theta_{jk}^{-1}\theta_{ij}^{-1} \\ &= \theta_{ij}(d\theta_{jk})\theta_{jk}^{-1}\theta_{ij}^{-1} + (d\theta_{ij})\theta_{ij}^{-1} \\ &= \theta_{ij}B_{jk}\theta_{ij}^{-1} + B_{ij}\ . \end{aligned}$$

From 2.1, it follows that (B_{ij}) represents an element of $H^1(E\otimes E^*\otimes\Omega_X)$. It is easy to see that it does not depend on the family (θ_{ij}) defining E. We denote by $\nabla_0(E)$ this element, and call it the *canonical class of* E.

If L is a line bundle on X, then we have $\nabla_0(L)\in H^1(\Omega_X)$.

3.2. Functorial properties

Let $f : X\to Y$ be a morphism of schemes over $\mathbb{C}$, and F a vector bundle on Y. We have a canonical morphism of sheaves $\eta : f^*(\Omega_Y)\to\Omega_X$, hence

$$\eta\otimes I : f^*(F\otimes F^*\otimes\Omega_Y)\longrightarrow f^*(F)\otimes f^*(F)^*\otimes\Omega_X.$$

Let ι be the canonical map $H^1(Y,F\otimes F^*\otimes\Omega_Y)\to H^1(X,f^*(F\otimes F^*\otimes\Omega_Y))$.

3.2.1. Lemma: *We have* $\nabla_0(f^*(F)) = H^1(\eta\otimes I)(\iota(\nabla_0(F)))$.

Proof. Let $(V_i)_{i\in I}$ be an open cover of Y such that for every $i\in I$ there is a trivialization $\theta_i : F_{V_i}\simeq\mathcal{O}_{V_i}\otimes\mathbb{C}^r$, so that, if $\theta_{ij}=\theta_i\theta_j^{-1} : \mathcal{O}_{V_{ij}}\otimes\mathbb{C}^r\to\mathcal{O}_{V_{ij}}\otimes\mathbb{C}^r$, then (θ_{ij}) is a cocycle representing F. We have then $f^*(\theta_{ij}) : \mathcal{O}_{f^{-1}(V_{ij})}\otimes\mathbb{C}^r\to\mathcal{O}_{f^{-1}(V_{ij})}\otimes\mathbb{C}^r$, and $(f^*(\theta_{ij}))$ is a cocycle representing $f^*(F)$, with respect to the open cover $(f^{-1}(V_i)_{i\in I})$ of X. The result follows immediately from the fact that, for every open subset $V\subset Y$, and $\alpha\in\mathcal{O}_Y(V)$ we have $\eta(d\alpha)=d(f\circ\alpha)$. □

Let E_1, E_2 be vector bundles on X. For $i=1,2$, let $\Psi_i : \mathcal{O}_X\to E_i\otimes E_i^*$ be the canonical morphism. If $1\leq i\neq j\leq 2$, we have then

$$I\otimes\Psi_j : E_i\otimes E_i^*\otimes\Omega_X\longrightarrow(E_1\otimes E_2)\otimes(E_1\otimes E_2)^*\otimes\Omega_X\ .$$

3.2.2. Lemma: *We have* $\nabla_0(E_1\otimes E_2) = (I\otimes\Psi_1)(\nabla_0(E_2)) + (I\otimes\Psi_2)(\nabla_0(E_1))$.

Proof. For $k=1,2$, let $r_k=\mathrm{rk}(E_k)$. Suppose that E_k is represented by a cocycle (θ_{ij}^k), where θ_{ij}^k is an automorphism of $\mathcal{O}_{U_{ij}}\otimes\mathbb{C}^{r_k}$. Then $E_1\otimes E_2$ is represented by the cocycle $(\theta_{ij}^1\otimes\theta_{ij}^2)$, $\theta_{ij}^1\otimes\theta_{ij}^2$ being an automorphism of $\mathcal{O}_{U_{ij}}\otimes\mathbb{C}^{r_1}\otimes\mathbb{C}^{r_2}$. We have then

$$\begin{aligned} \big(d(\theta_{ij}^1\otimes\theta_{ij}^2)\big)(\theta_{ij}^1\otimes\theta_{ij}^2)^{-1} &= \big(d\theta_{ij}^1\otimes\theta_{ij}^2 + \theta_{ij}^1\otimes d\theta_{ij}^2\big)(\theta_{ij}^1\otimes\theta_{ij}^2)^{-1} \\ &= (d\theta_{ij}^1)(\theta_{ij}^1)^{-1}\otimes I + I\otimes(d\theta_{ij}^2)(\theta_{ij}^2)^{-1}\ . \end{aligned}$$

The result follows immediately. □

3.2.3. Corollary: *Let L_1, L_2 be line bundles on X. Then we have* $\nabla_0(L_1 \otimes L_2) = \nabla_0(L_1) + \nabla_0(L_2)$.

3.3. The canonical class of the determinant of a vector bundle

Let U be a scheme over $\mathbb{C}$. Let

$$\begin{aligned} \eta_r : \mathrm{GL}(r, H^0(\mathcal{O}_U)) &\longrightarrow M(r, H^0(\Omega_U)) \\ M &\longmapsto M^{-1}dM \end{aligned}$$

We have, for any $M, N \in \mathrm{GL}(r, H^0(\mathcal{O}_U))$,

$$\eta_r(MN) \ = \ \eta_r(N) + N^{-1}\eta_r(M)N \ .$$

Let

$$\begin{aligned} T_r : \mathrm{GL}(r, H^0(\mathcal{O}_U)) &\longrightarrow H^0(\Omega_U) \\ M &\longmapsto tr(\eta_r(M)) \end{aligned}$$

(where tr is the trace morphism). We have

$$T_r(MN) \ = \ T_r(M) + T_r(N) \ . \tag{1}$$

3.3.1. Proposition: *For every $M \in \mathrm{GL}(r, H^0(\mathcal{O}_U))$ we have* $T_r(M) = T_1(\det(M))$.

Proof. It is true if $M \in \mathrm{GL}(r, \mathbb{C})$, because in this case $T_r(M) = T_1(\det(M))) = 0$.
Let's show that it is also true if M is a triangular matrix (upper or lower). Let $\lambda_1, \ldots, \lambda_r \in \mathcal{O}^*(U)$ be the diagonal elements of M. Then $M^{-1}dM$ is also triangular, and its diagonal elements are $\dfrac{d\lambda_i}{\lambda_i}$, $1 \leq i \leq r$. Hence we have

$$T_r(M) \ = \ \sum_{k=1}^{r} \frac{d\lambda_k}{\lambda_k} \ .$$

We have $\det(M) = \prod_{k=1}^r \lambda_k$. Hence

$$d(\det(M)) \ = \ \big(\prod_{k=1}^{r} \lambda_k\big) \sum_{k=1}^{r} \frac{d\lambda_k}{\lambda_k} \ ,$$

and $T_r(M) = T_1(\det(M)))$.
Note that for any $A, B \in \mathcal{O}^*(U)$, we have $T_1(AB) = T_1(A) + T_1(B)$. Hence it follows from (1) that it suffices to show that M is a product of matrices that are triangular or in $\mathrm{GL}(r, \mathbb{C})$. We can also replace M by the product of M with a triangular matrix, or by the product of M with an element of $\mathrm{GL}(r, \mathbb{C})$. Il particular, we can multiply rows or columns of M by elements of $\mathcal{O}^*(U)$, add a column (or row) to another one, or switch columns or rows.

We now prove proposition 3.3.1, more precisely the fact that M is a product of matrices that are triangular or in $\mathrm{GL}(r, \mathbb{C})$, by induction on r. The case $r = 1$ is obvious. Suppose that the theorem is true for $r - 1$.

In the first column of M there is at least one invertible element M_{i1}. By permuting rows of M we can assume that $i=1$. For $2\leq i\leq r$, by subtracting $M_{i1}M_{11}^{-1}$ times the first row to the i-th we can assume that $M_{i1}=0$. Similarly we can assume that $M_{1i}=0$. We have then

$$M \;=\; \begin{pmatrix} M_{11} & 0 \\ 0 & M' \end{pmatrix} \ ,$$

where $M'\in\mathrm{GL}(r-1,H^0(\mathcal{O}_U))$. From the induction hypothesis we can write

$$M' \;=\; M'_1\cdots M'_p \ ,$$

where, for $1\leq i\leq p$, M'_i is triangular or in $\mathrm{GL}(r-1,\mathbb{C})$. We have

$$M \;=\; \begin{pmatrix} M_{11} & 0 \\ 0 & I_{r-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & M'_1 \end{pmatrix}\cdots\begin{pmatrix} 1 & 0 \\ 0 & M'_p \end{pmatrix} \ ,$$

and $\begin{pmatrix} M_{11} & 0 \\ 0 & I_{r-1} \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & M'_1 \end{pmatrix},\ldots,\begin{pmatrix} 1 & 0 \\ 0 & M'_p \end{pmatrix}$ are triangular or in $\mathrm{GL}(r,\mathbb{C})$. □

Let $tr_E : H^1(\mathcal{E}nd(E)\otimes\Omega_X)\to H^1(\Omega_X)$ be the trace morphism.

3.3.2. Corollary: *We have* $tr_E(\nabla_0(E))=\nabla_0(\det(E))$.

We have a canonical isomorphism $\mathcal{E}nd(E)\simeq\mathcal{A}d(E)\oplus\mathcal{O}_X$, where $\mathcal{A}d(E)$ is the sheaf of endomorphisms of trace 0. Let $\nabla(E)$ be the component of $\nabla_0(E)$ in $H^1(\mathcal{A}d(E)\otimes\Omega_X)$. So we have

$$\nabla_0(E) \;=\; \nabla(E)+\frac{1}{r}\nabla_0(\det(E)) \ .$$

4. Primitive multiple schemes

4.1. Definitions

Let X be a smooth connected variety, and $d=\dim(X)$. A *multiple scheme with support* X is a Cohen-Macaulay scheme Y such that $Y_{red}=X$. If Y is quasi-projective we say that it is a *multiple variety with support* X. In this case Y is projective if X is.

Let n be the smallest integer such that $Y=X^{(n-1)}$, $X^{(k-1)}$ being the k-th infinitesimal neighborhood of X, i.e. $\mathcal{I}_{X^{(k-1)}}=\mathcal{I}_X^k$. We have a filtration $X=X_1\subset X_2\subset\cdots\subset X_n=Y$ where X_i is the biggest Cohen-Macaulay subscheme contained in $Y\cap X^{(i-1)}$. We call n the *multiplicity* of Y.

We say that Y is *primitive* if, for every closed point x of X, there exists a smooth variety S of dimension $d+1$, containing a neighborhood of x in Y as a locally closed subvariety. In this case, $L=\mathcal{I}_X/\mathcal{I}_{X_2}$ is a line bundle on X, X_j is a primitive multiple scheme of multiplicity j and we have $\mathcal{I}_{X_j}=\mathcal{I}_X^j$, $\mathcal{I}_{X_j}/\mathcal{I}_{X_{j+1}}=L^j$ for $1\leq j<n$. We call L the line bundle on X *associated* to Y. The ideal sheaf $\mathcal{I}_{X,Y}$ can be viewed as a line bundle on X_{n-1}.

Let $P \in X$. Then there exist elements $y_1, \ldots, y_d$, t of $m_{S,P}$ whose images in $m_{S,P}/m_{S,P}^2$ form a basis, and such that for $1 \leq i < n$ we have $\mathcal{I}_{X_i,P} = (t^i)$. In this case the images of $y_1, \ldots, y_d$ in $m_{X,P}/m_{X,P}^2$ form a basis of this vector space.

A *multiple scheme with support* X is primitive if and only if $\mathcal{I}_X/\mathcal{I}_X^2$ is zero or a line bundle on X (cf. [12], proposition 2.3.1).

Even if X is projective, we do not assume that Y is projective. In fact we will give examples of non quasi-projective Y.

The simplest case is when Y is contained in a smooth variety S of dimension $d+1$. Suppose that Y has multiplicity n. Let $P \in X$ and $f \in \mathcal{O}_{S,P}$ a local equation of X. Then we have $\mathcal{I}_{X_i,P} = (f^i)$ for $1 < j \leq n$ in S, in particular $\mathcal{I}_{Y,P} = (f^n)$, and $L = \mathcal{O}_X(-X)$.

For any $L \in \mathrm{Pic}(X)$, the *trivial primitive variety* of multiplicity n, with induced smooth variety X and associated line bundle L on X is the n-th infinitesimal neighborhood of X, embedded by the zero section in the dual bundle L^*, seen as a smooth variety.

4.2. Construction and parametrization of primitive multiple schemes

Let Y be a primitive multiple scheme of multiplicity n, $X = Y_{red}$. Let $\mathbf{Z}_n = \mathrm{spec}(\mathbb{C}[t]/(t^n))$. Then for every closed point $P \in X$, there exists an open neighborhood U of P in X, such that if $U^{(n)}$ is the corresponding neighborhood of P in Y, there exists a commutative diagram

$$\begin{tikzcd}
 & U \arrow[dl, hook'] \arrow[dr, hook] & \\
U^{(n)} \arrow[rr, "\simeq"] & & U \times \mathbf{Z}_n
\end{tikzcd}$$

i.e. Y is locally trivial ([8], théorème 5.2.1, corollaire 5.2.2).

It follows that we can construct a primitive multiple scheme of multiplicity n by taking an open cover $(U_i)_{i \in I}$ of X and gluing the varieties $U_i \times \mathbf{Z}_n$ (with automorphisms of the $U_{ij} \times \mathbf{Z}_n$ leaving U_{ij} invariant).

Let $\mathcal{A}_n = \mathcal{O}_{X \times \mathbf{Z}_n}$. So we have, for any open subset U of X

$$\mathcal{A}_n(U) \ = \ \mathcal{O}_X(U) \otimes_{\mathbb{C}} \mathbb{C}[t]/(t^n) \ .$$

We have a canonical morphism $\mathcal{A}_n \to \mathcal{O}_X$ of sheaves on X coming from the inclusion $X \subset X \times \mathbf{Z}_n$. Let $\mathcal{G}_n$ be the sheaf of groups of automorphisms of $\mathcal{A}_n$ leaving X invariant, i.e. $\mathcal{G}_n(U)$ is the group of automorphisms of $\mathbb{C}$-algebras $\theta : \mathcal{A}_n(U) \to \mathcal{A}_n(U)$ such that the following diagram is commutative

$$\begin{tikzcd}
\mathcal{A}_n(U) \arrow[rr, "\theta"] \arrow[dr] & & \mathcal{A}_n(U) \arrow[dl] \\
 & \mathcal{O}_X(U) &
\end{tikzcd}$$

Then the set of primitive multiple schemes Y of multiplicity n, such that $X = Y_{red}$, can be identified with the cohomology set $H^1(X, \mathcal{G}_n)$.

4.2.1. *Primitive multiple schemes and the associated cohomology classes* – For every open subset U of X, let $U^{(n)}$ be the corresponding open subset of Y. Let $(U_i)_{i\in I}$ be an affine open cover of X such that we have trivializations

$$\delta_i : U_i^{(n)} \xrightarrow{\simeq} U_i \times \mathbf{Z}_n,$$

and $\delta_i^* : \mathcal{O}_{U_i\times\mathbf{Z}_n} \to \mathcal{O}_{U_i^{(n)}}$ the corresponding isomorphism. Let

$$\delta_{ij} = \delta_j\delta_i^{-1} : U_{ij}\times\mathbf{Z}_n \xrightarrow{\simeq} U_{ij}\times\mathbf{Z}_n,$$

and $\delta_{ij}^* = \delta_i^{*-1}\delta_j^* \in \mathcal{G}_n(U_{ij})$. Then (δ_{ij}^*) is a cocycle which represents the element g_n of $H^1(X,\mathcal{G}_n)$ corresponding to Y.

4.2.2. *The ideal sheaf of* X – There exists $a_{ij} \in \mathcal{O}_X(U_{ij})\otimes_{\mathbb{C}}\mathbb{C}[t]/(t^{n-1})$ such that $\delta_{ij}^*(t) = a_{ij}t$. Let $\alpha_{ij} = a_{ij|X} \in \mathcal{O}_X(U_i)$. For every $i\in I$, $\delta_i^*(t)$ is a generator of $\mathcal{I}_{X,Y|U^{(n)}}$. So we have local trivializations

$$\begin{aligned}\lambda_i : \mathcal{I}_{X,Y|U_i^{(n-1)}} &\longrightarrow \mathcal{O}_{U_i^{(n-1)}}\\ \delta_i^*(t) &\longmapsto 1\end{aligned}$$

Hence $\lambda_{ij} = \lambda_i\lambda_j^{-1} : \mathcal{O}_{U_{ij}^{(n-1)}} \to \mathcal{O}_{U_{ij}^{(n-1)}}$ is the multiplication by $\delta_j^*(a_{ij})$. It follows that $(\delta_j^*(a_{ij}))$ (resp. (α_{ij})) is a cocycle representing the line bundle $\mathcal{I}_{X,Y}$ (resp. L) on X_{n-1} (resp. X).

We have a canonical morphism of sheaves of groups

$$\xi_n : \mathcal{G}_n \longrightarrow \mathcal{O}_X^*$$

defined as follows: if U is an open subset of X and $\phi\in\mathcal{G}_n(U)$, then there exists $\nu\in\mathcal{O}_X^*(U)$ such that $\phi(t) = \nu t$. Then $\xi_n(\phi) = \nu$. The map

$$H^1(\xi_n) : H^1(X,\mathcal{G}_n) \longrightarrow H^1(X,\mathcal{O}_X^*)$$

associates to the primitive scheme Y the associated line bundle L on X. This map is surjective: for every line bundle L on X, there exists a primitive variety of multiplicity n and associated line bundle L (the trivial primitive variety).

4.2.3. *Descriptions using the open cover* (U_i)

(i) *Construction of sheaves* – Let $\mathcal{E}$ be a coherent sheaf on X_n. We can define it in the usual way, starting with sheaves $\mathcal{F}_i$ on the open sets $U_i^{(n)}$ and gluing them. We take these sheaves of the form $\mathcal{F}_i = \delta_i^*(\mathcal{E}_i)$, where $\mathcal{E}_i$ is a sheaf on $U_i\times\mathbf{Z}_n$. To glue the sheaves $\mathcal{F}_i$ on the intersection $U_{ij}^{(n)}$ we use isomorphisms $\rho_{ij} : \mathcal{F}_{j|U_{ij}^{(n)}} \to \mathcal{F}_{i|U_{ij}^{(n)}}$, with the relations $\rho_{ik} = \rho_{ij}\rho_{jk}$. Let

$$\theta_{ij} = (\delta_i^*)^{-1}(\rho_{ij}) : \delta_{ij}^*(\mathcal{E}_{j|U_{ij}\times\mathbf{Z}_n}) \longrightarrow \mathcal{E}_{i|U_{ij}\times\mathbf{Z}_n}\ .$$

We have then the relations $\theta_{ik} = \theta_{ij}\circ\delta_{ij}^*(\theta_{jk})$. Conversely, starting with sheaves $\mathcal{E}_i$ and isomorphisms θ_{ij} satisfying the preceding relations, one obtains a coherent sheaf on X_n.

This applies to trivializations, i.e when $\mathcal{E}_i = \mathcal{O}_{U_i^{(n)}}\otimes\mathbb{C}^r$. We have then
$\theta_{ij} : \mathcal{O}_{U_{ij}\times\mathbf{Z}_n}\otimes\mathbb{C}^r \to \mathcal{O}_{U_{ij}\times\mathbf{Z}_n}\otimes\mathbb{C}^r$.

(ii) *Morphisms* – Suppose that we have another sheaf $\mathcal{E}'$ on X_n, defined by sheaves $\mathcal{E}'_i$ on $U_i \times \mathbf{Z}_n$ and isomorphisms θ'_{ij}. One can see easily that a morphism $\Psi : \mathcal{E} \to \mathcal{E}'$ is defined by morphisms $\Psi_i : \mathcal{E}_i \to \mathcal{E}'_i$ such that $\theta'_{ij} \circ \delta^*_{ij}(\Psi_j) = \Psi_i \circ \theta_{ij}$.

(iii) *Inverse images* – Let φ be an automorphism of X_n inducing the identity on X. Let $\varphi_i = \delta_i \circ \varphi \circ \delta_i^{-1} : U_i \times \mathbf{Z}_n \to U_i \times \mathbf{Z}_n$. We have then $\varphi_j \circ \delta_{ij} = \delta_{ij} \circ \varphi_i$. And conversely, given automorphisms $\varphi_i : U_i \times \mathbf{Z}_n \to U_i \times \mathbf{Z}_n$ inducing the identity on U_i and satisfying the preceding relations we can build a corresponding automorphism φ of X_n.

Suppose that $\mathcal{E}$ is defined by trivializations as in (i), and described with the morphisms θ_{ij}. In the same way $\varphi^*(\mathcal{E})$ is described by the morphisms $\varphi^*_i(\theta_{ij})$.

4.3. The case of double schemes

We suppose that $n = 2$. Then there exists a map $D_{ij} : \mathcal{O}_X(U_{ij}) \to \mathcal{O}_X(U_{ij})$ such that, for every $f \in \mathcal{O}_X(U_{ij})$ we have

$$\delta^*_{ij}(f) \ = \ f + tD_{ij}(f) \ ,$$

and it is easy to see that D_{ij} is a derivation of $\mathcal{O}_X(U_{ij})$, i.e. a section of $T_{X|U_{ij}}$. It follows that δ^*_{ij} can be represented as the matrix $\begin{pmatrix} 1 & 0 \\ D_{ij} & \alpha_{ij} \end{pmatrix}$. The formula $\delta^*_{ij}\delta^*_{jk} = \delta^*_{ik}$ implies that $D_{ij} + \alpha_{ij}D_{jk} = D_{ik}$. It follows from 2.1.2 that the family (D_{ij}) represents an element of $H^1(T_X \otimes L)$. This element does not depend on the choice of the automorphisms δ_i and the trivializations $\lambda_{i|X}$ of L : suppose that τ_i is an automorphism of $U_i \times \mathbf{Z}_n$, represented by the matrix $\begin{pmatrix} 1 & 0 \\ D_i & \beta_i \end{pmatrix}$, and that we replace δ_i with $\delta'_i = \tau_i\delta_i$. Then δ^*_{ij} is replaced with $\delta^{*\prime}_{ij}$, represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ D'_{ij} & \alpha_{ij}\beta_j\beta_i^{-1} \end{pmatrix} \ = \ \begin{pmatrix} 1 & 0 \\ D_i & \beta_i \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ D_{ij} & \alpha_{ij} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D_j & \beta_j \end{pmatrix} \ ,$$

with $D'_{ij} = -\beta_i^{-1}D_i + \beta_i^{-1}D_{ij} + \beta_i^{-1}\alpha_{ij}D_j$. It follows from 2.1.2 that (D'_{ij}) represents the same element of $H^1(T_X \otimes L)$ as (D_{ij}).

We have $\ker(\xi_2) \simeq T_X$, so we have an exact sequence of sheaves of groups on X

$$0 \longrightarrow T_X \longrightarrow \mathcal{G}_2 \xrightarrow{\xi_2} \mathcal{O}^*_X \longrightarrow 0.$$

Recall that $g_2 \in H^1(X, \mathcal{G}_2)$ is associated to Y. Let $g_1 = H^1(\xi_2)(g_2)$.

4.3.1. Lemma: 1 – *We have* $T_X^{g_1} \simeq T_X \otimes L$.

2 – *The action of* $H^0(X, \mathcal{O}^*_X) = \mathbb{C}^*$ *on* $H^1(X, T_X \otimes L)$ *is the multiplication.*

Proof. Let $U \subset X$ be an open subset, $D, D_0 \in H^0(U, T_X)$, $\alpha \in H^0(U, \mathcal{O}^*_X)$. The result follows easily from the formula

$$\begin{pmatrix} 1 & 0 \\ D & \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D & \alpha \end{pmatrix}^{-1} \ = \ \begin{pmatrix} 1 & 0 \\ \alpha D_0 & 1 \end{pmatrix}$$

and 2.1.1. □

4.3.2. Proposition: *The element of* $H^1(X, T_X \otimes L) = \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X, L)$ *corresponding to the canonical exact sequence*

$$0 \longrightarrow L \longrightarrow \Omega_{Y|X} \longrightarrow \Omega_X \longrightarrow 0$$

is g_2.

Proof. The vector bundle $\Omega_{Y|X}$ can be constructed, by the method of 2.1.4, using the local isomorphisms $U_i^{(2)} \simeq U_i \times \mathbf{Z}_2$, the bundles $\Omega_{U_i\times\mathbf{Z}_2|U_i}$, and the canonical automorphisms of $\Omega_{U_{ij}\times\mathbf{Z}_2|U_{ii}}$ defined by δ_{ij}. Let

$$\mu_{ij} : \delta_{ij}^*(\Omega_{U_{ij}\times\mathbf{Z}_2}) = \Omega_{U_{ij}\times\mathbf{Z}_2} \longrightarrow \Omega_{U_{ij}\times\mathbf{Z}_2}$$

be the canonical morphism. Then for every $\alpha \in \mathcal{O}_X(U_{ij})$ we have

$$\begin{aligned}
\mu_{ij}(d\alpha) &= d(\delta_{ij} \circ \alpha) \\
&= d(\alpha + D_{ij}(\alpha)t) \\
&= d\alpha + d(D_{ij}(\alpha))t + D_{ij}(\alpha)dt, \\
\mu_{ij}(dt) &= d(\delta_{ij} \circ t) \\
&= d(\alpha_{ij}t) \\
&= d(\alpha_{ij})t + \alpha_{ij}dt.
\end{aligned}$$

It follows that $\mu_{ij|U_{ij}} : \Omega_{U_{ij}\times\mathbf{Z}_2|U_{ij}} \to \Omega_{U_{ij}\times\mathbf{Z}_2|U_{ij}}$ is defined, with respect to the isomorphism $\Omega_{U_{ij}\times\mathbf{Z}_2|U_{ij}} \simeq \Omega_{U_{ij}} \oplus (\mathcal{O}_{U_{ij}} \otimes \mathbb{C}dt)$, by the matrix $\begin{pmatrix} I & 0 \\ D_{ij} & \alpha_{ij} \end{pmatrix}$. The result follows then easily from 2.1. $\square$

4.3.3. *Parametrization of primitive double schemes* – From lemma 4.3.1 it follows that the non trivial primitive double schemes Y such that $Y_{red} = X$ and with associated line bundle L are parametrized by $\mathbb{P}(H^1(X, T_X \otimes L))$. This result has been proved by another method in [4] (cf. 7.5).

4.4. The case of primitive schemes of multiplicity $n > 2$

There is a canonical obvious morphism $\rho_n : \mathcal{G}_n \to \mathcal{G}_{n-1}$.

4.4.1. Proposition: *We have* $\ker(\rho_n) \simeq T_X \oplus \mathcal{O}_X$.

Proof. Let $U \subset X$ be an open subset. Let ϕ be an automorphism of $U \times \mathbf{Z}_n$ leaving U invariant, and ϕ^* the associated automorphism of $\mathcal{O}_X(U)[t]/(t^n)$. It belongs to $\ker(\rho_n)(U)$ if and only if there is a derivation D of $\mathcal{O}_X(U)$ and $\alpha \in \mathcal{O}_X^*(U)$ such that $\phi^*(\lambda) = \lambda + D(\lambda)t^{n-1}$ for every $\lambda \in \mathcal{O}_X(U)$, and $\phi^*(t) = (1 + \alpha t^{t-2})t$. The result follows easily. $\square$

Let $g_n \in H^1(X, \mathcal{G}_n)$, and for $2 \leq k < n$, $g_k = \rho_{k+1} \circ \cdots \circ \rho_n(g_n)$. We denote by Y the double primitive scheme defined by g_2, and use the notations of 4.2 and 4.3. Suppose that g_n is represented by the cocycle (δ_{ij}) with respect to (U_i).

4.4.2. Proposition: *We have* $\ker(\rho_n)^{g_{n-1}} \simeq (\Omega_{Y|X})^* \otimes L^{n-1}$.

Proof. The sheaf $\ker(\rho_n)^{g_{n-1}} = (T_X \oplus \mathcal{O}_X)^{g_{n-1}}$ is constructed as follows: we glue the sheaves $(T_X \oplus \mathcal{O}_X)_{|U_i}$ using the automorphisms of $\ker(\rho_n)_{|U_{ij}}$: $\phi \mapsto \delta_{ij}^*\phi(\delta_{ij}^*)^{-1}$. We can write

$$\phi(\alpha) \;=\; \alpha + D(\alpha)t^{n-1}\ , \qquad\qquad \phi(t) \;=\; (1+\epsilon t^{n-2})t$$

for every $\alpha \in \mathcal{O}_X(U_{ij})$ (for some derivation D of $\mathcal{O}_X(U_{ij})$ and $\epsilon \in \mathcal{O}_X(U_{ij})$); D and ϵ are the components of ψ if $H^0(U_{ij}, T_X)$ and $\mathcal{O}_X(U_{ij})$ respectively. An easy computation shows then that

$$\begin{aligned}
\delta_{ij}^*\psi(\delta_{ij}^*)^{-1}(\alpha) &= \alpha + \big(D(\alpha) + D_{ij}(\alpha)\epsilon\big)\alpha_{ij}^{n-1}t^{n-1},\\
\delta_{ij}^*\psi(\delta_{ij}^*)^{-1}(t) &= (1 + \epsilon\alpha_{ij}^{n-2}t^{n-2})t,
\end{aligned}$$

i.e. we obtain the automorphism of $(T_X \oplus \mathcal{O}_X)_{|U_{ij}}$ defined by the matrix
$\begin{pmatrix} \alpha_{ij}^{n-1} & \alpha_{ij}^{n-1}D_{ij} \\ 0 & \alpha_{ij}^{n-2} \end{pmatrix}$. The result follows then from 2.1. $\square$

4.5. Primitive multiple schemes with extensions of the canonical ideal sheaf

We use the notations of 4.1 to 4.4. Suppose that $n \geq 2$, and let $Y = X_n$ be a primitive multiple scheme of multiplicity n such that $Y_{red} = X$, with associated line bundle $L \in \text{Pic}(X)$. The ideal sheaf $\mathcal{I}_{X,Y}$ is a line bundle on X_{n-1}. A necessary condition for the possibility to extend Y to a primitive multiple scheme X_{n+1} of multiplicity $n+1$ is that $\mathcal{I}_{X,Y}$ can be extended to a line bundle on Y, because in this case $\mathcal{I}_{X,X_{n+1}}$ is a line bundle on Y and we have $\mathcal{I}_{X,X_{n+1}|Y} = \mathcal{I}_{X,Y}$. This is why we will consider pairs $(Y, \mathbb{L})$, where $\mathbb{L}$ is a line bundle on Y such that $\mathbb{L}_{|X_{n-1}} \simeq \mathcal{I}_{Y,X}$.

4.5.1. *The corresponding sheaf of groups* – The sheaf of groups $\mathcal{H}_n$ on X (as $\mathcal{G}_n$ for primitive multiple schemes of multiplicity n) is defined as follows: for every open subset $U \subset X$, $\mathcal{H}_n(U)$ consists of pairs (ϕ, u), where $\phi \in \mathcal{G}_n(U)$, and $u \in \mathcal{O}_X(U)[t]/(t^n)$ is such that $\phi(t) = ut$. Note that u is then defined up to a multiple of t^{n-1}. The multiplication is defined by:

$$(\phi', u')(\phi, u) \;=\; (\phi'\phi, u'\phi'(u))$$

(cf. the formulas in 5.1). Associativity is easily verified. The identity element is $(I, 1)$ and the inverse of (ϕ, u) is $(\phi^{-1}, \phi^{-1}(\frac{1}{u}))$. The set of isomorphism classes of the above pairs $(Y, \mathbb{L})$ can then be identified with the cohomology set $H^1(X, \mathcal{H}_n)$. More precisely, given a cocycle $((\phi_{ij}, u_{ij}))$ of $\mathcal{H}_n$ associated to $(Y, \mathbb{L})$, (ϕ_{ij}) is a cocycle of $\mathcal{G}_n$ defining Y and (u_{ij}) a family defining $\mathbb{L}$ (according to 4.2.2 and 4.2.3).

We have canonical obvious morphisms

$$\epsilon_n : \mathcal{H}_n \longrightarrow \mathcal{G}_n\ , \qquad \tau_n : \mathcal{G}_{n+1} \longrightarrow \mathcal{H}_n\ .$$

4.5.2. Proposition: 1 – *We have* $\ker(\tau_n) \simeq T_X$.

2 – *Let* $g \in H^1(X, \mathcal{H}_n)$. *Then we have* $T_X^g \simeq T_X \otimes L^n$.

The proof is similar to those of propositions 4.4.1 and 4.4.2.

Finally we obtain commutative diagrams with exact rows

$$
\begin{tikzcd}
0 \arrow[r] & T_X \arrow[r] \arrow[d, hook] & \mathcal{G}_{n+1} \arrow[r, "\tau_n"] \arrow[d, equal] & \mathcal{H}_n \arrow[r] \arrow[d, "\epsilon_n"] & 0 \\
0 \arrow[r] & T_X \oplus \mathcal{O}_X \arrow[r] & \mathcal{G}_{n+1} \arrow[r, "\rho_{n+1}"] & \mathcal{G}_n \arrow[r] & 0
\end{tikzcd}
$$

(2)

$$
\begin{tikzcd}
0 \arrow[r] & T_X \otimes L^n \arrow[r] \arrow[d, "\iota"] & \mathcal{G}^g_{n+1} \arrow[r] \arrow[d, equal] & \mathcal{H}^g_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\Omega_{Y|X})^* \otimes L^n \arrow[r] & \mathcal{G}^g_{n+1} \arrow[r] & \mathcal{G}^g_n \arrow[r] & 0
\end{tikzcd}
$$

It is easy to see that ι is the injective morphism deduced from the exact sequence of proposition 4.4.2.

4.5.3. *The associated sheaf of groups* – Let $g \in H^1(X, \mathcal{H}_n)$, corresponding to the pair $(Y, \mathbb{L})$. Suppose that g is defined by the cocycle $((\delta^*_{ij}, u_{ij}))$, $(\delta^*_{ij}, u_{ij}) \in \mathcal{H}_n(U_{ij})$, so that (δ^*_{ij}) is the cocycle defining Y (cf. 4.2.1).

A global section of $\mathcal{H}^g_n$ is defined by a family $((\chi_i, v_i))_{i\in I}$, $(\chi_i, v_i) \in \mathcal{H}_n(U_i)$, such that

$$(\delta^*_{ij}, u_{ij})(\chi_j, v_j)(\delta^*_{ij}, u_{ij})^{-1} \ = \ (\chi_i, v_i) \ ,$$

which is equivalent to the two relations

$$
\begin{aligned}
\delta^*_{ij}\chi_j\delta^{*-1}_{ij} \ &= \ \chi_i \ , \\
u_{ij}\delta^*_{ij}(v_j) \ &= \ v_i\chi_i(u_{ij}) \ .
\end{aligned}
$$

The first one says that (χ_i) defines an automorphism χ of Y inducing the identity on X. The second that (v_i) defines an isomorphism $\mathbb{L} \to \chi^*(\mathbb{L})$. It follows that *we can identify $H^0(X, \mathcal{H}^g_n)$ with the set of pairs (χ, η), where χ is an automorphism of Y and $\eta : \mathbb{L} \to \chi^*(\mathbb{L})$ is an isomorphism.*

4.6. Ample line bundles and projectivity

We use the notations of 4.1 to 4.4. Suppose that $n \geq 2$, and let $Y = X_n$ be a primitive multiple scheme of multiplicity n such that $Y_{red} = X$, with associated line bundle $L \in \mathrm{Pic}(X)$. The following result follows from prop. 4.2 of [25]:

4.6.1. Proposition: *Let $\mathbb{D}$ be a line bundle on X_n, and $D = \mathbb{D}_{|X}$. Then $\mathbb{D}$ is ample if and only D is ample.*

4.6.2. Corollary: *The scheme X_n is quasi-projective if and only if there exists an ample line bundle on X that can be extended to a line bundle on X_n.*

In this case X_n is even projective.

5. Structure of $\mathcal{G}_n$

We use the notations of 4.1 and 4.2.

5.1. Description of local automorphisms

Let $U \subset X$ be a non empty open subset. Let ϕ be an automorphism of $U \times \mathbf{Z}_n$ leaving U invariant, that we can view ϕ as an automorphism of $\mathcal{O}_X(U)[t]/(t^n)$ such that for every $\alpha \in \mathcal{O}_X(U)$ we have $\phi(\alpha) = \alpha + t\eta(\alpha)$, for some map $\eta : \mathcal{O}_X(U) \to \mathcal{O}_X(U)/(t^{n-1})$. We have also $\phi(t) = \mu t$, for some $\mu \in \mathcal{O}_X(U)/(t^{n-1})$, invertible, and ϕ is completely defined by η and μ. The map η is $\mathbb{C}$-linear and has the following property: for every $\alpha, \beta \in \mathcal{O}_X(U)$ we have

$$\eta(\alpha\beta) \ = \ \eta(\alpha)\beta + \alpha\eta(\beta) + \eta(\alpha)\eta(\beta).t \ .$$

In particular, if $n = 2$ then η is a derivation of $\mathcal{O}_X(U)$. Conversely, given an invertible element $\mu \in \mathcal{O}_X(U)/(t^{n-1})$ and a $\mathbb{C}$-linear map η satisfying the preceding property, there is a unique automorphism ϕ of $\mathcal{O}_X(U)[t]/(t^n)$ such that for every $\alpha \in \mathcal{O}_X(U)$ we have $\phi(\alpha) = \alpha + t\eta(\alpha)$, and $\phi(t) = \mu t$. We will write

$$\phi \ = \ \phi_{\eta,\mu} \ .$$

Let $d = \dim(X)$. Suppose that there exist $x_1, \ldots, x_d \in \mathcal{O}_X(U)$ such that for every $P \in U$, $x_1 - x_1(P), \ldots, x_d - x_d(P)$ generate $m_{X,P}/m_{X,P}^2$. Then η is completely determined by μ and $\eta(x_i)$, $1 \leq i \leq d$. Conversely, any sequence $(\lambda_1, \ldots, \lambda_d, \mu)$ of elements of $\mu \in \mathcal{O}_X(U)/(t^{n-1})$ uniquely defines such a map η with the required properties. For every $\alpha \in \mathcal{O}_X(U)$, $\phi_{\eta,\mu}(\alpha)$ can be computed with the Taylor formula. For example, for $n = 3$

$$\phi_{\eta,\mu}(\alpha) \ = \ \alpha + t\sum_{i=1}^{d} \lambda_i \frac{\partial\alpha}{\partial x_i} + \frac{t^2}{2} \sum_{0 \leq i,j \leq d} \lambda_i\lambda_j \frac{\partial^2\alpha}{\partial x_i \partial x_j} \ .$$

For every $\beta \in \mathcal{O}_X(U)[t]/(t^n)$ and $0 \leq k < n$, let β_k be the coefficient of t^k in β, and if $k > 0$

$$[\beta]_k \ = \ \sum_{p=0}^{k-1} \beta_p t^p \ .$$

Let $\phi_{\eta,\mu}$ and $\phi_{\eta',\mu'}$ be automorphisms, and

$$\phi_{\eta'',\mu''} \ = \ \phi_{\eta',\mu'} \circ \phi_{\eta,\mu} \ .$$

Then we have, for every $\alpha \in \mathcal{O}_X(U)$

$$\eta''(\alpha) \ = \ \eta'(\alpha) + \nu'[\phi_{\eta',\nu'}(\eta(\alpha))]_{n-1} \ ,$$

and $\nu'' = [\phi_{\eta',\nu'}(\nu)]_{n-1}\nu'$.

5.2. THE CASE $n = 2$

Let $\phi_{D,\mu}$ be an automorphism of $\mathcal{O}_X(U)[t]/(t^2)$, where D is a derivation of $\mathcal{O}_X(U)$ and $\mu \in \mathcal{O}_X(U)$ is invertible.

5.2.1. *Composition and inverse* – Let $\Phi_{D,\mu}$, $\Phi_{D',\mu'}$, be automorphisms, and

$$\Phi_{D'',\mu''} \ = \ \Phi_{D',\mu'} \circ \Phi_{D,\mu} \ , \qquad \Phi_{\widehat{D},\widehat{\mu}} \ = \ (\Phi_{D,\mu})^{-1} \ .$$

Using the formulas of 5.1 or directly, it is easy to prove that

$$D'' \ = \ D' + \mu' D \ , \quad \mu'' \ = \ \mu\mu' \ ,$$
$$\widehat{D} \ = \ -\frac{1}{\mu_0} D \ , \quad \widehat{\mu} \ = \ \frac{1}{\mu} \ .$$

5.3. THE CASE $n = 3$

The following computations will be used in 7.3 to describe the obstruction to extend primitive double schemes to multiplicity 3.

5.3.1. We can write the automorphism Φ of $\mathcal{O}_X(U)[t]/(t^3)$ as follows: for every $\alpha \in \mathcal{O}_X(U)$

$$\Phi(\alpha) \ = \ \alpha + D(\alpha)t + E(\alpha)t^2 \ , \tag{3}$$

where D is a derivation of $\mathcal{O}_X(U)$ and E is a D-*operator*, i.e. a $\mathbb{C}$-linear map $\mathcal{O}_X(U) \to \mathcal{O}_X(U)$ such that

$$E(\alpha\beta) \ = \ \alpha E(\beta) + \beta E(\alpha) + D(\alpha)D(\beta)$$

for every $\alpha, \beta \in \mathcal{O}_X(U)$. We have also

$$\Phi(t) \ = \ \mu t \ ,$$

where $\mu = \mu_0 + \mu_1 t \in \mathcal{O}_X(U)[t]/(t^2)$, with $\mu_0, \mu_1 \in \mathcal{O}_X(U)$, μ_0 invertible.

5.3.2. *Properties of D-operators* - We have

(i) If E, E' are D-operators, then $E - E'$ is a derivation.
(ii) If E is a D-operator and D_0 a derivation, then $E + D_0$ is a D-operator.
(iii) $\frac{1}{2}D^2 = \frac{1}{2}D \circ D$ is a D-operator.

It follows that the D-operators are the maps of the form

$$E \ = \ \frac{1}{2}D^2 + D^{(1)} \ ,$$

where $D^{(1)}$ is a derivation.

5.3.3. *Notations* – We will denote the automorphism Φ defined as in (3), with $E = \frac{1}{2}D^2 + D^{(1)}$, as

$$\Phi \ = \ \Phi_{D,\mu,D^{(1)}} \ .$$

The automorphism of $\mathcal{O}_X(U)[t]/(t^2)$ induced by $\Phi_{D,\mu,D^{(1)}}$ is Φ_{D,μ_0}.

5.3.4. *Composition and inverse* – Let $\Phi_{D,\mu,D^{(1)}}$, $\Phi_{D',\mu',D'^{(1)}}$, be automorphisms, and

$$\Phi_{D'',\mu'',D''^{(1)}} \ = \ \Phi_{D',\mu',D'^{(1)}}\circ\Phi_{D,\mu,D^{(1)}} \ , \qquad \Phi_{\widehat{D},\widehat{\mu},\widehat{D}^{(1)}} \ = \ (\Phi_{D,\mu,D^{(1)}})^{-1} \ .$$

Using the formulas of 5.1 or directly, it is easy to prove the

5.3.5. Lemma: *We have*

$$D'' \ = \ D'+\mu'_0D \ ,$$

$$\mu''_0 \ = \ \mu_0\mu'_0 \ , \qquad \mu''_1 \ = \ \mu'_0D'(\mu_0)+{\mu'_0}^2\mu_1+\mu_0\mu'_1 \ ,$$

$$D''^{(1)} \ = \ D'^{(1)}+{\mu'_0}^2D^{(1)}+\mu'_1D-\frac{1}{2}(D'+\mu'_0D)(\mu'_0)D+\frac{1}{2}\mu'_0(D'D-DD') \ ,$$

$$\widehat{D} \ = \ -\frac{1}{\mu_0}D \ ,$$

$$\widehat{\mu}_0 \ = \ \frac{1}{\mu_0} \ , \quad \widehat{\mu}_1 \ = \ \frac{D(\mu_0)-\mu_1}{\mu_0^3} \ ,$$

$$\widehat{D}^{(1)} \ = \ -\frac{1}{\mu_0^2}D^{(1)}+\frac{1}{\mu_0^3}\big(\mu_1-\frac{1}{2}D(\mu_0)\big)D \ .$$

5.3.6. *Canonical extensions* – Let D be a derivation of $\mathcal{O}_X(U)$ and $\mu=\mu_0+\mu_1t\in\mathcal{O}_X[t]/(t^2)$, with μ_0 invertible. Let

$$D^{(1)} \ = \ \frac{\mu_1-D(\mu_0)}{2\mu_0}D$$

and

$$\Psi(D,\mu) \ = \ \Phi_{D,\mu,D^{(1)}} \ .$$

Let $\widehat{D}$ and $\widehat{\mu}$ be defined as in lemma 5.3.5. An easy computation using lemma 5.3.5 shows that

5.3.7. Lemma: *We have* $\Psi(D,\mu)^{-1}=\Psi(\widehat{D},\widehat{\mu})$.

So we have defined an extension of Φ_{D,μ_0} to an automorphism of $\mathcal{O}_X(U)[t]/(t^3)$ in such a way that the extension of the inverse is the inverse of the extension.

Let $\Psi(D,\mu)$, $\Psi(D',\mu')$, be automorphisms, and

$$\Phi_{D'',\mu'',D''^{(1)}} \ = \ \Psi(D',\mu')\circ\Psi(D,\mu)$$

(with D'', μ'' as in lemma 5.3.5). In general we don't have $\Phi_{D'',\mu'',D''^{(1)}}=\Psi(D'',\mu'')$. The following result can be computed using lemma 5.3.5, and will be used in 7.2.

5.3.8. Proposition: *We have*

$$\Psi(D',\mu')\circ\Psi(D,\mu)\circ\Psi(D'',\mu'')^{-1} \ = \ \Phi_{0,1,\widehat{D}^{(1)}} \ ,$$

with

$$\widehat{D}^{(1)} \ = \ \frac{1}{2}\Big(\mu'_1D+\big(D(\mu'_0)+\frac{\mu'_0}{\mu_0}D(\mu_0)-\frac{\mu'_0}{\mu_0}\mu_1\big)D'+\mu'_0(D'D-DD')\Big) \ .$$

5.4. THE CANONICAL SHEAF OF PRIMITIVE MULTIPLE SCHEMES AND EXTENSION IN HIGHER MULTIPLICITY

This section contains technical results that are used in 8. We use the notations of 4.1 to 4.4.

5.4.1. *Canonical sheaves and isomorphisms* – Let Y, Z be schemes and $\phi : Y \to Z$ an isomorphism. Then ϕ induces a canonical isomorphism

$$\theta : \phi^*(\Omega_Z) \longrightarrow \Omega_Y \ .$$

Let $U \subset Z$ be an open subset and $f, g \in \mathcal{O}_Z(U)$. Then $f.dg \in \Omega_Z(U) = \phi^*(\Omega_Z)(\phi^{-1}(U))$, and

$$\theta(f.dg) \ = \ f \circ \phi.d(g \circ \phi) \ = \ \phi^*(f).d(\phi^*(g)) \ .$$

We will also note $\theta = \phi^*$.

5.4.2. *Description of canonical sheaves using open covers* – Let X_n be a primitive multiple curve of multiplicity $n \geq 2$, such that $(X_n)_{red} = X$, with associated line bundle L on X. Let (U_i) be an open cover of X such that we have trivializations

$$\delta_i^{(n)} : U_i^{(n)} \longrightarrow U_i \times \mathbf{Z}_n$$

as in 4.2.

Let (τ_{ij}) be a cocycle representing $\tau \in H^1(\Omega_n)$ $(\tau_{ij} \in H^0(U_{ij}^{(n)}, \Omega_{X_n}))$. Let

$$\theta_i^{(n)} : ((\delta_i^{(n)})^{-1})^*(\Omega_{U_i^{(n)}}) \longrightarrow \Omega_{U_i \times \mathbf{Z}_n}$$

be the canonical morphism (cf. 5.4.1). Let

$$\rho_{ij} \ = \ H^0(\theta_i^{(n)})(\tau_{ij}) \ \in \ H^0(U_{ij} \times \mathbf{Z}_n, \Omega_{U_{ij} \times \mathbf{Z}_n}) \ .$$

We have

$$\rho_{ij} + \theta_{ij}^{(n)}(\rho_{jk}) \ = \ \rho_{ik} \ ,$$

where

$$\theta_{ij}^{(n)} : ((\delta_{ij}^{(n)})^{-1})^*(\Omega_{U_{ij} \times \mathbf{Z}_n}) \longrightarrow \Omega_{U_{ij} \times \mathbf{Z}_n}$$

is the morphism described in 5.4.1, corresponding to $\delta_{ij}^{(n)}$. Conversely, every family (ρ_{ij}) satisfying the preceding relation defines an element of $H^1(\Omega_{X_n})$.

5.4.3. *Extension to multiplicity* $n+1$ – Suppose that X_n can be extended to X_{n+1}, of multiplicity $n+1$, and let

$$\delta_i^{(n+1)} : U_i^{(n+1)} \longrightarrow U_i \times \mathbf{Z}_{n+1}$$

be trivializations extending $\delta_i^{(n)}$.

We can view every element of $H^0(\mathcal{O}_{U_{ij} \times \mathbf{Z}_n})$ as an element of $H^0(\mathcal{O}_{U_{ij} \times \mathbf{Z}_{n+1}})$ (with coefficient 0 in degree n with respect to t).

For every $\beta \in H^0(\mathcal{O}_{U_{ij} \times \mathbf{Z}_{n+1}})$, $\beta = \sum_{p=0}^{n} \beta_k t^k$, $\beta_k \in \mathcal{O}_X(U_{ij})$, let $[\beta]_n = \sum_{p=0}^{n-1} \beta_k t^k \in H^0(\mathcal{O}_{U_{ij} \times \mathbf{Z}_n})$.

Similarly, we can view every element of $H^0(\Omega_{U_{ij}\times\mathbf{Z}_n})$ as an element of $H^0(\Omega_{U_{ij}\times\mathbf{Z}_{n+1}})$. Let $\omega\in H^0(\Omega_{U_{ij}\times\mathbf{Z}_{n+1}})$. We can write

$$\omega \;=\; \sum_{k=0}^{n} t^k b_k + \big(\sum_{p=0}^{n-1} c_p t^p\big)dt \;, \tag{4}$$

with $b_k\in H^0(U_{ij},\Omega_X)$, $c_p\in\mathcal{O}_X(U_{ij})$. Let

$$[\omega]_n \;=\; \sum_{k=0}^{n-1} t^k b_k + \big(\sum_{p=0}^{n-2} c_p t^p\big)dt \;,$$

Then we have $\omega\in H^0(\Omega_{U_{ij}\times\mathbf{Z}_n})$ if and only if $\omega=[\omega]_n$, i.e. if and only if $b_n=0$ and $c_{n-1}=0$. Let

$$\theta_i^{(n+1)} : ((\delta_i^{(n+1)})^{-1})^*(\Omega_{U_i^{(n+1)}}) \longrightarrow \Omega_{U_i\times\mathbf{Z}_{n+1}} \;,$$
$$\theta_{ij}^{(n+1)} : ((\delta_{ij}^{(n+1)})^{-1})^*(\Omega_{U_{ij}\times\mathbf{Z}_{n+1}}) \longrightarrow \Omega_{U_{ij}\times\mathbf{Z}_{n+1}}$$

be the canonical morphisms. We have

$$\theta_{ij}^{(n+1)}(\omega) \;=\; \theta_{ij}^{(n)}([\omega]_n) + t^n\Upsilon_0(\omega) + \Upsilon_1(\omega)t^{n-1}dt \;,$$

with $\Upsilon_0^{ij}(\omega)\in H^0(U_{ij},\Omega_X)$ and $\Upsilon_1^{ij}(\omega)\in\mathcal{O}_X(U_{ij})$. If $R(\theta_{ij}^{(n+1)}(\omega))$ is the restriction of $\theta_{ij}^{(n+1)}(\omega)$ to $U_{ij}\times\mathbf{Z}_n$, we have

$$R(\theta_{ij}^{(n+1)}(\omega)) \;=\; \theta_{ij}^{(n)}([\omega]_n) + \Upsilon_1^{ij}(\omega)t^{n-1}dt \;.$$

Suppose that there exist $x_1,\ldots,x_d\in\mathcal{O}_X(U_{ij})$, $d=\dim(X)$, such that
$\Omega_X(U_{ij})=\mathbb{C}dx_1\oplus\cdots\oplus\mathbb{C}dx_d$.

5.4.4. *The case of a trivial X_n* – We suppose that X_n is the trivial primitive multiple scheme associated to L and that $n\geq 2$. We have then $(\delta_{ij}^{(n)})^*(\alpha)=\alpha$ for every $\alpha\in\mathcal{O}_X(U_{ij})$, and $(\delta_{ij}^{(n)})^*(t)=\alpha_{ij}t$, where $\alpha_{ij}\in\mathcal{O}_X(U_i)^*$ (and (α_{ij}) is a cocycle which defines L). We have then, for every $\alpha\in\mathcal{O}_X(U_{ij})$,

$$(\delta_{ij}^{(n+1)})^*(\alpha) \;=\; \alpha+\eta_{ij}(\alpha)t^n \;,$$

where η_{ij} is a derivation of $\mathcal{O}_X(U_{ij})$ and $(\delta_{ij}^{(n+1)})^*(t)=\alpha_{ij}(1+\epsilon_{ij}t^{n-1})t$, with $\epsilon_{ij}\in\mathcal{O}_X(U_{ij})$. Now let $\omega\in H^0(\Omega_{U_{ij}\times\mathbf{Z}_n})$, written as in 4, with $b_n=0$ and $c_{n-1}=0$. We have then

$$\theta_{ij}^{(n)}(\omega) \;=\; \sum_{k=0}^{n-1}\alpha_{ij}^k t^k b_k + \sum_{p=0}^{n-2} c_p\alpha_{ij}^p t^p(td\alpha_{ij}+\alpha_{ij}dt) \;,$$

$$\begin{aligned}\theta_{ij}^{(n+1)}(\omega) \;&=\; \theta_{ij}^{(n+1)}(b_0)+\alpha_{ij}(1+\epsilon_{ij}t^{n-1})t\theta_{ij}^{(n+1)}(b_1)+\sum_{k=2}^{n-1}\alpha_{ij}^k t^k\theta_{ij}^{(n+1)}(b_k)\\ &\quad+\big[c_0+\eta_{ij}(c_0)t^n+c_1\alpha_{ij}(1+\epsilon_{ij}t^{n-1})t+\sum_{p=2}^{n-2}c_p\alpha_{ij}^p t^p\big]d\big((\delta_{ij}^{(n+1)})^*(t),\end{aligned}$$

hence

$$R(\theta_{ij}^{(n+1)}(\omega)) \;=\; \sum_{k=0}^{n-1}\alpha_{ij}^k t^k\theta_{ij}^{(n+1)}(b_k) + \big(\sum_{p=0}^{n-2}c_p\alpha_{ij}^p t^p\big)\big(td\alpha_{ij}+(\alpha_{ij}+n\alpha_{ij}\epsilon_{ij}t^{n-1})dt\big) \;.$$

Suppose that $b_k = \sum_{q=1}^{d} \gamma_q^{(k)} dx_q$. Then

$$t^k \theta_{ij}^{(n+1)}(b_k) \ = \ t^k \sum_{q=1}^{d} (\gamma_q^{(k)} + \mu_{ij}(\gamma_q^{(k)})t^n)(dx_q + n\mu_{ij}(x_q)t^{n-1}dt + d(\mu_{ij}(x_q))t^n) \ .$$

Restricting to $U_{ij} \times \mathbf{Z}_n$ we have

$$\theta_{ij}^{(n+1)}(b_0)_{|U_{ij}\times\mathbf{Z}_n} \ = \ b_0 + n\big(\sum_{q=1}^{d} \gamma_q^{(0)}\mu_{ij}(x_q)\big)t^{n-1}dt \ ,$$

and $t^k\theta_{ij}^{(n+1)}(b_k)_{|U_{ij}\times\mathbf{Z}_n} = t^k b_k$ if $k \geq 1$. Finally

$$R(\theta_{ij}^{(n+1)}(\omega)) \ = \ \theta_{ij}^{(n)}(\omega) + n\Big(\big(\sum_{q=1}^{d} \gamma_q^{(0)}\mu_{ij}(x_q)\big) + c_0\alpha_{ij}\epsilon_{ij}\Big)t^{n-1}dt \ ,$$

whence

$$\Upsilon_1^{ij}(\omega) \ = \ n\Big(\big(\sum_{q=1}^{d} \gamma_q^{(0)}\mu_{ij}(x_q)\big) + c_0\alpha_{ij}\epsilon_{ij}\Big) \ = \ n(<\mu_{ij}, b_0> + c_0\alpha_{ij}\epsilon_{ij}) \ .$$

In particular, if $n = 1$ we have $\omega = b_0$ and $\Upsilon_1^{ij}(\omega) = <\mu_{ij}, \omega>$.

Let $\delta : H^1(\Omega_{X_n}) \to H^2(L^n)$ be the map deduced from the exact sequence $0 \to L^n \to \Omega_{X_{n+1}|X_n} \to \Omega_{X_n} \to 0$.

5.4.5. *Description of* δ – Let $\tau \in H^0(\Omega_{X_n})$ as in 5.4.2. Let $\overline{\rho}_{ij}$ be ρ_{ij} viewed as an element og $H^0(\Omega_{U_{ij}\times\mathbf{Z}_{n+1}})$, and

$$\tau'_{ij} \ = \ H^0((\theta_i(n+1))^{-1})(\overline{\rho}_{ij}) \ \in \ H^0(U_{ij}^{(n+1)}, \Omega_{X_{n+1}}) \ .$$

Let $\overline{\tau}_{ij} = \tau'_{ij|U_{ij}^{(n)}} \in H^0(U_{ij}^{(n)}, \Omega_{X_{n+1}|X_n})$, which is over $\tau_{ij} \in H^0(U_{ij}^{(n)}, \Omega_{X_n})$. Then

$$\omega_{ijk} \ = \ \overline{\tau}_{ij} + \overline{\tau}_{jk} - \overline{\tau}_{ik} \ \in \ H^0(U_{ijk}, L^n) \ ,$$

and $\omega_{ijk})$ is a cocycle which represents $\delta(\tau)$. It is also represented (as in 2.1.1) by $(\theta_i^{(n+1)}(\omega'_{ijk})_{|U_{ijk}\times\mathbf{Z}_n})$, where $\omega'_{ijk} = \tau'_{ij} + \tau'_{jk} - \tau'_{ik}$. We have

$$\begin{aligned} \theta_i^{(n+1)}(\omega'_{ijk})_{|U_{ijk}\times\mathbf{Z}_n} \ &= \ \big(\overline{\rho}_{ij} + \theta_{ij}^{n+1}(\overline{\rho}_{jk}) - \overline{\rho}_{ik}\big)_{|U_{ijk}\times\mathbf{Z}_n} \\ &= \ \Upsilon_1^{ij}(\rho_{jk})t^{n-1}dt \ . \end{aligned}$$

It follows that $\delta(\tau)$ is represented (in the sense of 2.1.1) by $(\Upsilon_1^{ij}(\rho_{jk}))$.

5.4.6. *Description of* δ *in the case of a trivial* X_n – We suppose also that:

- (i) τ is the canonical class of the line bundle $\mathcal{I}_{X,X_{n+1}}$ on X_n: $\tau = \nabla_0(\mathcal{I}_{X,X_{n+1}})$.
- (ii) $\mathcal{I}_{X,X_{n+1}}$ is the same as the ideal sheaf of X in the trivial extension of X_n in multiplicity $n+1$.

For example, condition (ii) will be satisfied if $h^1(L^{n-1})=0$. With the notations of 5.4.4, we have $b_0=\dfrac{d\alpha_{ij}}{\alpha_{ij}}$ and $c_0=0$. Hence $\delta(\tau)$ is represented by the cocycle $\Big(\dfrac{\mu_{ij}(\alpha_{jk})}{\alpha_{jk}}\Big)$.

6. Automorphisms of primitive multiple schemes

6.1. Derivations and automorphisms

Let U be a smooth affine variety, $U=\operatorname{spec}(A)$, of dimension $d>0$, such that the vector bundle Ω_U is trivial, generated by $dx_1,\ldots,dx_d$, with $x_1,\ldots,x_d\in A$. Then for every closed point $P\in U$, the images of $x_i-x_i(P)$, $1\leq i\leq d$, form a basis of m_P/m_P^2.

Let $n\geq 2$ be an integer, and $R_n=A[t]/(t^n)$.

6.1.1. *Derivations of R_n* – They are of the form

$$D \ = \ \sum_{i=1}^d a_i\frac{\partial}{\partial x_i}+bt\frac{\partial}{\partial t} \ , \tag{5}$$

with $a_1,\ldots,a_d,b\in R_n$. If $\alpha=\sum_{p=0}^{n-1}\alpha_p t^p$, with $\alpha_0,\ldots,\alpha_{n-1}\in A$, then

$$D(\alpha) \ = \ \sum_{i=1}^d a_i\Big(\sum_{p=0}^{n-1}\frac{\partial\alpha_p}{\partial x_i}t^p\Big)+b\sum_{p=1}^{n-1}p\alpha_p t^p \ .$$

Let $\mathrm{Der}_0(R_n)$ be the vector space of derivations D of R_n such that

$$D(R_n) \ \subset \ (t) \qquad \text{and} \qquad D((t)) \ \subset \ (t^2) \ ,$$

i.e. derivations (5) with $a_1,\ldots,a_d,b$ multiple of t. In other words, a derivation D belongs to $\mathrm{Der}_0(R_n)$ if and only if there exists another derivation D_0 of R_n such that $D=tD_0$.

For every $D\in\mathrm{Der}_0(R_n)$, and every integer $k\geq 1$, we have $D^k(R_n)\subset(t^k)$ (where D^k denotes the k times composition of D).

6.1.2. *Automorphisms from derivations* – Let $D\in\mathrm{Der}_0(R_n)$. Then we have $D^k=0$ if $k\geq n$, so we can define the map

$$\chi_D \ = \ \sum_{k\geq 0}\frac{1}{k!}D^k : R_n\longrightarrow R_n \ , \tag{6}$$

such that for every $\alpha\in R_n$, the terms of degree 0 of α and $\chi_D(\alpha)$, with respect to t, are the same.

Let D_0 be a derivation of R_n and $D\in\mathrm{Der}_0(R_n)$. Then $D+t^{n-1}D_0\in\mathrm{Der}_0(R_n)$, and we have $\chi_{D+t^{n-1}D_0}=\chi_D+t^{n-1}D_0$.

6.1.3. Proposition: **1** – *The map χ_D is an automorphism of $\mathbb{C}$-algebras.*

2 – *If $D, D' \in Der_0(R_n)$ commute, then* $\chi_{D+D'} = \chi_D \circ \chi_{D'}$.

Proof. The fact that χ_D is a morphism of $\mathbb{C}$-algebras and **2-** are immediate. From **2-** we see that χ_{-D} is the inverse of χ_D, so χ_D is an automorphism. ☐

6.1.4. Examples: If $n = 2$ and $D, D' \in \mathrm{Der}_0(R_2)$, then $DD' = D'D = 0$ (multiple of t^2).

If $n = 3$ and $D, D' \in \mathrm{Der}_0(R_3)$, then we have

$$\chi_D \circ \chi_{D'} \ = \ \chi_{D+D'} + \frac{1}{2}(D' \circ D - D \circ D') \ = \ \chi_{D+D'+\frac{1}{2}(D' \circ D - D \circ D')}$$

($D' \circ D - D \circ D'$ is a multiple of t^2).

For every $D \in \mathrm{Der}_0(R_n)$ have $\chi_D \in \mathcal{G}_n(U)$ (cf. 4.2). Let $\mathcal{G}_n^0(U)$ denote the subgroup of $\mathcal{G}_n(U)$ of automorphisms γ such that $\gamma(t) = (1 + t\epsilon)t$ for some $\epsilon \in R_n$. It is also the set of $\gamma \in \mathcal{G}_n(U)$ which can be written as $\gamma = I + t\psi$, for some map $\psi : R_n \to R_{n-1}$.

We have an obvious "division by t" map $(t) \to R_{n-1}$, that we can denote by multiplication by $\frac{1}{t}$. With the notations of 5.1 we have $\chi_D = \phi_{\eta,\mu}$, with

$$\eta \ = \ \frac{1}{t}\sum_{k\geq 1}\frac{1}{k!}D^k \ , \qquad \mu \ = \ 1 + \frac{1}{t}\sum_{k\geq 1}\frac{1}{k!}D^k(t) \ .$$

6.1.5. Theorem: *The map*

$$\begin{aligned} Der_0(R_n) &\longrightarrow \mathcal{G}_n^0(U) \\ D &\longmapsto \chi_D \end{aligned}$$

is a bijection.

Proof. It similar to the proof of theorem 4.2.5 of [8]. One can also define the inverse using the logarithm

$$I + t\psi \longmapsto \sum_{k=1}^{n-1}(-1)^{k-1}\frac{1}{k}(t\psi)^k.$$

☐

6.1.6. Remark: If we allow in (6) derivations D which are not multiple of t we could still get a convergent series but not in R_n (in the ring of holomorphic functions on U instead), or the terms of degree 0 of $\in R_n$ and $\chi_D(\alpha)$ can be distinct. For example in this way, with $A = \mathbb{C}[x]$, if we take $D = \dfrac{\partial}{\partial x}$, then for every $\alpha \in \mathbb{C}[x]$ we have $\chi_D(\alpha) = \alpha(x + 1)$.

6.2. Sheaves of differentials

Let X be a smooth connected variety of dimension $d>0$ and n an integer such that $n\geq 2$. Let X_n be a primitive multiple scheme, of multiplicity n, such that $(X_n)_{red}=X$. We give here some properties of the sheaf of differentials Ω_{X_n} and of its dual $\mathcal{T}_n=(\Omega_{X_n})^*$, that will be used in 8 and 9 . Recall that we have a canonical filtration by primitive multiple schemes

$$X_1=X\subset X_2\subset\cdots\subset X_{n-1}\subset X_n$$

(cf. 4.1).

The sheaves Ω_{X_n} and $\mathcal{T}_n$ are *quasi locally free* (cf. [7], [9], i.e. they are locally isomorphic to a direct sum of sheaves $\mathcal{O}_{X_i}$, $1\leq i\leq n$). They are locally isomorphic to $d.\mathcal{O}_{X_n}\oplus\mathcal{O}_{X_{n-1}}$: if $P\in X$ and $x_1,\ldots,x_d\in m_{X_nP}$ generate $m_{X,P}/m^2_{X,P}$, and $z\in\mathcal{O}_{X_nP}$ is a generator of the ideal of X, then

$$\Omega_{X_nP}\ \simeq\ \mathcal{O}_{X_nP}dx_1\oplus\cdots\oplus\mathcal{O}_{X_nP}dx_d\oplus\mathcal{O}_{X_nP}dz\ ,$$

and we have $z^{n-1}dz=\frac{1}{n}d(z^n)=0$ $(\mathcal{O}_{X_nP}dz\simeq\mathcal{O}_{X_{n-1,P}})$.

Let $\mathcal{E}$ be a coherent sheaf on X_n. For $0\leq i\leq n$, let $\mathcal{E}_i=\mathcal{I}_X^i\mathcal{E}\subset\mathcal{E}$, and $\mathcal{E}^{(i)}\subset\mathcal{E}$ the subsheaf of elements annihilated by $\mathcal{I}_X^i$. The filtration

$$0=\mathcal{E}_n\subset\mathcal{E}_{n-1}\subset\cdots\subset\mathcal{E}_1\subset\mathcal{E}=\mathcal{E}_0$$

is called the *first canonical filtration of* $\mathcal{E}$, and

$$0=\mathcal{E}^{(0)}\subset\mathcal{E}^{(1)}\subset\cdots\subset\mathcal{E}^{(n-1)}\subset\mathcal{E}^{(n)}=\mathcal{E}$$

the *second canonical filtration* of $\mathcal{E}$.

For $0\leq i<n$, let

$$G_i(\mathcal{E})\ =\ \mathcal{E}_i/\mathcal{E}_{i+1}\ ,\qquad G^{(i+1)}(\mathcal{E})=\mathcal{E}^{(i+1)}/\mathcal{E}^{(i)}\ .$$

If $1\leq i<n$ we have $(\mathcal{E}^*)^{(i)}=(\mathcal{E}/\mathcal{E}_i)^*$, and it follows easily that if $\mathcal{E}$ is quasi locally free, then

$$G^{(i+1)}(\mathcal{E}^*)\ =\ G_i(\mathcal{E})^*\otimes L^{n-1}\ .$$

In the simplest case, i.e. if $\mathcal{E}$ is locally free, the two filtrations are the same, and we have, if $E=\mathcal{E}_{|X}$, $G_i(\mathcal{E})\simeq E\otimes L^i$ for $0\leq i<n$.

6.2.1. *The case of* Ω_{X_n} *and* $\mathcal{T}_n$ – We have $\Omega_{X_n|X}\simeq\Omega_{X_2|X}$, and an exact sequence

$$0\longrightarrow L\longrightarrow\Omega_{X_2|X}\longrightarrow\Omega_X\longrightarrow 0\ ,$$

associated to $\sigma\in Ext^1_{\mathcal{O}_X}(\Omega_X,L)=H^1(T_X\otimes L)$. We have seen in 4.3 that $\sigma=0$ if and only X_2 is trivial, and that if $\sigma\neq 0$, then $\mathbb{C}\sigma\in\mathbb{P}(H^1(T_X\otimes L))$ defines completely X_2. We have

$$G_i(\Omega_{X_n})\ \simeq\ \Omega_{X_2|X}\otimes L^i\qquad\text{for } 1\leq i<n-1\ ,$$

and $G_{n-1}(\Omega_{X_n})\simeq\Omega_X\otimes L^{n-1}$.

For every $\nabla\in H^1(\Omega_X)=\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X,\Omega_X)$, let $0\longrightarrow\Omega_X\longrightarrow E_\nabla\longrightarrow\mathcal{O}_X\longrightarrow 0$ be the exact sequence associated to ∇, where E_∇ is a rank $d+1$ vector bundle on X. Recall that $\nabla_0(L)\in H^1(\Omega_X)$ denotes the canonical class of L (cf. 3).

6.2.2. Theorem: *We have* $\Omega^{(1)}_{X_n}\simeq E_{\nabla_0(L)}\otimes L^{n-1}$.

Proof. Suppose that X_n is constructed as in 4.2, using the open cover $(U_i)_{i\in I}$ of X. We suppose also that for every $i\in I$, there are $x_1,\ldots,x_d\in\mathcal{O}_X(U_i)$ such that $\Omega_{X|U_i}$ is generated by $dx_1,\ldots,dx_n$. We will first construct Ω_{X_n} by the method and notations of 4.2.3. We take $\mathcal{E}_i=\Omega_{U_i\times\mathbf{Z}_n}$. We have canonical isomorphisms induced by δ_{ij}

$$\Psi_{ij}:\delta_{ij}^*(\Omega_{U_{ij}\times\mathbf{Z}_n})\longrightarrow\Omega_{U_{ij}\times\mathbf{Z}_n}\ .$$

Suppose that $\delta_{ij}=\phi_{\eta^{ij},\mu^{ij}}$, cf. 5.1), and let $\lambda_k^{ij}=\eta^{ij}(x_k)$ for $1\leq k\leq d$. We have then

$$\begin{aligned}\Psi_{ij}(dx_k) &= d(x_k+\lambda_k^{ij}t)\\ &= dx_k+d(\lambda_k^{ij})t+\lambda_k^{ij}dt\ ,\\ \Psi_{ij}(dt) &= d(\mu^{ij}t)\\ &= d(\mu^{ij})t+\mu^{ij}dt\ .\end{aligned}$$

Let

$$\Psi_{ij}^{(1)}:\delta_{ij}^*(\Omega_{U_{ij}\times\mathbf{Z}_n})^{(1)}=(\Omega_{U_{ij}\times\mathbf{Z}_n})^{(1)}\longrightarrow(\Omega_{U_{ij}\times\mathbf{Z}_n})^{(1)}$$

be the isomorphism induced by Ψ_{ij}. We have $(\Omega_{U_{ij}\times\mathbf{Z}_n})^{(1)}\simeq\Omega_{X|U_{ij}}\oplus\mathcal{O}_{U_{ij}}$, and the sheaf $(\Omega_{U_{ij}\times\mathbf{Z}_n})^{(1)}$ is generated by $t^{n-1}dx_1,\ldots,t^{n-1}dx_d$ and $t^{n-2}dt$. We have

$$\begin{aligned}\Psi_{ij}^{(1)}(t^{n-1}dx_k) &= (\mu_0^{ij})^{n-1}t^{n-1}dx_k\ ,\\ \Psi_{ij}^{(1)}(t^{n-2}dt) &= (\mu_0^{ij})^{n-1}(\frac{d\mu_0^{ij}}{\mu_0^{ij}}t^{n-1}+t^{n-2}dt)\ .\end{aligned}$$

In other words, $\Psi_{ij}^{(1)}:\Omega_{X|U_{ij}}\oplus\mathcal{O}_{U_{ij}}\to\Omega_{X|U_{ij}}\oplus\mathcal{O}_{U_{ij}}$ is defined by the matrix $(\mu_0^{ij})^{n-1}\begin{pmatrix}I & \frac{d\mu_0^{ij}}{\mu_0^{ij}}\\ 0 & 1\end{pmatrix}$. Proposition 6.2.2 follows then from 2.1.5 and 4.2.2. $\square$

It follows easily that we have $G^{(i)}(\Omega_{X_n})\simeq E_{\nabla_0(L)}\otimes L^{n-i}$ for $1\leq i<n$ and $G^{(n)}(\Omega_{X_n})\simeq\Omega_X$.

6.2.3. Corollary: 1 – *We have* $G_i(\mathcal{T}_n)\simeq(E_{\nabla_0(L)})^*\otimes L^i$ *for* $0\leq i<n-1$ *and* $G_{n-1}(\mathcal{T}_n)\simeq T_X\otimes L^{n-1}$.

2 – *We have* $G^{(i)}(\mathcal{T}_n)\simeq(\Omega_{X_2|X})^*\otimes L^{n-i}$ *for* $1\leq i\leq n-1$ *and* $G^{(n)}(\mathcal{T}_n)\simeq T_X$.

6.3. Automorphisms of primitive multiple schemes

We use the notations of 6.2. Suppose that X_n corresponds to $\gamma\in H^1(X,\mathcal{G}_n)$. Let $\mathcal{A}ut(X_n)$ be the sheaf of groups of automorphisms of X_n leaving X invariant, which can be identified with $\mathcal{G}_n^\gamma$ (cf. 2.2). Suppose that $n\geq 3$. From lemma 4.4.2 we have an exact sequence of sheaves of groups on X

$$0\longrightarrow(\Omega_{X_2|X})^*\otimes L^{n-1}\longrightarrow\mathcal{A}ut(X_n)\longrightarrow\mathcal{A}ut(X_{n-1})\longrightarrow 0\ .$$

Let $\mathrm{Aut}(X_n)=H^0(\mathcal{A}ut(X_n))$ be the group of global automorphisms of X_n leaving X invariant. From 4.2.2 we have a canonical morphism $\xi_n:\mathcal{G}_n\to\mathcal{O}_X^*$, inducing $\xi_n^\gamma:\mathcal{G}_n^\gamma\to(\mathcal{O}_X^*)^\gamma=\mathcal{O}_X^*$ and

$$H^0(\xi_n^\gamma):\mathrm{Aut}(X_n)\longrightarrow\mathbb{C}^*\ ,$$

sending to an automorphism of X_n the induced automorphism of L. Let
$\mathrm{Aut}_0(X_n) = \ker(H^0(\xi_n^\gamma))$.

6.3.1. *Automorphisms from derivations 2* – We have $\mathcal{I}_X\mathcal{T}_n \simeq \mathcal{T}_{n-1}\otimes\mathcal{I}_X$: this follows easily from the fact that $\mathcal{I}_X$ is a line bundle on X_{n-1}. Let $D\in H^0(X_{n-1},\mathcal{I}_X\mathcal{T}_n)$. Then for every affine open subset U of X_n such that $\mathcal{I}_{X|U}\simeq\mathcal{O}_{X_{n-1}|U}$, $D_{|U}$ is a derivation of $\mathcal{O}_{X_n}(U)$ such that $\mathrm{im}(D_{|U})\subset\mathcal{I}_{X|U}$ and $D_{|U}(\mathcal{I}_{C|U})\subset\mathcal{I}^2_{X|U}$, and we obtain from 6.1 an automorphism $\chi_{D_{|U}}$ of U. These automorphisms glue to define an automorphism χ_D of X_n. From theorem 6.1.5 we deduce

6.3.2. Theorem: *For every $\phi\in\mathrm{Aut}_0(X_n)$, there is an unique $D\in H^0(X_{n-1},\mathcal{I}_X\mathcal{T}_n)$ such that $\phi=\chi_D$.*

Of course, if $n>2$ the map $D\to\chi_D$ is not a morphism of groups, so we have
$\mathrm{Aut}_0(X_n)\simeq H^0(X_{n-1},\mathcal{I}_X\mathcal{T}_n)$ as sets only. But if $n=2$, it is actually a morphism of groups, and $\mathcal{I}_X\mathcal{T}_n = T_X\otimes L$.

6.3.3. *Automorphisms of double primitive schemes* – Suppose first that X_2 is not trivial. Then the automorphisms of X_2 leaving X invariant are of the form $\Theta_D = I_{X_2}+D$, where D is a sheaf of derivations with value in $\mathcal{I}_X$. With the notations of 6.2, at $P\in X$, we have

$$\mathcal{T}_{2,P} \ \simeq\ \mathcal{O}_{X_2,P}\frac{\partial}{\partial x_1}\oplus\cdots\oplus\mathcal{O}_{X_2,P}\frac{\partial}{\partial x_1}\oplus\mathcal{O}_{X_2,P}.z\frac{\partial}{\partial z}\ .$$

It follows that we must have $D\in H^0(X,G^{(1)}(\mathcal{T}_2)) = H^0(X,(\Omega_{X_2|X})^*\otimes L)$. From the
exact sequence of lemma 4.3.2 and the fact that X_2 is not trivial, we see that
$H^0(X,(\Omega_{X_2|X})^*\otimes L)\simeq H^0(X,T_X\otimes L)$. It follows that

6.3.4. Corollary: *If X_2 is not trivial, then* $\mathrm{Aut}(X_2)=\mathrm{Aut}_0(X_2)=H^0(T_X\otimes L)$.

If X_2 is trivial, see 6.3.8.

6.3.5. *The general case* – We suppose now that $n\geq 2$, and that $\mathrm{Aut}_0(X_n)$ is trivial. This is the case for example if $h^0(T_X\otimes L^p)=0$ for $1\leq p\leq n-1$ and $h^0(L^p)=0$ for $1\leq p\leq n-2$ (using corollary 6.2.3)

6.3.6. Theorem: *If $\mathrm{im}(H^0(\xi_n^\gamma))$ contains a number λ such that $\lambda^p\neq 1$ for every integer p such that $1\leq p\leq n-1$, then X_n is trivial.*

Proof. By induction on n. The result is true for $n=2$ by corollary 6.3.4.

Suppose that $n\geq 3$ and that it is true for $n-1$. Suppose that $\mathrm{im}(H^0(\xi_n^\gamma))$ contains a number λ such that $\lambda^i\neq 1$ for every integer i such that $1\leq i\leq n-1$. Then, by the induction hypothesis, X_{n-1} is trivial, it is the $(n-1)$th infinitesimal neighborhood of X in L^*.

We now use the notations of 4.2 and 5.1. Let (ν_{ij}) $(\nu_{ij}\in\mathbb{C}^*)$ be a cocycle which defines L. Then X_{n-1} is defined by the cocycle $(\phi_{0,\nu_{ij}})$ of $\mathcal{G}_{n-1}$. According to proposition 4.4.1, X_n is defined by the cocycle δ^*_{ij}, with $\delta^*_{ij}\in H^0(U_{ij},T_X\oplus\mathcal{O}_X)\subset H^0(U_{ij},\mathcal{G}_n)$. We can write

$$\delta^*_{ij} \;=\; \phi_{D_{ij}t^{n-2},\nu_{ij}(1+\rho_{ij}t^{n-2})}\ ,$$

i.e.

$$\delta^*_{ij} \;=\; \phi_{D_{ij}t^{n-2},1+\rho_{ij}t^{n-2}}\circ\phi_{0,\nu_{ij}}\ ,$$

where $((D_{ij},\rho_{ij}))$ is a cocyle which represents $\sigma\in H^1(X,(\Omega_{X_2|X})^*\otimes L^{n-1})$ (cf. 7.2.2), in the way of 2.1.. To prove that X_n is trivial, we must show that $\sigma=0$, i.e. that there exist derivations F_i of $\mathcal{O}_X(U_i)$, $f_i\in\mathcal{O}_X(U_i)$, such that

$$D_{ij} \;=\; F_i-\nu_{ij}^{n-1}F_j\ , \qquad \rho_{ij} \;=\; f_i-\nu_{ij}^{n-2}f_j\ . \tag{7}$$

Let χ be an automorphism of X_n, such that, if $\lambda^p\neq 1$ for every integer p such that $1\leq p\leq n-1$. From theorem 6.3.2 and the hypothesis, $\chi_{|X_{n-1}}$ is induced by the multiplication by λ in L^*. If $\chi_i=\delta_i\circ\chi\circ\delta_i^{-1}\in\mathrm{Aut}(U_i\times\mathbf{Z}_n)$, we can write

$$\chi^*_i \;=\; \phi_{t^{n-2}E_i,\lambda(1+\epsilon_i t^{n-2})}\ ,$$

where E_i is a derivation of $\mathcal{O}_X(U_i)$ and $\epsilon_i\in\mathcal{O}_X(U_i)$. On can then see easily that the equations (7) follow from the equality $\delta^*_{ij}\circ\chi^*_j=\chi^*_i\circ\delta^*_{ij}$, with

$$F_i \;=\; \frac{1}{1-\lambda_{n-1}}E_i\ , \qquad f_i \;=\; \frac{1}{1-\lambda_{n-2}}\epsilon_i\ .$$

$\square$

6.3.7. Corollary: *Suppose that* $h^0(T_X\otimes L^p)=0$ *for* $1\leq p\leq n-1$ *and* $h^0(L^p)=0$ *for* $1\leq p\leq n-2$. *If* X_n *is non trivial, then* $\mathrm{Aut}(X_n)$ *is finite.*

6.3.8. *Automorphisms of trivial primitive multiple schemes* – Suppose that X_n is trivial. then $\mathrm{Aut}_0(X_n)$ is a normal subgroup of $\mathrm{Aut}(X_n)$. In this case X_n is the nth-infinitesimal neighborhood of $X\subset L^*$ (via the zero section), and for every $\lambda\in\mathbb{C}^*$ the multiplication $\times\lambda: L^*\to L^*$ induces a corresponding automorphism of X_n. We can see in this way $\mathbb{C}^*$ as a subgroup of $\mathrm{Aut}(X_n)$. We have $\mathrm{Aut}(X_n)=\mathrm{Aut}_0(X_n).\mathbb{C}^*$ and $\mathrm{Aut}_0(X_n)\cap\mathbb{C}^*=\{I_{X_n}\}$. Hence $\mathrm{Aut}(X_n)$ *is the semi-direct product of* $\mathbb{C}^*$ *and* $\mathrm{Aut}_0(X_n)$. For example, if $n=2$, as a set, $\mathrm{Aut}(X_2)=\mathrm{Aut}_0(X_2)\times\mathbb{C}^*$, with the group law

$$\big((\sigma,\lambda),(\sigma',\lambda')\big)\mapsto(\sigma+\lambda\sigma',\lambda\lambda')\ .$$

6.4. The case of pairs $(X_n,\mathbb{L})$

With the notations of 4.5, let $g\in\mathcal{H}_n$, corresponding to a pair $(X_n,\mathbb{L})$, where X_n is a primitive multiple scheme such that $(X_n)_{red}=X$ and $\mathbb{L}$ is a vector bundle on X_n such that $\mathbb{L}_{X_{n-1}}=\mathcal{I}_{X,X_n}$. Then it is easy to see, using 4.2.3, that $H^0(X,\mathcal{H}_n^g)$ is isomorphic to the group of automorphisms χ of X_n leaving X invariant and such that $\chi^*(\mathbb{L})\simeq\mathbb{L}$.

6.5. Parametrizations of the extensions of trivial double schemes to multiplicity 3

We use the notations of 4.1 to 4.4. Let X_2 be the trivial primitive double scheme such that $(X_2)_{red} = X$, with associated line bundle $L \in \mathrm{Pic}(X)$. Then X_2 can be extended to a primitive multiple scheme X_3 of multiplicity 3 (for example the trivial one). Recall that $\rho_3 : \mathcal{G}_3 \to \mathcal{G}_2$ is the canonical morphism. Let $g_i \in H^1(X, \mathcal{G}_i)$, $i = 2, 3$, correspond to X_i. We have $H^1(\rho_3)(g_3) = g_2$. From 2.2.3 and 4.4.2 we have a canonical surjective map

$$\lambda_{g_3} : H^1(X, (\Omega_{X_2|X})^* \otimes L^2) \longrightarrow H^1(\rho_3)^{-1}(g_2)$$

which sends 0 to g_3. Note that we have $(\Omega_{X_2|X})^* \simeq L^{-1} \oplus T_X$.

There is an action of the group $\mathrm{Aut}(X_2)$ on $H^1(X, (\Omega_{X_2|X})^* \otimes L^2)$ such that the fibers of λ_{g_3} are the orbits of this action, so that there is a bijection between the set of isomorphism classes of primitive schemes of multiplicity 3 extending X_2 and the quotient $H^1(X, (\Omega_{X_2|X})^* \otimes L^2)/\mathrm{Aut}(X_2)$. The action is as follows: let (g_{ij}) be a cocycle of $\mathcal{G}_3$ (with respect to $(U_i)_{i\in I}$) representing g_3, $\gamma \in H^1(X, (\Omega_{X_2|X})^* \otimes L^2)$, represented by (γ_{ij}) (seen as a cocycle in $\mathcal{G}_3$, cf. 4.4.1), and $\chi \in \mathrm{Aut}(X_2)$. Let $\delta_i : U_i^{(3)} \to U_i \times \mathbf{Z}_3$ be isomorphisms such that $g_{ij} = (\delta_j^*)^{-1}\delta_i^*$. The induced isomorphisms $U_i^{(2)} \to U_i \times \mathbf{Z}_2$ will also be denoted by δ_i. Let $\chi_i = (\delta_j^*)^{-1}\chi^*\delta_i^* \in \mathrm{Aut}(U_i \times \mathbf{Z}_2) = \mathcal{G}_2(U_i)$. Suppose that χ_i can be extended to $c_i \in \mathcal{G}_3(U_i)$. Then we have $\chi\gamma = (\theta_{ij})$, with

$$\theta_{ij} \;=\; c_i\gamma_{ij}g_{ij}c_j^{-1}g_{ij}^{-1} \; . \tag{8}$$

We use the notations of 5.3. Let (ν_{ij}) be a cocycle representing L, $\nu_{ij} \in \mathcal{O}_X(U_{ij})^*$. Then we can take

$$g_{ij} \;=\; \Phi_{0,\nu_{ij},0} \; .$$

Let $\gamma = (\eta, \epsilon)$, with $\eta \in H^1(X, L)$ and $\epsilon \in H^1(X, T_X \otimes L^2)$, represented by cocycles (η_{ij}), (ϵ_{ij}), $\eta_{ij} \in \mathcal{O}_X(U_{ij})$, $\epsilon_{ij} \in H^0(U_{ij}, T_X)$, with the cocycle relations

$$\eta_{ik} \;= \eta_{ij} + \nu_{ij}\eta_{jk} \; , \qquad \epsilon_{ik} \;=\; \epsilon_{ij} + \nu_{ij}^2\epsilon_{jk}$$

(cf. 2.1.1). Using 4.4.2, we see that

$$\gamma_{ij} \;=\; \Phi_{0,1+\rho_{ij}t,\epsilon_{ij}} \; .$$

According to 6.3, as a set we have $\mathrm{Aut}(X_2) = H^0(T_X \otimes L) \times \mathbb{C}^*$.

6.5.1. *The action of $H^0(T_X \otimes L)$* – We suppose that $\chi \in \mathrm{Aut}_0(X_2) \simeq H^0(T_X \otimes L)$. In this case we can take $\chi_i = \Phi_{D_i,1}$ (with the notations of 5.2), where (D_i) represents χ (with the cocycle relation $D_i = \nu_{ij}D_j$). We can take

$$c_i \;=\; \Phi_{D_i,1,K_i} \; ,$$

where K_i is a derivation of $\mathcal{O}_X(U_i)$. A lengthy calculation of formula (8) shows then that $\chi\gamma = (\eta', \epsilon')$, where η' (resp. ϵ') is represented by the cocycle (η'_{ij}) (resp. (ϵ'_{ij})), with

$$\eta'_{ij} \;=\; \eta_{ij} + D_j(\nu_{ij}) \; , \qquad \epsilon'_{ij} \;=\; \epsilon_{ij} - \nu_{ij}\big(\eta_{ij} + \frac{1}{2}D_j(\nu_{ij})\big)D_j + K_i - \nu_{ij}^2K_j \; . \tag{9}$$

The duality $\Omega_X \simeq T_X^*$ induces a canonical bilinear map

$$\begin{array}{ccc} H^0(T_X \otimes L) \times H^1(\Omega_X) & \longrightarrow & H^1(L) \\ (\chi, \tau) & \longmapsto & <\chi, \tau> \; , \end{array}$$

and we have a canonical product $H^1(L) \times H^0(T_X \otimes L) \to H^1(T_X \otimes L^2)$. Recall that $\nabla_0(L)$ denotes the canonical class of L in $H^1(\Omega_X)$ (cf. 3).

6.5.2. Proposition: *We have*

$$\eta' \ = \ \eta + <\chi, \nabla_0(L)> \ , \qquad \epsilon' \ = \ \epsilon - \eta\chi - \frac{1}{2} <\chi, \nabla_0(L)> \chi \ .$$

The proof follows easily from formulas (9), with the conventions of 2.1. See also the similar more detailed proof of theorem 7.1.2.

Remarks: 1 – The term "$K_i - \nu_{ij}^2 K_j$" in ϵ'_{ij} can be suppressed (as expected), since $(K_i - \nu_{ij}^2 K_j)$ is a boundary.

2 – Let $\chi' \in \mathrm{Aut}_0(X_2)$. The formulas of proposition 6.5.2 do not show directly that $\chi'(\chi(\eta,\epsilon)) = (\chi'\chi)(\eta,\epsilon)$. We find that

$$\chi'(\chi(\eta,\epsilon)) - (\chi'\chi)(\eta,\epsilon) \ = \ \big(0, \frac{1}{2}(<\chi', \nabla_0(L)> \chi - <\chi, \nabla_0(L)> \chi')\big) \ = \ (0,\beta) \ .$$

If $\chi_i = (\delta_j^*)^{-1}\chi^*\delta_i^* = \phi_{D'_i,1}$, then β is represented by the cocycle $\big(\frac{\nu_{ij}}{2}(D'_j(\nu_{ij})D_j - D_j(\nu_{ij})D'_j))\big)$. But we have

$$\frac{\nu_{ij}}{2}(D'_j(\nu_{ij})D_j - D_j(\nu_{ij})D'_j) \ = \ K_i - \nu_{ij}^2 K_j \ ,$$

with $K_i = \frac{1}{2}(D'_i D_i - D_i D'_i)$, so actually $\beta = 0$ in $H^1(T_X \otimes L^2)$.

6.5.3. *The action of* $\mathbb{C}^*$ – Let $\lambda \in \mathbb{C}^*$. For the corresponding element χ of $\mathrm{Aut}(X_2)$ we have $\chi_i = \Phi_{0,\lambda}$, and we can take $c_i = \Phi_{0,\lambda,0}$. Let $(\eta', \epsilon') = \chi\gamma$. Then it follows easily that

6.5.4. Proposition: *We have* $\eta' = \lambda\eta$ *and* $\epsilon' = \lambda^2\epsilon$.

6.6. Parametrizations of the extensions of trivial schemes of multiplicity $n > 2$

Let X_n be the trivial primitive multiple scheme of multiplicity n such that $(X_n)_{red} = X$, with associated line bundle $L \in \mathrm{Pic}(X)$. Then X_n can be extended to a primitive multiple scheme X_{n+1} of multiplicity $n+1$ (for example the trivial one). Let $\rho_{n+1} : \mathcal{G}_{n+1} \to \mathcal{G}_n$ be the canonical morphism. Let $g_i \in H^1(X, \mathcal{G}_i)$, $i = n, n+1$, correspond to X_i. We have $H^1(\rho_{n+1})(g_{n+1}) = g_n$. From 2.2.3 and 4.4.2 we have a canonical surjective map

$$\lambda_{g_{n+1}} : H^1(X, (\Omega_{X_2|X})^* \otimes L^n) \longrightarrow H^1(\rho_{n+1})^{-1}(g_n)$$

which sends 0 to g_{n+1}. We have $(\Omega_{X_2|X})^* \simeq T_X \oplus L^{-1}$. There is an action of the group $\mathrm{Aut}(X_n)$ on $H^1(X, (\Omega_{X_2|X})^* \otimes L^n)$ such that the fibers of $\lambda_{g_{n+1}}$ are the orbits of this action, so that there is a bijection between the set of isomorphism classes of primitive schemes of multiplicity $n+1$ extending X_n and the quotient $H^1(X, (\Omega_{X_2|X})^* \otimes L^n)/\mathrm{Aut}(X_n)$.

We will consider only the action of the subgroup $\mathbb{C}^* \subset \mathrm{Aut}(X_n)$ on $H^1(X, (\Omega_{X_2|X})^* \otimes L^n)$ (cf. 6.3.8). We have

$$H^1(X, (\Omega_{X_2|X})^* \otimes L^n) \ \simeq \ H^1(X, L^{n-1}) \times H^1(X, T_X \otimes L^n) \ .$$

Then the following result is an easy generalization of proposition 6.6.1:

6.6.1. Proposition: *For every* $(\eta, \epsilon) \in H^1(X, L^{n-1}) \times H^1(X, T_X \otimes L^n)$ *and* $\lambda \in \mathbb{C}^*$, *we have* $\lambda.(\eta, \epsilon) = (\lambda^{n-1}\eta, \lambda^n \epsilon)$.

7. Extension to higher multiplicity

Let $n \geq 1$ be an integer, X a smooth projective variety, and X_n a primitive multiple scheme of multiplicity n such that $(X_n)_{red} = X$. Let L be the associated line bundle on X. We will see when it is possible to extend X_n in multiplicity $n+1$, i.e. to construct a primitive multiple scheme X_{n+1} of multiplicity $n+1$ whose canonical subscheme of multiplicity n is isomorphic to X_n. If such a X_{n+1} exists, we will also see when it is possible to extend a vector bundle on X_n to a vector bundle on X_{n+1}. The two problems are related: $\mathcal{I}_{X,X_n}$ is a line bundle on X_{n-1}, and a necessary condition for the existence of X_{n+1} is the extension of $\mathcal{I}_{X,X_n}$ to a line bundle on X_n (that will be $\mathcal{I}_{X,X_{n+1}}$).

7.1. Extension of vector bundles

Let r be a positive integer. If Y is a primitive multiple scheme, let $\mathrm{GL}(r, \mathcal{O}_Y)$ denote the sheaf of groups on Y of invertible $r \times r$-matrices with coefficients in $\mathcal{O}_Y$ (the group law being the multiplication of matrices), and if $\mathcal{E}$ is a coherent sheaf on Y, let $M(r, \mathcal{E})$ denote the sheaf of groups on Y of $r \times r$-matrices with coefficients in $\mathcal{E}$ (the group law being the addition of matrices). There is a canonical bijection between $H^1(Y, \mathrm{GL}(r, \mathcal{O}_Y))$ and the set of isomorphism classes of rank r vector bundles on Y.

Suppose that X_n, of multiplicity n, can be extended to X_{n+1} of multiplicity $n+1$. For every integer p such that $1 \leq p \leq n+1$, we can view the sheaf of groups $\mathrm{GL}(r, \mathcal{O}_{X_p})$ on X_p as a sheaf of groups on X_{n+1}, and we have a canonical surjective morphism $\mathrm{GL}(r, \mathcal{O}_{X_{n+1}}) \to \mathrm{GL}(r, \mathcal{O}_{X_n})$. Its kernel is isomorphic to $M(r, L^n)$. So we have an exact sequence of sheaves of groups on X_{n+1}

$$0 \longrightarrow M(r, L^n) \longrightarrow \mathrm{GL}(r, \mathcal{O}_{X_{n+1}}) \xrightarrow{\ p_n\ } \mathrm{GL}(r, \mathcal{O}_{X_n}) \longrightarrow 0 \ .$$

Let $g \in H^1(X_n, \mathrm{GL}(r, \mathcal{O}_{X_n}))$, $\mathbb{E}$ the corresponding vector bundle on X_n and $E = \mathbb{E}_{|X}$.

7.1.1. Proposition: *We have* $M(r, L^n)^g \simeq E^* \otimes E \otimes L^n$.

Proof. The action of $\mathrm{GL}(r, \mathcal{O}_{X_n})$ is the action by conjugation after restriction to X. The result follows then easily from 2.1.3. $\square$

Recall that we have a canonical map

$$\Delta : H^1(X_n, \mathrm{GL}(r, \mathcal{O}_{X_n})) \longrightarrow H^2(E^* \otimes E \otimes L^n)$$

such that $g \in \mathrm{im}(H^1(p_n))$ if and only $\Delta(g) = 0$ (cf. 2.2.4). Hence $\Delta(g)$ *is the obstruction to the extension of* $\mathbb{E}$ *to a vector bundle on* X_{n+1}. In particular, if $\mathbb{E}$ is a line bundle then this obstruction lies in $H^2(L^n)$.

We have a canonical exact sequence of sheaves on X_n

$$0 \longrightarrow L^n \longrightarrow \Omega_{X_{n+1}|X_n} \longrightarrow \Omega_{X_n} \longrightarrow 0 \ ,$$

corresponding to $\sigma_n \in \mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\Omega_{X_n}, L^n)$, inducing $\overline{\sigma}_n \in \mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\mathbb{E} \otimes \Omega_{X_n}, \mathbb{E} \otimes L^n)$. In 3 we have defined $\nabla_0(\mathbb{E}) \in \mathrm{Ext}^1(\mathbb{E}, \mathbb{E} \otimes \Omega_{X_n})$, the canonical class of $\mathbb{E}$. We have a canonical product $\mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\mathbb{E}\otimes\Omega_{X_n}, \mathbb{E}\otimes L^n) \times \mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\mathbb{E}, \mathbb{E}\otimes\Omega_{X_n}) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_{X_n}}(\mathbb{E}, \mathbb{E}\otimes L^n) = H^2(X, E^*\otimes E\otimes L^n)$.

7.1.2. Theorem: *We have* $\Delta(g) = \overline{\sigma}_n \nabla_0(\mathbb{E})$.

Proof. For every subset U of X, let $U^{(n)}$ (resp. $U^{(n+1)}$) denote the corresponding open subset of X_n (resp. X_{n+1}). Let $(U_i)_{i\in I}$ be an open cover of X such that L is trivial on every U_i, $U_i^{(n+1)} \simeq U_i \times \mathbf{Z}_{n+1}$, and that $\mathbb{E}$ is represented by a cocycle (θ_{ij}), $\theta_{ij} \in \mathrm{Aut}(\mathcal{O}^{(n)}_{U_{ij}} \otimes \mathbb{C}^r)$. Let $t_i : \mathcal{O}_{U_i} \to L_{|U_i}$ be an isomorphism. For every $i, j \in I$ such that $i \neq j$, let $\overline{\theta}_{ij}$ be an extension of θ_{ij} to an automorphism of $\mathcal{O}^{(n+1)}_{U_{ij}} \otimes \mathbb{C}^r$. Using the isomorphisms $U_i^{(n+1)} \simeq U_i \times \mathbf{Z}_{n+1}$ and 2.3, we can assume that $\overline{\theta}_{ji} = \overline{\theta}_{ij}^{-1}$. Then by 2.2, $\Delta(g)$ is represented by the family $(\overline{\theta}_{ij}\overline{\theta}_{jk}\overline{\theta}_{ki})$. We can write

$$\overline{\theta}_{ij}\overline{\theta}_{jk}\overline{\theta}_{ki} \ = \ I_{U_{ijk}^{(n+1)}\otimes\mathbb{C}^r} + t_i^n \rho_{ijk} \ ,$$

and $\Delta(g)$ is also represented by the family $(t_i^n \rho_{ijk})$. More precisely, let $\theta_i : \mathbb{E}_{|U_i^{(n)}} \to \mathcal{O}_{U_i^{(n)}} \otimes \mathbb{C}^r$ be isomorphisms such that $\theta_{ij} = \theta_i\theta_j^{-1}$. Then $\Delta(g)$ is represented by the true cocycle $(t_i^n.\theta_i^{-1}\rho_{ijk}\theta_i)$.

On the other hand, consider the morphism

$$\delta : H^1(\mathbb{E}^* \otimes \mathbb{E} \otimes \Omega_{X_n}) \longrightarrow H^2(\mathbb{E}^* \otimes \mathbb{E} \otimes L^n)$$

coming from the exact sequence

$$0 \longrightarrow \mathbb{E}^* \otimes \mathbb{E} \otimes L^n \longrightarrow \mathbb{E}^* \otimes \mathbb{E} \otimes \Omega_{X_{n+1}|X_n} \longrightarrow \mathbb{E}^* \otimes \mathbb{E} \otimes \Omega_{X_n} \longrightarrow 0 \ .$$

Then we have

$$\overline{\sigma}_n.\nabla_0(\mathbb{E}) \ = \ \delta(\nabla_0(\mathbb{E})) \ .$$

The class $\nabla_0(\mathbb{E})$ is represented by the cocycle $((d\theta_{ij})\theta_{ij}^{-1})$ in the way of 2.1. In the ordinary way (local sections of $\mathbb{E}^* \otimes \mathbb{E} \otimes \Omega_{X_n}$), it is represented by the cocycle $(\theta_i^{-1}(d\theta_{ij})\theta_j)$. To describe $\delta(\nabla_0(E))$ we first take $\theta_i^{-1}(d\overline{\theta}_{ij})\theta_j \in H^0(U_{ij}^{(n)}, \mathbb{E}^* \otimes \mathbb{E} \otimes \Omega_{X_{n+1}|X_n})$, which is over $\theta_i^{-1}(d\theta_{ij})\theta_j$. Then we have

$$\tau_{ijk} \ = \ \theta_i^{-1}(d\overline{\theta}_{ij})\theta_j - \theta_i^{-1}(d\overline{\theta}_{ik})\theta_k + \theta_j^{-1}(d\overline{\theta}_{jk})\theta_k \ \in \ H^0(U_{ijk}^{(n)}, \mathbb{E}^* \otimes \mathbb{E} \otimes L^n) \ ,$$

and (τ_{ijk}) is a cocycle representing $\delta(\nabla_0(E))$. We have

$$\overline{\theta}_{ij}\overline{\theta}_{jk} \ = \ (I_{U_{ijk}^{(n+1)}\otimes\mathbb{C}^r} + t_i^n \rho_{ijk})\overline{\theta}_{ik} \ ,$$

hence

$$\overline{\theta}_{ik} \;=\; \overline{\theta}_{ij}\overline{\theta}_{jk} - t_i^n\rho_{ijk}\theta_{ij}^0\theta_{jk}^0$$

(where the exponent 0 means restriction to X). Restricting to X_n we have

$$d\overline{\theta}_{ik} \;=\; \theta_{ij}.d\overline{\theta}_{jk} + d\overline{\theta}_{ij}.\theta_{jk} - nt_i^{n-1}\theta_{ij}^0\theta_{jk}^0\rho_{ijk}dt_i \ .$$

An easy computation shows then that

$$\tau_{ijk} \;=\; nt_i^{n-1}dt_i.(\theta_i^{-1}\rho_{ijk}\theta_i) \ ,$$

i.e. it is the image in $H^0(U_{ijk}^{(n)},\mathbb{E}^*\otimes\mathbb{E}\otimes\Omega_{X_{n+1}|X_n})$ of $t_i^n\rho_{ijk}\in H^0(U_{ijk}^{(n)},\mathbb{C}^{r*}\otimes\mathbb{C}^r\otimes L^n)$. This proves theorem 7.1.2. $\square$

7.1.3. *Other construction of the obstruction* – To get an extension of $\mathbb{E}$ to a vector bundle on X_{n+1} we can try to build an extension

$$0\longrightarrow E\otimes L^n\longrightarrow\mathcal{E}\longrightarrow\mathbb{E}\longrightarrow 0$$

on X_{n+1} such that $\mathcal{E}$ is locally free. We have an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^1(\mathcal{H}om(\mathbb{E},E\otimes L^n)) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E},E\otimes L^n) \arrow[r, "\zeta"] & H^0(\mathcal{E}xt^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E},E\otimes L^n)) \arrow[r, "\nabla'"] \arrow[d, equal] & {} \\ & H^1(E\otimes E^*\otimes L^n) & & \mathrm{End}(E) & \end{tikzcd}$$

$$\longrightarrow H^2(\mathcal{H}om(\mathbb{E},E\otimes L^n)) \;=\; H^2(E\otimes E^*\otimes L^n) \ .$$

Suppose that the preceding extension corresponds to $\sigma\in\mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E},E\otimes L^n$“). Then one can prove that $\mathcal{E}$ is locally free if and only $\zeta(\sigma)$ is an automorphism. So it can be possible to build a locally free $\mathcal{E}$ if and only if $\nabla'(I_E)=0$. One can then show that $\nabla'(I_E)=\Delta(g)$.

7.2. Extension of multiple schemes

We use the notations of 4. Let $g_n\in H^1(X,\mathcal{G}_n)$ and X_n be the corresponding multiple scheme of multiplicity n. Recall that we have an exact sequence

$$0\longrightarrow T_X\oplus\mathcal{O}_X\longrightarrow\mathcal{G}_{n+1}\xrightarrow{\ \rho_{n+1}\ }\mathcal{G}_n\longrightarrow 0\ ,$$

that $(T_X\oplus\mathcal{O}_X)^{g_n}\simeq(\Omega_{X_2|X})^*\otimes L^n$, and that we have a canonical map

$$\Delta_n : H^1(X,\mathcal{G}_n)\longrightarrow H^2((\Omega_{X_2|X})^*\otimes L^n)$$

such that $g_n\in\mathrm{im}(H^1(\rho_{n+1}))$ if and only $\Delta_n(g_n)=0$ (cf. 2.2.4). Hence $\Delta_n(g_n)$ *is the obstruction to the extension of X_n to a primitive multiple scheme of multiplicity $n+1$.*

Suppose that $\Delta_n(g_n)=0$. By 2.2.3, given an extension of X_n in multiplicity $n+1$, corresponding to $g_{n+1}\in H^1(\mathcal{G}_{n+1})$, there is a canonical surjective map

$$\lambda_{g_{n+1}} : H^1((\Omega_{X_2|X})^*\otimes L^n)\longrightarrow H^1(\rho_{n+1})^{-1}(g_n)$$

such that $\lambda_{g_{n+1}}(0)=g_{n+1}$.

By 7.1, we have also a canonical map

$$\Delta'_n : H^1(X_{n-1}, \mathcal{O}^*_{X_{n-1}}) = \mathrm{Pic}(X_{n-1}) \longrightarrow H^2(L^{n-1})$$

such that a line bundle $\mathcal{L}$ on X_{n-1} can be extended to a line bundle on X_n if and only if $\Delta'_n(\mathcal{L}) = 0$.

By 4.3 we have an exact sequence

$$0 \longrightarrow T_X \xrightarrow{\iota} (\Omega_{X_2|X})^* \xrightarrow{\upsilon} L^{-1} \longrightarrow 0 \ .$$

Let $\Upsilon_n : H^1(X, \mathcal{G}_n) \to \mathrm{Pic}(X_{n-1})$ be the map defined as follows: if $g_n \in H^1(X, \mathcal{G}_n)$ and X_n is the corresponding multiple scheme of multiplicity n, then $\Upsilon_n(g_n) = \mathcal{I}_{X,X_n}$.

The following result can be proved easily using the methods of 4.4:

7.2.1. Proposition: *The following diagram*

$$\begin{tikzcd}
H^1(X, \mathcal{G}_n) \arrow[r, "\Delta_n"] \arrow[d, "\Upsilon_n"] & H^2((\Omega_{X_2|X})^* \otimes L^n) \arrow[d, "H^2(\upsilon \otimes I_{L^n})"] \\
\mathrm{Pic}(X_{n-1}) \arrow[r, "\Delta'_n"] & H^2(L^{n-1})
\end{tikzcd}$$

is commutative.

This result is not surprising: if $g_n \in H^1(X, \mathcal{G}_n)$ and X_n is the corresponding primitive multiple scheme, then $\Delta_n(g_n)$ is the obstruction to its extension in multiplicity $n+1$, and $\Delta'_n(\Upsilon_n(g_n))$ is the obstruction to the extension of $\mathcal{I}_{X,X_n}$ to a line bundle on X_n. It follows that $\Delta_n(g_n) = 0$ implies that $\Delta'_n(\Upsilon_n(g_n)) = 0$.

Suppose that $\Delta'_n(\Upsilon_n(g_n)) = 0$. It follows that there exists $\eta \in H^2(T_X \otimes L^n)$ such that $\Delta_n(g_n) = H^2(\iota \otimes I_{L^n})(\eta)$. This can be seen more precisely using 4.5. From (2) we have a canonical map

$$\Delta''_n : H^1(X, \mathcal{H}_n) \longrightarrow H^2(T_X \otimes L^n)$$

such that for any pair $(X_n, \mathbb{L})$ (where $\mathbb{L}$ is an extension of $\mathcal{I}_{X,X_n}$ to a line bundle on X_n), corresponding to $h_n \in H^1(X, \mathcal{H}_n)$, there exists an extension of X_n to a primitive multiple scheme X_{n+1} of multiplicity $n+1$ such that $\mathcal{I}_{X,X_{n+1}} = \mathbb{L}$ if and only if $\Delta''_n(h_n) = 0$. We have also a commutative diagram

$$\begin{tikzcd}
H^1(X, \mathcal{H}_n) \arrow[r, "\Delta''_n"] \arrow[d, "H^1(\epsilon_n)"] & H^2(T_X \otimes L^n) \arrow[d, "H^2(\iota \otimes I_{L^n})"] \\
H^1(X, \mathcal{G}_n) \arrow[r, "\Delta_n"] & H^2((\Omega_{Y|X})^* \otimes L^n)
\end{tikzcd}$$

Since $\Delta'_n(\Upsilon_n(g_n)) = 0$, $\mathcal{I}_{X,X_n}$ can be extended to a line bundle $\mathbb{L}$ on X_n, and $(X_n, \mathbb{L})$ corresponds to $h_n \in H^1(X, \mathcal{H}_n)$. It follows that we can take $\eta = \Delta''_n(h_n)$.

Let $\mathcal{X}$ be the set of extensions of X_n to a primitive multiple scheme X_{n+1} of multiplicity $n+1$ such that $\mathcal{I}_{X,X_{n+1}} \simeq \mathbb{L}$. Then there exists a canonical surjective map

$$\lambda : H^1(T_X \otimes L^n) \longrightarrow \mathcal{X}$$

such that the fibers of λ are the orbits of the group of automorphisms χ of X_n leaving X invariant and such that $\chi^*(\mathbb{L}) \simeq \mathbb{L}$.

7.2.2. *Parametrization of the extensions* – We use the notations of 4.2 and 5.1. Suppose that $g_n = H^1(\rho_{n+1})(g_{n+1})$, and that g_{n+1} is defined by the cocycle (δ^*_{ij}), $\delta^*_{ij} = \phi_{\eta_{ij},\mu_{ij}}$. Then the elements of $H^1(\rho_{n+1})^{-1}(g_n)$ are represented by the cocycles of the form $(\tau_{ij}\delta^*_{ij})$, with

$$\tau_{ij} \ = \ \phi_{D_{ij}t^{n-1},1+\epsilon_{ij}t^{n-1}} \ ,$$

where D_{ij} is a derivation of $\mathcal{O}_X(U_{ij})$ and $\epsilon_{ij} \in \mathcal{O}_X(U_{ij})$. The cocyle relation satisfied by $(\tau_{ij}\delta^*_{ij})$ (to ensure that it defines an element of $H^1(X, \mathcal{G}_{n+1})$) is equivalent to a cocyle relation satisfied by (D_{ij}, ϵ_{ij}), so that it defines an element of $H^1(X, (\Omega_{X_2|X})^* \otimes L^n)$ in the way described in 2.1 (cf. 2.2.3).

7.2.3. *Extensions of trivial primitive multiple schemes* – Let X^0_n be trivial primitive multiple scheme of multiplicity $n \geq 2$, with associated smooth variety X and associated line bundle L. We suppose that $h^0(X_n, \mathcal{I}_X\mathcal{T}_n) = 0$. Since $\mathrm{Aut}_0(X^0_n) = \{I_{X^0_n}\}$, we have $\mathrm{Aut}(X^0_n) = \mathbb{C}^*$. It follows from proposition 6.6.1 that the non trivial extensions of X^0_n to a primitive multiple scheme of multiplicity $n+1$ can be identified with the quotient $\big(H^1(X, L^{n-1}) \times H^1(X, T_X \otimes L^n)\big)/\mathbb{C}^*$ with the action

$$\begin{array}{ccc} \mathbb{C}^* \times \big(H^1(X, L^{n-1}) \times H^1(X, T_X \otimes L^n)\big) & \longrightarrow & H^1(X, L^{n-1}) \times H^1(X, T_X \otimes L^n) \\ (\lambda, (\eta, \epsilon)) & \longmapsto & (\lambda^{n-1}\eta, \lambda^n\epsilon) \end{array}$$

i.e. it is a *weighted projective space.*

7.3. Extension of double schemes

We use the notations of 4.2 and 4.3: X_2 is a double scheme with underlying smooth variety X and associated line bundle L on X, (U_i) is an open affine cover of X, and X_2 is obtained by gluing the varieties $U_i \times \mathbf{Z}_2$ using the automorphisms $\delta_{ij} : U_{ij} \times \mathbf{Z}_2 \to U_{ij} \times \mathbf{Z}_2$. We can write (using the notations of 5)

$$\delta^*_{ij} \ = \ \Phi_{D_{ij},\alpha_{ij}} \ ,$$

where D_{ij} is a derivation of $\mathcal{O}_X(U_{ij})$ and $\alpha_{ij} \in \mathcal{O}_X(U_{ij})$ is invertible. The cocycle (α_{ij}) defines the line bundle L, and we have $D_{ik} = D_{ij} + \alpha_{ij}D_{jk}$.

Suppose that L can be extended to a line bundle on X_2, and let $(\alpha_{ij} + \beta_{ij}t)$ $(\alpha_{ij} + \beta_{ij}t \in \mathcal{O}_X(U_{ij})[t]/(t^2))$ be a family defining $\mathbb{L}$ (the cocycle defining $\mathbb{L}$ on X_2 is $(\delta^*_i(\alpha_{ij} + \beta_{ij}t))$). The pair $(X_2, \mathbb{L})$ corresponds to $\eta \in H^1(X, \mathcal{H}_2)$ (cf. 4.5). The cocycle $((\Phi_{D_{ij},\alpha_{ij}}, \alpha_{ij} + \beta_{ij}t))$ represents η. We have a canonical map

$$\Delta''_2 : H^1(X, \mathcal{H}_2) \longrightarrow H^2(T_X \otimes L^2)$$

such that there exists an extension of X_2 to a primitive multiple scheme X_3 of multiplicity 3 such that $\mathcal{I}_{X,X_3} = \mathbb{L}$ if and only if $\Delta''_2(\eta) = 0$.

7.3.1. Theorem: *The cocycle (ρ_{ijk}), with*

$$\rho_{ijk} \ = \ \frac{1}{2}\Big(\beta_{ij}D_{jk} + \big(D_{jk}(\alpha_{ij}) + \frac{\alpha_{ij}}{\alpha_{jk}}D_{jk}(\alpha_{jk}) - \frac{\alpha_{ij}}{\alpha_{jk}}\beta_{jk}\big)D_{ij} + \alpha_{ij}(D_{ij}D_{jk} - D_{jk}D_{ij})\Big)$$

represents $\Delta_2''(\eta)$.

Proof. The construction of $\Delta_2''(\eta)$ follows 2.2.4, using the exact sequence of (2)

$$0 \longrightarrow T_X \otimes L^2 \longrightarrow \mathcal{G}_3^\eta \longrightarrow \mathcal{H}_2^\eta \longrightarrow 0 \ .$$

We take $\Psi_{ij} = \Psi(D_{ij}, \alpha_{ij} + \beta_{ij}t) \in \mathcal{G}_3(U_{ij})$ as a pull-back of $(\Phi_{D_{ij},\alpha_{ij}}, \alpha_{ij} + \beta_{ij}t) \in \mathcal{H}_2(U_{ij})$. The family (Ψ_{ij}) has the correct property, i.e. $\Psi_{ij}^{-1} = \Psi_{ij}$, by lemma 5.3.7, and $(\Psi_{ij}\Psi_{jk}\Psi_{ki})$ represents $\Delta_2''(\eta)$. We have

$$\Psi_{ij}\Psi_{jk}\Psi_{ki} \ = \ \Phi_{0,1,\rho_{ijk}} \ ,$$

from proposition 5.3.8, and the result follows. □

7.3.2. *The case* $L = \mathcal{O}_X$ – In this case we take $\alpha_{ij} = 1$, and (D_{ij}) is a cocycle representing the element $\sigma \in H^1(T_X)$ which defines X_2 (cf. 4.3.3). On the other hand, (β_{ij}) is a cocycle representing $\theta_{\mathbb{L}} \in H^1(\mathcal{O}_X)$ associated to $\mathbb{L}$. We have a canonical bilinear map (*Poisson bracket*)

$$\begin{aligned} H^1(T_X) \times H^1(T_X) &\longrightarrow H^2(T_X) \\ (u,v) &\longmapsto [u,v] \end{aligned}$$

and it follows easily from theorem 7.3.1 that

$$\Delta_2''(\eta) \ = \ \theta_{\mathbb{L}}\sigma + \frac{1}{2}[\sigma,\sigma] \ .$$

7.4. The case of surfaces – Double primitives schemes with canonical associated sheaf

We suppose that X is a surface and $L = \omega_X$. The non trivial primitive double schemes with support X and associated line bundle ω_X are parametrized by $\mathbb{P}(H^1(T_X \otimes \omega_X)) = \mathbb{P}(H^1(\Omega_X))$. Suppose that $\sigma \in H^1(\Omega_X)$, $\sigma \neq 0$, and let X_2 be the primitive double scheme corresponding to $\mathbb{C}\sigma$. The first necessary condition to extend X_2 to multiplicity 3 is that ω_X can be extended to a line bundle on X_2. This is equivalent to

$$\sigma.\nabla_0(\omega_X) \ = \ 0$$

in $H^2(\omega_X) \simeq \mathbb{C}$. If this is true, a sufficient condition to extend X_2 to multiplicity 3 is that

$$h^2(T_X \otimes \omega_X^2) \ = \ h^0(T_X) \ = \ 0 \ .$$

In other words

7.4.1. Proposition: *If $H^0(T_X) = \{0\}$, then X_2 can be extended to a primitive multiple scheme of multiplicity 3 if and only if $\sigma.\nabla_0(\omega_X) = 0$.*

7.5. Relations with the construction of D. Bayer and D. Eisenbud

D. Bayer and D. Eisenbud gave in [4] the construction and parametrization of primitive double schemes, also called *ribbons*. These constructions can be generalized to higher multiplicities, as we will see (mostly without proofs).

Given X_n, if X_{n+1} exists, then we have an exact sequence of sheaves on X_n

$$(10) \qquad 0 \longrightarrow L^n \longrightarrow \Omega_{X_{n+1}|X_n} \xrightarrow{p} \Omega_{X_n} \longrightarrow 0 \ ,$$

and $\Omega_{X_{n+1}|X_n}$ is locally free. Let

$$d_n : \mathcal{O}_{X_n} \longrightarrow \Omega_{X_n} \ , \quad d_{n+1} : \mathcal{O}_{X_{n+1}} \longrightarrow \Omega_{X_{n+1}}$$

be the canonical derivations, and $\rho : \Omega_{X_{n+1}} \to \Omega_{X_{n+1}|X_n}$ the restriction morphism. Then we have a commutative diagram

$$\begin{tikzcd} \mathcal{O}_{X_{n+1}} \arrow[r] \arrow[d, "\rho d_{n+1}"] & \mathcal{O}_{X_n} \arrow[d, "d_n"] \\ \Omega_{X_{n+1}|X_n} \arrow[r, "p"] & \Omega_{X_n} \end{tikzcd}$$

inducing, for every $x \in X_n$, a bijection between $\mathcal{O}_{X_{n+1},x}$ and the set of pairs $(\alpha,\beta) \in \mathcal{O}_{X_n,x} \times \Omega_{X_{n+1}|X_n,x}$ such that $d_{n,x}(\alpha) = \rho_x d_{n+1,x}(\beta)$. It follows that *given* $\sigma \in \mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\Omega_{X_n}, L^n)$*, there exists at most one extension* X_{n+1} *such that* σ *is associated to* (10).

Conversely, given an exact sequence of sheaves on X_n

$$(11) \qquad 0 \longrightarrow L^n \longrightarrow \mathcal{E} \xrightarrow{p} \Omega_{X_n} \longrightarrow 0 \ ,$$

where $\mathcal{E}$ is locally free, we can define a sheaf of abelian groups $\mathcal{C}$ to be the pullback

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{O}_{X_n} \arrow[d, "d_n"] \\ \mathcal{E} \arrow[r, "p"] & \Omega_{X_n} \end{tikzcd}$$

and make $\mathcal{C}$ into a sheaf of $\mathbb{C}$-algebras with an appropriate product law (cf. [4], proof of theorem 1.2). We can then define X_{n+1} as $X_{n+1} = \mathrm{spec}(\mathcal{C})$. It follows that X_n *can be extended to a primitive multiple scheme of multiplicity* $n+1$ *if and only if there exists an extension* (11) *such that* $\mathcal{E}$ *is locally free.*

To study the extensions of X_n in multiplicity $n+1$ we have to consider extensions (11). We have from [22], 7.3, a canonical exact sequence

$$(12) \qquad 0 \longrightarrow H^1(\mathcal{H}om(\Omega_{X_n}, L^n)) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\Omega_{X_n}, L^n) \longrightarrow$$
$$H^0(\mathcal{E}xt^1_{\mathcal{O}_{X_n}}(\Omega_{X_n}, L^n)) \longrightarrow H^2(\mathcal{H}om(\Omega_{X_n}, L^n)) \ .$$

7.5.1. Lemma: *We have* $\mathcal{E}xt^1_{\mathcal{O}_{X_n}}(\Omega_{X_n}, L^n) \simeq \mathcal{O}_C$.

Proof. We use the exact sequence

$$L^n \longrightarrow \Omega_{X_{n+1}|X_n} \longrightarrow \Omega_{X_n} \longrightarrow 0 \ .$$

For every $x \in X$, there is an open neighborhood U of x in X_n such that $\mathcal{I}_{X,X_n|U}$ can be extended to a line bundle $\mathbb{L}$ of U. We obtain a locally free resolution of $\Omega_{X_n|U}$:

$$\cdots \longrightarrow \mathbb{L}^{n+1} \longrightarrow \mathbb{L}^n \longrightarrow \Omega_{X_{n+1}|U} \longrightarrow \Omega_{X_n|U} \longrightarrow 0 \ .$$

It follows that we have an isomorphism $\mathcal{E}xt^1_{\mathcal{O}_U}(\Omega_{X_n|U}, L^n_{|U}) \simeq \mathcal{O}_U$. It is easy to see that all these isomorphisms glue and thus define the isomorphism of lemma 7.5.1. $\square$

7.5.2. Lemma: *Let $x \in X$ and $\alpha \in \mathrm{Ext}^1_{\mathcal{O}_{X_n,x}}(\Omega_{X_n,x}, L^n_x) \simeq \mathcal{O}_{X,x}$. Let*

$$0 \longrightarrow L^n_x \longrightarrow M \longrightarrow \Omega_{X_n,x} \longrightarrow 0$$

be the corresponding extension of $\mathcal{O}_{X_n,x}$-modules. Then M is free if and only if α is invertible.

Proof. We use the free resolution

$$\cdots \longrightarrow \mathbb{L}^{n+1}_x \longrightarrow \mathbb{L}^n_x \xrightarrow{\beta} \Omega_{X_{n+1}|U,x} \longrightarrow \Omega_{X_n,x} \longrightarrow 0 \ ,$$

where β induces $\overline{\beta} : L^n_x \to \Omega_{X_{n+1}|U,x}$. Then, by the usual construction of extensions, M is isomorphic to the cokernel of

$$\alpha \oplus \overline{\beta} : L^n_x \longrightarrow L^n_x \oplus \Omega_{X_{n+1}|U,x} \ .$$

Lemma 7.5.2 follows easily. $\square$

We have $\mathcal{H}om(\Omega_{X_n}, L^n) \simeq (\Omega_{X_2|X})^* \otimes L^n$. Hence we have an exact sequence

$$0 \longrightarrow H^1((\Omega_{X_2|X})^* \otimes L^n) \xrightarrow{\nu} \mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\Omega_{X_n}, L^n) \xrightarrow{\tau} \mathbb{C} \xrightarrow{\delta} H^2((\Omega_{X_2|X})^* \otimes L^n) \ .$$

it follows that X_n can be extended to primitive multiple scheme of multiplicity $n+1$ if and only if $\delta(1) = 0$, and that the set of isomorphism classes of extensions of X_n in multiplicity $n+1$ can be identified with $\tau^{-1}(1)$.

Let X_{n+1} be an extension of X_n in multiplicity $n+1$ and $\sigma \in \tau^{-1}(1)$ the corresponding element. We have a bijection

$$\begin{aligned} \overline{\nu} : H^1((\Omega_{X_2|X})^* \otimes L^n) &\longrightarrow \tau^{-1}(1) \\ u &\longmapsto \sigma + \nu(u) \ . \end{aligned}$$

With the notations of 7.2, $\tau^{-1}(1)$ can be identified with $H^1(\rho_{n+1})^{-1}(g_n)$, and we have a canonical map

$$\lambda_{g_{n+1}} : H^1((\Omega_{X_2|X})^* \otimes L^n) \longrightarrow H^1(\rho_{n+1})^{-1}(g_n) \ .$$

The following result is proved in [11], 3.1, when X is a curve, but the proof is similar in any dimension:

7.5.3. Theorem: *We have, for every $u \in H^1((\Omega_{X_2|X})^* \otimes L^n)$, $\lambda_{g_{n+1}}(u) = \overline{\nu}(nu)$.*

8. The case of projective spaces

We use the notations of 4.1 to 4.4.

Let V be a complex vector space of dimension $m+1$, $m \geq 2$. We suppose that $X = \mathbb{P}(V) = \mathbb{P}_m$ (the set of lines in V). Let k be an integer, and $L = \mathcal{O}_{\mathbb{P}_m}(k)$. We will prove

8.0.1. Theorem: 1 − *If $m > 2$, all primitive multiple schemes Y such that $Y_{red} = \mathbb{P}_m$ are trivial.*

2 − *There are only two non trivial primitive multiple schemes Y such that $Y_{red} = \mathbb{P}_2$, one in multiplicity 2, with $L = \mathcal{O}_{\mathbb{P}_2}(-3)$, and the other in multiplicity 4, with $L = \mathcal{O}_{\mathbb{P}_2}(-1)$.*

Proof. There exists a non trivial primitive double scheme X_2 with underlying smooth variety X and associated line bundle L if and only if $h^1(T_X \otimes L) \neq 0$. This is true only if $m = 2$ and $k = -3$.

Suppose that $m > 2$. According to 7.2, there can be non trivial extensions of a trivial primitive multiple scheme X_n to multiplicity $n+1$ only if $h^1(\mathbb{P}_m, (\Omega_{X_2|X})^* \otimes L^n) \neq 0$. This never happens if $m > 2$. So **1** is proved.

Suppose now that $m = 2$ and $k = -3$. In this case, according to 4.3, the non trivial primitive double schemes are parametrized by $\mathbb{P}(H^1(T_{\mathbb{P}_2}(-3)))$. Since $h^1(T_{\mathbb{P}_2}(-3)) = 1$, there is just only one non trivial primitive double scheme, that we will denote by $\mathbf{X}_2$. We will prove

8.0.2. Theorem: *The only line bundle on $\mathbf{X}_2$ is the trivial bundle $\mathcal{O}_{\mathbf{X}_2}$.*

It follows that $\mathbf{X}_2$ is not quasi-projective and cannot be extended to a primitive scheme of multiplicity 3. The scheme $\mathbf{X}_2$ appears also in [2] (it is an example of *non projective K3 carpet*).

Hence any non trivial primitive multiple scheme with underlying smooth variety $X = \mathbb{P}_2$ and associated line bundle L, and of multiplicity $n+1 > 2$ is an extension of a trivial multiple scheme of multiplicity $n \geq 2$. This can only happen if $h^1(T_X \otimes L^n) \neq 0$ (cf. 7.2.2). In this case we have $L^n = \mathcal{O}_{\mathbb{P}_2}(-3)$, hence $n = 3$ and $L = \mathcal{O}_{\mathbb{P}_2}(-1)$. Starting with the trivial primitive multiple scheme X_3 of multiplicity 3 (with $L = \mathcal{O}_{\mathbb{P}_2}(-1)$), we get a non trivial primitive multiple scheme $\mathbf{X}_4$ of multiplicity 4. Since $h^1(T_X \otimes L^3) = 1$, proposition 6.5.4 implies that $\mathbf{X}_4$ is unique. We will prove

8.0.3. Theorem: *The only line bundle on $\mathbf{X}_4$ is the trivial bundle $\mathcal{O}_{\mathbf{X}_4}$.*

It follows that $\mathbf{X}_4$ is not quasi-projective and cannot be extended to a primitive scheme of multiplicity 5. Hence **2** is proved. □

8.1. Description of some sheaves on the projective plane

We assume that $m = 2$. Let $x_0, x_1, x_2 \in V^*$ be coordinates in $\mathbb{P}_2$, and for i=0,1,2, let

$$U_i \;=\; \{\mathbb{C}(u_0, u_1, u_2) \in \mathbb{P}_2 ; u_i \neq 0\} \ .$$

We will take indices in $\mathbb{Z}/3\mathbb{Z}$, so that $x_3 = x_0, x_4 = x_1, \cdots$.
We have $\mathcal{O}_{\mathbb{P}_2}(U_i) = \mathbb{C}[\frac{x_{i+1}}{x_i}, \frac{x_{i+2}}{x_i}]$.

8.1.1. *Line bundles* – For every $x \in \mathbb{P}_2$, we have $\mathcal{O}_{\mathbb{P}_2}(-1)_x = x$. For $i = 0, 1, 2$ we have a trivialization of $\mathcal{O}_{\mathbb{P}_2}(-1)$ on U_i, $\alpha_i : \mathcal{O}_{\mathbb{P}_2}(-1)_{|U_i} \to \mathcal{O}_{U_i} \otimes \mathbb{C}$, such that for every $x = \mathbb{C}(u_0, u_1, u_2)$ in U_i, we have $\alpha_{i,x}(u_0, u_1, u_2) = u_i$. Hence $\mathcal{O}_{\mathbb{P}_2}(-1)$ is defined by the cocycle (α_{ij}), where $\alpha_{ij} : U_{ij} \to \mathbb{C}^*$, and for every $x = \mathbb{C}(u_0, u_1, u_2)$ in U_{ij}, we have $\alpha_{ij,x}(x) = \dfrac{u_i}{u_j}$.

Similarly, let $k \in \mathbb{Z}$. Then $\mathcal{O}_{\mathbb{P}_2}(k)$ is defined by the cocycle $(\alpha_{ij}^{(k)})$, where $\alpha_{ij}^{(k)} : U_{ij} \to \mathbb{C}^*$, and for every $x = \mathbb{C}(u_0, u_1, u_2)$ in U_{ij}, we have $\alpha_{ij,x}^{(k)}(x) = \big(\dfrac{u_i}{u_j}\big)^{-k}$.

Let $U \subset \mathbb{P}_2$ be a nonempty open subset. Then $H^0(U, \mathcal{O}_{\mathbb{P}_2}(k))$ can be identified with the set of rational functions $\dfrac{P}{Q}$, $P, Q \in \mathbb{C}[X_0, X_1, X_2]$, such that P, Q are homogeneous, Q does not vanish on U and $\deg(P) - \deg(Q) = k$. If $k < 0$, then for every $x \in U$, $x = \mathbb{C}.u$, $\mathcal{O}_{\mathbb{P}_2}(k)_x = x^{-k} \in S^{-k}V$, and $\dfrac{P}{Q}(x) = \dfrac{P(u)}{Q(u)}.u^{-k}$.

For example, if $k = -3$, these rational functions $\dfrac{P}{Q}$ such that Q does not vanish on U_{012} are the linear combinations of $\dfrac{1}{X_0X_1X_2}$, $\dfrac{X_k^a}{X_i^bX_j^c}$, $\dfrac{X_i^dX_j^e}{X_k^f}$, $0 \leq i < j \leq 2$, $0 \leq k \leq 2$, $k \neq i, j$ $a, b, c, d, e, f \geq 0$, $a - b - c = 3$, $d + e - f = 3$. We have $H^2(\mathcal{O}_{\mathbb{P}_2}(-3)) \simeq \mathbb{C}$, generated by the element corresponding to the cocycle $\Big(\dfrac{1}{X_0X_1X_2}\Big)$.

8.1.2. *Canonical bundle and tangent bundle* – We have $H^1(\Omega_{\mathbb{P}_2}) \simeq \mathbb{C}$, and it is generated by the canonical class of $\mathcal{O}_{\mathbb{P}_2}(-1)$, which is represented by the cocycle (ρ_{ij}), with

$$\rho_{ij} \ = \ \frac{x_j}{x_i} d\Big(\frac{x_i}{x_j}\Big) \ .$$

The $\mathcal{O}_{\mathbb{P}_2}(U_i)$-module $H^0(T_{\mathbb{P}_2} \otimes \mathcal{O}_{\mathbb{P}_2}(-3))$ is generated by the derivations $\dfrac{\partial}{\partial(x_{i+1}/x_i)}, \dfrac{\partial}{\partial(x_{i+2}/x_i)}$.
We have $\Omega_{\mathbb{P}_2} \simeq T_{\mathbb{P}_2} \otimes \mathcal{O}_{\mathbb{P}_2}(-3)$, and $H^1(T_{\mathbb{P}_2} \otimes \mathcal{O}_{\mathbb{P}_2}(-3)) \simeq \mathbb{C}$ is generated by the element ρ' corresponding to the cocycle (ρ'_{ij}) with

$$\rho'_{i,i+1} \ = \ \frac{x_i}{x_{i+1}} \frac{\partial}{\partial(x_{i+2}/x_i)} \otimes \frac{1}{x_i^3}$$

(where $\dfrac{1}{x_i^3}$ corresponds to a section of $\mathcal{O}_{\mathbb{P}_2}(-3)$ on U_i, cf. 8.1.1).

8.2. The primitive scheme $\mathbf{X}_2$ – Proof of theorem 8.0.2

It is the primitive double scheme defined by $\rho' \in H^1(T_{\mathbb{P}_2} \otimes \mathcal{O}_{\mathbb{P}_2}(-3))$ (cf. 8.1.2). It is obtained from the cocycle (δ^*_{ij}), with $\delta^*_{ij} = \phi_{D_{ij},\nu_{ij}}$, where

$$D_{i,i+1} \;=\; \frac{x_i}{x_{i+1}} \frac{\partial}{\partial(x_{i+2}/x_i)} \;, \qquad \nu_{ij} \;=\; \Big(\frac{x_i}{x_j}\Big)^3$$

(with the notations of 5.4.2).

Let p be an integer, $\neq 0$. The line bundle $\mathcal{O}_{\mathbb{P}_2}(p)$ is defined by the cocycle $\Big(\Big(\dfrac{x_i}{x_j}\Big)^{-p}\Big)$.

Consider the exact sequence $0 \to \mathcal{O}_{\mathbb{P}_2}(-3) \to \Omega_{\mathbf{X}_2|\mathbb{P}_2} \to \Omega_{\mathbb{P}_2} \to 0$, and the associated map $\delta : H^1(\Omega_{\mathbb{P}_2}) \to H^2(\mathcal{O}_{P_2}(-3))$. According to theorem 7.1.2, $\mathcal{O}_{\mathbb{P}_2}(p)$ can be extended to a line bundle on $\mathbf{X}_2$ if and only if $\delta(\nabla_0(\mathcal{O}_{\mathbb{P}_2}(p))) \neq 0$.

The class $\nabla_0(\mathcal{O}_{\mathbb{P}_2}(p))$ is represented by the cocycle $\Big(\Big(-p\dfrac{x_j}{x_i} d(x_i/x_j)\Big)\Big)$. From 5.4.6, $\delta(\nabla_0(\mathcal{O}_{\mathbb{P}_2}(p)))$ is represented by the cocycle (λ_{012}), with

$$\lambda_{012} \;=\; \frac{1}{x_0^3}\frac{x_0}{x_1}\left(\frac{\frac{\partial}{\partial x_2/x_0}\Big(\Big(\frac{x_1}{x_2}\Big)^{-p}\Big)}{\Big(\Big(\frac{x_1}{x_2}\Big)^{-p}\Big)}\right) \;=\; \frac{p}{x_0x_1x_2} \;.$$

It follows from 8.1.1 that $\delta(\nabla_0(\mathcal{O}_{\mathbb{P}_2}(p))) \neq 0$. This proves theorem 8.0.2.

8.3. The primitive scheme $\mathbf{X}_4$ – Proof of theorem 8.0.3

The computations are exactly the same as in 8.2.

9. The case of projective bundles over curves

Let C be a smooth irreducible projective curve of genus g and E a rank 2 vector bundle on C. Let $X = \mathbb{P}(E)$ be the associated projective bundle and $\pi : \mathbb{P}(E) \to C$ the canonical projection. For every closed point $x \in C$, $\pi^{-1}(x)$ is the projective line $\mathbb{P}(E_x)$ of lines in E_x.

9.1. Preliminaries

The line bundles on X are of the form

$$\mathcal{L} \;=\; \pi^*(D) \otimes \mathcal{O}_{\mathbb{P}(E)}(k) \;,$$

for some uniquely determined $D \in \mathrm{Pic}(C)$ and integer k. We have then

$$\begin{aligned}
\pi_*(\mathcal{L}) \;&=\; (S^kE^*) \otimes D \quad \text{if} \quad k \geq 0, \\
&=\; 0 \quad \text{if} \quad k < 0, \\
R^1\pi_*(\mathcal{L}) \;&=\; (S^{-2-k}E) \otimes \det(E) \otimes D \quad \text{if} \quad k \leq -2, \\
&=\; 0 \quad \text{if} \quad k > -2.
\end{aligned}$$

it follows that

$$\begin{array}{rcl} H^0(\mathcal{L}) & = & H^0((S^kE^*)\otimes D) \quad \text{if} \quad k\geq 0, \\ & = & \{0\} \quad \text{if} \quad k<0, \\ H^1(\mathcal{L}) & = & H^1((S^kE^*)\otimes D) \quad \text{if} \quad k\geq 0, \\ & = & H^0((S^{-2-k}E)\otimes \det(E)\otimes D) \quad \text{if} \quad k\leq -2, \\ & = & \{0\} \quad \text{if} \quad k=-1, \\ H^2(\mathcal{L}) & = & H^1((S^{-2-k}E)\otimes \det(E)\otimes D) \quad \text{if} \quad k\leq -2, \\ & = & \{0\} \quad \text{if} \quad k>-2. \end{array}$$

We have canonical exact sequences

$$0\longrightarrow \Omega_{\mathbb{P}(E)/C}\longrightarrow \pi^*(E^*)\otimes \mathcal{O}_{\mathbb{P}(E)}(-1)\longrightarrow \mathcal{O}_{\mathbb{P}(E)}\longrightarrow 0\ , \tag{13}$$

$$0\longrightarrow \pi^*(\omega_C)\longrightarrow \Omega_{\mathbb{P}(E)}\longrightarrow \Omega_{\mathbb{P}(E)/C}\longrightarrow 0\ .$$

9.1.1. Proposition: *If E is simple and $g\geq 2$, we have $H^0(T_{\mathbb{P}(E)})=\{0\}$.*

Proof. Since $g\geq 2$, we have $H^0(\omega_C^*)=\{0\}$, and $H^0(T_{\mathbb{P}(E)})\simeq H^0((\Omega_{\mathbb{P}(E)/C})^*)$ (by the second exact sequence). Applying π_* to the dual of the first exact sequence we find

$$0\longrightarrow \mathcal{O}_C\longrightarrow E\otimes E^*\longrightarrow \pi_*((\Omega_{\mathbb{P}(E)/C})^*)\simeq \mathrm{Ad}(E)\longrightarrow 0\ ,$$

hence $H^0((\Omega_{\mathbb{P}(E)/C})^*)=H^0(\mathrm{Ad}(E))=\{0\}$ because E is simple. □

Using proposition 7.4.1 we get

9.1.2. Corollary: *Suppose that E is simple. Let X_2 be a non trivial primitive double scheme with support $\mathbb{P}(E)$ and associated line bundle $\omega_{\mathbb{P}(E)}$. Let $\mathbb{C}\sigma$ be the element of $\mathbb{P}(H^1(\Omega_{\mathbb{P}(E)}))$ associated do X_2. Then X_2 can be extended to a primitive multiple scheme of multiplicity 3 if and only if $\sigma.\nabla_0(\omega_{\mathbb{P}(E)})=0$ in $H^2(\omega_{\mathbb{P}(E)})$.*

9.2. Annulation of obstructions

Let $\Gamma, F\in \mathrm{Pic}(C)$, $\gamma=\deg(\Gamma)$, n, k, p integers, with $n,k>0$. We will consider 3 cases:

(i) E is semi-stable and $\deg(E)=0$,
(ii) E is semi-stable and $\deg(E)=-1$,
(iii) E is not semi-stable of degree $\epsilon=0$ or -1. Let $L_1\subset E$ be the Harder-Narasimhan filtration of E, where L_1 is a line bundle of degree $\epsilon_1>\epsilon$.

9.2.1. Lemma: *We have $h^0((S^{kn+p}E)\otimes F\otimes \Gamma^n)=0$ if $\gamma<\gamma_0$, and $\gamma_0=-\dfrac{\deg(F)}{n}$ in case (i), $\gamma_0=\dfrac{k}{2}-\dfrac{\deg(F)}{n}+\dfrac{p}{2n}$ in case (ii), $\gamma_0=-\dfrac{\deg(F)}{n}-\dfrac{p\epsilon_1}{n}-k\epsilon_1$ in case (iii).*

Proof. In case (i), $S^{kn+p}E$ is semi-stable of degree 0, so $h^0((S^{kn+p}E)\otimes F\otimes\Gamma^n)=0$ if $\deg(F\otimes\Gamma^n)<0$, which is equivalent to $\gamma<-\dfrac{\deg(F)}{n}$. The proof in case (ii) is similar. To prove case (iii) we use the fact that there is a filtration of $S^{kn+p}E$ with graduates $L_1^a\otimes(\mathcal{O}_C/L_1)^{kn+p-a}$, $0\leq a\leq kn+p$. □

9.2.2. Lemma: *We have* $h^0((S^{kn+p}E)\otimes E\otimes F\otimes\Gamma^n)=0$ *if* $\gamma<\gamma_0$, *and* $\gamma_0=-\dfrac{\deg(F)}{n}$ *in case (i),* $\gamma_0=\dfrac{k}{2}-\dfrac{\deg(F)}{n}+\dfrac{p+1}{2n}$ *in case (ii),* $\gamma_0=-\dfrac{\deg(F)}{n}-\dfrac{(p+1)\epsilon_1}{n}-k\epsilon_1$ *in case (iii).*

Proof. Similar to that of lemma 9.2.1. □

Let n, k be positive integers, and $L=\pi^*(D)\otimes\mathcal{O}_{\mathbb{P}(E)}(-k)$ a line bundle on $\mathbb{P}(E)$. If we want to study the primitive multiple schemes with associated smooth variety $\mathbb{P}(E)$ and associated line bundle L, we need to consider the cohomology groups $H^2(L^n)$, $H^2(T_X\otimes L^n)$, $n>0$.

9.2.3. *Obstructions to the extension of line bundles in higher multiplicity* – We have $H^0(L^n)=\{0\}$, $H^1(L^n)\simeq H^0((S^{kn-2}E)\otimes\det(E)\otimes D^n)$, $H^2(L^n)\simeq H^1((S^{kn-2}E)\otimes\det(E)\otimes D^n)$, and by Serre duality

$$h^1((S^{kn-2}E)\otimes\det(E)\otimes D^n) \;=\; h^0((S^{kn-2}E^*)\otimes\det(E^*)\otimes D^{-n}\otimes\omega_C) \;=\; h^0((S^{kn-2}E)\otimes F\otimes\Gamma^n)\ ,$$

with $F=\det(E)\otimes\omega_C$, $\Gamma=D^*\otimes\det(E)^{-k}$. Hence, by lemma 9.2.1, $h^2(L^n)=0$ if $\deg(D)>\delta_0$, with $\delta_0=\dfrac{2g-2}{n}$ in case (i), $\delta_0=\dfrac{k}{2}+\dfrac{2g-2}{n}$ in case (ii), and $\delta_0=\dfrac{2g-2+\deg(E)-2\epsilon_1}{n}+k\epsilon_1-k\deg(E)$ in case (iii).

9.2.4. *Obstructions to the extension of the schemes in higher multiplicity* – We have exact sequences

$$0\longrightarrow L^n\longrightarrow\pi^*(E)\otimes\mathcal{O}_{\mathbb{P}(E)}(1)\otimes L^n\longrightarrow\Omega^*_{\mathbb{P}(E)/C}\otimes L^n\longrightarrow 0\ , \tag{14}$$

$$0\longrightarrow\Omega^*_{\mathbb{P}(E)/C}\otimes L^n\longrightarrow T_{\mathbb{P}(E)}\otimes L^n\longrightarrow\pi^*(\omega_C^*)\otimes L^n\longrightarrow 0\ .$$

Hence we have $H^2(T_{\mathbb{P}(E)}\otimes L^n)=\{0\}$ whenever

$$h^2(\pi^*(E)\otimes\mathcal{O}_{\mathbb{P}(E)}(1)\otimes L^n) \;=\; h^2(\pi^*(\omega_C^*)\otimes L^n) \;=\; 0.$$

We have

$$h^2(\pi^*(E)\otimes\mathcal{O}_{\mathbb{P}(E)}(1)\otimes L^n) \;=\; h^1((S^{kn-3}E)\otimes E\otimes\det(E)\otimes D^n)\ ,$$

and by Serre duality

$$\begin{aligned} h^1((S^{kn-3}E)\otimes E\otimes\det(E)\otimes D^n) &= h^0((S^{kn-3}E^*)\otimes E^*\otimes\det(E^*)\otimes D^{-n}\otimes\omega_C)\\ &= h^0((S^{kn-3}E)\otimes E\otimes F\otimes\Gamma^n)\ , \end{aligned}$$

with $F=\det(E)\otimes\omega_C$, $\Gamma=D^*\otimes\det(E)^{-k}$. Hence, by lemma 9.2.2, $h^2(\pi^*(E)\otimes\mathcal{O}_{\mathbb{P}(E)}(1)\otimes L^n)=0$ if the same conditions as in 9.2.3 are satisfied.

We have $H^2(\pi^*(\omega_C^*)\otimes L^n)\simeq H^1((S^{kn-2}E)\otimes\det(E)\otimes\omega_C^*\otimes D^n)$, and

$$h^1((S^{kn-2}E)\otimes\det(E)\otimes\omega_C^*\otimes D^n) \;=\; h^0((S^{kn-2}E)\otimes F\otimes\Gamma^n)\ ,$$

with $F=\det(E)\otimes\omega_C^2$, $\Gamma=D^*\otimes\det(E)^{-k}$. Hence, by lemma 9.2.1, $h^2(\pi^*(\omega_C^*)\otimes L^n)=0$ if $\deg(D)>\delta_0$, with $\delta_0=\dfrac{4g-4}{n}$ in case (i), $\delta_0=\dfrac{k}{2}+\dfrac{4g-4}{n}$ in case (ii), and
$\delta_0=\dfrac{4g-4+\deg(E)-2\epsilon_1}{n}+k\epsilon_1-k\deg(E)$ in case (iii).

9.3. Construction of primitive multiple schemes

Suppose that $L=\pi^*(D)\otimes\mathcal{O}_{\mathbb{P}(E)}(-k)$, with $k\geq 3$. Let $d=\deg(D)$. The following is a consequence of 9.2:

9.3.1. Lemma: *We have $H^2(L^n)=\{0\}$ for every $n\geq 1$ and $H^2(T_{\mathbb{P}(E)}\otimes L^n)=\{0\}$ for every $n\geq 2$ if*

In case (i):

- *If $g=0$, $d\geq 0$.*
- *If $g=1$, $d>0$.*
- *If $g\geq 2$, $d>2g-2$.*

In case (ii):

- *If $g=0$, $d\geq\dfrac{k}{2}$.*
- *If $g=1$, $d>\dfrac{k}{2}$.*
- *If $g\geq 2$, $d>\dfrac{k}{2}+2g-2$.*

In case (iii):

- *If $g=0$ or 1, $d\geq k(\epsilon_1-\deg(E))$.*
- *If $g\geq 2$, $d\geq k(\epsilon_1-\deg(E))+2g-2$.*

Recall that to extend a primitive scheme X_n of multiplicity $n\geq 2$ to one of multiplicity $n+1$, we must first extend the ideal sheaf of $\mathbb{P}(E)$ in X_n to a line bundle $\mathbb{L}$ on X_n (with an obstruction in $H^2(L^{n-1})$), and then to extend X_n to X_{n+1} such that the ideal sheaf of $\mathbb{P}(E)$ in X_{n+1} is $\mathbb{L}$, we have another obstruction in $H^2(T_{\mathbb{P}(E)}\otimes L^n)$. Hence if the conditions of lemma 9.3.1 are satisfied, the obstructions vanish, and it is possible to extend a primitive multiple scheme of multiplicity n to one of multiplicity $n+1$.

Similarly we have

9.3.2. Lemma: *We have $H^2(T_{\mathbb{P}(E)}\otimes L)=\{0\}$ if*

In case (i):

- *If $g=0$, $d>-2$.*

– *If* $g = 1$, $d > 0$.
– *If* $g \geq 2$, $d > 4g - 4$.

In case (ii):

– *If* $g = 0$, $d > \frac{k}{2} - 2$
– *If* $g = 1$, $d > \frac{k}{2}$
– *If* $g \geq 2$, $d > \frac{k}{2} + 4g - 4$

In case (iii):

– *If* $g = 0$, $d > k(\epsilon_1 - \deg(E)) + \deg(E) - 2 - 2\epsilon_1$.
– *If* $g = 1$, $d > k(\epsilon_1 - \deg(E)) + \deg(E) - 2\epsilon_1$.
– *If* $g \geq 2$, $d > k(\epsilon_1 - \deg(E)) + \deg(E) - 2\epsilon_1 + 4g - 4$.

9.3.3. Lemma: *Suppose that* $n \geq 1$. *Then the map* $H^1(L^n) \to H^1(\pi^*(E) \otimes \mathcal{O}_{\mathbb{P}(E)}(1) \otimes L^n)$ *induced by* (14) *is injective.*

Proof. This map is the canonical one

$$H^0(D^n \otimes (S^{kn-2}E) \otimes \det(E)) \longrightarrow H^0(D^n \otimes E \otimes (S^{kn-3}E) \otimes \det(E)) \ ,$$

which is clearly injective. $\square$

Suppose that the conditions of lemmas 9.3.1 and 9.3.2 are satisfied and $n \geq 1$. We have $H^0(\Omega^*_{\mathbb{P}(E)/C} \otimes L^n) = \{0\}$, because for every $x \in C$, $\Omega^*_{\mathbb{P}(E)/C} \otimes L^n_{|\pi^{-1}(x)} \simeq \mathcal{O}_{\mathbb{P}(E_x)}(2 - kn)$. Hence

$$h^1(\Omega^*_{\mathbb{P}(E)/C} \otimes L^n) \ = \ h^1(\pi^*(E) \otimes \mathcal{O}_{\mathbb{P}(E)}(1) \otimes L^n) - h^1(L^n) \ .$$

We have then

$$\begin{aligned}
h^1(T_{\mathbb{P}(E)} \otimes L^n) \ &= \ h^1(\Omega^*_{\mathbb{P}(E)/C} \otimes L^n) + h^1(\pi^*(\omega^*_C) \otimes L^n) \\
&= \ h^1(\pi^*(E) \otimes \mathcal{O}_{\mathbb{P}(E)}(1) \otimes L^n) - h^1(L^n) + h^1(\pi^*(\omega^*_C) \otimes L^n) \\
&= \ h^0(D^n \otimes E \otimes (S^{kn-3}E) \otimes \det(E)) - h^0(D^n \otimes (S^{kn-2}E) \otimes \det(E)) \\
&\quad + h^0(D^n \otimes \omega^*_C \otimes (S^{kn-2}E) \otimes \det(E)) \\
&= \ \chi(D^n \otimes E \otimes (S^{kn-3}E) \otimes \det(E)) - \chi(D^n \otimes (S^{kn-2}E) \otimes \det(E)) \\
&\quad + \chi(D^n \otimes \omega^*_C \otimes (S^{kn-2}E) \otimes \det(E)) \\
&= \ n(kn-2)(2d + k\deg(E)) + 2(1-g)(2kn-3) \ .
\end{aligned}$$

It is easily verified that $h^1(T_{\mathbb{P}(E)} \otimes L^n) > 0$. It follows that *there exist infinite sequences*

$$X_1 = \mathbb{P}(E) \subset X_2 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots \ ,$$

where X_n *is a non trivial projective primitive multiple scheme with associated smooth variety* $\mathbb{P}(E)$ *and associated line bundle* L.

We have also

$$\begin{aligned} h^1(L^n) &= \chi(D^n\otimes(S^{kn-2}E)\otimes\det(E)) \\ &= (kn-1)\big(\frac{kn}{2}\deg(E)+nd+1-g\big) \ . \end{aligned}$$

The space $\mathbb{P}(H^1(T_{\mathbb{P}(E)}\otimes L))$ parametrizes the primitive double schemes (cf. 4.3.3).

Let X_n be a non trivial primitive multiple scheme of multiplicity n. Let $\mathcal{X}$ be the set of extensions of X_n to a primitive multiple scheme X_{n+1} of multiplicity $n+1$. Then from 7.2, there is a canonical surjective map

$$H^1((\Omega_{X_2|\mathbb{P}(E)})^*\otimes L^n)\longrightarrow\mathcal{X}$$

whose fibers are the orbits of an action of $\mathrm{Aut}(X_n)$. As a set, we have $\mathrm{Aut}_0(X_n)\simeq H^0(\mathcal{T}_{n-1}\otimes\mathcal{I}_{\mathbb{P}(E),X_n})$ (cf. theorem 6.3.2). Using corollary 6.2.3 and the exact sequence $0\to T_{\mathbb{P}(E)}\to(\Omega_{X_2|\mathbb{P}(E)})^*\to L^{-1}\to 0$, it is easy to see that $H^0(\mathcal{T}_{n-1}\otimes\mathcal{I}_{\mathbb{P}(E),X_n})=\{0\}$. From 6.3.7, $\mathrm{Aut}(X_n)$ is finite. Hence $\mathcal{X}$ is parametrized, up to the action of a finite group, by $H^1((\Omega_{X_2|\mathbb{P}(E)})^*\otimes L^n)$. We have

$$\begin{aligned} h^1((\Omega_{X_2|\mathbb{P}(E)})^*\otimes L^n) &= h^1(T_{\mathbb{P}(E)}\otimes L^n)+h^1(L^{n-1}) \\ &= (3kn^2-5n-2kn+k+1)\big(\frac{k}{2}\deg(E)+d\big)+2(1-g)(5kn-7-k) \ . \end{aligned}$$

9.4. The case of $\mathbb{P}_1\times\mathbb{P}_1$

We have here $C=\mathbb{P}_1$, $E=\mathcal{O}_{\mathbb{P}_1}\otimes\mathbb{C}^2$, π is the projection $\mathbb{P}_1\times\mathbb{P}_1\to\mathbb{P}_1$ on the first factor, and, with the usual notations, $\mathcal{O}_{\mathbb{P}(E)(1)}=\mathcal{O}(0,1)$, $\pi^*(\mathcal{O}_{\mathbb{P}_1}(1))=\mathcal{O}(1,0)$.

Suppose that $k\geq 3$, and $L=\mathcal{O}(d,-k)$. The conditions of lemmas 9.3.1 and 9.3.2 are: $d>0$.

There exist infinite sequences

$$X_1=\mathbb{P}_1\times\mathbb{P}_1\subset X_2\subset\cdots\subset X_n\subset X_{n+1}\subset\cdots \ ,$$

where X_n is a non trivial projective primitive multiple scheme with associated smooth variety $\mathbb{P}_1\times\mathbb{P}_1$ and associated line bundle L.

The extensions of X_n to a primitive multiple scheme of multiplicity $n+1$ form a family of dimension $d(3kn^2-5n-2kn+k+1)+5kn-7-k$.

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