MODULI OF VECTOR BUNDLES ON PRIMITIVE MULTIPLE SCHEMES

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RESUME. A primitive multiple scheme is a Cohen-Macaulay scheme Y such that the associated reduced scheme $X = Y_{red}$ is smooth, irreducible, and that Y can be locally embedded in a smooth variety of dimension $\dim(X) + 1$. If n is the multiplicity of Y, there is a canonical filtration $X = X_1 \subset X_2 \subset \cdots \subset X_n = Y$, such that X_i is a primitive multiple scheme of multiplicity i. The simplest example is the trivial primitive multiple scheme of multiplicity n associated to a line bundle L on X: it is the n-th infinitesimal neighborhood of X, embedded in the line bundle L^* by the zero section.

The main subject of this paper is the construction and properties of fine moduli spaces of vector bundles on primitive multiple schemes. Suppose that $Y = X_n$ is of multiplicity n, and can be extended to X_{n+1} of multiplicity $n + 1$, and let M_n be a fine moduli space of vector bundles on X_n. With suitable hypotheses, we construct a fine moduli space M_{n+1} for the vector bundles on X_{n+1} whose restriction to X_n belongs to M_n. It is an affine bundle over the subvariety $N_n \subset M_n$ of bundles that can be extended to X_{n+1}. In general this affine bundle is not banal. This applies in particular to Picard groups.

We give also many new examples of primitive multiple schemes Y such that the dualizing sheaf ω_Y is trivial.

SUMMARY

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1. Introduction

In this paper a *scheme* is a noetherian separated scheme over $\mathbb{C}$. A *primitive multiple scheme* is a Cohen-Macaulay scheme Y over $\mathbb{C}$ such that:

- $Y_{red} = X$ is a smooth connected variety,
- for every closed point $x \in X$, there exists a neighbourhood U of x in Y, and a smooth variety S of dimension $\dim(X) + 1$ such that U is isomorphic to a closed subscheme of S.

We call X the *support* of Y. It may happen that Y is quasi-projective, and in this case it is projective if X is.

For every closed subscheme $Z \subset Y$, let $\mathcal{I}_Z$ (or $\mathcal{I}_{Z,Y}$) denote the ideal sheaf of Z in Y. For every positive integer i, let X_i be the closed subscheme of Y corresponding to the ideal sheaf $\mathcal{I}_X^i$. The smallest integer n such that $X_n = Y$ is called the *multiplicity* of Y. For $1 \leq i \leq n$, X_i is a primitive multiple scheme of multiplicity i, $L = \mathcal{I}_X/\mathcal{I}_{X_2}$ is a line bundle on X, and we have $\mathcal{I}_{X_i}/\mathcal{I}_{X_{i+1}} = L^i$. We call L the line bundle on X *associated* to Y. The ideal sheaf $\mathcal{I}_X$ can be viewed as a line bundle on X_{n-1}. If $n = 2$, Y is called a *primitive double scheme*.

The simplest case is when Y is contained in a smooth variety S of dimension $\dim(X) + 1$. Suppose that Y has multiplicity n. Let $P \in X$ and $f \in \mathcal{O}_{S,P}$ a local equation of X. Then we have $\mathcal{I}_{X_i,P} = (f^i)$ for $1 < j \leq n$ in S, in particular $\mathcal{I}_{Y,P} = (f^n)$, and $L = \mathcal{O}_X(-X)$.

For any $L \in \mathrm{Pic}(X)$, the *trivial primitive variety* of multiplicity n, with induced smooth variety X and associated line bundle L on X is the n-th infinitesimal neighborhood of X, embedded by the zero section in the dual bundle L^*, seen as a smooth variety.

The primitive multiple curves where defined in [18], [2]. Primitive double curves were parametrized and studied in [4] and [17]. More results on primitive multiple curves can be found in [8], [9], [10], [11], [12], [13], [14], [15], [5], [27], [28], [29]. Some primitive double schemes are studied in [3], [20] and [22]. The case of varieties of any dimension is studied in [16], where the following subjects were treated:

- construction and parametrization of primitive multiple schemes,
- obstructions to the extension of a vector bundle on X_m to X_{m+1},
- obstructions to the extension of a primitive multiple scheme of multiplicity n to one of multiplicity $n + 1$.

The main subject of this paper is the construction and properties of moduli spaces of vector bundles on X_n. If X_{n+1} is an extension of X_n to a primitive multiple scheme of multiplicity $n + 1$, and $\mathbf{M}$ is a moduli space of vector bundles on X_n, we will see how to construct a moduli space $\mathbf{M}'$ for the vector bundles on X_{n+1} whose restriction to X_n belongs to $\mathbf{M}$. It is an affine bundle over the closed subvariety $\mathbf{N} \subset \mathbf{M}$ of bundles that can be extended to X_{n+1}. This applies in particular to Picard groups.

1.1. Fine moduli spaces

Let χ be a set of isomorphism classes of vector bundles on a scheme Z. Suppose that χ is *open*, i.e. for every scheme V and vector bundle $\mathbb{E}$ on $Z \times V$, if $v \in V$ is a closed point such that $\mathbb{E}_v \in \chi$, then there exists an open neighbourhood U of v such that $\mathbb{E}_u \in \chi$ for every closed point $u \in U$. A *fine moduli space for* χ is the data of a scheme M and of

– a bijection

$$\begin{aligned} M^0 &\longrightarrow \chi \\ m &\longmapsto E_m \end{aligned}$$

(where M^0 denotes the set of closed points of M),

– an open cover $(M_i)_{i\in I}$ of M, and for every $i \in I$, a vector bundle $\mathcal{E}_i$ on $X \times M_i$ such that for every $m \in M_i$, $\mathcal{E}_{i,m} \simeq E_m$,

such that: for any scheme S, any vector bundle $\mathcal{F}$ on $Z \times S$ such that for any closed point $s \in S$, $\mathcal{F}_s \in \chi$, there is a morphism $f_{\mathcal{F}} : S \to M$ such that for every $s \in S$, if $m = f_{\mathcal{F}}(s)$ then $\mathcal{F}_s \simeq E_m$, and if $m \in M_i$, then there exists an open neighbourhood U of s such that $f_{\mathcal{F}}(U) \subset M_i$ and $(I_X \times f_{\mathcal{F}|U})^*(\mathcal{E}_i) \simeq \mathcal{F}_{|X\times U}$.

1.1.1. *Extensions of a vector bundle to higher multiplicity* – Let $\mathbb{E}$ be a vector bundle on X_n and $E = \mathbb{E}_{|X}$. In [16], 7.1, a class $\Delta(\mathbb{E}) \in \operatorname{Ext}^2_{\mathcal{O}_X}(E, E \otimes L^n)$ is defined, such that $\mathbb{E}$ can be extended to a vector bundle on X_{n+1} if and only if $\Delta(\mathbb{E}) = 0$.

If $\mathcal{E}$ is a vector bundle on X_{n+1} such that $\mathcal{E}_{|X_n} = \mathbb{E}$, the kernel of the restriction morphism $\mathcal{E} \to \mathbb{E}$ is isomorphic to $E \otimes L^n$, and we have an exact sequence on X_{n+1}

$$0 \longrightarrow E \otimes L^n \longrightarrow \mathcal{E} \longrightarrow \mathbb{E} \longrightarrow 0 \ . \tag{1}$$

Hence if we want to extend $\mathbb{E}$ to X_{n+1}, it is natural to study $\operatorname{Ext}^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E}, E \otimes L^n)$. Using the local-to-global Ext spectral sequence (cf. [21], Corollary of Theorem 7.3.3), and the fact that $\mathcal{E}xt^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E}, E \otimes L^n) \simeq \mathcal{E}nd(E)$, we obtain a canonical exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_X}(E, E \otimes L^n) \xrightarrow{\psi} \operatorname{Ext}^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E}, E \otimes L^n) \xrightarrow{\phi} \operatorname{End}(E) \xrightarrow{\delta_{\mathbb{E}}} \operatorname{Ext}^2_{\mathcal{O}_X}(E, E \otimes L^n) \ ,$$

and we have, from [16], $\Delta(\mathbb{E}) = \delta_{\mathbb{E}}(I_E)$. In [16] is also given the following description of $\Delta(\mathbb{E})$: we have a canonical exact sequence of sheaves on X_n: $0 \to L^n \to \Omega_{X_{n+1}|X_n} \to \Omega_{X_n} \to 0$, associated to $\sigma_{\Omega_{X_{n+1}}} \in \operatorname{Ext}^1_{\mathcal{O}_{X_n}}(\Omega_{X_n}, L^n)$, inducing $\overline{\sigma}_{\mathbb{E},X_{n+1}} \in \operatorname{Ext}^1_{\mathcal{O}_{X_n}}(\mathbb{E} \otimes \Omega_{X_n}, \mathbb{E} \otimes L^n)$. We have then

$$\Delta(\mathbb{E}) \ = \ \overline{\sigma}_{\mathbb{E},X_{n+1}} \nabla_0(\mathbb{E}) \ , \tag{2}$$

where $\nabla_0(\mathbb{E}) \in \operatorname{Ext}^1_{\mathcal{O}_{X_n}}(\mathbb{E}, \mathbb{E} \otimes \Omega_{X_n})$ is the *fundamental class* of $\mathbb{E}$.

Let $\mathbb{E}$ be a vector bundle on X_n. Then we have $H^0(\mathcal{O}_{X_n}) \subset \operatorname{End}(\mathbb{E})$. We say that $\mathbb{E}$ is *simple* if $\operatorname{End}(\mathbb{E}) = H^0(\mathcal{O}_{X_n})$.

Let X_n be a primitive multiple scheme of support X and multiplicity n, that can be extended to a primitive multiple scheme X_{n+1} of multiplicity $n+1$. Suppose that we have a nonempty open set χ_n of isomorphism classes of simple vector bundles on X_n such that there is a fine moduli space M_n for χ_n. Then there exists a closed subvariety $N_n \subset M_n$ such that for every closed point $t \in N_n$, the corresponding bundle can be extended to X_{n+1}, and that for every scheme T and every vector bundle $\mathbb{E}$ on $X_{n+1} \times T$ such that for every closed point $t \in T$, $\mathbb{E}_{t|X_n} \in \chi_n$, then if $\mathcal{F} = \mathbb{E}_{|X_n \times T}$ and $f_{\mathcal{F}} : T \to M_n$ is the associated morphism, we have $f_{\mathcal{F}}(T) \subset N_n$.

Let χ_{n+1} be the set of isomorphism classes of vector bundles E on X_{n+1} such that $E_{|X_n} \in \chi_n$. We will show that with suitable hypotheses, there is also a fine moduli space M_{n+1} for χ_{n+1}, and that the restriction morphism $M_{n+1} \to N_n$ is an affine bundle.

1.1.2. *Construction of fine moduli spaces* – Let χ_n be an open set of isomorphism classes of vector bundles on X_n. Suppose that there is a fine moduli space for χ_n, defined by a smooth irreducible variety M_n, and that the subvariety $N_n \subset M_n$ of bundles that can be extended to X_{n+1} is smooth (this variety is defined in 6.5). We suppose that for every $\mathcal{E} \in \chi_n$, if $E = \mathcal{E}_{|X}$,

(i) $\mathcal{E}$ and E are simple,
(ii) $\dim(\mathrm{Ext}^1_{\mathcal{O}_X}(E, E \otimes L^n))$ and $\dim(\mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\mathcal{E}, \mathcal{E} \otimes \Omega_{X_n}))$ are independent of $\mathcal{E}$,
(iii) $\dim(\mathrm{Ext}^2_{\mathcal{O}_X}(E, E \otimes L^n))$ is independent of $\mathcal{E}$.
(iv) $\Delta(\mathcal{E}) = 0$.
(v) every vector bundle $\mathcal{F}$ on X_{n+1} such that $\mathcal{F}_{|X_n} \simeq \mathcal{E}$ is simple.

Let χ_{n+1} be the set of isomorphism classes of vector bundles $\mathbb{E}$ on X_{n+1} such that $\mathbb{E}_{|X_n} \in \chi_n$. We have then (cf. 6)

1.1.3. Theorem: *The set χ_{n+1} is open, and there is a fine moduli space M_{n+1} for χ_{n+1}, which is an affine bundle over N_n.*

We have an exact sequence $0 \to L^n \to \mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n} \to 0$, whence a connecting homomorphism $\delta^0 : H^0(X_n, \mathcal{O}_{X_n}) \to H^1(X, L^n)$. Let $\mathcal{E} \in N_n$, $E = \mathcal{E}_{|X}$. Since $E^* \otimes E \otimes L^n \simeq L^n \oplus (\mathcal{A}d(E) \otimes L^n)$ (where $\mathcal{A}d(E)$ is the subbundle of $E^* \otimes E$ of trace zero endomorphisms), we can see $\mathrm{im}(\delta^0)$ as a subspace of $\mathrm{Ext}^1_{\mathcal{O}_X}(E, E \otimes L^n)$. Let $\mathbf{A}$ be the vector bundle on N_n associated to the affine bundle $M_{n+1} \to N_n$. Then we have a canonical isomorphism

$$\mathbf{A}_{\mathcal{E}} \ \simeq \ \mathrm{Ext}^1_{\mathcal{O}_X}(E, E \otimes L^n)/\, \mathrm{im}(\delta_0) \ .$$

Suppose that for some open subset $S \in M_n$, there is a universal vector bundle $\mathbb{F}$ on $X_n \times S$. Let S' be the inverse image of S in M_{n+1}. We have then an obstruction class $\Delta(\mathbb{F}) \in H^2(X_n \times S, \mathcal{H}om(\mathbb{F}, E \otimes p_X^*(L^n)))$ such that $\Delta(\mathbb{F}) = 0$ if and only if $\mathbb{F}$ can be extended to a vector bundle on $X_{n+1} \times S$. Then we have

$$\mathbf{A}_{|S} \ = \ R^1 p_{S*}(\mathcal{H}om(\mathbb{F}, E \otimes p_X^*(L^n))/(\mathcal{O}_S \otimes \mathrm{im}(\delta_0)) \ .$$

Suppose that $p_{S*}(\mathcal{H}om(\mathbb{F}, E \otimes p_X^*(L^n))) = 0$ and $\delta^0 = 0$. We have an injective canonical map

$$\lambda : H^1(S, R^1 p_{S*}(\mathcal{H}om(\mathbb{F}, E \otimes p_X^*(L^n)))) \longrightarrow H^2(X_n \times S, \mathcal{H}om(\mathbb{F}, E \otimes p_X^*(L^n)))$$

coming from the Leray spectral sequence. Since $\Delta(\mathbb{F}_s) = 0$ for every $s \in S$, we will see that

$$\Delta(\mathbb{F}) \ \in \ H^1(S, R^1 p_{S*}(\mathcal{H}om(\mathbb{F}, E \otimes p_X^*(L^n)))) \ = \ H^1(\mathbf{A}_{|S}) \ .$$

Suppose that $\eta \in H^1(\mathbf{A}_{|S})$ corresponds to the affine bundle $S' \to S$. Then (cf. 6.2.10)

1.1.4. Theorem: *We have* $\mathbb{C}.\Delta(\mathbb{F}) = \mathbb{C}\eta$.

In other words there is a link between the possibility to extend a whole family in higher multiplicity and the structure of the affine bundle. The affine bundle is actually a vector bundle if and only if the family can be extended in multiplicity $n + 1$.

The hypothesis (v) is satisfied if L is a non trivial ideal sheaf and if the restrictions to X of the bundles of χ_n are simple (Proposition 4.8.2).

The assumption on N_n is satisfied in the the following cases:

- The case of Picard groups (cf. 1.2), i.e. χ_n consists of line bundles.
- When $N_n = M_n$, in particular if X is a curve, because in this case, all the obstructions to extend vector bundles on X_n to X_{n+1} vanish.

1.2. Picard groups

Let $\mathbf{P}$ be an irreducible component of $\mathrm{Pic}(X)$ and $\mathbf{P}_0$ the component that contains $\mathcal{O}_X$. Let $\mathcal{L}_{n,\mathbf{P}}$ be the set of line bundles $\mathbb{L}$ on X_n such that $\mathbb{L}_{|X} \in \mathbf{P}$. We prove by induction on n that if $\mathcal{L}_{n,\mathbf{P}}$ is not empty, then there is a fine moduli space $\mathrm{Pic}^{\mathbf{P}}(X_n)$ for $\mathcal{L}_{n,\mathbf{P}}$.

Suppose that some $\mathcal{B} \in \mathcal{L}_{n,\mathbf{P}}$ can be extended to X_{n+1}. Then the product with $\mathcal{B}$ defines an isomorphism $\eta_{\mathcal{B}} : \mathrm{Pic}^{\mathbf{P}_0}(X_n) \to \mathrm{Pic}^{\mathbf{P}}(X_n)$. The set of line bundles in $\mathcal{L}_{n,\mathbf{P}}$ that can be extended to X_{n+1} is a smooth closed subvariety

$$\Gamma^{\mathbf{P}}(X_{n+1}) \ \subset \ \mathrm{Pic}^{\mathbf{P}}(X_n) \ .$$

We have $\Gamma^{\mathbf{P}}(X_{n+1}) = \eta_{\mathcal{B}}\big(\Gamma^{\mathbf{P}_0}(X_{n+1})\big)$, and $\Gamma^{\mathbf{P}_0}(X_{n+1})$ is a subgroup of $\mathrm{Pic}^{\mathbf{P}_0}(X_n)$: it is the kernel of the morphism of groups

$$\Delta : \mathrm{Pic}^{\mathbf{P}_0}(X_n) \to H^2(X, L^n) \ ,$$

(the *obstruction morphism*). The case $n = 1$ is simpler: since the fundamental class of a line bundle on X is a discrete invariant, we have either $\Gamma^{\mathbf{P}}(X_2) = \emptyset$ or $\Gamma^{\mathbf{P}}(X_2) = \mathrm{Pic}^{\mathbf{P}}(X)$. We have (cf. 7.1.2)

1.2.1. Theorem: *Suppose that $\mathcal{L}_{n+1,\mathbf{P}}$ is nonempty. Then $\mathrm{Pic}^{\mathbf{P}}(X_{n+1})$ is an affine bundle over $\Gamma^{\mathbf{P}}(X_{n+1})$ with associated vector bundle $\mathcal{O}_{\Gamma^{\mathbf{P}}(X_{n+1})} \otimes \big(H^1(X, L^n)/\operatorname{im}(\delta^0)\big)$.*

1.3. The case of curves

Suppose that X is a curve. In this case the obstructions to extend a primitive multiple curve or a vector bundle to higher multiplicity vanish (from [16] they lie in the H^2 of some vector bundles on X).

Suppose that $\deg(L) < 0$. Let r, d be integers such that $r > 0$ and r, d are coprime. We take for M_n the set of isomorphism classes of vector bundles $\mathcal{E}$ on X_n such that $\mathcal{E}_{|X}$ is stable of rank r and degree d. The hypotheses of 1.1.2 are satisfied, and we obtain by induction on n

1.3.1. Theorem: *There is a fine moduli space $M_{X_n}(r,d)$ for M_n, which is a smooth irreducible variety. It is an affine bundle over $M_{X_{n-1}}(r,d)$.*

Of course for $n = 1$, $M_{X_n}(r,d)$ is the well known moduli space of stable vector bundles of rank r and degree d on X. From [11], the vector bundles of $M_{X_n}(r,d)$, $n \geq 2$, are also stable. Hence $M_{X_n}(r,d)$ is an open subset of a moduli space of stable sheaves on X_n.

Let $\mathbb{E}$ be a Poincaré bundle on $X \times M_X(r,d)$. Let $p_X : X \times M_X(r,d) \to X$, $p_M : X \times M_X(r,d) \to M_X(r,d)$ be the projections. Then if $n \geq 2$,

$$\mathbf{A}_{n-1} \ = \ R^1 p_{M*}(\mathbb{E} \otimes \mathbb{E}^* \otimes p_X^*(L^{n-1}))$$

is a vector bundle on $M_X(r,d)$. Let $p_{n-1} : M_{X_{n-1}}(r,d) \to M_X(r,d)$ be the restriction morphism. Then $p_{n-1}^*(\mathbf{A}_{n-1})$ is the vector bundle on $M_{X_{n-1}}(r,d)$ associated to the affine bundle $M_{X_n}(r,d) \to M_{X_{n-1}}(r,d)$.

1.3.2. *The case of Picard groups* – We don't require here that $\deg(L) < 0$. The irreducible components of $\mathrm{Pic}(X)$ are the smooth projective varieties $\mathbf{P}_d = \mathrm{Pic}^d(X)$ of line bundles of degree d.

Then if follows from 1.2 that

1.3.3. Theorem: *The irreducible components of* $\mathrm{Pic}(X_n)$ *are the* $\mathrm{Pic}^{\mathbf{P}_d}(X_n)$, $d \in \mathbb{Z}$, *and* $\mathrm{Pic}^{\mathbf{P}_d}(X_{n+1})$ *is an affine bundle over* $\mathrm{Pic}^{\mathbf{P}_d}(X_n)$, *with associated vector bundle* $\mathcal{O}_{\mathrm{Pic}^{\mathbf{P}_d}(X_n)} \otimes H^1(L^n)$.

These affine bundles need not to be banal: (cf. 9.2.1)

1.3.4. Theorem : $\mathrm{Pic}^d(X_2)$ *is not a vector bundle over* $\mathrm{Pic}^d(X)$ *if* Y *is not trivial and either* X *is not hyperelliptic and* $\deg(L) \leq 2-2g$, *or* $L = \omega_X$.

Using some parts of the proof of Theorem 1.3.4 we obtain

1.3.5. Theorem : *Suppose that* C *is not hyperelliptic,* X_2 *not trivial and* $\deg(L) \leq 2-2g$. *Then* $\mathbf{M}_{X_2}(r,-1)$ *is not a vector bundle over* $M_X(r,-1)$.

1.4. Primitive double schemes with trivial canonical line bundle

Since X_n is a locally complete intersection, it has a *dualizing sheaf* ω_{X_n} which is a line bundle on X_n. We call also ω_{X_n} the canonical line bundle of X_n. We have

$$\omega_{X_n|X} \ \simeq \ \omega_X \otimes L^{1-n} \ .$$

In particular we have $\omega_{X_2|X} \simeq \omega_X \otimes L^{-1}$. We have an exact sequence

$$0 \longrightarrow L \longrightarrow \mathcal{O}_{X_2} \xrightarrow{\ \zeta\ } \mathcal{O}_X \longrightarrow 0 \ .$$

We will prove (cf. 5.3.1)

1.4.1. Theorem: *The map* $H^1(\zeta) : H^1(X_2, \mathcal{O}_{X_2}) \to H^1(X, \mathcal{O}_X)$ *is surjective.*

which will imply (cf. 5.3.2)

1.4.2. Corollary: *We have* $\omega_{X_2} \simeq \mathcal{O}_{X_2}$ *if and only if* $\omega_X \simeq L$ *.*

If X is a surface, $h^1(X_2, \mathcal{O}_{X_2}) = 0$ and $\omega_{X_2} \simeq \mathcal{O}_{X_2}$, X_2 is called a *K3-carpet* (cf. [3], [20]).

Suppose that $L \simeq \omega_X$. Then according to the exact sequence $0 \to \omega_X \to \mathcal{O}_{X_2} \to \mathcal{O}_X \to 0$, we have $h^1(X_2, \mathcal{O}_{X_2}) = 0$ if and only if $h^1(X, \mathcal{O}_X) = 0$ (cf. also [20], Proposition 1.6).

1.5. Examples

Let C and D be smooth projective irreducible curves, $X = C \times D$ and $\pi_C : X \to C$, $\pi_D : X \to D$ the projections. Let g_C (resp. g_D) be the genus of C (resp. D). Suppose that $g_C \geq 2$, $g_D \geq 2$. Let L_C (resp. L_D) be a line bundle on C (resp. D) and $L = \pi_C^*(L_C) \otimes \pi_D^*(L_D)$. The non trivial double schemes X_2 with associated line bundle L are parametrized by $\mathbb{P}(H^1(X, T_X \otimes L))$, where T_X is the tangent bundle of X (cf. [4] and [16]). We have

$$\begin{aligned} H^1(X, T_X \otimes L) \ = \ & \big(H^0(\omega_C^* \otimes L_C) \otimes H^1(L_D)\big) \oplus \big(H^1(\omega_C^* \otimes L_C) \otimes H^0(L_D)\big) \oplus \\ & \big(H^0(L_C) \otimes H^1(\omega_D^* \otimes L_D)\big) \oplus \big(H^1(L_C) \otimes H^0(\omega_D^* \otimes L_D)\big) \ , \end{aligned}$$

and using this decomposition, the double schemes X_2 are defined by quadruplets $(\eta_1, \eta_2, \eta_3, \eta_4)$.

We denote by ϕ_{L_C}, ϕ_{L_D} the canonical maps $H^0(\omega_C^* \otimes L_C) \otimes H^1(\omega_C) \to H^1(L_C)$ and $H^0(\omega_D^* \otimes L_D) \otimes H^1(\omega_D) \to H^1(L_D)$ respectively.

Let M_C be a line bundle on C (resp. D), and $M = \pi_C^*(M_C) \otimes \pi_D^*(M_D)$. Using (2) we will see that

$$\Delta(M) \ = \ (\phi_{L_C} \otimes I_{H^1(L_D)})(\eta_1 \otimes \nabla_0(M_C)) + (I_{H^1(L_C)} \otimes \phi_{L_D})(\eta_4 \otimes \nabla_0(M_D)) \ .$$

The canonical class of a line bundle on a smooth projective curve is in fact an integer. For every line bundle $\mathcal{L}$ on C we have $\nabla_0(\mathcal{L}) = \deg(\mathcal{L})c$, where $c = \nabla_0(\mathcal{O}_C(P))$, for any $P \in C$.

For example let Θ be a theta characteristic on C such that $h^0(C, \Theta) > 0$. We take $L_C = \Theta$, $L_D = \omega_D$. We have $\eta_1 = 0$. Suppose that $\eta_4 = 0$. Such non trivial double schemes are parametrized by $\mathbb{P}\big((H^1(\Theta^{-1}) \otimes H^0(\omega_D)) \oplus (H^0(\Theta) \otimes H^1(\mathcal{O}_D))\big)$ (pairs (η_2, η_3)). Suppose that $\eta_3 \neq 0$. Then every line bundle on X can be extended to X_2, which is projective, and (using Theorem 1.6.2 below) there exists an extension of X_2 to a primitive multiple scheme X_3 of multiplicity 3, and X_3 is projective.

1.5.1. *The case* $L = \omega_X$ – This happens if and only if $L_C = \omega_C$ and $L_D = \omega_D$. Let $\eta = (\eta_1, \eta_2, \eta_3, \eta_4) \in H^1(X, T_X \otimes \omega_X)$. Here we have

$$\eta_1 \in \mathbb{C} \ , \quad \eta_2 \in H^1(C, \mathcal{O}_C) \otimes H^0(D, \omega_D) \ , \quad \eta_3 \in H^0(C, \omega_C) \otimes H^1(D, \mathcal{O}_D) \ , \quad \eta_4 \in \mathbb{C} \ .$$

Let M_C (resp. M_D) be a line bundle on C (resp. D). If $M = \pi_C^*(M_C) \otimes \pi_D^*(M_D)$, then

$$\Delta(M) \ = \ \eta_1 \nabla_0(M_C) + \eta_4 \nabla_0(M_D) \ ,$$

hence M can be extended to a line bundle on X_2 if and only if $\eta_1 \deg(M_C) + \eta_4 \deg(M_D) = 0$.

It follows that *the scheme* X_2 *is projective if and only if* $\eta_1=\eta_4=0$ *or* $\eta_1\eta_4<0$ *and* $\dfrac{\eta_1}{\eta_4}$ *is rational.* We will see that X_2 can be extended to a primitive multiple scheme of multiplicity 3 if and only ω_X can be extended to a line bundle on X_2. This is the case if and only $(g_C-1)\eta_1+(g_D-1)\eta_4=0$, and then X_2 is projective.

We will see also that there may exist components $\mathbf{P}$ of $\mathrm{Pic}(X)$ such that the affine bundle $\mathrm{Pic}^{\mathbf{P}}(X_2)$ is not empty, and not a vector bundle over $\mathbf{P}$.

1.6. Improvements of results of [16]

1.6.1. *Extensions of* X_n *to multiplicity* $n+1$ – If X_n can be extended to a primitive multiple scheme X_{n+1} of multiplicity $n+1$, then $\mathcal{I}_{X,X_n}$, which is a line bundle on X_{n-1}, can be extended to a line bundle on X_n, namely $\mathcal{I}_{X,X_{n+1}}$. Conversely, given an extension of $\mathcal{I}_{X,X_n}$ to a line bundle $\mathbb{L}$ on X_n, an invariant $\Delta''_n(X_n,\mathbb{L})\in H^2(X,T_X\otimes L^n)$ is defined in [16], such that X_n can be extended to X_{n+1} with $\mathcal{I}_{X,X_{n+1}}\simeq\mathbb{L}$ if and only if $\Delta''_n(X_n,\mathbb{L})=0$.

By 1.1.1, another extension $\mathbb{L}'$ of $\mathcal{I}_{X,X_n}$ to X_{n+1} comes from some $\eta\in H^1(X,L^{n-1})$. Let $\boldsymbol{\zeta}\in H^1(X,T_X\otimes L)$ correspond to the exact sequence $0\to L\to\Omega_{X_2|X}\to\Omega_X\to 0$. We have a canonical product

$$H^1(X,T_X\otimes L)\otimes H^1(X,L^{n-1})\longrightarrow H^2(X,T_X\otimes L^n)$$

and (cf. 4.5.1)

1.6.2. Theorem: *We have* $\Delta''_n(X_n,\mathbb{L}')-\Delta''_n(X_n,\mathbb{L})=\boldsymbol{\zeta}\eta$.

This implies that if the product is a surjective map, then there exists an extension of X_n in multiplicity $n+1$.

1.6.3. *Extensions of a line bundle on* X_n *to* X_{n+1} – Let $\mathbb{L}=\mathcal{I}_{X,X_{n+1}|X_2}$, which is a line bundle on X_2. Let $\delta^1_{\mathbb{L}^{n-1}}$ be the connecting morphism $H^1(X,L^{n-1})\to H^2(X,L^n)$ coming from the exact sequence $0\to L^n\to\mathbb{L}^{n-1}\to L^{n-1}\to 0$.

Suppose that X_n can be extended to X_{n+1}, and let $\mathbb{D}$ be a line bundle on X_n. By 1.1.1, $\mathbb{D}$ can be extended to a line bundle on X_{n+1} if and only if $\Delta(\mathbb{D})=0$. Let $\mathbb{D}'\in\mathrm{Pic}(X_n)$ be such that $\mathbb{D}'_{|X_{n-1}}\simeq\mathbb{D}_{|X_{n-1}}$. If $\mathbb{E}=\mathbb{D}_{|X_{n-1}}$, $E=\mathbb{D}_{|X}$, we have exact sequences

$$0\to E\otimes L^{n-1}\to\mathbb{D}\to\mathbb{E}\to 0\ ,\qquad 0\to E\otimes L^{n-1}\to\mathbb{D}'\to\mathbb{E}\to 0\ ,$$

corresponding to $\eta,\eta'\in\mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\mathbb{E},E\otimes L^{n-1})=H^1(X,L^{n-1})$ respectively. Then (cf. 7.2.1)

1.6.4. Theorem: *We have* $\Delta(\mathbb{D}')-\Delta(\mathbb{D})=n\delta^1_{\mathbb{L}^{n-1}}(\eta'-\eta)$.

Theorem 1.6.4 implies for Picard groups (cf. 1.2, 7.2.2)

1.6.5. Proposition: *Let Z be the image of he restriction morphism* $\Gamma^{\mathbf{P}}(X_{n+1}) \to \Gamma^{\mathbf{P}}(X_n)$. *Then Z is smooth and $\Gamma^{\mathbf{P}}(X_{n+1})$ is an affine bundle over Z with associated vector bundle* $\mathcal{O}_Z \otimes \ker(\delta^1_{\mathbb{L}^{n-1}})$.

1.7. Outline of the paper

In Chapter 2, we give several ways to use Čech cohomology, and recall some basic definitions about fine moduli spaces and affine bundles.

In the very technical Chapter 3, we give a description of Leray's spectral sequence and the exact sequence of Ext's in terms on Čech cohomology (which is our main tool).

The Chapter 4 is devoted to the definitions and properties of primitive multiple schemes, with the improvement 1.6.1 of [16]. It contains also supplementary results on vector bundles on primitive multiple schemes, that are used in the following chapters.

In Chapter 5 the definition and properties of the fundamental class of a vector bundles are recalled. We also give here the results of 1.4.

Chapter 6 contains the proofs of the main results (the construction and properties of fine moduli spaces of vector bundles on primitive multiple schemes).

In Chapter 7 we treat the special case of Picard groups (cf. 1.2), and prove the improvement 1.6.3 of [16].

In Chapter 8 we give the examples 1.5, when X is a product of curves.

Chapter 9 is devoted to moduli spaces of vector bundles on primitive multiple curves.

Notations and terminology: – An *algebraic variety* is a quasi-projective scheme over $\mathbb{C}$. A *vector bundle* on a scheme is an algebraic vector bundle.

– If X is a scheme and $P \in X$ is a closed point, we denote by $m_{X,P}$ (or m_P) the maximal ideal of P in $\mathcal{O}_{X,P}$.

– If X is a scheme and $Z \subset X$ is a closed subscheme, $\mathcal{I}_Z$ (or $\mathcal{I}_{Z,X}$) denotes the ideal sheaf of Z in X.

– If V is a finite dimensional complex vector space, $\mathbb{P}(V)$ denotes the projective space of lines in V, and we use a similar notation for projective bundles.

– Let X be a set and $(X_i)_{i\in I}$ a family of subsets of X. If $i, j \in I$, X_{ij} denotes the intersection $X_i \cap X_j$, and similarly for more indices.

– If X and Y are two schemes and $f : X \to Y$ is a morphism, $\mathbf{d}(f) : f^*(\Omega_Y) \to \Omega_X$ is the corresponding canonical morphism.

– If R is a commutative ring, n a positive integer, $\mathbf{a} \in R[t]/(t^n)$ and i an integer such that $0 \leq i < n$, let $\mathbf{a}^{(i)}$ denote the coefficient of t^i on $\mathbf{a}$, so that we have $\mathbf{a} = \displaystyle\sum_{i=0}^{n-1} \mathbf{a}^{(i)} t^i$.

2. Preliminaries

2.1. Čech cohomology

Let X be a scheme over $\mathbb{C}$ and E, V vector bundles on X of rank r. Let $(U_i)_{i\in I}$ be an open cover of X such that we have isomorphisms:

$$\alpha_i : E_{|U_i} \xrightarrow{\simeq} V_{|U_i}.$$

Let

$$\alpha_{ij} = \alpha_i \circ \alpha_j^{-1} : V_{|U_{ij}} \xrightarrow{\simeq} V_{|U_{ij}},$$

so that we have the relation $\alpha_{ij}\alpha_{jk} = \alpha_{ik}$ on U_{ijk}.

2.1.1. Let n be a positive integer, and for every sequence $(i_0, \ldots, i_n)$ of distinct elements of I, $\sigma_{i_0\cdots i_n} \in H^0(U_{i_0\cdots i_n}, V)$. Let

$$\theta_{i_0\cdots i_n} \;=\; \alpha_{i_0}^{-1}\sigma_{i_0\cdots i_n} \;\in\; H^0(U_{i_0\cdots i_n}, E)\ .$$

The family $(\theta_{i_0\cdots i_n})$ represents an element of $H^n(X, E)$ if the cocycle relations are satisfied: for every sequence $(i_0, \ldots, i_{n+1})$ of distinct elements of I,

$$\sum_{k=0}^{n}(-1)^k\theta_{i_0\cdots\widehat{i_k}\cdots i_{n+1}} \;=\; 0\ ,$$

which is equivalent to

$$\alpha_{i_0i_1}\sigma_{i_1\cdots i_{n+1}} + \sum_{k=1}^{n+1}(-1)^k\sigma_{i_0\cdots\widehat{i_k}\cdots i_{n+1}} \;=\; 0\ .$$

For $n = 1$, this gives that elements of $H^1(X, E)$ are represented by families (σ_{ij}), $\sigma_{ij} \in H^0(U_{ij}, V)$, such that

$$\alpha_{ij}\sigma_{jk} + \sigma_{ij} - \sigma_{ik} \;=\; 0$$

(instead of $(\alpha_i^{-1}\sigma_{ij})$ with the usual cocycle relations). In Čech cohomology, it is generally assumed that $\theta_{ji} = -\theta_{ij}$. This implies that $\sigma_{ji} = -\alpha_{ji}\sigma_{ij}$.

2.1.2. Let F be a vector bundle on X. Similarly, an element of $H^n(X, E\otimes F)$ is represented by a family $(\mu_{i_0\cdots i_n})$, with $\mu_{i_0\cdots i_n} \in H^0(U_{i_0\cdots i_n}, V\otimes F)$, satisfying the relations

$$(\alpha_{i_0i_1}\otimes I_F)(\mu_{i_1\cdots i_{n+1}}) + \sum_{k=1}^{n+1}(-1)^k\mu_{i_0\cdots\widehat{i_k}\cdots i_{n+1}} \;=\; 0\ .$$

The corresponding element of $H^0(U_{i_0\cdots i_n}, E\otimes F)$ is $\theta_{i_0\cdots i_n} = (\alpha_{i_0}\otimes I_F)^{-1}(\mu_{i_0\cdots i_n})$.

2.1.3. *Coboundaries* – In the situation of 2.1.2, the family $(\theta_{i_0\cdots i_n})$ represents 0 in $H^n(X, E\otimes F)$ if and only if there exists a family $(\tau_{j_0\cdots j_{n-1}})$, with $\tau_{j_0\cdots j_{n-1}} \in H^0(U_{j_0\cdots j_{n-1}}, E\otimes F)$, such that

$$\theta_{i_0\cdots i_n} \;=\; \sum_{k=0}^{n}(-1)^k\tau_{i_0\cdots\widehat{i_k}\cdots i_n}\ .$$

Similarly, the family $(\mu_{i_0\cdots i_n})$ represents 0 if and only there exists a family $(\nu_{j_0\cdots j_{n-1}})$, with $\nu_{j_0\cdots j_{n-1}} \in H^0(U_{j_0\cdots j_{n-1}}, V\otimes F)$, such that

$$\mu_{i_0\cdots i_n} \;=\; (\alpha_{i_0i_1}\otimes I_F)(\nu_{i_1\cdots i_n}) + \sum_{k=1}^{n}(-1)^k \nu_{i_0\cdots \widehat{i_k}\cdots i_n} \; .$$

2.1.4. *Representation of morphisms* – Suppose that we have also local trivializations of F:

$$\beta_i : F_{|U_i} \xrightarrow{\simeq} \mathcal{O}_{U_i}\otimes \mathbb{C}^s.$$

We have then local trivializations of $\mathcal{H}om(E,F)$

$$\begin{aligned} \Delta_i : \mathcal{H}om(E,F)_{|U_i} &\xrightarrow{\simeq} \mathcal{O}_{U_i}\otimes L(\mathbb{C}^r,\mathbb{C}^s) \\ \phi &\longmapsto \beta_i\circ\phi\circ\alpha_i^{-1} \end{aligned}$$

such that, for every $\lambda\in H^0(\mathcal{O}_{U_{ij}}\otimes L(\mathbb{C}^r,\mathbb{C}^s))$, we have

$$\Delta_{ij}(\lambda) \;=\; \Delta_i\Delta_j^{-1}(\lambda) \;=\; \beta_{ij}\lambda\alpha_{ij}^{-1} \; .$$

2.1.5. *Products* – Let $\chi_{E,F} : H^1(X,E)\otimes H^1(X,F)\to H^2(X,E\otimes F)$ be the canonical map. Let $\epsilon\in H^1(X,E)$, $\phi\in H^1(X,F)$, represented by cocycles (ϵ_{ij}), (ϕ_{ij}), $\epsilon_{ij}\in H^0(U_{ij},E)$, $\phi_{ij}\in H^0(U_{ij},F)$. Then $\chi_{E,F}(\epsilon\otimes\phi)$ is represented by the cocycle $(\epsilon_{ij}\otimes\phi_{jk})$.

With respect to the canonical isomorphism $F\otimes E\simeq E\otimes F$ we have $\chi_{F,E}(\phi\otimes\epsilon) = -\chi_{E,F}(\epsilon\otimes\phi)$.

Suppose that we have vector bundles V, W on X with isomorphisms

$$\alpha_i : E_{|U_i}\longrightarrow V_{|U_i}\ , \qquad \beta_i : F_{|U_i}\longrightarrow W_{|U_i}\ .$$

Let $e_{ij} = \alpha_i\epsilon_{ij}$, $f_{ij} = \beta_i\phi_{ij}$. We have then

$$\epsilon_{ij}\otimes\phi_{jk} \;=\; (\alpha_i^{-1}\otimes\beta_j^{-1})(e_{ij}\otimes f_{jk})\ ,$$

and

$$(\alpha_i\otimes\beta_i)(\epsilon_{ij}\otimes\phi_{jk}) \;=\; (I\otimes\beta_{ij})(e_{ij}\otimes f_{jk})\ .$$

It follows that the family $(e_{ij}\otimes\beta_{ij}f_{jk})$ represents $\chi_{E,F}(\epsilon\otimes\phi)$ in the sense of 2.1.1, with respect to the local isomorphisms $\alpha_i\otimes\beta_i : (E\otimes F)_{|U_i}\longrightarrow (V\otimes W)_{|U_i}$.

2.1.6. *Construction of vector bundles via local isomorphisms* – Let Z be a scheme over $\mathbb{C}$, $(Z_i)_{i\in I}$ an open cover of Z, and for every $i\in I$, a scheme U_i, with an isomorphism $\delta_i : Z_i\to U_i$. Let $\delta_{ij} = \delta_j\delta_i^{-1} : \delta_i(Z_{ij})\to\delta_j(Z_{ij})$, which is an isomorphism. Then a vector bundle of X can be constructed (in an obvious way) using vector bundles E_i on U_i and isomorphisms

$$\Theta_{ij} : \delta_{ij}^*(E_{j|\delta_j(Z_{ij})})\longrightarrow E_{i|\delta_i(Z_{ij})}$$

such that $\Theta_{ij}\circ\delta_{ij}^*(\Theta_{jk}) = \Theta_{ik}$ on $\delta_i(Z_{ijk})$.

2.1.7. *Representation of extensions* – Let E, F be coherent sheaves on X, and $\sigma \in H^1(X, \mathcal{H}om(F,E)) \subset \mathrm{Ext}^1_{\mathcal{O}_X}(F,E)$. Let

$$0 \longrightarrow E \longrightarrow \mathcal{E} \longrightarrow F \longrightarrow 0$$

be the corresponding exact sequence. Suppose that σ is represented by a cocycle (σ_{ij}), $\sigma_{ij} : F_{|U_{ij}} \to E_{|U_{ij}}$. Then $\mathcal{E}$ can be constructed by gluing the sheaves $(E \oplus F)_{|U_i}$ using the automorphisms of $(E \oplus F)_{|U_{ij}}$ defined by the matrices $\begin{pmatrix} I_E & \sigma_{ij} \\ 0 & I_F \end{pmatrix}$.

2.2. Moduli spaces of sheaves

(cf. [7])

Let X be a scheme over $\mathbb{C}$.

2.2.1. *Families and sets of vector bundles* – Let S be a scheme over $\mathbb{C}$.

A *family of vector bundles on X parametrized by S* is a vector bundle on $X \times S$. If $f : T \to S$ is a morphism of schemes and $\mathcal{E}$ a family of vector bundles on X parametrized by S, we note

$$f^\sharp(\mathcal{E}) \ = \ (I_X \times f)^*(\mathcal{E}) \ ,$$

which is a family of vector bundles on X parametrized by T.

Let χ be a nonempty set of isomorphism classes of vector bundles on X. A *family of vector bundles of χ parametrized by χ* is a family of vector bundles $\mathcal{E}$ on X parametrized by S such that for every closed point $s \in S$, $\mathcal{E}_s$ belongs to χ.

We say that χ is *open* if for every S and every family of vector bundles $\mathcal{E}$ on X parametrized by S, if $s \in S$ is a closed point such that $\mathcal{E}_s \in \chi$, then there exists a neighbourhood U of s such that $\mathcal{E}_u \in \chi$ for every closed point $u \in U$.

2.2.2. *Fine moduli spaces* – Let χ be a nonempty open set of isomorphism classes of vector bundles on X. A *fine moduli space for χ* is the data of a scheme M and of

– a bijection

$$\begin{aligned} M^0 &\longrightarrow \chi \\ m &\longmapsto E_m \end{aligned}$$

(where M^0 denotes the set of closed points of M),

– An open cover $(M_i)_{i \in I}$ of M, and for every $i \in I$, a vector bundle $\mathcal{E}_i$ on $X \times M_i$ such that for every $m \in M_i$, $\mathcal{E}_{i,m} \simeq E_m$,

such that for any scheme S, any family $\mathcal{F}$ of vector bundles of χ parametrized by S, there exists a morphism $f_{\mathcal{F}} : S \to M$ such that: for every closed point $s \in S$, if $\mathcal{F}_s \simeq E_m$, with $m \in M_i$, then there exists an open neighbourhood U of s such that $f_{\mathcal{F}}(U) \subset M_i$ and $f^\sharp_{\mathcal{F}|U}(\mathcal{E}_i) \simeq \mathcal{F}_{|X \times U}$.

If χ is an open set, then for every closed point $m \in M$, if $m \in M_i$, then $\mathcal{E}_i$ is a semi-universal deformation of $\mathcal{E}_{i,m}$, hence the Kodaïra-Spencer map $T_m M_i \to \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}_{i,m}, \mathcal{E}_{i,m})$ is an isomorphism.

Fine moduli spaces are unique.

2.3. Affine bundles

Let $f : \mathcal{A} \to S$ be a morphism of schemes, and $r \geq 0$ an integer. We say that f (or $\mathcal{A}$) is an *affine bundle* of rank r over S if there exists an open cover $(S_i)_{i \in I}$ of S such that for every $i \in I$ there is an isomorphism $\tau_i : f^{-1}(S_i) \to S_i \times \mathbb{C}^r$ over S_i such that for every distinct $i, j \in I$, $f_j \circ f_i^{-1} : S_{ij} \times \mathbb{C}^r \to S_{ij} \times \mathbb{C}^r$ is of the form

$$(x, u) \longmapsto (x, A_{ij}(x)u + b_{ij}(x)) \ ,$$

where A_{ij} is an $r \times r$-matrix of elements of $\mathcal{O}_S(S_{ij})$ and b_{ij} is a morphism from S_{ij} to $\mathbb{C}^r$. We have then the cocycle relations

$$A_{ij} A_{jk} \ = \ A_{ik} \ , \quad b_{ik} \ = \ A_{ij} b_{jk} + b_{ij} \ .$$

The first relation shows that the family (A_{ij}) defines a vector bundle $\mathbb{A}$ on S, and the second that (b_{ij}) defines $\lambda \in H^1(S, \mathbb{A})$ (according to [16], 2.1.1).

The vector bundle $\mathbb{A}$ is uniquely defined, as well as $\eta(\mathcal{A}) = \mathbb{C}\lambda \in \big(\mathbb{P}(H^1(S, \mathbb{A})) \cup \{0\}\big) / \operatorname{Aut}(\mathbb{A})$.

We say that $\mathcal{A}$ *is a vector bundle* if f has a section. This is the case if and only $\lambda = 0$, and then a section of f induces an isomorphism $\mathcal{A} \simeq \mathbb{A}$ over S.

For every closed point $s \in S$ there is a canonical action of the additive group $\mathbb{A}_s$ on $\mathcal{A}_s$

$$\begin{aligned} \mathbb{A}_s \times \mathcal{A}_s &\longrightarrow \mathcal{A}_s \\ (u, a) &\longmapsto a + u \end{aligned}$$

such that for every $a \in \mathcal{A}_s$, $u \mapsto a + u$ is an isomorphism $\mathbb{A}_s \simeq \mathcal{A}_s$.

2.3.1. *Quotients* – Let $\mathbb{B} \subset \mathbb{A}$ be a subbundle of $\mathbb{A}$, $q : \mathbb{A} \to \mathbb{B}/\mathbb{A}$ the quotient morphism and $R : H^1(S, \mathbb{A}) \to H^1(S, \mathbb{A}/\mathbb{B})$ the induced morphism. Let $\lambda \in H^1(S, \mathcal{A})$ and $\lambda'' = R(\lambda)$. Let $f'' : \mathcal{A}'' \to S$ be the affine bundle corresponding to λ''. We will also use the notation $\mathcal{A}'' = \mathcal{A}/\mathbb{B}$. Then there is an obvious S-morphism $p : \mathcal{A} \to \mathcal{A}''$, such that for every closed point $s \in S$, $a \in \mathcal{A}_s$ and $u \in \mathbb{A}_s$, we have $p_s(a + u) = p_s(a) + q_s(u)$. Moreover p is an affine bundle with associated vector bundle $f''^*(\mathbb{B})$.

3. Canonical exact sequences in Čech cohomology

The Ext spectral sequence allows to compute Ext groups in terms of cohomology groups and Leray's spectral sequence to compute the cohomology groups of a sheaf in terms of the cohomology groups of its direct images by a suitable morphism. We will describe simple examples of consequences of these spectral sequences in terms of Čech cohomology (this provides also elementary proofs of these consequences).

3.1. Leray's spectral sequence

Let $\pi : X \to S$ be a proper flat morphism of sheaves, and $\mathcal{E}$ a coherent sheaf on X such that $\pi_*(\mathcal{E}) = 0$. Leray's spectral sequence implies that we have an exact sequence

$$0 \longrightarrow H^1(S, R^1\pi_*(\mathcal{E})) \xrightarrow{\lambda} H^2(X, \mathcal{E}) \xrightarrow{\eta} H^0(S, R^2\pi_*(\mathcal{E})) \longrightarrow 0 \ .$$

We will describe λ and η in terms of Čech cohomology.

Let $(S_m)_{m\in M}$ (resp. $(X_i)_{i\in I}$) be an affine cover of S (resp. X), such that for every $i \in I$, $m \in M$, $\pi^{-1}(S_m) \cap X_i$ is affine.

3.1.1. *Description of* η – Let $\Sigma \in H^2(X, \mathcal{E})$, represented by the cocycle $(B_{ijk})_{i,j,k\in I}$, $B_{ijk} \in H^0(X_{ijk}, \mathcal{E})$. For every $m \in M$, $(B_{ijk|\pi^{-1}(S_m)\cap X_{ijk}})_{i,j,k\in I}$ defines $\Sigma_m \in H^2(\pi^{-1}(S_m), \mathcal{E})$, and $(\Sigma_m)_{m\in M}$ defines $\eta(\Sigma) \in H^0(S, R^2\pi_*(\mathcal{E}))$.

3.1.2. *Description of* λ – Let $\theta \in H^1(S, R^1\pi_*(\mathcal{E}))$, represented by the cocycle $(A_{mn})_{m,n\in M}$, $A_{mn} \in H^1(\pi^{-1}(S_{mn}), \mathcal{E})$. Suppose that A_{mn} is represented by the cocycle $(\alpha^{mn}_{ij})_{i,j\in I}$, $\alpha^{mn}_{ij} \in H^0(X_{ij} \cap \pi^{-1}(S_{mn}), \mathcal{E})$. The cocycle condition for (A_{mn}) implies that $(\alpha^{mn}_{ij} + \alpha^{np}_{ij} - \alpha^{mp}_{ij})_{i,j\in I}$ represents 0 in $H^1(\pi^{-1}(S_{mnp}), \mathcal{E})$. Hence we can write

$$\alpha^{mn}_{ij} + \alpha^{np}_{ij} - \alpha^{mp}_{ij} \ = \ \beta^{mnp}_i - \beta^{mnp}_j \ ,$$

where $\beta^{mnp}_k \in H^0(\pi^{-1}(S_{mnp}) \cap X_k, \mathcal{E})$ for every $k \in I$. We have, for every distinct $m, n, p, q \in M$, $i, j \in I$

$$\beta^{mnp}_i - \beta^{mnq}_i + \beta^{mpq}_i - \beta^{npq}_i \ = \ \beta^{mnp}_j - \beta^{mnq}_j + \beta^{mpq}_j - \beta^{npq}_j \ .$$

Hence there exists $\Delta^{mnpq} \in H^0(\pi^{-1}(S_{mnpq}), \mathcal{E})$ such that

$$\beta^{mnp}_i - \beta^{mnq}_i + \beta^{mpq}_i - \beta^{npq}_i \ = \ \Delta^{mnpq}_{|\pi^{-1}(S_{mnpq})\cap V_i}$$

for every $i \in I$. Since $\pi_*(\mathcal{E}) = 0$, we have $\Delta^{mnpq} = 0$. Hence for every $i \in I$, $(\beta^{mnp}_i)_{m,n,p\in M}$ satisfies the cocycle condition, and defines an element of $H^2(X_i, \mathcal{E}) = 0$ (because X_i is affine). So we can write, for every distinct $m, n, p \in M$, $i \in I$,

$$\beta^{mnp}_i \ = \ \Gamma^{mn}_i + \Gamma^{np}_i - \Gamma^{mp}_i \ ,$$

with $\Gamma^{mn}_i \in H^0(\pi^{-1}(S_{mn}) \cap X_i, \mathcal{E})$. Let

$$a^{mn}_{ij} \ = \ \alpha^{mn}_{ij} - \Gamma^{mn}_i + \Gamma^{mn}_j \ .$$

Then A_{mn} is also represented by $(a^{mn}_{ij})_{i,j\in I}$. For every distinct $i, j \in I$, we have $a^{mn}_{ij} + a^{np}_{ij} - a^{mp}_{ij} = 0$, hence $(a^{mn}_{ij})_{m,n\in M}$ represents an element of $H^1(X_{ij}, \mathcal{E})$, which is zero (because X_{ij} is affine). So we can write

$$a^{mn}_{ij} \ = \ \mu^m_{ij} - \mu^n_{ij} \ ,$$

$\mu^m_{ij} \in H^0(\pi^{-1}(S_m) \cap X_{ij})$. We have, for every distinct $i, j, k \in I$, $m, n \in M$,

$$\mu^m_{ij} + \mu^m_{jk} - \mu^m_{ik} \ = \ \mu^n_{ij} + \mu^n_{jk} - \mu^n_{ik}$$

on $\pi^{-1}(S_{mn}) \cap X_{ijk}$. Hence there exists $B_{ijk} \in H^0(X_{ijk}, \mathcal{E})$ such that $\mu^m_{ij} + \mu^m_{jk} - \mu^m_{ik} = B_{ijk|\pi^{-1}(S_m)\cap X_{ijk}}$. Then $(B_{ijk})_{i,j,k\in I}$ verifies the cocycle condition, and defines $\lambda(\theta) \in H^2(X, \mathcal{E})$.

With these definitions, it is easy to see that λ is injective, η surjective and $\ker(\eta) = \operatorname{im}(\lambda)$.

3.2. The exact sequence of Ext's

Let X be a scheme and E, F be coherent sheaves on X. From [21], 7.3 (Ext spectral sequence), we have an exact sequence $\mathbf{\Gamma}_{E,F}$

$$0 \to H^1(X, \mathcal{H}om(E,F)) \xrightarrow{\alpha_X} \operatorname{Ext}^1_{\mathcal{O}_X}(E,F) \xrightarrow{\beta_X} H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(E,F)) \xrightarrow{\delta_X} H^2(X, \mathcal{H}om(E,F))$$

We will give a simple description of this exact sequence using Čech cohomology.

3.2.1. *Description of* α_X – Let $(X_i)_{i\in I}$ be an open affine cover of X, and $u \in H^1(X, \mathcal{H}om(E,F))$, represented by the cocycle (ϕ_{ij}) with respect to the cover (X_i), with $\phi_{ij} : E_{|X_{ij}} \to F_{|X_{ij}}$. Let $\mathcal{E}$ be the coherent sheaf on X obtained by gluing the sheaves $(E \oplus F)_{|X_i}$ using the automorphisms of $(E \oplus F)_{|X_{ij}}$ defined by the matrices $\begin{pmatrix} I_E & 0 \\ \phi_{ij} & I_F \end{pmatrix}$. Then we have an obvious exact sequence $0 \to F \to \mathcal{E} \to E \to 0$, whose associated element in $\operatorname{Ext}^1_{\mathcal{O}_X}(E,F)$ is $\alpha_X(u)$.

Let $\sigma \in \operatorname{Ext}^1_{\mathcal{O}_X}(E,F)$, corresponding to the extension

$$0 \longrightarrow F \xrightarrow{\lambda} \mathcal{E} \xrightarrow{\mu} E \longrightarrow 0.$$

Let $u \in H^1(X, \mathcal{H}om(E,F))$, represented by the cocycle (ϕ_{ij}). Let $\sigma' = \sigma + \alpha_X(u)$, corresponding to the extension $0 \to F \to \mathcal{E}' \to E \to 0$. Then $\mathcal{E}'$ is obtained by gluing the sheaves $\mathcal{E}_{|X_i}$ using the automorphisms $I + \lambda \circ \phi_{ij} \circ \mu : \mathcal{E}_{|X_{ij}} \to \mathcal{E}_{|X_{ij}}$.

3.2.2. *Description of* β_X – Let $\sigma \in \operatorname{Ext}^1_{\mathcal{O}_X}(E,F)$, corresponding to an exact sequence $0 \to F \to \mathcal{E} \to E \to 0$, whose restriction to X_i defines $\sigma_i \in \operatorname{Ext}^1_{\mathcal{O}_{X_i}}(E_{|X_i}, F_{|X_i}) = \mathcal{E}xt^1_{\mathcal{O}_X}(E,F)(X_i)$. then $\beta_X(\sigma)$ is the section s of $\mathcal{E}xt^1_{\mathcal{O}_X}(E,F)$ such that for every $i \in I$, $s_{|X_i} = \sigma_i$.

3.2.3. *Description of* δ_X – Let $s \in H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(E,F))$. For every $i \in I$, we have $s_{|X_i} \in \operatorname{Ext}^1_{\mathcal{O}_{X_i}}(E_{|X_i}, F_{|X_i})$, which defines an exact sequence

$$\Sigma_i : 0 \longrightarrow F_{|X_i} \xrightarrow{\gamma_i} \mathcal{E}_i \xrightarrow{p_i} E_{|X_i} \longrightarrow 0\ .$$

The restrictions of Σ_i and Σ_j to X_{ij} define the same element of $\operatorname{Ext}^1_{\mathcal{O}_{X_{ij}}}(E_{|X_{ij}}, F_{|X_{ij}})$, i.e. we have a commutative diagram

$$\begin{tikzcd}
\Sigma_i : \ 0 \arrow[r] & F_{|X_{ij}} \arrow[r, "\gamma_i"] \arrow[d, equal] & \mathcal{E}_{i|X_{ij}} \arrow[r, "p_i"] \arrow[d, "\psi_{ij}"] & E_{|X_{ij}} \arrow[r] \arrow[d, equal] & 0 \\
\Sigma_j : \ 0 \arrow[r] & F_{|X_{ij}} \arrow[r, "\gamma_j"] & \mathcal{E}_{j|X_{ij}} \arrow[r, "p_j"] & E_{|X_{ij}} \arrow[r] & 0
\end{tikzcd}$$

We have then $\psi_{jk}\psi_{ij} - \psi_{ik} : \mathcal{E}_{i|X_{ijk}} \to \mathcal{E}_{k|X_{ijk}}$, and there exists $\sigma_{ijk} : E_{|X_{ijk}} \to F_{|X_{ijk}}$ such that $\psi_{jk}\psi_{ij} - \psi_{ik} = \gamma_k\sigma_{ijk}p_i$. It is easily checked that the family (σ_{ijk}) satisfies the cocycle relation, and then defines an element of $H^2(X, \mathcal{H}om(E,F))$, which is $\delta_X(s)$.

3.2.4. *Local extensions* – Let $s \in H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(E,F))$. Let $(U_m)_{m\in M}$ be an open cover of X such that for every $i \in I$ and $m \in M$, the intersection $X_i \cap U_m$ is affine. For every $i, j \in I$ and $m, n \in M$, let $Z_i^m = X_i \cap U_m$, $Z_{ij}^m = X_{ij} \cap U_m$, etc.

Given two extensions

$$\Sigma : 0 \longrightarrow F \xrightarrow{\lambda} \mathcal{E} \xrightarrow{\mu} E \longrightarrow 0, \qquad \Sigma' : 0 \longrightarrow F \xrightarrow{\lambda'} \mathcal{E}' \xrightarrow{\mu'} E \longrightarrow 0 ,$$

an isomorphism $\phi : \mathcal{E} \to \mathcal{E}'$ is called a *I-isomorphism* (with respect to Σ, Σ') if the following diagram

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r, "\lambda"] \arrow[d, equal] & \mathcal{E} \arrow[r, "\mu"] \arrow[d, "\phi"] & E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F \arrow[r, "\lambda'"] & \mathcal{E}' \arrow[r, "\mu'"] & E \arrow[r] & 0
\end{tikzcd}$$

is commutative. If ϕ' is another I-isomorphism $\mathcal{E} \to \mathcal{E}'$, there exists a morphism $\tau : E \to F$ such that $\phi - \phi' = \lambda'\tau\mu$.

Suppose that for every $m \in M$ there exists $\sigma_m \in \mathrm{Ext}^1_{\mathcal{O}_{U_m}}(E,F)$ such that $\delta_{U_m}(\sigma_m) = s_{|U_m}$. Let

$$0 \longrightarrow F_{|U_m} \xrightarrow{c_m} \mathcal{F}_m \xrightarrow{\pi_m} E_{|U_m} \longrightarrow 0$$

be the extension corresponding to σ_m. Then for every $i \in I$, $m \in M$, there exists an I-isomorphism $\alpha_i^m : \mathcal{E}_{i|Z_i^m} \to \mathcal{F}_{m|Z_i^m}$. Let $j \in I$. Then $(\alpha_j^m)^{-1}\alpha_i^m$ is an I-isomorphism $\mathcal{E}_{i|Z_{ij}^m} \to \mathcal{E}_{j|Z_{ij}^m}$. So we can write

$$\psi_{ij} - (\alpha_j^m)^{-1}\alpha_i^m \;=\; \gamma_j\mu_{ij}^m p_i \ ,$$

with $\mu_{ij}^m : E_{|Z_{ij}^m} \to F_{|Z_{ij}^m}$. It follows that

$$\big(\psi_{jk}\psi_{ij} - \psi_{ik}\big)_{|Z_{ijk}^m} \;=\; \gamma_k(\mu_{jk}^m + \mu_{ij}^m - \mu_{ik}^m)_{|Z_{ijk}^m} p_i \ ,$$

so we have

$$\sigma_{ijk|Z_{ijk}^m} \;=\; (\mu_{jk}^m + \mu_{ij}^m - \mu_{ik}^m)_{|Z_{ijk}^m} \ .$$

Let $a_{ij}^{mn} = \mu_{ij}^m - \mu_{ij}^n$. Then for every $m, n \in M$, $(a_{ij}^{mn})_{i,j\in I}$ is a cocycle and thus defines $A_{mn} \in H^1(U_{mn}, \mathcal{H}om(E,F))$.

3.2.5. *The case of families* – We suppose now that X is a product : $X = Y \times S$, and that $p_{S*}(\mathcal{H}om(E,F)) = 0$, where $p_S : X \to S$ is the projection. We have from 3.1 a canonical injection

$$\lambda : H^1(S, R^1p_{S*}(\mathcal{H}om(E,F))) \longrightarrow H^2(X, \mathcal{H}om(E,F)) \ .$$

Let $(S_m)_{m\in M}$ be an affine open cover of S. We suppose that $U_m = p_S^{-1}(S_m)$ for every $m \in M$. Then $(A_{mn})_{m,n\in M}$ defines $\epsilon \in H^1(S, R^1p_{S*}(\mathcal{H}om(E,F)))$. It follows from 3.1 that

3.2.6. Proposition: *We have* $\delta_X(s) = \lambda(\epsilon)$.

3.2.7. *Generalization* – If we don't assume that $p_{S*}(\mathcal{H}om(E,F))=0$, we have still an injective morphism

$$\lambda' : H^1(S, R^1p_{S*}(\mathcal{H}om(E,F))) \longrightarrow H^2(X, \mathcal{H}om(E,F))/H^0(S, p_{S*}(\mathcal{H}om(E,F)) \ ,$$

and we have an analogous result (using a more complicated proof), i.e. $\lambda'(\epsilon)$ is the class of $\delta_X(s)$.

4. Primitive multiple schemes

4.1. Definition and construction

Let X be a smooth connected variety, and $d=\dim(X)$. A *multiple scheme with support* X is a Cohen-Macaulay scheme Y such that $Y_{red}=X$. If Y is quasi-projective we say that it is a *multiple variety with support* X. In this case Y is projective if X is.

Let n be the smallest integer such that $Y=X^{(n-1)}$, $X^{(k-1)}$ being the k-th infinitesimal neighborhood of X, i.e. $\mathcal{I}_{X^{(k-1)}}=\mathcal{I}_X^k$. We have a filtration $X=X_1\subset X_2\subset\cdots\subset X_n=Y$ where X_i is the biggest Cohen-Macaulay subscheme contained in $Y\cap X^{(i-1)}$. We call n the *multiplicity* of Y.

We say that Y is *primitive* if, for every closed point x of X, there exists a smooth variety S of dimension $d+1$, containing a neighborhood of x in Y as a locally closed subvariety. In this case, $L=\mathcal{I}_X/\mathcal{I}_{X_2}$ is a line bundle on X, X_j is a primitive multiple scheme of multiplicity j and we have $\mathcal{I}_{X_j}=\mathcal{I}_X^j$, $\mathcal{I}_{X_j}/\mathcal{I}_{X_{j+1}}=L^j$ for $1\leq j<n$. We call L the line bundle on X *associated* to Y. The ideal sheaf $\mathcal{I}_{X,Y}$ can be viewed as a line bundle on X_{n-1}.

Let $P\in X$. Then there exist elements $y_1,\ldots,y_d$, t of $m_{S,P}$ whose images in $m_{S,P}/m_{S,P}^2$ form a basis, and such that for $1\leq i<n$ we have $\mathcal{I}_{X_i,P}=(t^i)$. In this case the images of $y_1,\ldots,y_d$ in $m_{X,P}/m_{X,P}^2$ form a basis of this vector space.

A *multiple scheme with support* X is primitive if and only if $\mathcal{I}_X/\mathcal{I}_X^2$ is zero or a line bundle on X (cf. [13], Proposition 2.3.1).

Even if X is projective, we do not assume that Y is projective. In fact we will see examples of non quasi-projective Y.

The simplest case is when Y is contained in a smooth variety S of dimension $d+1$. Suppose that Y has multiplicity n. Let $P\in X$ and $f\in\mathcal{O}_{S,P}$ a local equation of X. Then we have $\mathcal{I}_{X_i,P}=(f^i)$ for $1<j\leq n$ in S, in particular $\mathcal{I}_{Y,P}=(f^n)$, and $L=\mathcal{O}_X(-X)$.

For any $L\in\operatorname{Pic}(X)$, the *trivial primitive variety* of multiplicity n, with induced smooth variety X and associated line bundle L on X is the n-th infinitesimal neighborhood of X, embedded by the zero section in the dual bundle L^*, seen as a smooth variety.

4.1.1. *Construction of primitive multiple schemes* – Let Y be a primitive multiple scheme of multiplicity n, $X=Y_{red}$. Let $\mathbf{Z}_n=\operatorname{spec}(\mathbb{C}[t]/(t^n))$. Then for every closed point $P\in X$, there exists an open neighborhood U of P in X, such that if $U^{(n)}$ is the corresponding neighborhood of P in Y, there exists a commutative diagram

$$\begin{tikzcd} & U \arrow[dl, hook'] \arrow[dr, hook] & \\ U^{(n)} \arrow[rr, "\simeq"] & & U \times \mathbf{Z}_n \end{tikzcd}$$

i.e. Y is locally trivial ([9], Théorème 5.2.1, Corollaire 5.2.2).

It follows that we can construct a primitive multiple scheme of multiplicity n by taking an open cover $(U_i)_{i\in I}$ of X and gluing the varieties $U_i \times \mathbf{Z}_n$ (with automorphisms of the $U_{ij} \times \mathbf{Z}_n$ leaving U_{ij} invariant).

Let $(U_i)_{i\in I}$ be an affine open cover of X such that we have trivializations

$$\delta_i : U_i^{(n)} \xrightarrow{\simeq} U_i \times \mathbf{Z}_n,$$

and $\delta_i^* : \mathcal{O}_{U_i\times\mathbf{Z}_n} \to \mathcal{O}_{U_i^{(n)}}$ the corresponding isomorphism. Let

$$\delta_{ij} = \delta_j\delta_i^{-1} : U_{ij} \times \mathbf{Z}_n \xrightarrow{\simeq} U_{ij} \times \mathbf{Z}_n \ .$$

Then $\delta_{ij}^* = \delta_i^{*-1}\delta_j^*$ is an automorphism of $\mathcal{O}_{U_i\times Z_n} = \mathcal{O}_X(U_{ij})[t]/(t^n)$, such that for every $\phi \in \mathcal{O}_X(U_{ij})$, seen as a polynomial in t with coefficients in $\mathcal{O}_X(U_{ij})$, the term of degree zero of $\delta_{ij}^*(\phi)$ is the same as the term of degree zero of ϕ.

4.1.2. *The ideal sheaf of X* – There exists $\alpha_{ij} \in \mathcal{O}_X(U_{ij}) \otimes_{\mathbb{C}} \mathbb{C}[t]/(t^{n-1})$ such that $\delta_{ij}^*(t) = \alpha_{ij}t$. Let $\alpha_{ij}^{(0)} = \alpha_{ij|X} \in \mathcal{O}_X(U_i)$. For every $i \in I$, $\delta_i^*(t)$ is a generator of $\mathcal{I}_{X,Y|U^{(n)}}$. So we have local trivializations

$$\begin{aligned} \lambda_i : \mathcal{I}_{X,Y|U_i^{(n-1)}} &\longrightarrow \mathcal{O}_{U_i^{(n-1)}} \\ \delta_i^*(t) &\longmapsto 1 \end{aligned}$$

Hence $\lambda_{ij} = \lambda_i\lambda_j^{-1} : \mathcal{O}_{U_{ij}^{(n-1)}} \to \mathcal{O}_{U_{ij}^{(n-1)}}$ is the multiplication by $\delta_j^*(\alpha_{ij})$. It follows that $(\delta_j^*(\alpha_{ij}))$ (resp. $(\alpha_{ij}^{(0)})$) is a cocycle representing the line bundle $\mathcal{I}_{X,Y}$ (resp. L) on X_{n-1} (resp. X).

4.1.3. *The associated sheaves of non abelian groups and obstructions to extension in higher multiplicity* – For every open subset U of X, we have $\mathcal{O}_{X\times\mathbf{Z}_n}(U) = \mathcal{O}_X(U)[t]/(t^n)$. Let $\mathcal{G}_n$ be the sheaf of (non abelian) groups on X defined by: for every open subset U of X, $\mathcal{G}_n(U)$ is the group of automorphisms θ of the $\mathbb{C}$-algebra $\mathcal{O}_X(U)[t]/(t^n)$ such that for every $\alpha \in \mathcal{O}_X(U)[t]/(t^n)$, $\theta(\alpha)_{|U} = \alpha_{|U}$. We have $\mathcal{G}_1 = \mathcal{O}_X^*$. Then (δ_{ij}^*) is a cocycle of $\mathcal{G}_n$, which describes completely X_n.

In this way we see that there is a canonical bijection between the cohomology set $H^1(X, \mathcal{G}_n)$ and the set of isomorphism classes of primitive multiple schemes X_n such that $X = (X_n)_{red}$. There is an obvious surjective morphism $\rho_{n+1} : \mathcal{G}_{n+1} \to \mathcal{G}_n$, such that if $n \geq 2$,

$$H^1(\rho_{n+1}) : H^1(X, \mathcal{G}_{n+1}) \longrightarrow H^1(X, \mathcal{G}_n)$$

sends a primitive multiple scheme of multiplicity $n+1$ to the underlying scheme of multiplicity n, whereas

$$H^1(\rho_2) : H^1(X, \mathcal{G}_2) \longrightarrow H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X)$$

sends X_2 to L.

We have $\ker(\rho_2)\simeq T_X$ and $\ker(\rho_{n+1})\simeq T_X\oplus\mathcal{O}_X$ if $n\geq 2$. The fact that they are sheaves of *abelian* groups allows to compute *obstructions*. Let $g_n\in H^1(X,\mathcal{G}_n)$, corresponding to the primitive multiple scheme X_n, and for $1\leq i<n$, g_i the image of g_n in $H^1(X,\mathcal{G}_i)$. Let $\ker(\rho_{n+1})^{g_n}$ be the associated sheaf of groups (cf. [16], 2.2, [19]). We have then

$$\ker(\rho_2)^{g_1}\ \simeq\ T_X\otimes L$$

By cohomology theory, we find that there is a canonical surjective map

$$H^1(X,T_X\otimes L)\longrightarrow H^1(\rho_2)^{-1}(L)$$

sending 0 to the trivial primitive scheme, and whose fibers are the orbits of the action of $\mathbb{C}^*$ on $H^1(X,T_X\otimes L)$ by multiplication. Hence there is a bijection between the set of non trivial double schemes with associated line bundle L, and $\mathbb{P}(H^1(X,T_X\otimes L))$. For a non trivial double scheme X_2, we have an exact sequence

$$0\longrightarrow L\longrightarrow\Omega_{X_2|X}\longrightarrow\Omega_X\longrightarrow 0\ ,$$

corresponding to $\sigma\in\operatorname{Ext}^1_{\mathcal{O}_X}(\Omega_X,L)=H^1(T_X\otimes L)$, and $\mathbb{C}\sigma$ is the element of $\mathbb{P}(H^1(T_X\otimes L))$ corresponding to X_2. We can write for $n=2$

$$(\delta^*_{ij})_{|\mathcal{O}_X(U_{ij})}\ =\ I_{|\mathcal{O}_X(U_{ij})}+tD_{ij}\ ,$$

where D_{ij} is a derivation of $\mathcal{O}_X(U_{ij})$. Then the family (D_{ij}) represents σ in the sense of 2.1.

If $n>2$ we have

$$\ker(\rho_{n+1})^{g_n}\ \simeq\ (\Omega_{X_2|X})^*\otimes L^n$$

We have then (by the theory of cohomology of sheaves of groups) an *obstruction map*

$$\Delta_n:H^1(X,\mathcal{G}_n)\longrightarrow H^2((\Omega_{X_2|X})^*\otimes L^n)$$

such that $g_n\in\operatorname{im}(H^1(\rho_{n+1}))$ if and only $\Delta_n(g_n)=0$.

4.1.4. *Extensions of the ideal sheaf of X* – The ideal sheaf $\mathcal{I}_{X,X_n}$ is a line bundle on X_{n-1}. A necessary condition to extend X_n to a primitive multiple scheme X_{n+1} of multiplicity $n+1$ is that $\mathcal{I}_{X,X_n}$ can be extended to a line bundle on X_n (namely $\mathcal{I}_{X,X_{n+1}}$). This is why we can consider pairs $(X_n,\mathbb{L})$, where $\mathbb{L}$ is a line bundle on X_n such that $\mathbb{L}_{|X_{n-1}}\simeq\mathcal{I}_{X,X_n}$.

The corresponding sheaf of groups on X is defined as follows: for every open subset $U\subset X$, $\mathcal{H}_n(U)$ is the set of pairs (ϕ,u), where $\phi\in\mathcal{G}_n(U)$, and $u\in\mathcal{O}_X(U)[t]/(t^n)$ is such that $\phi(t)=ut$ (cf. [16], 4.5). The set of isomorphism classes of the above pairs $(X_n,\mathbb{L})$ can then be identified with the cohomology set $H^1(X,\mathcal{H}_n)$.

There is an obvious morphism $\tau_n:\mathcal{G}_{n+1}\to\mathcal{H}_n$, such that

$$H^1(\tau_n):H^1(X,\mathcal{G}_{n+1})\longrightarrow H^1(X,\mathcal{H}_n)$$

sends X_{n+1} to $(X_n,\mathcal{I}_{X,X_{n+1}})$. Let $g\in H^1(X,\mathcal{H}_n)$. Then $\ker(\tau_n)\simeq T_X$ and

$$\ker(\tau_n)^g\ \simeq\ T_X\otimes L^n\ .$$

Consequently there is again, by cohomology theory, an *obstruction map*

$$\Delta''_n:H^1(X,\mathcal{H}_n)\longrightarrow H^2(T_X\otimes L^n)$$

such that, if $(X_n,\mathbb{L})$ corresponds to g, there is an extension of X_n to a primitive multiple scheme X_{n+1} of multiplicity $n+1$ with $\mathcal{I}_{X,X_{n+1}}\simeq\mathbb{L}$ if and only if $\Delta''_n(g)=0$.

4.2. Descriptions using the open cover (U_i)

(i) *Construction of sheaves and morphisms* – (cf. [16], 2.1.3). Let $\mathcal{E}$ be a coherent sheaf on X_n. We can define it in the usual way, starting with sheaves $\mathcal{F}_i$ on the open sets $U_i^{(n)}$ and gluing them. We take these sheaves of the form $\mathcal{F}_i = \delta_i^*(\mathcal{E}_i)$, where $\mathcal{E}_i$ is a sheaf on $U_i \times \mathbf{Z}_n$. To glue the sheaves $\mathcal{F}_i$ on the intersection $U_{ij}^{(n)}$ we use isomorphisms $\rho_{ij} : \mathcal{F}_{j|U_{ij}^{(n)}} \to \mathcal{F}_{i|U_{ij}^{(n)}}$, with the relations $\rho_{ik} = \rho_{jk}\rho_{ij}$. Let

$$\theta_{ij} = (\delta_i^*)^{-1}(\rho_{ij}) : \delta_{ij}^*(\mathcal{E}_{j|U_{ij}\times\mathbf{Z}_n}) \longrightarrow \mathcal{E}_{i|U_{ij}\times\mathbf{Z}_n} \ .$$

We have then the relations $\theta_{ik} = \theta_{ij} \circ \delta_{ij}^*(\theta_{jk})$. Conversely, starting with sheaves $\mathcal{E}_i$ and isomorphisms θ_{ij} satisfying the preceding relations, one obtains a coherent sheaf on X_n.

This applies to trivializations, i.e when $\mathcal{E}_i = \mathcal{O}_{U_i^{(n)}} \otimes \mathbb{C}^r$. We have then $\theta_{ij} : \mathcal{O}_{U_{ij}\times\mathbf{Z}_n} \otimes \mathbb{C}^r \to \mathcal{O}_{U_{ij}\times\mathbf{Z}_n} \otimes \mathbb{C}^r$. In particular, $\mathcal{I}_{X,X_n}$ is represented by (α_{ij}).

Suppose that we have another sheaf $\mathcal{E}'$ on X_n, defined by sheaves $\mathcal{E}'_i$ on $U_i \times \mathbf{Z}_n$ and isomorphisms θ'_{ij}. One can see easily that a morphism $\Psi : \mathcal{E} \to \mathcal{E}'$ is defined by morphisms $\Psi_i : \mathcal{E}_i \to \mathcal{E}'_i$ such that $\theta'_{ij} \circ \delta_{ij}^*(\Psi_j) = \Psi_i \circ \theta_{ij}$.

(ii) *Other construction of sheaves* – Let E be a vector bundle on X. For every open subset $i \in I$, let $p_i : U_i \times \mathbf{Z}_n \to U_i$ be the projection. Then we have $\delta_i^*(p_i^*(E_{|U_i}))_{|U_i} = E_{|U_i}$. We construct a vector bundle $\mathbb{E}$ on X_n by gluing the vector bundles $\delta_i^*(p_i^*(E_{|U_i}))$ on $U_i^{(n)}$. For this we take isomorphisms

$$\epsilon_{ij} : \delta_j^*(p_j^*(E_{|U_{ij}}))_{U_{ij}^{(n)}} \longrightarrow \delta_i^*(p_i^*(E_{|U_{ij}}))_{U_{ij}^{(n)}}$$

satisfying $\epsilon_{ij}\epsilon_{jk} = \epsilon_{ik}$. Let

$$\tau_{ij} = \delta_i^{*-1}(\epsilon_{ij}) : \delta_{ij}^*(p_j^*(E_{|U_{ij}})) = p_i^*(E_{|U_{ij}}) \longrightarrow p_i^*(E_{|U_{ij}}) \ .$$

Then we have $\tau_{ij} \circ \delta_{ij}^*(\tau_{jk}) = \tau_{ik}$. Conversely, given automorphisms τ_{ij} satisfying the preceding relation, we can define ϵ_{ij} and the vector bundle $\mathbb{E}$ on X_n.

Suppose that the τ_{ij} are homotheties (multiplication by $\nu_{ij} \in \mathcal{O}(U_{ij} \times \mathbf{Z}_n)^*$). Then the family (ν_{ij}) defines a line bundle $\mathbb{L}$ on X_n (by (i)), and we have $\mathbb{E}_{|X} \simeq E \otimes \mathbb{L}_{|X}$.

(iii) *Cohomology of vector bundles* – Suppose that a vector bundle E on X_n is built using isomorphisms $\theta_{ij} : \mathcal{O}_{U_{ij}\times\mathbf{Z}_n} \otimes \mathbb{C}^r \to \mathcal{O}_{U_{ij}\times\mathbf{Z}_n} \otimes \mathbb{C}^r$ as in (i). Any $\beta \in H^1(X_n, E)$ is represented by a family (β_{ij}), $\beta_{ij} \in H^0(\mathcal{O}_{U_{ij}\times\mathbf{Z}_n} \otimes \mathbb{C}^r)$, with the cocycle relations $\beta_{ik} = \beta_{ij} + \theta_{ij}\delta_{ij}^*(\beta_{jk})$.

4.3. Connecting morphisms

Recall that on X_2 we can write $\delta^*_{ij|\mathcal{O}_X(U_{ij})} = I + tD_{ij}$, where D_{ij} is a derivation of $\mathcal{O}_X(U_{ij})$ (cf. 4.1.3).

Let $\mathbb{E}$ be a vector bundle on X_2 of rank r, defined as in 4.2, (i), using isomorphisms $\theta_{ij} : \mathcal{O}_{U_{ij}\times\mathbf{Z}_2} \otimes \mathbb{C}^r \to \mathcal{O}_{U_{ij}\times\mathbf{Z}_2} \otimes \mathbb{C}^r$. We can write

$$\theta_{ij} \ = \ \theta_{ij}^{(0)} + \theta_{ij}^{(1)} t \ ,$$

where $\theta_{ij}^{(0)}$, $\theta_{ij}^{(1)}$ are matrices with coefficients in $\mathcal{O}_X(U_{ij})$.

Let $E = \mathbb{E}_{|X}$. We have a canonical exact sequence

$$0 \longrightarrow E \otimes L \longrightarrow \mathbb{E} \longrightarrow E \longrightarrow 0 \ ,$$

whence a connecting morphism

$$\delta_{\mathbb{E}}^1 : H^1(X,E) \longrightarrow H^2(X, E \otimes L) \ .$$

Let $\beta \in H^1(X,E)$, represented by a family (β_{ij}), $\beta_{ij} \in H^0(\mathcal{O}_{U_{ij}} \otimes \mathbb{C}^r)$, in the sense of 2.1.

4.3.1. Lemma: *$\delta_{\mathbb{E}}^1(\beta)$ is represented, in the sense of 4.2, (iii), by the family (ν_{ijk}), with*

$$\nu_{ijk} \ = \ \theta_{ij}^{(0)} D_{ij}(\beta_{jk}) + \theta_{ij}^{(1)}\beta_{jk} \ .$$

Proof. We construct $\delta_{\mathbb{E}}^1(\beta)$ as follows: we view β_{ij}, β_{kj}, β_{ik} as elements of $H^0(\mathcal{O}_{U_{ij}\times \mathbf{Z}_n} \otimes \mathbb{C}^r)$. Then we can take

$$\nu_{ijk} \ = \ \beta_{ij} - \beta_{ik} + \theta_{ij}\delta_{ij}^*(\beta_{jk})$$

which gives Lemma 4.3.1. □

Suppose that X_n can be extended to a primitive multiple scheme X_{n+1} of multiplicity $n+1$.

We have an exact sequence of sheaves on X_{n+1}

$$0 \longrightarrow L^n \longrightarrow \mathcal{O}_{X_{n+1}} \longrightarrow \mathcal{O}_{X_n} \longrightarrow 0 \ ,$$

whence a connecting morphism

$$\delta^0 : H^0(\mathcal{O}_{X_n}) \longrightarrow H^1(X, L^n) \ .$$

We have $H^0(X, L^{n-1}) \subset H^0(\mathcal{O}_{X_n})$. Let $\eta \in H^0(X, L^{n-1})$, represented by a cocycle (η_i), $\eta_i \in \mathcal{O}_X(U_i)$ (in the sense of 2.1.1, i.e. the cocycle relation is $\eta_i = (\alpha_{ij}^{(0)})^{n-1}\eta_j$). Recall that $\mathcal{I}_{X,X_{n+1}}$ is represented by the cocycle (α_{ij}), where α_{ij} is an invertible element of $\mathcal{O}_X(U_{ij})[t]/(t^n)$.

The proof of the following lemma is similar to that of Lemma 4.3.1:

4.3.2. Lemma: *$\delta^0(\eta)$ is represented by the cocycle (γ_{ij}), with*

$$\gamma_{ij} \ = \ (\alpha_{ij}^{(0)})^{n-1} D_{ij}(\eta_j) + (n-1)(\alpha_{ij}^{(0)})^{n-2}\alpha_{ij}^{(1)}\eta_j \ .$$

4.4. The sheaf of differentials

We use the notations of 4.1.1. We will give a construction of Ω_{X_n} using 4.2 : it is defined by the sheaves $\mathcal{E}_i = \Omega_{U_i \times \mathbf{Z}_n}$, and

$$\mathbf{d}(\delta_{ij}) : \delta_{ij}^*(\Omega_{U_i\times\mathbf{Z}_n}) = \Omega_{U_i\times\mathbf{Z}_n} \longrightarrow \Omega_{U_i\times\mathbf{Z}_n}$$

(more precisely $\mathbf{d}(\delta_{ij})(a.db) = \delta_{ij}^*(a)d(\delta_{ij}^*(b))))$.

Let $\sigma \in H^1(X_n, \Omega_{X_n})$, represented by the cocycle (σ_{ij}), $\sigma_{ij} \in \Omega_{U_{ij}^{(n)}}$. Then

$\sigma'_{ij} = \mathbf{d}(\delta_i^{-1})(\sigma_{ij}) \in \Gamma(\Omega_{U_{ij}\times\mathbf{Z}_n})$, and we have the relations

$$\sigma'_{ik} \ = \ \sigma'_{ij} + \mathbf{d}(\delta_{ij})(\sigma'_{jk}) \ .$$

Conversely, every family (σ'_{ij}) satisfying these relations defines an element of $H^1(X_n, \Omega_{X_n})$.

4.4.1. *The dualizing sheaf* – Suppose that X is projective. Then X_n is a proper Cohen-Macaulay scheme. There exists a dualizing sheaf ω_{X_n} for X_n, which is a line bundle (because X_n is locally a complete intersection).

4.5. Extensions of primitive multiple schemes

Let X_n a primitive multiple scheme of multiplicity $n \geq 2$, with underlying smooth variety X projective, irreducible, and associated line bundle L on X. For every open subset $U \subset X$, let $U^{(n)}$ denote the corresponding open subset of X_n. Let $\boldsymbol{\zeta} \in H^1(X, T_X \otimes L)$, corresponding to the exact sequence $0 \to L \to \Omega_{X_2|X} \to \Omega_X \to 0$.

If X_n can be extended to a primitive multiple scheme X_{n+1} of multiplicity $n+1$, then $\mathcal{I}_{X,X_n}$, which is a line bundle on X_{n-1}, can be extended to a line bundle on X_n, namely $\mathcal{I}_{X,X_{n+1}}$. We have exact sequences

$$0 \longrightarrow L^n \longrightarrow \mathcal{I}_{X,X_{n+1}} \longrightarrow \mathcal{I}_{X,X_n} \longrightarrow 0 \ , \tag{3}$$

$$0 \longrightarrow H^1(X, L^{n-1}) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_{X_n}}(\mathcal{I}_{X,X_n}, L^n) \xrightarrow{\ \phi\ } \operatorname{End}(L) \xrightarrow{\ \delta\ } H^2(X, L^{n-1})$$

(cf 4.6.3), and given any exact sequence

$$0 \longrightarrow L^n \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{X,X_n} \longrightarrow 0 \ ,$$

corresponding to $\sigma \in \operatorname{Ext}^1_{\mathcal{O}_{X_n}}(\mathcal{I}_{X,X_n}, L^n)$, the sheaf $\mathcal{E}$ is a line bundle on X_n if and only if $\phi(\sigma)$ is an automorphism. In this case we can suppose that $\phi(\sigma) = I_L$. In this way we can see that, given (3), the extensions of $\mathcal{I}_{X,X_n}$ to a line bundle on X_n are parametrized by $H^1(X, L^{n-1})$.

More precisely, using the notations of 4.1.1 and 4.2, let $\mathbb{L}_0 \in \operatorname{Pic}(X_n)$ be an extension of $\mathcal{I}_{X,X_n}$. Suppose that it is defined by the family (α_{ij}), with $\alpha_{ij} \in [\mathcal{O}_X(U_{ij})[t]/(t^n)]^*$, satisfying the relations $\alpha_{ik} = \alpha_{ij}\delta^*_{ij}(\alpha_{jk})$, so that L is defined by the family $\big(\alpha^{(0)}_{ij}\big)$. Let $\mathbb{L}$ be an extension of $\mathcal{I}_{X,X_n}$ to a line bundle on X_n, corresponding to $\eta \in H^1(X, L^{n-1})$. Then $\mathbb{L}$ is defined by a cocycle (θ_{ij}), with

$$\theta_{ij} \ = \ \alpha_{ij} + \beta_{ij} t^{n-1} \ ,$$

$\beta_{ij} \in \mathcal{O}_X(U_{ij})$. The cocycle relation $\theta_{ik} = \theta_{ij}\delta^*_{ij}(\theta_{jk})$ is equivalent to

$$\frac{\beta_{ik}}{\alpha^{(0)}_{ik}} \ = \ \frac{\beta_{ij}}{\alpha^{(0)}_{ij}} + \big(\alpha^{(0)}_{ij}\big)^{n-1} \frac{\beta_{jk}}{\alpha^{(0)}_{jk}} \ . \tag{4}$$

According to 2.1, this means that $\left(\dfrac{\beta_{ij}}{\alpha^{(0)}_{ij}}\right)$ defines an element of $H^1(X, L^{n-1})$, which is η.

Now we use the results of 4.1.4. The family $(\delta^*_{ij}, \alpha_{ij})$ defines an element $\mathbf{h}$ of $H^1(X, \mathcal{H}_n)$. The family $(\delta^*_{ij}, \theta_{ij})$ defines an element $\mathbf{k}$ of $H^1(X, \mathcal{H}_n)$ (corresponding to $(X_n, \mathbb{L})$), and we have $\Delta''_n(\mathbf{k}) = 0$ if and only there exists an extension X'_{n+1} of X_n in multiplicity $n+1$ such that $\mathcal{I}_{X,X'_{n+1}} \simeq \mathbb{L}$. We will compute $\Delta''_n(\mathbf{k} - \mathbf{h})$.

We can write

$$(\delta^*_{ij})_{|\mathcal{O}_X(U_{ij})} \ = \ I_{|\mathcal{O}_X(U_{ij})} + tD_{ij} + t^2\Phi_{ij} \ ,$$

where D_{ij} is a derivation of $\mathcal{O}_X(U_{ij})$. The family (D_{ij}) represents the element $\boldsymbol{\zeta}$ of $H^1(X, T_X \otimes L)$ associated to X_2 (cf. 4.1.3) in the sense of 2.1 (i.e. the cocycle relations are $D_{ik} = D_{ij} + \alpha_{ij}^{(0)} D_{jk}$). We have a canonical product

$$H^1(X, T_X \otimes L) \otimes H^1(X, L^{n-1}) \longrightarrow H^2(X, T_X \otimes L^n)$$

and

4.5.1. Theorem: *We have* $\Delta_n''(\mathbf{k} - \mathbf{h}) = \boldsymbol{\zeta}\eta$.

Proof. To construct $\Delta_n''(\mathbf{k} - \mathbf{h})$ we start from the exact sequence

$$0 \longrightarrow T_X \longrightarrow \mathcal{G}_{n+1} \xrightarrow{\tau_n} \mathcal{H}_n \longrightarrow 0$$

(cf. [16], 2.2, [19]). We take $\psi_{ij} \in \mathrm{Aut}(\mathcal{O}_X(U_{ij})[t]/(t^n))$ over $(\delta_{ij}^*, \theta_{ij})$ (such that $\psi_{ji} = \psi_{ij}^{-1}$). Then $\psi_{ij}\psi_{jk}\psi_{ki}$ is of the form

$$\psi_{ij}\psi_{jk}\psi_{ki} \ = \ I + \nu_{ijk} t^n \ ,$$

where ν_{ijk} is a derivation of $\mathcal{O}_X(U_{ij})$. The family (ν_{ijk}) represents $\Delta_n''(\mathbf{k})$ in the sense of 2.1, i.e. we have the cocycle relations $\nu_{ijk} - \nu_{ijl} + \nu_{ikl} - \big(\alpha_{ij}^{(0)}\big)^n \nu_{jkl} = 0$.

Similarly we take $\phi_{ij} \in \mathrm{Aut}(\mathcal{O}_X(U_{ij})[t]/(t^n))$ over $(\delta_{ij}^*, \alpha_{ij})$ (such that $\phi_{ji} = \phi_{ij}^{-1}$). Then $\phi_{ij}\phi_{jk}\phi_{ki}$ is of the form

$$\phi_{ij}\phi_{jk}\phi_{ki} \ = \ I + \tau_{ijk} t^n \ ,$$

where τ_{ijk} is a derivation of $\mathcal{O}_X(U_{ij})$. The family (τ_{ijk}) represents $\Delta_n''(\mathbf{h})$ in the sense of 2.1.

We can take

$$\psi_{ij} - \phi_{ij} \ = \ t^n \Delta_{ij} \ ,$$

where, if we want that ψ_{ij} is a morphism of rings, $\Delta_{ij} : \mathcal{O}_X(U_{ij})[t]/(t^{n+1}) \to \mathcal{O}_X(U_{ij})$ is such that there exist a derivation E_{ij} of $\mathcal{O}_X(U_{ij})$ and $\mu_{ij} \in \mathcal{O}_X(U_{ij})$ such that for every $u \in \mathcal{O}_X(U_{ij})[t]/(t^{n+1})$ we have

$$\Delta_{ij}(u) \ = \ E_{ij}(u^{(0)}) + u^{(1)}\mu_{ij} \ .$$

Since ψ_{ij} is over $(\delta_{ij}^*, \theta_{ij})$ we have $\mu_{ij} = \beta_{ij}$. It is easy to see that if $\phi_{ji} = \phi_{ij}^{-1}$, the condition $\psi_{ji} = \psi_{ij}^{-1}$ is equivalent to

(5) $$\beta_{ji} \ = \ -\frac{\beta_{ij}}{(\alpha_{ij}^{(0)})^{n+1}}$$

which is already fulfilled by (4) and

(6) $$E_{ji} \ = \ \frac{\beta_{ij}}{(\alpha_{ij}^{(0)})^{n+1}} D_{ij} - \frac{1}{(\alpha_{ij}^{(0)})^{n}} E_{ij} \ .$$

We have then

$$\nu_{ijk} - \tau_{ijk} \ = \ t^n \Delta_{ij}\phi_{jk}\phi_{ki} + \phi_{ij}(t^n \Delta_{jk})\phi_{ki} + \phi_{ij}\phi_{jk}(t^n \Delta_{ki}) \ .$$

We have, for every $u \in \mathcal{O}_X(U_{ij})[t]/(t^{n+1})$

$$\begin{aligned}
t^n \Delta_{ij} \phi_{jk} \phi_{ki}(u) &= \big(E_{ij}(u^{(0)}) + (D_{ji}(u^{(0)}) + \alpha_{ji}^{(0)} u^{(1)})\beta_{ij}\big) t^n \ , \\
\phi_{ij}(t^n \Delta_{jk}) \phi_{ki}(u) &= \big(E_{jk}(u^{(0)}) + (D_{ki}(u^{(0)}) + \alpha_{ki}^{(0)} u^{(1)})\beta_{jk}\big) \big(\alpha_{ij}^{(0)}\big)^n t^n \ , \\
\phi_{ij} \phi_{jk}(t^n \Delta_{ki})(u) &= \big(\alpha_{ik}^{(0)}\big)^n \big(E_{ki}(u^{(0)}) + \beta_{ki} u^{(1)}\big) t^n \ .
\end{aligned}$$

Hence we have

$$\begin{aligned}
\nu_{ijk}(u) - \tau_{ijk}(u) = \ & \big(E_{ij} + \big(\alpha_{ij}^{(0)}\big)^n E_{jk} + \big(\alpha_{ik}^{(0)}\big)^n E_{ki}\big)(u^{(0)}) + \\
& \big(\beta_{ij} D_{ji} + \big(\alpha_{ij}^{(0)}\big)^n \beta_{jk} D_{ki}\big)(u^{(0)}) + \\
& \big(\alpha_{ji}^{(0)} \beta_{ij} + \big(\alpha_{ij}^{(0)}\big)^n \alpha_{ki}^{(0)} \beta_{jk} + \big(\alpha_{ik}^{(0)}\big)^n \beta_{ki}\big) u^{(1)} \ .
\end{aligned}$$

By (4) and (5) we have $\alpha_{ji}^{(0)} \beta_{ij} + \big(\alpha_{ij}^{(0)}\big)^n \alpha_{ki}^{(0)} \beta_{jk} + \big(\alpha_{ik}^{(0)}\big)^n \beta_{ki} = 0$. On the other hand, by 2.1.3 the family $\big(\big(\alpha_{ij}^{(0)}\big)^n E_{jk} - E_{ik} + E_{ij}\big)$ is a coboundary, and by (6) we have $E_{ki} = \dfrac{\beta_{ik}}{(\alpha_{ik}^{(0)})^{n+1}} D_{ik} - \dfrac{1}{(\alpha_{ik}^{(0)})^n} E_{ik}$. Hence we can assume that

$$\begin{aligned}
\nu_{ijk} &= \frac{\beta_{ik}}{\alpha_{ik}^{(0)}} D_{ik} + \beta_{ij} D_{ji} + \big(\alpha_{ij}^{(0)}\big)^n \beta_{jk} D_{ki} \\
&= \Big(\frac{\beta_{ik}}{\alpha_{ik}^{(0)}} - \frac{\big(\alpha_{ij}^{(0)}\big)^n}{\alpha_{ik}^{(0)}} \beta_{jk}\Big) D_{ik} + \beta_{ij} D_{ji} \ .
\end{aligned}$$

But $\dfrac{\beta_{ik}}{\alpha_{ik}^{(0)}} - \dfrac{\big(\alpha_{ij}^{(0)}\big)^n}{\alpha_{ik}^{(0)}} \beta_{jk} = \dfrac{\beta_{ij}}{\alpha_{ij}^{(0)}}$ by (4), and $D_{ik} = D_{ij} + \alpha_{ij}^{(0)} D_{jk}$, $D_{ji} = -\dfrac{1}{\alpha_{ij}^{(0)}} D_{ij}$. Hence

$$\nu_{ijk} = \beta_{ij} D_{jk} = \alpha_{ij}^{(0)} \Big(\frac{\beta_{ij}}{\alpha_{ij}^{(0)}}\Big) D_{jk} \ .$$

Since $\Big(\dfrac{\beta_{ij}}{\alpha_{ij}^{(0)}}\Big)$ represents η and (D_{ij}) represents $\boldsymbol{\zeta}$, the result follow from 2.1.5. □

4.6. Extensions of vector bundles to higher multiplicity

Suppose that X_n can be extended to a primitive multiple scheme X_{n+1} of multiplicity $n+1$. Let $\mathbb{E}$ be a vector bundle on X_n, and $E = \mathbb{E}_{|X}$. If $\mathbb{E}$ can be extended to a vector bundle $\mathbb{E}_{n+1}$ on X_{n+1}, then we have an exact sequence

$$0 \longrightarrow E \otimes L^n \longrightarrow \mathbb{E}_{n+1} \longrightarrow \mathbb{E} = \mathbb{E}_{n+1|X_n} \longrightarrow 0 \ .$$

4.6.1. *Obstruction to the extension of a vector bundle in higher multiplicity* – In [16], 7.1, a class $\Delta(\mathbb{E}) \in \mathrm{Ext}^2_{\mathcal{O}_X}(E, E \otimes L^n)$ is defined, such that $\mathbb{E}$ can be extended to a vector bundle on X_{n+1} if and only $\Delta(\mathbb{E}) = 0$.

The canonical exact sequence $0 \to L^n \to \Omega_{X_{n+1}|X_n} \to \Omega_{X_n} \to 0$ induces $\overline{\sigma}_{\mathbb{E},X_{n+1}} \in \mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\mathbb{E}\otimes\Omega_{X_n}, \mathbb{E}\otimes L^n)$. We have a canonical product

$$\mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\mathbb{E}\otimes\Omega_{X_n}, \mathbb{E}\otimes L^n) \times \mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\mathbb{E}, \mathbb{E}\otimes\Omega_{X_n}) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_{X_n}}(\mathbb{E}, \mathbb{E}\otimes L^n) = \mathrm{Ext}^2_{\mathcal{O}_X}(E, E\otimes L^n) ,$$

and

$$\Delta(\mathbb{E}) = \overline{\sigma}_{\mathbb{E},X_{n+1}}\nabla_0(\mathbb{E}) , \tag{7}$$

where $\nabla_0(\mathbb{E})$ is the *canonical class* of $\mathbb{E}$ (cf. 5, [16], Theorem 7.1.2).

4.6.2. Lemma: *We have* $\mathcal{E}xt^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E}, E\otimes L^n) \simeq \mathcal{E}nd(E)$.

Proof. Let $U \subset X$ be a nonempty subset and U_n, U_{n+1} the corresponding open subsets of X_n, X_{n+1} respectively. Suppose that U is affine. It follows from 4.6 that $\mathbb{E}_{|U_n}$ (resp. $\mathcal{I}_{U,U_{n+1}}$) can be extended to a vector bundle $\mathbb{F}$ (resp. $\mathbb{L}$) on U_{n+1}. We have a canonical morphism $\mathbb{L} \to \mathcal{O}_{U_{n+1}}$:

$$\mathbb{L} \longrightarrow \mathbb{L}_{|U_n} = \mathcal{I}_{U,U_{n+1}} \hookrightarrow \mathcal{O}_{U_{n+1}} ,$$

inducing $\alpha : \mathbb{L}^{n+1} \to \mathbb{L}^n$ and $\beta : \mathbb{L}^n \to \mathcal{O}_{n+1}$. We have then an obvious locally free resolution of $\mathbb{E}_{|U_{n+1}}$:

$$\cdots \longrightarrow \mathbb{F}\otimes\mathbb{L}^{n+1} \xrightarrow{I_F\otimes\alpha} \mathbb{F}\otimes\mathbb{L}^n \xrightarrow{I_F\otimes\beta} \mathbb{F} \longrightarrow \mathbb{E}_{|U_n} \longrightarrow 0 .$$

Using this resolution it follows that $\mathcal{E}xt^1_{\mathcal{O}_{U_{n+1}}}(\mathbb{E}_{U_{n+1}}, E_{|U}\otimes L^n) \simeq \mathcal{E}nd(E_{|U})$. It is easy to see that these isomorphisms can be glued together to define the isomorphism of Lemma 4.6.2. $\square$

4.6.3. *Other description of* $\Delta(\mathbb{E})$ *and of the extensions* – From 3.2 and Lemma 4.6.2, we have the exact sequence $\mathbf{\Gamma}_{\mathbb{E},E\otimes L^n}$ on X_{n+1}

$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(E, E\otimes L^n) \xrightarrow{\psi} \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E}, E\otimes L^n) \xrightarrow{\phi} \mathrm{End}(E) \xrightarrow{\delta_{\mathbb{E}}} \mathrm{Ext}^2_{\mathcal{O}_X}(E, E\otimes L^n) ,$$

and we have $\Delta(\mathbb{E}) = \delta_{\mathbb{E}}(I_E)$ (cf. [16]).

4.6.4. *Other description of the extensions* – Let

$$0 \longrightarrow E\otimes L^n \longrightarrow \mathcal{E} \to \mathbb{E} \longrightarrow 0$$

be an extension on X_{n+1}, with $\mathcal{E}$ locally free, and $\beta \in \mathrm{Ext}^1_{\mathcal{O}_X}(E, E\otimes L^n)$. There is an open cover $(U_i)_{i\in I}$ of X such that there are trivializations

$$\theta_i : \mathcal{E}_{|U_i^{(n+1)}} \longrightarrow \mathcal{O}_{U_i^{(n+1)}} \otimes \mathbb{C}^r$$

so that $\mathcal{E}$ is represented by the family (θ_{ij}), $\theta_{ij} \in \mathrm{Aut}(\mathcal{O}_{U_{ij}^{(n+1)}}\otimes\mathbb{C}^r)$. Recall that $\mathcal{I}_{X_n,X_{n+1}} = L^n$, viewed as a sheaf on X_{n+1}. Let $\beta \in \mathrm{Ext}^1_{\mathcal{O}_X}(E, E\otimes L^n)$. Then (for a suitable cover (U_i)) β is represented by (β_{ij}), with

$$\beta_{ij} \ \in\ H^0(\mathcal{O}_{U_{ij}}\otimes\mathrm{End}(\mathbb{C}^r)\otimes L^n) \ \subset\ H^0(\mathcal{O}_{U_{ij}^{(n+1)}}\otimes\mathrm{End}(\mathbb{C}^r)) ,$$

in the sense of 2.1.1, i.e. the cocycle relation is $\beta_{ik} = \beta_{ij} + \theta_{ij}\beta_{jk}\theta_{ij}^{-1}$. Let

$$0 \longrightarrow E\otimes L^n \longrightarrow \mathcal{E}' \longrightarrow \mathbb{E} \longrightarrow 0$$

be the extension, corresponding to $\sigma + \Psi(\beta)$. Then $\mathcal{E}'$ is represented by the family $((1+\beta_{ij})\theta_{ij})$. This description is similar to that of 2.1.7.

If $\Delta(\mathbb{E}) = 0$ then ϕ is surjective (because $\mathrm{im}(\phi)$ contains all the automorphisms of E).

This is also a consequence of

4.6.5. Lemma: *Let $0 \to E \otimes L^n \to \mathcal{E} \to \mathbb{E} \longrightarrow 0$ be an extension, corresponding to $\sigma \in \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E}, E \otimes L^n)$. Then the sheaf $\mathcal{E}$ is locally free if and only if $\phi(\sigma)$ is an isomorphism.*

Proof. We need only to prove the analogous following result: let U be a nonempty open subset of X, U_n, U_{n+1} the corresponding open subsets of X_n, X_{n+1} respectively, such that $\mathcal{I}_{U,U_{n+1}}$ can be extended to a line bundle $\mathbb{L}$ on U_{n+1}. Let r be a positive integer and $0 \to L^n_{|U} \otimes \mathbb{C}^r \to \mathcal{E} \to \mathcal{O}_{U_n} \otimes \mathbb{C}^r \to 0$ an exact sequence of sheaves on U_{n+1}, and $\phi \in \mathrm{End}(L^n_{|U} \otimes \mathbb{C}^r)$ the associated morphism. Then we have $\mathcal{O} \simeq \mathcal{O}_{U_{n+1}} \otimes \mathbb{C}^r$ if and only ϕ is an isomorphism.

The result follows easily from the following fact, using the locally free resolution of Lemma 4.6.2: $\mathcal{E}$ is isomorphic to the cokernel of the morphism

$$L^n_{|U} \otimes \mathbb{C}^r \xrightarrow{(\phi,i)} (L^n_{|U} \otimes \mathbb{C}^r) \oplus (\mathcal{O}_{U_{n+1}} \otimes \mathbb{C}^r) \ ,$$

where i is the canonical inclusion (cf. [6], 4.2). □

It follows that when $\mathcal{E}$ is locally free we can always assume that $\phi(\sigma) = I_E$.

4.6.6. *Isomorphism classes* – We have an exact sequence $0 \to L^n \to \mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n} \to 0$, whence a connecting homomorphism

$$\delta^0 : H^0(X_n, \mathcal{O}_{X_n}) \longrightarrow H^1(X, L^n) \ .$$

Since $E^* \otimes E \otimes L^n \simeq L^n \oplus (\mathcal{A}d(E) \otimes L^n)$, we can see $\mathrm{im}(\delta^0)$ as a subspace of $\mathrm{Ext}^1_{\mathcal{O}_X}(E, E \otimes L^n)$.

Recall that $\mathbb{E}$ is called *simple* if the only endomorphisms of $\mathbb{E}$ are the homotheties, i.e. if the canonical morphism $H^0(\mathcal{O}_{X_n}) \to \mathrm{End}(\mathbb{E})$ is an isomorphism.

4.6.7. Proposition: *Suppose that $\mathbb{E}$ is simple. Let*

$$0 \longrightarrow E \otimes L^n \longrightarrow \mathcal{E} \to \mathbb{E} \longrightarrow 0$$

be an extension on X_{n+1}, with $\mathcal{E}$ locally free, corresponding to $\sigma \in \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E}, E \otimes L^n)$, and

$$0 \longrightarrow E \otimes L^n \longrightarrow \mathcal{E}' \longrightarrow \mathbb{E} \longrightarrow 0$$

another extension, corresponding to $\sigma + \Psi(\beta)$, $\beta \in \mathrm{Ext}^1_{\mathcal{O}_X}(E, E \otimes L^n)$. Then $\mathcal{E}' \simeq \mathcal{E}$ if and only if $\beta \in \mathrm{im}(\delta^0)$.

Proof. We use the notations of 4.6.3. Let $\theta'_i : \mathcal{E}'_{|U_i^{(n+1)}} \to \mathcal{O}_{U_i^{(n+1)}} \otimes \mathbb{C}^r$ be trivializations, with $\theta'_{ij} = (1+\beta_{ij})\theta_{ij}$.

There is an automorphism $f : \mathcal{E} \to \mathcal{E}'$ if and only for every $i \in I$ there exists an automorphism f_i of $\mathcal{O}_{U_i^{(n+1)}} \otimes \mathbb{C}^r$ such that the following square is commutative

$$\begin{tikzcd} \mathcal{E}_{|U_i^{(n+1)}} \arrow[r, "\theta_i"] \arrow[d, "f"] & \mathcal{O}_{U_i^{(n+1)}} \otimes \mathbb{C}^r \arrow[d, "f_i"] \\ \mathcal{E}'_{|U_i^{(n+1)}} \arrow[r, "\theta'_i"] & \mathcal{O}_{U_i^{(n+1)}} \otimes \mathbb{C}^r \end{tikzcd}$$

Suppose that f exists. Then since $\mathbb{E}$ is simple the morphism $\mathcal{E}_{|X_n} = \mathbb{E} \to \mathcal{E}'_{|X_n} = \mathbb{E}$ induced by f is the multiplication by $\mu \in H^0(\mathcal{O}_{X_n})^*$. Hence we can write $f_i = \mu_i.I + \eta_i$, where $\mu_i \in H^0(\mathcal{O}_{X_{n+1}}(U_i^{(n+1)}))$ is such that $\mu_{i|U_i^{(n)}} = \mu$ and $\eta_i : \mathcal{O}_{U_i^{(n+1)}} \otimes \mathbb{C}^r \to \mathcal{I}_{X_n, X_{n+1}|U_i^{(n+1)}}$. Let $\mu_0 = \mu_{|X} \in \mathbb{C}^*$. We have on $U_{ij}^{(n+1)}$, ${\theta'}_i^{-1} f_i\theta_i = {\theta'}_j^{-1} f_j\theta_j = f$, i.e. $f_i\theta_{ij} = \theta'_{ij}f_j$, which gives

$$\beta_{ij} \;=\; \frac{\mu_i - \mu_j}{\mu_j} + \frac{\eta_i - \theta_{ij}\eta_j\theta_{ij}^{-1}}{\mu_0} \;.$$

By 2.1.3, the family $(\eta_i - \theta_{ij}\eta_j\theta_{ij}^{-1})$ is a coboundary, hence we can suppose that $\beta_{ij} = \dfrac{\mu_i - \mu_j}{\mu_j}$.

Then, since $\mu_i - \mu_j \in H^0(U_{ij}^{(n+1)}, \mathcal{I}_{X_n, X_{n+1}})$, we have $\dfrac{\mu_i - \mu_j}{\mu_j} = \dfrac{\mu_i}{\mu_0} - \dfrac{\mu_j}{\mu_0}$. Hence $\beta = \delta^0\Big(\dfrac{\mu}{\mu_0}\Big)$.

The converse is proved in the same way. □

4.7. Canonical filtrations, quasi locally free sheaves and Serre duality

Let $P \in X$ be a closed point, $z \in \mathcal{O}_{Y,P}$ an equation of X and M a $\mathcal{O}_{X_n,P}$-module of finite type. Let $\mathcal{E}$ be a coherent sheaf on X_n.

The two *canonical filtrations* are useful tools to study the coherent sheaves on primitive multiple curves.

4.7.1. *First canonical filtration* – The first canonical filtration of M is

$$M_n = \{0\} \subset M_{n-1} \subset \cdots \subset M_1 \subset M_0 = M$$

where for $0 \leq i < n$, M_{i+1} is the kernel of the surjective canonical morphism $M_i \to M_i \otimes_{\mathcal{O}_{n,P}} \mathcal{O}_{X,P}$. So we have

$$M_i/M_{i+1} \;=\; M_i \otimes_{\mathcal{O}_{n,P}} \mathcal{O}_{X,P}, \qquad M/M_i \;\simeq\; M \otimes_{\mathcal{O}_{n,P}} \mathcal{O}_{X_i,P}, \qquad M_i \;=\; z^iM.$$

One defines similarly the *first canonical filtration of* $\mathcal{E}$: it is the filtration

$$\mathcal{E}_n = 0 \subset \mathcal{E}_{n-1} \subset \cdots \subset \mathcal{E}_1 \subset \mathcal{E}_0 = \mathcal{E}$$

such that for $0 \leq i < n$, $\mathcal{E}_{i+1}$ is the kernel of the canonical surjective morphism $\mathcal{E}_i \to \mathcal{E}_{i|X}$. So we have $\mathcal{E}_i/\mathcal{E}_{i+1} = \mathcal{E}_{i|X}$, $\mathcal{E}/\mathcal{E}_i = \mathcal{E}_{|X_i}$.

4.7.2. *Second canonical filtration* – One defines similarly the *second canonical filtration of* M: it is the filtration

$$M^{(0)} = \{0\} \subset M^{(1)} \subset \cdots \subset M^{(n-1)} \subset M^{(n)} = M$$

with $M^{(i)} = \big\{u \in M; z^i u = 0\big\}$. If $M_n = \{0\} \subset M_{n-1} \subset \cdots \subset M_1 \subset M_0 = M$ is the first canonical filtration of M, we have $M_i \subset M^{(n-i)}$ for $0 \leq i \leq n$.

One defines in the same way the *second canonical filtration of* $\mathcal{E}$:

$$\mathcal{E}^{(0)} = \{0\} \subset \mathcal{E}^{(1)} \subset \cdots \subset \mathcal{E}^{(n-1)} \subset \mathcal{E}^{(n)} = \mathcal{E}.$$

Let P be a closed point of X. Let M be a $\mathcal{O}_{X_n,P}$-module of finite type. Then M is called *quasi free* if there exist non negative integers $m_1, \ldots, m_n$ and an isomorphism $M \simeq \oplus_{i=1}^n m_i \mathcal{O}_{X_i,P}$. The integers $m_1, \ldots, m_n$ are uniquely determined: it is easy to recover them from the first canonical filtration of M. We say that $(m_1, \ldots, m_n)$ is the *type* of M.

Let $\mathcal{E}$ be a coherent sheaf on X_n. We say that $\mathcal{E}$ is *quasi free at* P if $\mathcal{E}_P$ is quasi free, and that $\mathcal{E}$ is *quasi locally free* on a nonempty open subset $U \subset X$ if it is quasi free at every point of U. If $U = X$ we say that $\mathcal{E}$ is quasi locally free. In this case the types of the modules $\mathcal{E}_x$, $x \in X$, are the same, and for every $x \in X$ there exists a neighbourhood $V \subset X_n$ of x and an isomorphism

$$\mathcal{E}_{|V} \;\simeq\; \sum_{n=1}^{n} m_i \mathcal{O}_{X_i \cap V} \ . \tag{8}$$

The proof of the following theorem is the same as in the case $\dim(X) = 1$ (th. 5.1.3 of [8]).

4.7.3. Theorem: *The following two assertions are equivalent:*

(i) *The* $\mathcal{O}_{X_n,P}$*-module* M *is quasi free.*
(ii) *All the* M_i/M_{i+1} *are free* $\mathcal{O}_{X,P}$*-modules.*

It follows that

4.7.4. Theorem: *The following two assertions are equivalent:*

(i) $\mathcal{E}$ *is quasi locally free.*
(ii) *All the* $\mathcal{E}_i/\mathcal{E}_{i+1}$ *are locally free on* X.

For every coherent sheaf $\mathcal{E}$ on X_n there exists a nonempty open subset $U \subset X$ such that $\mathcal{E}$ is quasi locally free on U.

4.7.5. Proposition: *Let* $\mathcal{E}$ *be a quasi locally free sheaf on* X_n. *Then*

(i) $\mathcal{E}$ *is reflexive,*

(ii) *for every positive integer* i *we have* $\mathcal{E}xt^i_{\mathcal{O}_{X_n}}(\mathcal{E}, \mathcal{O}_{X_n}) = 0$.

(iii) *for every vector bundle* $\mathbb{F}$ *on* X_n *and every integer* i *we have*
$\mathrm{Ext}^i_{\mathcal{O}_{X_n}}(\mathcal{E}, \mathbb{F}) \simeq H^i(X_n, \mathcal{E}^\vee \otimes \mathbb{F})$.

Proof. The assertion (iii) follows easily from (i), with the Ext spectral sequence. Assertions (i) and (ii) are local, so we can replace X_n with $U\times\mathbf{Z}_n$, where U is an open subset of X_n. Let $U_i=U\times\mathbf{Z}_i$, for $1\leq i\leq n$. The results follow from (8) and the free resolutions

$$\ldots\ldots \longrightarrow \mathcal{O}_{U_n} \xrightarrow{\times t^i} \mathcal{O}_{U_n} \xrightarrow{\times t^{n-i}} \mathcal{O}_{U_n} \xrightarrow{\times t^i} \mathcal{O}_{U_i}$$

□

4.7.6. *Serre duality for quasi locally free sheaves* – Suppose that X is projective of dimension d. It follows from Proposition 4.7.5 that

4.7.7. Proposition: *For every $i\geq 0$ and every quasi locally free sheaf $\mathcal{E}$ on X_n, there is a functorial isomorphism*

$$H^i(X_n,\mathcal{E}) \ \simeq \ H^{d-i}(X_n,\mathcal{E}^\vee\otimes\omega_{X_n})^* \ .$$

4.7.8. Corollary: *We have* $\omega_{X_n|X}\simeq\omega_X\otimes L^{1-n}$.

The proof is similar to that of Proposition 7.2, III, of [23].

4.8. Simple vector bundles

We use the notations of 4.1.

4.8.1. Lemma: *Suppose that the restriction map* $H^0(\mathcal{O}_{X_n})\to H^0(\mathcal{O}_{X_{n-1}})$ *is surjective, that* $\mathbb{E}_{|X_{n-1}}$ *is simple on* X_{n-1} *and* $\mathrm{Hom}(E,E\otimes L^{n-1})=H^0(X,L^{n-1})$. *Then* $\mathbb{E}$ *is simple.*

Proof. Let $f\in\mathrm{End}(\mathbb{E})$. Then the induced morphism $\mathbb{E}_{|X_{n-1}}\to\mathbb{E}_{|X_{n-1}}$ is the multiplication by some $\lambda\in H^0(\mathcal{O}_{X_{n-1}})$. Let $\mu\in H^0(\mathcal{O}_{X_n})$ be such that $\mu_{|X_{n-1}}=\lambda$. Then the restriction of $f-\mu I_{X_n}$ to X_{n-1} vanishes. It follows that $f-\mu I_{X_n}$ can be factorized in the following way

$$\mathbb{E} \twoheadrightarrow \mathbb{E}_{|X}=E \longrightarrow E\otimes L^{n-1}=\mathbb{E}^{(1)} \hookrightarrow \mathbb{E}$$

Since $\mathrm{Hom}(E,E\otimes L^{n-1})=H^0(X,L^{n-1})$, $f-\mu I_{X_n}$ is the multiplication by some $\eta\in H^0(X,L^{n-1})\subset H^0(\mathcal{O}_{X_n})$, and $f=(\mu+\eta)I_{X_n}$. □

It follows easily by induction on n that

4.8.2. Corollary: *Suppose that L is a non trivial ideal sheaf, or* $\deg(L)<0$ *if X is a curve, and that $E=\mathbb{E}_{|X}$ is simple. Then $\mathbb{E}$ is simple.*

5. Canonical class associated to a vector bundle

5.1. Definition

Let Z be a scheme over $\mathbb{C}$ and E a vector bundle on Z of rank r. To E one associates an element $\nabla_0(E)$ of $H^1(Z, E \otimes E^* \otimes \Omega_Z)$, called the *canonical class of* E (cf. [16], 3-). If L is a line bundle on Z, then $\nabla_0(L) \in H^1(Z, \Omega_Z)$. If Z is smooth and projective, $L = \mathcal{O}_Z(Y)$, where $Y \subset Z$ is a smooth hypersurface, then $\nabla_0(L)$ is the cohomology class of Y.

Let $(Z_i)_{i \in I}$ be an open cover of Z such that E is defined by a cocycle (θ_{ij}), $\theta_{ij} \in \mathrm{GL}(r, \mathcal{O}_{Z_{ij}})$. Then $((d\theta_{ij})\theta_{ij}^{-1})$ is a cocycle (in the sense of 2.1.4) which represents $\nabla_0(E)$.

We have $E \otimes E^* \simeq \mathcal{A}d(E) \oplus \mathcal{O}_Z$, so $H^1(E \otimes E^* \otimes \Omega_Z) \simeq H^1(Z, \mathcal{A}d(E) \otimes \Omega_Z) \oplus H^1(Z, \Omega_Z)$. The component of $\nabla_0(E)$ in $H^1(Z, \Omega_Z)$ is $\dfrac{1}{r}\nabla_0(\det(E))$.

If L_1, L_2 are line bundles on Z, then

$$\nabla_0(L_1 \otimes L_2) \ = \ \nabla_0(L_1) + \nabla_0(L_2) \ . \tag{9}$$

Let S be a scheme over $\mathbb{C}$ and $s \in S$ a closed point. Let Z be a scheme and $p_Z : Z \times S \to Z$, $p_S : Z \times S \to S$ the projections. Let $\mathbb{E}$ be a vector bundle over $Z \times S$. We have

$$\nabla_0(\mathbb{E}) \ \in \ H^1(Z \times S, \mathbb{E} \otimes \mathbb{E}^* \otimes \Omega_{Z \times S}) = H^1(Z \times S, \mathbb{E} \otimes \mathbb{E}^* \otimes p_Z^*(\Omega_Z)) \oplus H^1(Z \times S, \mathbb{E} \otimes \mathbb{E}^* \otimes p_S^*(\Omega_S)) \ .$$

By restricting to $Z \times \{s\}$ we get a canonical map

$$r_s : H^1(Z \times S, \mathbb{E} \otimes \mathbb{E}^* \otimes \Omega_{Z \times S}) \longrightarrow H^1(Z, \mathbb{E}_s \otimes \mathbb{E}_s^* \otimes \Omega_Z) \ .$$

From Lemma 3.2.1 of [16] we have

$$r_s(\nabla_0(\mathbb{E})) \ = \ \nabla_0(\mathbb{E}_s) \ . \tag{10}$$

5.2. The case of smooth projective varieties

Let Z be a scheme over $\mathbb{C}$, T a projective integral variety and $\mathcal{L}$ a line bundle on $Z \times T$, viewed as a family of line bundles on Z parametrized by T. For any closed point $t \in T$, let $\mathcal{L}_t = \mathcal{L}_{|Z \times \{t\}} \in \mathrm{Pic}(Z)$.

5.2.1. Proposition: *The map*

$$\begin{aligned} \nabla_{\mathcal{L}} : T &\longrightarrow H^1(Z, \Omega_Z) \\ t &\longmapsto \nabla_0(\mathcal{L}_t) \end{aligned}$$

is constant.

Proof. Let $t \in T$ and $S \subset T$ an affine open neighbourhood of t. Then we have $H^1(Z \times S, \Omega_{Z \times S}) \simeq (\mathcal{O}_S(S) \otimes H^1(Z, \Omega_Z) \oplus (H^0(S, \Omega_S) \otimes H^1(T, \mathcal{O}_T))$. Let

$$\nabla_0(\mathcal{L}_{|Z \times S}) \ = \ \sum_{k=1}^{p} f_k \otimes \nabla_k + \eta \ ,$$

with $f_1,\ldots,f_p \in \mathcal{O}_S(S)$, $\nabla_1,\ldots,\nabla_p \in H^1(Z,\Omega_Z)$ and $\eta \in H^0(S,\Omega_S)\otimes H^1(T,\mathcal{O}_T))$. Then by 5.1 we have $\nabla_{\mathcal{L}}(s) = \sum_{k=1}^{p} f_k(s)\nabla_k$ for every $s \in S$. Hence $\nabla_{\mathcal{L}}$ is a regular map to a finite dimensional vector space. Since T is projective, $\nabla_{\mathcal{L}}$ must be constant. □

5.2.2. Corollary: *If Z is smooth and projective, and $L \in \mathrm{Pic}(Z)$, then $\nabla_0(L)$ is invariant by deformation of L.*

Proof. This follows from the fact that the connected component of $\mathrm{Pic}(Z)$ containing L parametrizes a projective family of line bundles containing L. □

5.2.3. *The differential morphism* – Let $\eta \in H^1(Z,\mathcal{O}_Z)$, represented by a cocycle (η_{ij}) (with respect to an open cover $(Z_i)_{i\in I}$ of Z), $\eta_{ij} \in \mathcal{O}_Z(Z_{ij})$. Then $(d\eta_{ij})$ is a cocycle which represents $d_Z(\eta) \in H^1(\mathbb{Z},\Omega_Z)$ (which depends only on η). This defines a linear map

$$d_Z : H^1(Z,\mathcal{O}_Z) \longrightarrow H^1(Z,\Omega_Z)\ .$$

The following result and its proof are classical:

5.2.4. Proposition: *If Z is smooth and projective, then* $d_Z = 0$.

Proof. Let $\mathcal{L}$ be a Poincaré bundle on $\mathrm{Pic}^0(Z)\times Z$. We can compute the canonical classes in analytic cohomology. In particular, there exist an open cover (Z_i) of Z and a neighbourhood S of $\mathcal{O}_Z \in \mathrm{Pic}^0(Z)$ (for the usual topology) such that $\mathcal{L}_{|Z\times S}$ is represented by a cocycle (θ_{ij}), where θ_{ij} is an invertible analytic function on $Z_{ij}\times S$. Let $\boldsymbol{u} \in H^1(Z,\mathcal{O}_Z) = T_{\mathcal{O}_Z}(\mathrm{Pic}^0(Z))$. There exists an analytic map $\iota : U \to S$ (where U is a neighbourhood of 0 in $\mathbb{C}$), which is an embedding such that $\phi'(0) : \mathbb{C} \to T_{\mathcal{O}_Z}(\mathrm{Pic}^0(Z))$ sends 1 to $\boldsymbol{u}$. Let

$$\alpha_{ij}^U \ = \ \alpha_{ij|Z_{ij}\times U}\ , \qquad \alpha_{ij}^0 \ = \ \alpha_{ij|Z_{ij}\times\{0\}}\ .$$

we can write

$$\alpha_{ij}^U(x,t) \ = \ \alpha_{ij}^0(x) + t\alpha_{ij}^0(x)\beta_{ij}(x,t)\ ,$$

where β_{ij} is an analytic map. Let $\beta_{ij}^0 \ = \ \beta_{ij|Z_{ij}\times\{0\}}$. We have $\beta_{ij}^0+\beta_{jk}^0=\beta_{ik}^0$ on Z_{ijk}, and the family (β_{ij}^0) is a cocycle which represents $\boldsymbol{u}$.

A simple computation shows that we can write

$$\frac{d\alpha_{ij}^U}{\alpha_{ij}^U} \ = \ \frac{d\alpha_{ij}^0}{\alpha_{ij}^0} + td\beta_{ij} + \rho_{ij}dt + t^2\mu_{ij}\ ,$$

where ρ_{ij} is an analytic function on $Z_{ij}\times U$ and $\mu_{ij} \in H^0(Z_{ij}\times U, \Omega_{Z_{ij}\times U})$. It follows that

$$d(\boldsymbol{u}) \ = \ (\nabla_{\mathcal{L}}\circ\iota)'(1)\ .$$

Since $\nabla_{\mathcal{L}}\circ\iota$ is constant by Proposition 5.2.1, we have $d_Z(\boldsymbol{u}) = 0$. □

Similarly, in the general case, let Y be a scheme and $\mathcal{L}$ a line bundle on $Z\times Y$. Let $y \in Y$ be a closed point and $\omega_y : T_yY \to H^1(Z,\mathcal{O}_Z)$ the Kodaïra-Spencer map. Then we have

5.2.5. Proposition: *We have* $T\nabla_{\mathcal{L},y} = d_Z \circ \omega_y$.

5.3. The dualizing sheaf of a primitive double scheme

Let X be a smooth, projective and irreducible variety. Let X_2 be a primitive double scheme, with underlying smooth variety X, and associated line bundle L on X. We have a canonical exact sequence

$$0 \longrightarrow L \longrightarrow \mathcal{O}_{X_2} \xrightarrow{\zeta} \mathcal{O}_X \longrightarrow 0$$

associated with $\eta \in H^1(X, T_X \otimes L)$ (cf. 4.1.3). We will prove

5.3.1. Theorem: *The map* $H^1(\zeta) : H^1(X_2, \mathcal{O}_{X_2}) \to H^1(X, \mathcal{O}_X)$ *is surjective.*

The following result generalizes Proposition 1.5 of [20]:

5.3.2. Corollary: *We have* $\omega_{X_2} \simeq \mathcal{O}_{X_2}$ *if and only if* $\omega_X \simeq L$ *.*

Proof. If $\omega_{X_2} \simeq \mathcal{O}_{X_2}$ then $\omega_X \simeq L$ by Corollary 4.7.8. Conversely, suppose that $\omega_X \simeq L$. Then $\omega_{X_2|X} \simeq \mathcal{O}_X$. We have an exact sequence

$$0 \longrightarrow \omega_X = L \longrightarrow \omega_{X_2} \xrightarrow{\rho} \mathcal{O}_X \longrightarrow 0 \ .$$

and the associated long exact sequence

$$0 \longrightarrow H^0(X, \omega_X) \longrightarrow H^0(X_2, \omega_{X_2}) \xrightarrow{H^0(\rho)} H^0(X, \mathcal{O}_X) \xrightarrow{\delta} H^1(X, L) \ .$$

On the other hand, from the exact sequence

$$0 \longrightarrow L \longrightarrow \mathcal{O}_{X_2} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

we deduce the following

$$H^1(X_2, \mathcal{O}_{X_2}) \xrightarrow{H^1(\xi)} H^1(X, \mathcal{O}_X) \xrightarrow{\delta'} H^2(X, L) \ .$$

From Serre duality on X_2 we get the commutative square

$$\begin{tikzcd} H^0(X, \mathcal{O}_X) \arrow[r, "\delta"] \arrow[d, "\simeq"] & H^1(X, L) \arrow[d, "\simeq"] \\ H^2(X, L)^* \arrow[r, "{}^t\delta'"] & H^1(X, \mathcal{O}_X)^* \end{tikzcd}$$

From Theorem 5.3.1 we have $\delta' = 0$. Hence $\delta = 0$ and $H^0(\rho)$ is surjective. If $\sigma \in H^0(X_2, \omega_{X_2})$ is such that $H^0(\rho)(\sigma) = 1$, the corresponding morphism $\mathcal{O}_{X_2} \to \omega_{X_2}$ is an isomorphism, so $\omega_{X_2} \simeq \mathcal{O}_{X_2}$. □

We now prove Theorem 5.3.1, i.e. that the connecting morphism $\delta : H^1(X, \mathcal{O}_X) \to H^2(X, L)$ is null. We use the notations of 4.1. Let $\beta \in H^1(X, \mathcal{O}_X)$, defined by a cocycle (β_{ij}), $\beta_{ij} \in \mathcal{O}_X(U_{ij})$.

We can view β_{ij} as an element of $H^0(\mathcal{O}_{U_{ij}\times Z_2})$. Then $\delta(\beta)$ is represented by the cocycle (σ_{ijk}), with

$$\sigma_{ijk} \ = \ \beta_{ij} + \delta^*_{ij}(\beta_{jk}) - \beta_{ik}$$

(cf. 4.3). We have then

$$\sigma_{ijk} \ = \ D_{ij}(\beta_{jk}) \ . \tag{11}$$

Let

$$\Theta : H^1(X, T_X \otimes L) \otimes H^1(X, \Omega_X) \longrightarrow H^2(X, L)$$

be the canonical map. Then it follows from (11) that $\delta(\beta) = \Theta(\eta \otimes d_X(\beta))$. Since $d_X(\beta) = 0$ by Proposition 5.2.4, we have $\delta(\beta) = 0$.

6. Extensions of families of vector bundles to higher multiplicity

Let $Y = X_n$ be a primitive multiple scheme of multiplicity n, with underlying smooth projective irreducible variety X, and associated line bundle L on X.

If X_n can be extended to a primitive multiple scheme X_{n+1} of multiplicity $n+1$, we have a canonical exact sequence of sheaves on X_n

$$\Sigma_{X_{n+1}} : \qquad 0 \longrightarrow L^n \longrightarrow \Omega_{X_{n+1}|X_n} \longrightarrow \Omega_{X_n} \longrightarrow 0 \ , \tag{12}$$

corresponding to $\sigma_{\Omega_{X_{n+1}}} \in \mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\Omega_{X_n}, L^n)$.

Let S be a smooth algebraic variety over $\mathbb{C}$, and $p_{X_n} : X_n \times S \to X_n$, $p_S : X_n \times S \to S$ the projections. We can see $X_n \times S$ as a primitive multiple scheme of multiplicity n, with induced smooth variety $X \times S$ and associated line bundle $p^*_{X_n}(L)_{|X\times S}$. If X_n is extended to X_{n+1}, $X_{n+1} \times S$ as a primitive multiple scheme of multiplicity $n+1$ extending $X_n \times S$. We have $\Omega_{X_n\times S} = p^*_{X_n}(\Omega_{X_n}) \oplus p^*_S(\Omega_S)$ and

$$\sigma_{X_{n+1}\times S} \ \in \ \mathrm{Ext}^1_{\mathcal{O}_{X_n\times S}}(p^*_{X_n}(\Omega_{X_n}), p^*_{X_n}(L^n)) \oplus \mathrm{Ext}^1_{\mathcal{O}_{X_n\times S}}(p^*_S(\Omega_S), p^*_{X_n}(L^n)) \ .$$

Then $\Sigma_{X_{n+1}\times S}$ is $p^*_{X_n}(\Sigma_{X_n}) \oplus p^*_S(\Sigma)$, where Σ is the exact sequence

$$0 \longrightarrow 0 \longrightarrow \Omega_S = \Omega_S \longrightarrow 0 \ .$$

It follows that the component of $\sigma_{X_{n+1}\times S}$ in $\mathrm{Ext}^1_{\mathcal{O}_{X_n\times S}}(p^*_S(\Omega_S), p^*_{X_n}(L^n))$ vanishes.

6.1. The family of extensions to X_{n+1} of a vector bundle on X_n

Let $\mathbb{E}$ be a vector bundle on X_n, and $E = \mathbb{E}_{|X}$. We keep the notations of 4.6.

Let $V = \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E}, E \otimes L^n)$, and $\mathbb{P} = \mathbb{P}(V)$. Let $p : X_{n+1} \times \mathbb{P} \to X_{n+1}$, $q : X_{n+1} \times \mathbb{P} \to \mathbb{P}$ be the projections. If

$$0 \longrightarrow p^*(E \otimes L^n) \otimes q^*(\mathcal{O}_{\mathbb{P}}(1)) \longrightarrow \mathcal{E} \longrightarrow p^*(\mathbb{E}) \longrightarrow 0$$

is an exact sequence of sheaves on $X_{n+1} \times \mathbb{P}$, the induced sequence on X_{n+1}

$$\begin{tikzcd} 0 \arrow[r] & E \otimes L^n \otimes V^* \arrow[r] \arrow[d, equal] & p_*(\mathcal{E}) \arrow[r] & \mathbb{E} \arrow[r] \arrow[d, equal] & 0 \\ & p_*(p^*(E \otimes L^n) \otimes q^*(\mathcal{O}_{\mathbb{P}}(1))) & & p_*(p^*(\mathbb{E})) & \end{tikzcd}$$

is also exact. This defines a linear map

$$\mathbf{F} : \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}\times\mathbb{P}}}(p^*(\mathbb{E}), p^*(E \otimes L^n) \otimes q^*(\mathcal{O}_{\mathbb{P}}(1))) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E}, E \otimes L^n) \otimes V^* = V \otimes V^* .$$

6.1.1. Proposition: $\mathbf{F}$ *is an isomorphism.*

Proof. We prove first the surjectivity of $\mathbf{F}$. Let $\phi \in V^*$, inducing $\overline{\phi} \in H^0(q^*(\mathcal{O}_{\mathbb{P}}(1)))$, $\sigma \in V$, corresponding to the exact sequence

$$0 \longrightarrow E \otimes L^n \longrightarrow \mathcal{E} \longrightarrow \mathbb{E} \longrightarrow 0 .$$

We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & E \otimes L^n \arrow[r] \arrow[d, "I\otimes\phi"] & \mathcal{E} \arrow[r] \arrow[d] & \mathbb{E} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & E \otimes \mathbb{L}^n \otimes V^* \arrow[r] & \mathcal{F} \arrow[r] & \mathbb{E} \arrow[r] & 0 , \end{tikzcd}$$

where the exact sequence in the bottom corresponds to $\sigma \otimes \phi \in V \otimes V^*$ (cf. [6], prop. 4.3.2). This diagram is the image (by p_*) of the following commutative diagram on $X_{n+1} \times \mathbb{P}$

$$\begin{tikzcd} 0 \arrow[r] & p^*(E \otimes L^n) \arrow[r] \arrow[d, "I\otimes\overline{\phi}"] & p^*(\mathcal{E}) \arrow[r] \arrow[d] & p^*(\mathbb{E}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & p^*(E \otimes L^n) \otimes q^*(\mathcal{O}_{\mathbb{P}}(1)) \arrow[r] & p^*(\mathcal{F}) \arrow[r] & p^*(\mathbb{E}) \arrow[r] & 0 , \end{tikzcd}$$

where the exact sequence in the bottom is associated to $\sigma' \in \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}\times\mathbb{P}}}(p^*(\mathbb{E}), p^*(E \otimes L^n) \otimes q^*(\mathcal{O}_{\mathbb{P}}(1)))$ such that $\mathbf{F}(\sigma') = \sigma \otimes \phi$. It follows that $\sigma \otimes \phi \in \mathrm{im}(\mathbf{F})$, and that $\mathbf{F}$ is surjective.

Now we prove that $\mathbf{F}$ is injective. For any coherent sheaves $\mathcal{E}$, $\mathcal{F}$ on a scheme Y, let $\Gamma_{\mathcal{E},\mathcal{F}}$ be the natural exact sequence

$$0 \longrightarrow H^1(Y, \mathcal{H}om(\mathcal{E}, \mathcal{F})) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{E}, \mathcal{F}) \longrightarrow H^0(Y, \mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{E}, \mathcal{F}))$$

which is a part of $\mathbf{\Gamma}_{\mathcal{E},\mathcal{F}}$ (cf. 3.2). For $\Gamma_{\mathbb{E},E\otimes L^n\otimes V^*}$ on X_{n+1} we have

$$H^1(\mathcal{H}om(\mathbb{E}, E \otimes L^n \otimes V^*)) \ = \ \mathrm{Ext}^1_{\mathcal{O}_X}(E, E \otimes L^n) \otimes V^* ,$$

$$H^0(\mathcal{E}xt^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E}, E \otimes L^n \otimes V^*)) \ = \ \mathrm{End}(E) \otimes V^* .$$

For $\Gamma_{p^*(\mathbb{E}),p^*(E\otimes L^n)\otimes q^*(\mathcal{O}_{\mathbb{P}}(1))}$ on $X_{n+1} \times \mathbb{P}$ we have

$$\mathcal{H}om(p^*(\mathbb{E}), p^*(E \otimes L^n) \otimes q^*(\mathcal{O}_{\mathbb{P}}(1))) \ = \ p^*(\mathcal{H}om(E, E \otimes L^n)) \otimes q^*(\mathcal{O}_{\mathbb{P}}(1)) ,$$

whence

$$H^1(\mathcal{H}om(p^*(\mathbb{E}), p^*(E \otimes L^n) \otimes q^*(\mathcal{O}_{\mathbb{P}}(1)))) \ = \ \mathrm{Ext}^1_{\mathcal{O}_X}(E, E \otimes L^n) \otimes V^* .$$

We have

$$\mathcal{E}xt^1_{\mathcal{O}_{X_{n+1}\times\mathbb{P}}}(p^*(\mathbb{E}), p^*(E\otimes L^n)\otimes q^*(\mathcal{O}_{\mathbb{P}}(1))) \ = \ \mathcal{E}nd(p^*(E))\otimes q^*(\mathcal{O}_{\mathbb{P}}(1)) \ ,$$

whence

$$H^0(\mathcal{E}xt^1_{\mathcal{O}_{X_{n+1}\times\mathbb{P}}}(p^*(\mathbb{E}), p^*(E\otimes L^n)\otimes q^*(\mathcal{O}_{\mathbb{P}}(1)))) \ = \ \mathrm{End}(E)\otimes V^* \ .$$

Let $W = \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}\times\mathbb{P}}}(p^*(\mathbb{E}), p^*(E\otimes L^n)\otimes q^*(\mathcal{O}_{\mathbb{P}}(1)))$. We have then a commutative diagram (a morphism from $\Gamma_{p^*(\mathbb{E}),p^*(E\otimes L^n)\otimes q^*(\mathcal{O}_{\mathbb{P}}(1))}$ to $\Gamma_{\mathbb{E},E\otimes L^n\otimes V^*}$)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_{\mathcal{O}_X}(E,E\otimes L^n)\otimes V^* \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d, "\mathbf{F}"] & \mathrm{End}(E)\otimes V^* \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Ext}^1_{\mathcal{O}_X}(E,E\otimes L^n)\otimes V^* \arrow[r] & \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E},E\otimes L^n)\otimes V^* \arrow[r] & \mathrm{End}(E)\otimes V^*
\end{tikzcd}$$

which implies easily the injectivity of $\mathbf{F}$. $\square$

6.1.2. *The universal family of extensions of* $\mathbb{E}$ – Let

$$\boldsymbol{\sigma} \in \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}\times\mathbb{P}}}(p^*(\mathbb{E}), p^*(E\otimes L^n)\otimes q^*(\mathcal{O}_{\mathbb{P}}(1))) \ \simeq \ V\otimes V^* \ ,$$

corresponding to I_V, and

$$0 \longrightarrow p^*(E\otimes L^n)\otimes q^*(\mathcal{O}_{\mathbb{P}}(1)) \longrightarrow \mathcal{E} \longrightarrow p^*(\mathbb{E}) \longrightarrow 0$$

the corresponding extension on $X_{n+1}\times\mathbb{P}$. The sheaf $\mathcal{E}$ is flat on $\mathbb{P}$, and for every $y\in\mathbb{P}$, the restriction of this exact sequence to $X_{n+1}\times\{y\}$

$$0 \longrightarrow E\otimes L^n\otimes y^* \longrightarrow \mathcal{E}_y \longrightarrow \mathbb{E} \longrightarrow 0$$

is exact (by Lemma 6.2.1), associated to the inclusion $i : y \hookrightarrow V$,
$i \in y^*\otimes V = \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E}, E\otimes L^n\otimes y^*)$.

Let $\delta : \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E}, E\otimes L^n) \to \mathrm{End}(E)$ be the canonical map, and $V_{\mathbb{E}} = \delta^{-1}(I_E)$. Then $\mathcal{E}_{\mathbb{E}} = \mathcal{E}_{|X_{n+1}\times V_{\mathbb{E}}}$ is a family of locally free sheaves on X_{n+1} parametrized by $V_{\mathbb{E}}$, flat on $V_{\mathbb{E}}$. Hence it is a vector bundle on $X_{n+1}\times V_{\mathbb{E}}$.

Let $p_X : X_{n+1}\times S \to X_{n+1}$, $p_S : X_{n+1}\times S \to S$ be the projections. Let

$$\begin{aligned}
A &= H^1(X_{n+1}\times S, \mathcal{H}om(p_X^*(\mathbb{E}), p_X^*(E\otimes L^n))),\\
B &= H^0(X_{n+1}\times S, \mathcal{E}xt^1(p_X^*(\mathbb{E}), p_X^*(E\otimes L^n))),\\
A' &= H^0(\mathcal{O}_S)\otimes \mathrm{Ext}^1_{\mathcal{O}_X}(E, E\otimes L^n),\\
B' &= H^0(\mathcal{O}_S)\otimes \mathrm{End}(E).
\end{aligned}$$

We have canonical isomorphisms $\alpha : A \to A'$, $\beta : B\to B'$.

6.1.3. Lemma: *If S is affine, then we have a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}\times S}}(p_X^*(\mathbb{E}), p_X^*(E\otimes L^n)) \arrow[r] \arrow[d, "\gamma_S"] & B \arrow[d, "\beta"] \\
0 \arrow[r] & A' \arrow[r] & H^0(\mathcal{O}_S)\otimes \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E}, E\otimes L^n) \arrow[r] & B',
\end{tikzcd}$$

where the bottom exact sequence is $H^0(\mathcal{O}_S)\otimes\Gamma_{\mathbb{E},E\otimes L^n}$*, and* γ_S *is an isomorphism.*

Proof. Analogous to the proof of Proposition 6.1.1. $\square$

6.1.4. Proposition: *Let* $\mathcal{E}$ *be a vector bundle on* $X_{n+1}\times S$ *such that for every closed point* $s\in S$*,* $\mathcal{E}_{s|X_n}\simeq\mathbb{E}$*. Let* $s_0\in S$ *be a closed point. Then there exists a neighbourhood* $S_0\subset S$ *of* s_0 *and a morphism* $f:S_0\to V_{\mathbb{E}}$ *such that* $f^\sharp(\mathcal{E}_{\mathbb{E}})\simeq\mathcal{E}_{|X_{n+1}\times S_0}$.

Proof. Let $\mathbb{W}=\mathcal{H}om(p_X^*(\mathbb{E}),\mathcal{E}_{|X_n\times S})$. The sheaf $p_{S*}(\mathbb{W})$ is a vector bundle on S. For every $s\in S$ we have $p_{S*}(\mathbb{W})_s=\mathrm{Hom}(\mathbb{E},\mathcal{E}_{s|X_n})$. There exists an affine neighbourhood S_0 of s_0 and $\psi\in H^0(S_0,p_{S*}(\mathbb{W}))$ such that for every $s\in S_0$, $\psi(s)$ is an isomorphism. This section induces an isomorphism $\boldsymbol{\psi}:p_X^*(\mathbb{E})_{|X_n\times S_0}\to\mathcal{E}_{|X_n\times S_0}$.

Let $W=\mathcal{H}om(p_X^*(E),\mathcal{E}_{|X\times S})$; $p_{S*}(W)$ is a vector bundle on S, and for every $s\in S$, $p_{S*}(W)_s=\mathrm{Hom}(E,\mathcal{E}_{s|X})$. Let $\overline{\psi}\in H^0(S_0,p_{S*}(W))$ induced by ψ. This section induces an isomorphism $\overline{\boldsymbol{\psi}}:p_X^*(E\otimes L^n)_{|X\times S_0}\to\mathcal{E}_{|X_\times S_0}\otimes p_X^*(L^n)_{|X_\times S_0}$.

From the canonical exact sequence

$$0\longrightarrow\mathcal{E}_{|X\times S}\otimes p_X^*(L^n)\xrightarrow{\ i\ }\mathcal{E}\xrightarrow{\ p\ }\mathcal{E}_{|X_n\times S}\longrightarrow 0$$

we get

$$0\longrightarrow p_X^*(E\otimes L^n)_{|X\times S_0}\xrightarrow{\quad i\circ\overline{\boldsymbol{\psi}}\quad}\mathcal{E}_{|X_{n+1}\times S_0}\xrightarrow{\quad\overline{\boldsymbol{\psi}}^{-1}\circ p\quad}p_X^*(\mathbb{E})_{|X_{n+1}\times S_0}\longrightarrow 0\ ,$$

associated to $\sigma\in\mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}\times S_0}}(p_X^*(\mathbb{E})_{|X_{n+1}\times S_0},p_X^*(E\otimes L^n)_{|X\times S_0})$. Then $f=\gamma_{S_0}(\sigma)$ is a morphism $S_0\to V_{\mathbb{E}}$ such that $f^\sharp(\mathcal{E}_{\mathbb{E}})\simeq\mathcal{E}_{|X_{n+1}\times S_0}$. $\square$

6.2. Families and their extension to higher multiplicity

Let $\mathbb{E}$ be a vector bundle on $X_n\times S$, and $E=\mathbb{E}_{|X\times S}$. Let $p_X:X_{n+1}\times S\to X_{n+1}$, $p_S:X_{n+1}\times S\to S$ be the projections.

6.2.1. Lemma: *Let* $s\in S$ *be a closed point. Then we have* $\mathrm{Tor}^1_{\mathcal{O}_{X_{n+1}\times S}}(\mathbb{E},\mathcal{O}_{X_{n+1}\times\{s\}})=0$.

Proof. Let $x\in X$ and $W\subset X_{n+1}\times S$ a neighbourhood of (x,s) such that $\mathbb{E}$ is free on W. It suffices to prove that

$$\mathrm{Tor}^1_{\mathcal{O}_W}(\mathcal{O}_{W\cap(X_n\times S)},\mathcal{O}_{W\cap(X_{n+1}\times\{s\})})\ =\ 0.$$

For this it suffices to show that

$$\mathrm{Tor}^1_{\mathcal{O}_{X_{n+1}\times S}}(\mathcal{O}_{X_n\times S},\mathcal{O}_{X_{n+1}\times\{s\}})\ =\ 0\ .$$

Let U_{n+1} a neighbourhood of x in X_{n+1}, U_n the corresponding neighbourhood in X_n, such that $\mathcal{I}_{X,X_{n+1}|U_{n+1}}$ is globally generated by $t\in\mathcal{O}_{X_{n+1},x}(U_{n+1})$. The result follows from the locally free resolution of $\mathcal{O}_{U_n\times S}$:

$$\cdots\mathcal{O}_{U_{n+1}\times S}\xrightarrow{\quad\times t^n\quad}\mathcal{O}_{U_{n+1}\times S}\xrightarrow{\quad\times t\quad}\mathcal{O}_{U_n\times S}\longrightarrow 0\ .$$

$\square$

It follows that an extension $0 \to E \otimes p_X^*(L^n) \to \mathcal{E} \to \mathbb{E} \to 0$ restricts on $X_{n+1} \times \{s\}$ to an extension $0 \to E_s \otimes L^n \to \mathcal{E}_s \to \mathbb{E}_s \to 0$.

Let $\mathbb{E}$ be a vector bundle on $X_n \times S$, and $E = \mathbb{E}_{|X \times S}$. We have

$$\nabla_0(\mathbb{E}) \ \in \ \mathrm{Ext}^1_{\mathcal{O}_{X_n \times S}}(\mathbb{E}, \mathbb{E} \otimes p_{X_n}^*(\Omega_{X_n})) \oplus \mathrm{Ext}^1_{\mathcal{O}_{X_n \times S}}(\mathbb{E}, \mathbb{E} \otimes p_S^*(\Omega_S)) \ .$$

We have $\Delta(\mathbb{E}) = \overline{\sigma}_{X_{n+1} \times S} \nabla_0(\mathbb{E})$. The component of $\overline{\sigma}_{X_{n+1} \times S}$ in $\mathrm{Ext}^1_{\Omega_{X_n} \times S}(\mathbb{E} \otimes p_S^*(\Omega_S), \mathbb{E} \otimes p_{X_n}^*(L^n))$ vanishes. Let ∇_1 be the component of $\nabla_0(\mathbb{E})$ in $\mathrm{Ext}^1_{\mathcal{O}_{X_n \times S}}(\mathbb{E}, \mathbb{E} \otimes p_{X_n}^*(\Omega_{X_n}))$. We have then

$$\Delta(\mathbb{E}) \ = \ \overline{\sigma}_{X_{n+1} \times S} \nabla_1 \ . \tag{13}$$

6.2.2. *Obstruction to the extension of the family* – Let $p_{S,n}$, $p_{S,1}$ denote the projections $X_n \times S \to S$, $X \times S \to S$ respectively. We suppose that for $s \in S$,

(i) $\dim(\mathrm{Hom}(\mathbb{E}_s, \mathbb{E}_s \otimes \Omega_{X_n}))$ is independent of s.
(ii) $\dim(\mathrm{End}(E_s))$ is independent of s,
(iii) $\dim(\mathrm{Ext}^1_{\mathcal{O}_X}(E_s, E_s \otimes L^n))$ and $\dim(\mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\mathbb{E}_s, \mathbb{E}_s \otimes \Omega_{X_n}))$ are independent of s,
(iv) $\dim(\mathrm{Ext}^2_{\mathcal{O}_X}(E_s, E_s \otimes L^n))$ is independent of s.

It follows that $\mathbb{B} = R^1 p_{S,n*}(\mathbb{E}^* \otimes \mathbb{E} \otimes p_{X_n}^*(\Omega_{X_n}))$ and $\mathbb{D} = R^2 p_{S,1*}(E^* \otimes E \otimes L^n)$ are vector bundles on S, with

$$\mathbb{B}_s = \mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\mathbb{E}_s, \mathbb{E}_s \otimes \Omega_{X_n}) \ , \ \mathbb{D}_s = \mathrm{Ext}^2_{\mathcal{O}_X}(E_s, E_s \otimes L^n) \ .$$

Let $s \in S$ be a closed point and

$$R_s : \mathrm{Ext}^2_{\mathcal{O}_{X \times S}}(E, E \otimes p_X^*(L^n)) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_X}(E_s, E_s \otimes L^n)$$

be the canonical map.

6.2.3. Proposition: *We have $R_s(\Delta(\mathbb{E})) = \Delta(\mathbb{E}_s)$.*

Proof. Let $\mathbf{B} = \mathrm{Ext}^1_{\mathcal{O}_{X_n \times S}}(\mathbb{E}, \mathbb{E} \otimes p_{X_n}^*(\Omega_{X_n}))$, $\mathbf{D} = \mathrm{Ext}^2_{\mathcal{O}_{X \times S}}(E, E \otimes p_X^*(L^n))$. We have a canonical map $\phi : \mathbf{B} \to \mathbf{D}$ (multiplication by $\sigma_{X_{n+1} \times S}$), and $\psi : \mathbb{B} \to \mathbb{D}$, where for every $s \in S$, ψ_s is the canonical map

$$\mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\mathbb{E}_s, \mathbb{E}_s \otimes \Omega_{X_n}) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_{X_n \times S}}(\mathbb{E}_s, \mathbb{E}_s \otimes L^n)$$

(multiplication by $\sigma_{X_{n+1}}$), and a canonical diagram

$$\begin{tikzcd}
\mathbf{B} \arrow[r, "\Phi"] \arrow[d] & \mathbf{D} \arrow[d] \\
H^0(S, \mathbb{B}) \arrow[r] & H^0(S, \mathbb{D})
\end{tikzcd}$$

where the vertical arrows come from Leray's spectral sequence. Evaluation at s gives the diagram

$$\begin{tikzcd}
\mathbf{B} \arrow[r, "\Phi"] \arrow[d] & \mathbf{D} \arrow[d] \\
\mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\mathbb{E}_s, \mathbb{E}_s \otimes \Omega_{X_n}) \arrow[r] & \mathrm{Ext}^2_{\mathcal{O}_X}(E_s, E_s \otimes L^n) \ .
\end{tikzcd}$$

The result follows from (13), (10) and (7). $\square$

6.2.4. Corollary: *Suppose that S is affine, and that for every closed point $s \in S$ we have $\Delta(\mathbb{E}_s) = 0$. Then we have $\Delta(\mathbb{E}) = 0$.*

Proof. We have, since S is affine, by Leray's spectral sequence

$$\mathrm{Ext}^2_{\mathcal{O}_{X \times S}}(E, E \otimes p_X^*(L^n)) \ = \ H^0(S, \mathbb{D}) \ ,$$

and the result follows from Proposition 6.2.3. $\square$

In other words, if S is affine, and if for every $s \in S$, $\mathbb{E}_s$ can be extended to a vector bundle on X_{n+1}, then $\mathbb{E}$ can be extended to a vector bundle on $X_{n+1} \times S$.

6.2.5. *The universal family of extensions* – We suppose that the hypotheses of 6.2.2 are satisfied, that S is affine, and that for every $s \in S$, $\Delta(\mathbb{E}_s) = 0$. From the exact sequence of 4.6.3 it follows that $\dim(\mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E}_s, E_s \otimes L^n))$ is independent of s.

The relative Ext-sheaf (cf. [25]) $\mathbb{V} = \mathcal{E}xt^1_{p_S}(\mathbb{E}, E \otimes p_X^*(L^n))$ is locally free, with, for every closed point $s \in S$, $\mathbb{V}_s = \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E}_s, E_s \otimes L^n))$. Let $\mathcal{P} = \mathbb{P}(\mathbb{V})$. Let $p : X_{n+1} \times \mathcal{P} \to X_{n+1} \times S$, $q : X_{n+1} \times \mathcal{P} \to \mathcal{P}$ be the projections. If

$$0 \longrightarrow p^*(E \otimes p_X^*(L^n)) \otimes q^*(\mathcal{O}_{\mathcal{P}}(1)) \longrightarrow \mathcal{E} \longrightarrow p^*(\mathbb{E}) \longrightarrow 0$$

is an exact sequence of sheaves on $X_{n+1} \times \mathcal{P}$, the induced sequence on $X_{n+1} \times S$

$$\begin{tikzcd}
0 \arrow[r] & E \otimes L^n \otimes \mathbb{V}^* \arrow[r] \arrow[d, equal] & p_*(\mathcal{E}) \arrow[r] & \mathbb{E} \arrow[r] \arrow[d, equal] & 0 \\
& p_*(p^*(E \otimes p_X^*(L^n)) \otimes q^*(\mathcal{O}_{\mathcal{P}}(1))) & & p_*(p^*(\mathbb{E})) &
\end{tikzcd}$$

is also exact. This defines a linear map

$$\mathbf{\Phi} : \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1} \times \mathcal{P}}}(p^*(\mathbb{E}), p^*(E \otimes p_X^*(L^n)) \otimes q^*(\mathcal{O}_{\mathcal{P}}(1))) \longrightarrow H^0(S, \mathbb{V} \otimes \mathbb{V}^*) \ .$$

The proof of the following result is similar to that of Proposition 6.1.1:

6.2.6. Proposition: *$\mathbf{\Phi}$ is an isomorphism.*

Let

$$\boldsymbol{\sigma} \in \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1} \times \mathcal{P}}}(p^*(\mathbb{E}), p^*(E \otimes p_X^*(L^n)) \otimes q^*(\mathcal{O}_{\mathcal{P}}(1))) \ \simeq \ H^0(\mathbb{V} \otimes \mathbb{V}^*) \ ,$$

corresponding to $I_{\mathbb{V}}$, and

$$0 \longrightarrow p^*(E \otimes p_X^*(L^n)) \otimes q^*(\mathcal{O}_{\mathcal{P}}(1)) \longrightarrow \boldsymbol{\mathcal{E}} \longrightarrow p^*(\mathbb{E}) \longrightarrow 0$$

the corresponding extension on $X_{n+1} \times \mathcal{P}$. The sheaf $\boldsymbol{\mathcal{E}}$ is flat on $\mathcal{P}$. For every $s \in S$, and for every $y \in \mathcal{P}_s$, the restriction of this exact sequence to $X_{n+1} \times \{y\}$

$$0 \longrightarrow E_s \otimes L^n \otimes y^* \longrightarrow \boldsymbol{\mathcal{E}}_y \longrightarrow \mathbb{E}_s \longrightarrow 0$$

is exact (by Lemma 6.2.1), associated to the inclusion $i : y \hookrightarrow \mathbb{V}_s$,
$i \in y^* \otimes \mathbb{V}_s = \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E}_s, E_s \otimes L^n \otimes y^*)$.

The sheaf $\mathbb{V}$ is associated to the presheaf $\mathcal{V}$ defined by

$$\mathcal{V}(U) \ = \ \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}\times U}}(p^*(\mathbb{E})_{|X_{n+1}\times U}, p^*(E \otimes p_X^*(L^n))_{|X_{n+1}\times U})$$

for every open subset $U \subset S$. We have a canonical morphism

$$\mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}\times U}}(\mathbb{E}_{|X_{n+1}\times U}, (E \otimes p_X^*(L^n))_{|X_{n+1}\times U}) \longrightarrow$$
$$\to H^0(\mathcal{E}xt^1_{\mathcal{O}_{X_{n+1}\times U}}(\mathbb{E}_{|X_{n+1}\times U}, (E \otimes p_X^*(L^n))_{|X_{n+1}\times U})) = \mathrm{End}(E_{|X_{n+1}\times U})$$

(coming from the Ext spectral sequence) inducing a morphism $\boldsymbol{\delta} : \mathbb{V} \to p_*(\mathcal{E}nd(E))$. For every $s \in S$, $\boldsymbol{\delta}_s$ is the canonical map

$$\mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E}_s, E_s \otimes L^n) \longrightarrow H^0(\mathcal{E}xt^1_{\mathcal{O}_{X_{n+1}}}(\mathbb{E}_s, E_s \otimes L^n)) = \mathrm{End}(E_s)$$

We have a canonical section μ of $p_*(\mathcal{E}nd(E))$, such that $\mu(s) = I_{E_s}$ for every $s \in S$. Let

$$\mathbb{V}_{\mathbb{E}} \ = \ \boldsymbol{\delta}^{-1}(\mathrm{im}(\mu)) \ ,$$

and $\mathcal{P}_{\mathbb{E}}$ the image of $\mathbb{V}_{\mathbb{E}}$ in $\mathcal{P}$, which is a smooth subvariety of $\mathcal{P}$. The projection $\mathbb{V}_{\mathbb{E}} \to \mathcal{P}_{\mathbb{E}}$ is an isomorphism. The sheaf $\boldsymbol{\mathcal{E}}_{\mathbb{E}} = \mathcal{E}_{|X_{n+1}\times\mathcal{P}_{\mathbb{E}}}$ is a vector bundle on $X_{n+1} \times \mathcal{P}_{\mathbb{E}}$ (and $X_{n+1} \times \mathbb{V}_{\mathbb{E}}$). We have a canonical exact sequence on $X_{n+1} \times \mathbb{V}_{\mathbb{E}}$:

$$0 \longrightarrow \pi^\sharp(E \otimes p_X^*(L^n)) \longrightarrow \mathcal{E}_{\mathbb{E}} \longrightarrow \pi^\sharp(\mathbb{E}) \longrightarrow 0$$

(where π is the projection $\mathbb{V}_{\mathbb{E}} \to S$).

6.2.7. *The case of non affine S* – We don't suppose as in Corollary 6.2.4 that S is affine. It is easy to see that the schemes $\mathbb{V}_{\mathbb{E}_{|U}}$, $\mathcal{P}_{\mathbb{E}_{|U}}$ on affine open subsets $U \subset S$, can be glued in an obvious way to define the schemes $\mathbb{V}_{\mathbb{E}}$, $\mathcal{P}_{\mathbb{E}}$ over S. Similarly the vector bundles $\mathcal{E}_{\mathbb{E}_{|\mathcal{P}_{\mathbb{E}_{|U}}}}$ define the vector bundle $\mathcal{E}_{\mathbb{E}}$ on $X_{n+1} \times \mathcal{P}_{\mathbb{E}}$.

6.2.8. Proposition: *Let $f : T \to S$ be a morphism of schemes and $\mathcal{F}$ a vector bundle on $X_{n+1} \times T$ such that $\mathcal{F}_{|X_n\times T} \simeq f^\sharp(\mathbb{E})$. Then there exists a morphism $\delta : T \to \mathcal{P}_{\mathbb{E}}$ over S such that for every closed point $t \in T$ there exists an open neighbourhood T_0 ot t such that $\delta^\sharp(\mathcal{E}_{\mathbb{E}})_{|X_{n+1}\times T_0} \simeq \mathcal{F}_{|X_{n+1}\times T_0}$.*

Proof. Let $\mathbb{V}_f = \mathcal{E}xt^1_f(f^\sharp(\mathbb{E}), f^\sharp(E \otimes p_X^*(L^n))$ be the relative Ext-sheaf. By [1], Theorem 1.9, for every affine open subset $U \subset S$ containing $f(t)$, the canonical morphism

$$\mathcal{E}xt^1_{\mathcal{O}_{X_{n+1}\times U}}(\mathbb{E}_{|X_{n+1}\times U}, E_{|X_{n+1}\times U} \otimes p_X^*(L^n)) \otimes_{\mathcal{O}_U} \mathcal{O}_{f^{-1}(U)} \longrightarrow$$
$$\mathcal{E}xt^1_{\mathcal{O}_{X_{n+1}\times f^{-1}(U)}}(f^\sharp(\mathbb{E}_{|X_{n+1}\times U}), f^\sharp(E_{|X_{n+1}\times U}))$$

is an isomorphism. It follows that we have a canonical isomorphism

$$\mathbb{V}_f \ \simeq \ f^\sharp(\mathbb{V}) \ . \tag{14}$$

We define in an obvious way the schemes $\mathbb{V}_{f^\sharp(\mathbb{E})} \simeq \mathcal{P}_{f^\sharp(\mathbb{E})}$ analogous to $\mathbb{V}_\mathbb{E} \simeq \mathcal{P}_\mathbb{E}$ respectively, which are closed subschemes of $\mathbb{V}_f$. We have a canonical morphism of schemes $\phi : \mathbb{V}_f \to \mathbb{V}$, coming from (14), and we have $\phi(\mathbb{V}_{f^\sharp(\mathbb{E})}) \subset V_\mathbb{E}$. In the same way, we have a canonical exact sequence

$$0 \longrightarrow {\pi'}^\sharp(E \otimes p_X^*(L^n)) \longrightarrow \mathcal{E}_{f^\sharp(\mathbb{E})} \longrightarrow {\pi'}^\sharp(\mathbb{E}) \longrightarrow 0 \tag{15}$$

on $X_{n+1} \times \mathbb{V}_{f^\sharp(\mathbb{E})}$ (where π' is the projection $\mathbb{V}_{f^\sharp(\mathbb{E})} \to T$), and $\mathcal{E}_{f^\sharp(\mathbb{E})} \simeq \phi^\sharp(\mathcal{E}_\mathbb{E})$. The canonical exact sequence

$$\begin{tikzcd} 0 \arrow[r] & f^\sharp(E \otimes p_X^*(L^n)) \arrow[r] & \mathcal{F} \arrow[r] & f^\sharp(\mathbb{E}) \arrow[r] \arrow[d, equal] & 0 \\ & & & \mathcal{F}_{|X_n \times T} & \end{tikzcd}$$

defines a section $\sigma : T \to \mathbb{V}_{f^\sharp(\mathbb{E})}$ of $\mathbb{V}_{f^\sharp(\mathbb{E})} \to T$. Let

$$\delta = \phi \circ \sigma : T \longrightarrow \mathcal{P}_\mathbb{E} = \mathbb{V}_\mathbb{E} \ .$$

The inverse image of (15) by ϕ is an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & f^\sharp(E \otimes p_X^*(L^n)) \arrow[r] & \sigma^\sharp(\mathcal{E}_{f^\sharp(\mathbb{E})}) \arrow[r] \arrow[d, equal] & f^\sharp(\mathbb{E}) \arrow[r] & 0 \ . \\ & & \delta^\sharp(\mathcal{E}_\mathbb{E}) & & \end{tikzcd}$$

Over $f^{-1}(U)$ this exact sequence corresponds to the same element of $\mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}\times f^{-1}(U)}}(f^\sharp(\mathbb{E})_{|X_{n+1}\times f^{-1}(U)}, f^\sharp(E \otimes p_X^*(L^n))_{|X_{n+1}\times f^{-1}(U)})$, and this implies that there exists a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & f^\sharp(E \otimes p_X^*(L^n))_{|X_{n+1}\times f^{-1}(U)} \arrow[r] \arrow[d, equal] & \delta^\sharp(\mathcal{E}_\mathbb{E}))_{|X_{n+1}\times f^{-1}(U)} \arrow[r] \arrow[d, "\psi_U"] & f^\sharp(\mathbb{E})_{|X_{n+1}\times f^{-1}(U)} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & f^\sharp(E \otimes p_X^*(L^n))_{|X_{n+1}\times f^{-1}(U)} \arrow[r] & \mathcal{F}_{|X_{n+1}\times f^{-1}(U)} \arrow[r] & f^\sharp(\mathbb{E})_{|X_{n+1}\times f^{-1}(U)} \arrow[r] & 0 \end{tikzcd}$$

where ψ_U is an isomorphism. We take $T_0 = f^{-1}(U)$. □

6.2.9. *The affine bundle structure of* $\mathcal{P}_\mathbb{E}$ – Let $\mathbb{A} = R^1p_{S*}\big(\mathcal{H}om(\mathbb{E}, E \otimes p_X^*(L^n))\big)$. It is a vector bundle on S (by 6.2.2, (ii)), and for every closed point $s \in S$ we have $\mathbb{A}_s = \mathrm{Ext}^1_{\mathcal{O}_X}(E_s, E_s \otimes L^n)$. For every open subset $U \subset S$, we have a canonical injective map

$$\mathbb{A}(U) \hookrightarrow \mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}\times U}}(p^*(\mathbb{E})_{|X_{n+1}\times U}, p^*(E \otimes p_X^*(L^n))_{|X_{n+1}\times U}) = \mathcal{V}(U) \ ,$$

which defines an injective morphism of vector bundles $\mathbb{A} \to \mathbb{V}$. It follows that $\mathcal{P}_\mathbb{E}$ has a natural structure of affine bundle over S, with associated vector bundle $\mathbb{A}$. By Corollary 6.2.4, for every affine open subset $U \subset S$, $\mathcal{P}_{\mathbb{E}|U}$ has a section, hence is actually a vector bundle isomorphic to $\mathbb{A}_{|U}$.

Suppose that $\mathrm{Hom}(E_s, E_s \otimes L^n) = \{0\}$ for every $s \in S$. Since $\Delta(\mathbb{E}_s) = 0$ for every $s \in S$, we have $\Delta(\mathbb{E}) \in \mathrm{im}(\lambda)$, where λ is the canonical map

$$H^1(S, R^1 p_{S*}(\mathcal{H}om(\mathbb{E}, E \otimes p_X^*(L^n)))) \longrightarrow H^2(X_n \times S, \mathcal{H}om(\mathbb{E}, E \otimes p_X^*(L^n)))$$

(cf. 3.1).

Recall that $\eta(\mathcal{P}_{\mathbb{E}})$ is the element of $H^1(S, \mathbb{A})$ associated to $\mathcal{P}_{\mathbb{E}}$ (cf. 2.3). The following is a consequence of Proposition 3.2.6:

6.2.10. Theorem: *We have* $\eta(\mathcal{P}_{\mathbb{E}}) = \mathbb{C}.\Delta(\mathbb{E})$.

If we don't assume that $\mathrm{Hom}(E_s, E_s \otimes L^n) = \{0\}$ for every $s \in S$, then we get still an analogous more complicated result, using 3.2.7.

6.3. Families of simple vector bundles

We keep the notations of 4.6 and 6.2. We don't suppose that S is affine. To the hypotheses of 6.2.2 we add the following: for every $s \in S$, $\mathbb{E}_s$ is simple, and for every $m \in \mathcal{P}_{\mathbb{E}}$, $\mathcal{E}_{\mathbb{E},m}$ is simple.

We have obvious subbundles of the bundle $\mathbb{A}$ of 6.2.9:

$$\mathcal{O}_S \otimes \mathrm{im}(\delta^0) \ \subset \ \mathcal{O}_S \otimes H^1(X, L^n) \ \subset \ \mathbb{A}$$

(cf. 4.6.6). Let $\mathcal{Q}_{\mathbb{E}} = \mathcal{P}_{\mathbb{E}}/(\mathcal{O}_S \otimes \mathrm{im}(\delta^0))$ (cf. 2.3), and $\omega : \mathcal{P}_{\mathbb{E}} \to \mathcal{Q}_{\mathbb{E}}$ the projection, which is an affine bundle with associated vector bundle $\mathcal{O}_{\mathcal{Q}_{\mathbb{E}}} \otimes \mathrm{im}(\delta^0)$.

6.3.1. *Invariance of* $\mathcal{Q}_{\mathbb{E}}$ – Let $\mathbb{E}'$ be a vector bundle on $X_n \times S$, $E' = \mathbb{E}'_{|X \times S}$, and $f : \mathbb{E} \to \mathbb{E}'$ an isomorphism, inducing obvious isomorphisms $p_f : \mathcal{P}_{\mathbb{E}} \to \mathcal{P}_{\mathbb{E}'}$, $q_f : \mathcal{Q}_{\mathbb{E}} \to \mathcal{Q}_{\mathbb{E}'}$. Let $\mathbb{A}'$ be the vector bundle on S analogous to $\mathbb{A}$, corresponding to $\mathbb{E}'$, i.e. for every $s \in S$, $\mathbb{A}'_s = \mathrm{Ext}^1_{\mathcal{O}_X}(E'_s, E'_s \otimes L^n)$. Then there are canonical obvious isomorphisms $\mathbb{A}' \simeq \mathbb{A}$, $\mathcal{Q}'_{\mathbb{E}} \simeq \mathcal{Q}_{\mathbb{E}}$, an they do not depend on f.

6.3.2. Let $s \in S$. Then there exists an open neighbourhood S_0 of s such that:

– $\mathbb{A}_{|S_0}$ is trivial and we have a subbundle $\mathbb{D} \subset \mathbb{A}_{|S_0}$ such that $\mathbb{A}_{|S_0} = (\mathcal{O}_{S_0} \otimes \mathrm{im}(\delta^0)) \oplus \mathbb{D}$,
– there exists a section $\nu : S_0 \to \mathcal{P}_{\mathbb{E}|S_0}$ of $\pi : \mathcal{P}_{\mathbb{E}|S_0} \to S_0$.

It follows that the affine bundle $\omega : \mathcal{P}_{\mathbb{E}|S_0} \to \mathcal{Q}_{\mathbb{E}|S_0}$ has a section τ, so that we can view $\mathcal{Q}_{\mathbb{E}|S_0}$ as a closed subvariety of $\mathcal{P}_{\mathbb{E}|S_0}$. Let $\eta : \mathcal{Q}_{\mathbb{E}|S_0} \to S_0$ be the projection.

6.3.3. Proposition: *Let* $m_0 \in \pi^{-1}(s)$. *There exists an open affine neighbourhood* $M(m_0) \subset \pi^{-1}(S_0)$ *of* m *such that* $\mathcal{E}_{\mathbb{E}|X_{n+1} \times M(m_0)} \simeq \omega^*(\mathcal{E}_{\mathbb{E}|X_{n+1} \times \mathcal{Q}_{\mathbb{E}|S_0}})_{|X_{n+1} \times M(m_0)}$.

Proof. Let

$$\mathbb{U} \ = \ \mathcal{E}_{\mathbb{E}|X_{n+1} \times \pi^{-1}(S_0)} \ , \quad \mathbb{V} \ = \ \omega^\sharp(\mathcal{E}_{\mathbb{E}|X_{n+1} \times \mathcal{Q}_{\mathbb{E}|S_0}}) \ .$$

For every $m \in \pi^{-1}(S_0)$ we have $\mathbb{U}_m \simeq \mathbb{V}_m$ (by Proposition 4.6.7), and $\mathrm{Hom}(\mathbb{U}_m, \mathbb{V}_m) \simeq H^0(\mathcal{O}_{X_{n+1}})$.

Let $p_0 : X_{n+1} \times \pi^{-1}(S_0) \to \pi^{-1}(S_0)$ be the projection. Then $p_{0*}(\mathcal{H}om(\mathbb{U}, \mathbb{V}))$ is a vector bundle. It follows that there exists an open affine neighbourhood $M(m_0) \subset \pi^{-1}(S_0)$ of m such that there exists a section $\sigma \in H^0(M(m_0), p_{0*}(\mathcal{H}om(\mathbb{U}, \mathbb{V})))$ such that for every $m \in M(m_0)$, $\sigma(m) \in \mathrm{Hom}(\mathbb{U}_m, \mathbb{V}_m)$ is an isomorphism. We obtain in this way an isomorphism $\mathbb{U}_{|X_{n+1}\times M(m_0)} \simeq \mathbb{V}_{|X_{n+1}\times M(m_0)}$. □

By Proposition 6.3.3 we can find finite open covers $(S_i)_{1\leq i\leq k}$ of S, $(P_i)_{1\leq i\leq k}$ of $\mathcal{P}_{\mathbb{E}}$, such that $P_i \subset \pi^{-1}(S_i)$, and vector bundles $\mathbb{F}_i$ on $X_{n+1} \times \eta^{-1}(S_i)$ such that $\mathcal{E}_{\mathbb{E}|X_{n+1}\times P_i} \simeq \omega^\sharp(\mathbb{F}_i)_{|X_{n+1}\times P_i}$.

6.3.4. Theorem: *Let $f : T \to S$ be a morphism of schemes and $\mathcal{F}$ a vector bundle on $X_{n+1} \times T$ such that $\mathcal{F}_{|X_n\times T} \simeq f^\sharp(\mathbb{E})$. Then there exists a unique morphism $\gamma_{\mathcal{F}} : T \to \mathcal{Q}_{\mathbb{E}}$ such that for every closed point $t \in T$, there exists i, $1 \leq i \leq k$, such that $f(t) \in S_i$, a neighbourhood $U \subset T$ of t such that $\gamma_{\mathcal{F}}(U) \subset \eta^{-1}(S_i)$ and $\mathcal{F}_{|X_{n+1}\times U} \simeq \gamma_{\mathcal{F}}^\sharp(\mathbb{F}_i)_{|X_{n+1}\times U}$.*

Proof. This follows from Propositions 6.2.8 and 6.3.3. The unicity of $\gamma_{\mathcal{F}}$ comes from the fact that if $1 \leq i, j \leq k$, and $q_i \in \eta^{-1}(S_i)$, $q_j \in \eta^{-1}(S_j)$ are distinct, then by Proposition 4.6.7, the vector bundles $\mathbb{F}_{i,q_i}$ and $\mathbb{F}_{j,q_j}$ on X_{n+1} are not isomorphic. □

Since points in $\mathcal{Q}_{\mathbb{E}_s}$ correspond exactly to extensions of $\mathbb{E}_s$ to X_{n+1}, we have

6.3.5. Proposition: *Let $f : T \to S$ be a morphism of schemes and $\mathcal{F}_1$, $\mathcal{F}_2$ vector bundles on $X_{n+1} \times T$, and $\theta_i : \mathcal{F}_{i|X_n\times T} \simeq f^\sharp(\mathbb{E})$ for $i = 1, 2$. Then the induced isomorphism $\theta = \theta_2^{-1}\theta_1 : \mathcal{F}_{1|X_n\times T} \to \mathcal{F}_{2|X_n\times T}$ induces and isomorphism of affine bundles over T $\mathcal{Q}_{\mathcal{F}_2} \simeq \mathcal{Q}_{\mathcal{F}_1}$ which is independent of θ_1, θ_2.*

6.4. Moduli spaces of vector bundles

We use the notations of 6.2 and 6.3.

Let χ_n be a set of isomorphism classes of vector bundles on X_n. Suppose that there is a fine moduli space for χ_n, defined by a smooth irreducible variety M_n, an open cover $(\mathcal{M}_i)_{i\in I}$ of M_n, and for every $i \in I$,a vector bundle $\mathbb{E}_i$ on $X_n \times \mathcal{M}_i$ (cf. 2.2.2). We suppose that for every $\mathcal{E} \in \chi_n$, if $E = \mathcal{E}_{|X}$,

(i) $\mathcal{E}$ and E are simple,
(ii) $\dim(\mathrm{Ext}^1_{\mathcal{O}_X}(E, E \otimes L^n))$ and $\dim(\mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\mathcal{E}, \mathcal{E} \otimes \Omega_{X_n}))$ are independent of $\mathcal{E}$,
(iii) $\dim(\mathrm{Ext}^2_{\mathcal{O}_X}(E, E \otimes L^n))$ is independent of $\mathcal{E}$.
(iv) $\Delta(\mathcal{E}) = 0$.
(v) every vector bundle $\mathcal{F}$ on X_{n+1} such that $\mathcal{F}_{|X_n} \simeq \mathcal{E}$ is simple.

In particular the conditions of 6.2.2 are verified (for the vector bundles $\mathbb{E}_i$). Hence the smooth variety $\mathcal{P}_{\mathbb{E}_i}$ and the vector bundle $\mathcal{E}_{\mathbb{E}_i}$ are defined. Let χ_{n+1} the set of isomorphism classes of vector bundles $\mathcal{F}$ on X_{n+1} such that $\mathcal{F}_{|X_n} \in \chi_n$.

The condition (v) is satisfied if the hypotheses of Lemma 4.8.1 are satisfied by every $\mathcal{E} \in \chi_n$. This is the case if L is a non trivial ideal sheaf. Condition (v) is also satisfied if χ_n consists of line bundles.

For every $i \in I$, we will apply Theorem 6.3.4 to $\mathbb{E}_i$: we obtain an affine bundle $\eta_i : \mathcal{Q}_{\mathbb{E}_i} \to \mathcal{M}_i$, an open cover $(\mathcal{M}_{i,k})_{k\in J_i}$ of $\mathcal{M}_i$, and for every $k \in J_i$ a vector bundle $\mathbb{F}_{i,k}$ on $X_{n+1} \times \eta_i^{-1}(\mathcal{M}_{i,k})$.

Finally we get a finer open cover $(\mathcal{N}_j)_{j\in J}$ of M_n, and for every $j \in J$, a vector bundle $\mathbb{G}_j$ on $X_n \times \mathcal{N}_j$ satisfying the conditions of 2.2.2). In particular, for every $j,k \in J$, we can find an open cover of $\mathcal{N}_j \cap \mathcal{N}_k$ such that for every U in this cover, we have $\mathbb{G}_{j|\mathcal{N}_i\cap\mathcal{N}_k\cap U} \simeq \mathbb{G}_{k|\mathcal{N}_i\cap\mathcal{N}_k\cap U}$. Moreover, for every $j \in J$, we have a vector bundle $\mathbb{A}_j$ on $\mathcal{N}_j$, an affine bundle $\eta_j : \mathcal{Q}_{\mathbb{G}_j} \to \mathcal{N}_j$ with associated vector bundle $\mathbb{A}_j$, and a vector bundle $\mathbb{H}_j$ on $X_{n+1} \times \mathcal{Q}_{\mathbb{G}_i}$.

From Proposition 6.3.5, all the vector bundles $\mathbb{A}_j$, the affine bundles $\mathcal{Q}_{\mathbb{G}_j}$ can be glued to define respectively a vector bundle $\mathbf{A}$ on M_n and an affine bundle $\mathcal{Q}$ on M_n with associated vector bundle $\mathbf{A}$.

6.4.1. Lemma: *The scheme $\mathcal{Q}$ is separated.*

Proof. According to [24], Proposition 2.6.1, we need to prove the following statement: let Z be a smooth variety, U, V affine open subsets of Z such that $U \cap V$ is affine, E (resp. F) a vector bundle on U (resp. V), and $\phi : E_{|U\cap V} \to F_{|U\cap V}$ an isomorphism. Let Y be the scheme obtained by gluing E and F using ϕ. Then Y is separated. But this is obvious. □

Let χ_{n+1} denote the set of isomorphism classes of vector bundles on X_{n+1} whose restriction to X_n belongs to χ_n. From Theorem 6.3.4 follows easily

6.4.2. Theorem: *The smooth variety $\mathcal{Q}$, the open cover $(\mathcal{Q}_{\mathbb{G}_j})$ of $\mathcal{Q}$ and the vector bundles $\mathbb{H}_j$ on $X_{n+1} \times \mathcal{Q}_{\mathbb{G}_j}$ define a fine moduli space for χ_{n+1}.*

This proves Theorem 1.1.3 if $N_n = M_n$. The proof in the general case is similar, using the appropriate definition of N_n given in 6.5.

6.5. The subvariety of extensible vector bundles

We use the notations of 6.4, except hypothesis (iv). We will indicate briefly how to define the subvariety $N_n \subset M_n$ of bundles that can be extended to X_{n+1}. If this variety is smooth then one can easily modify the preceding proofs to obtain a fine moduli space for the vector bundles on X_{n+1} whose restriction to X_n belongs to χ, and this moduli space is an affine bundle over N_n.

Let $p_X : X \times \mathcal{M}_i \to X$, $p_M : X \times \mathcal{M}_i \to \mathcal{M}_i$ be the projections, and $E_i = \mathbb{E}_{i|X\times\mathcal{M}_i}$. Then

$$\mathbb{W}_i \;=\; R^1 p_{M*}(E_i^* \otimes E_i \otimes p_X^*(L^n))$$

is a vector bundle on $\mathcal{M}_i$, and since all the vector bundles in χ are simple, these vector bundles can be glued naturally to define a vector bundle $\mathbb{W}$ on M_n. We have a canonical section of each $\mathbb{W}_i$

$$\begin{aligned} \sigma_i : \mathcal{M}_i &\longrightarrow \mathbb{W}_i \\ m &\longmapsto \Delta(\mathbb{E}_{i,m}). \end{aligned}$$

These sections can be glued to define $\sigma \in H^0(M_n, \mathbb{W})$. The variety N_n is the zero subscheme scheme of σ.

7. Picard groups

We use the notations of 4. Let $Y = X_n$ be a primitive multiple scheme of multiplicity $n \geq 2$, with underlying smooth projective irreducible variety X, and associated line bundle L on X.

7.1. Inductive definition

If $\mathbf{P} = \mathrm{Pic}^d(X)$ is an irreducible component of $\mathrm{Pic}(X)$, let $\mathcal{L}_{n,\mathbf{P}}$ be the set of line bundles $\mathcal{B}$ on X_n such that $\mathcal{B}_{|X} \in \mathbf{P}$. Let $\mathbf{P}_0$ be the component that contains $\mathcal{O}_X$. Then $\mathcal{L}_{n,\mathbf{P}_0} \neq \emptyset$ (it contains $\mathcal{O}_{X_n}$). Suppose that for every $\mathbf{P}$ such that $\mathcal{L}_{n,\mathbf{P}}$ is not empty, there exists a fine moduli space $\mathrm{Pic}^{\mathbf{P}}(X_n)$ for $\mathcal{L}_{n,\mathbf{P}}$, which is a smooth variety, defined by an open cover $(P_i)_{i\in I}$ of $\mathrm{Pic}^{\mathbf{P}}(X_n)$, and for every $i \in I$, a line bundle $\mathbb{L}_i$ on $X_n \times P_i$.

The component $\mathrm{Pic}^{\mathbf{P}_0}(X_n)$ is an algebraic group. The multiplication with an element of $B \in \mathcal{L}_{n,\mathbf{P}}$ defines an isomorphism $\eta_B : \mathrm{Pic}^{\mathbf{P}_0}(X_n) \to \mathrm{Pic}^{\mathbf{P}}(X_n)$.

Suppose that X_n can be extended to a primitive multiple scheme X_{n+1} of multiplicity $n + 1$. From 4.6.1, we have a map $\Delta : \mathrm{Pic}(X_n) \to H^2(X, L^n)$ such that $\mathcal{B} \in \mathrm{Pic}(X_n)$ can be extended to a line bundle on X_{n+1} if and only if $\Delta(\mathcal{B}) = 0$. Suppose that some $\mathcal{B} \in \mathcal{L}_{n,\mathbf{P}}$ can be extended to X_{n+1}. The set of line bundles in $\mathcal{L}_{n,\mathbf{P}}$ that can be extended to X_{n+1} is a smooth closed subvariety

$$\Gamma^{\mathbf{P}}(X_{n+1}) \ \subset \ \mathrm{Pic}^{\mathbf{P}}(X_n) \ .$$

We have $\Gamma^{\mathbf{P}}(X_{n+1}) = \eta_{\mathcal{B}}\big(\Gamma^{\mathbf{P}_0}(X_{n+1})\big)$, and $\Gamma^{\mathbf{P}_0}(X_{n+1})$ is a subgroup of $\mathrm{Pic}^{\mathbf{P}_0}(X_n)$: from 4.6.1 and 5, (9), $\Gamma^{\mathcal{O}_X}(X_{n+1})$ is the kernel of the morphism of groups

$$\Delta : \mathrm{Pic}^{\mathbf{P}_0}(X_n) \to H^2(X, L^n) \ ,$$

(the *obstruction morphism*).

Let Z be a scheme and $\mathbb{L}$ a line bundle on $X_{n+1} \times Z$. Let $z \in Z$ be a closed point. Then there is an open neighbourhood U of z, $i \in I$, and a morphism $\phi : U \to P_i$, such that $\mathbb{L}_{|X_n\times U} \simeq \phi^\sharp(\mathbb{L}_i)$.

7.1.1. Lemma: *We have* $\mathrm{im}(\phi) \subset \Gamma^{\mathbf{P}}(X_{n+1})$.

Proof. Let $p_X : X \times U \to X$, $p_U : X \times U \to U$ be the projections. As in [16], 7.1, we can associate to every line bundle $\mathcal{L}_0$ on $X_n \times U$ an element $\Delta(\mathcal{L}_0)$ of $H^2(X \times U, p_X^*(L^n))$ such that $\mathcal{L}_0$ can be extended to a line bundle on $X_{n+1} \times U$ if and only if $\Delta(\mathcal{L}_0) = 0$. So we have $\delta(\mathbb{L}_{|X_{n+1}\times U}) = 0$. Consider the canonical morphism

$$\rho : H^2(X \times U, p_X^*(L^n)) \longrightarrow H^0(U, R^2p_{U*}(p_X^*(L^n))) \ = \ H^0(\mathcal{O}_U) \otimes H^2(X, L^n) \ .$$

As in Proposition 6.2.3, $\rho(\Delta(\mathcal{L}_0)) = \Delta \circ \phi$. Hence $\Delta \circ \phi = 0$, i.e. $\mathrm{im}(\phi) \subset \Gamma^{\mathbf{P}}(X_{n+1})$. □

From Theorem 6.4.2 we have

7.1.2. Theorem: *Suppose that $\mathcal{L}_{n+1,\mathbf{P}}$ is nonempty. Then there exists a fine moduli space* $\mathrm{Pic}^{\mathbf{P}}(X_{n+1})$ *for $\mathcal{L}_{n+1,\mathbf{P}}$. It is a smooth irreducible variety, and an affine bundle over $\Gamma^{\mathbf{P}}(X_{n+1})$ with associated vector bundle* $\mathcal{O}_{\Gamma^{\mathbf{P}}(X_{n+1})}\otimes\big(H^1(X,L^n)/\operatorname{im}(\delta^0)\big)$.

7.2. The kernel of the obstruction morphism

As in 4 we assume that we have an affine cover $(U_i)_{i\in I}$ of X and trivializations $\delta_i : U_i^{(n)} \to U_i\times\mathbf{Z}_n$. We can write

$$(\delta_{ij}^*)_{|\mathcal{O}_X(U_{ij})} \;=\; I_{|\mathcal{O}_X(U_{ij})} + tD_{ij} + t^2\Phi_{ij} \ ,$$

where D_{ij} is a derivation of $\mathcal{O}_X(U_{ij})$.

Suppose that X_n can be extended to a primitive multiple scheme X_{n+1} of multiplicity $n+1$. Let (δ'_{ij}) be a family which defines X_{n+1}, δ'_{ij} being an automorphism of $U_{ij}\times\mathbf{Z}_{n+1}$ inducing δ_{ij}. We have then

$$\delta'_{ij}(t) \;=\; \alpha_{ij}t \ ,$$

for some $\alpha_{ij}\in[\mathcal{O}_X(U_{ij})[t]/(t^n)]^*$.

Let $\mathbb{D}$ be a line bundle on X_n that can be extended to a line bundle on X_{n+1}. According to 4.6.1 this is equivalent to $\Delta(\mathbb{D})=0$ (cf. 4.6.1). Now let $\mathbb{D}'$ be a line bundle on X_n such that $\mathbb{D}'_{|X_{n-1}}\simeq\mathbb{D}_{|X_{n-1}}$. Then by formula (7), $\mathbb{D}'$ can be extended to a line bundle on X_{n+1} if and only if

$$\overline{\sigma}_{X_{n+1}}(\nabla_0(\mathbb{D}')-\nabla_0(\mathbb{D})) \;=\; 0 \ .$$

Suppose that $\mathbb{D}$ is defined by a cocycle (θ_{ij}), with $\theta_{ij}\in[\mathcal{O}_X(U_{ij})(t)/(t^n)]^*$, which is the set of elements $a\in\mathcal{O}_X(U_{ij})(t)/(t^n)$ such that $a^{(0)}\in\mathcal{O}_X(U_{ij})$ is invertible. The cocycle relation is

$$\theta_{ik} \;=\; \theta_{ij}\delta_{ij}^*(\theta_{jk})$$

(cf. 4.2). We can assume that $\mathbb{D}'$ is defined by a cocycle (θ'_{ij}), with θ'_{ij} of the form

$$\theta'_{ij} \;=\; \theta_{ij}+\rho_{ij}t^{n-1} \ ,$$

with $\rho_{ij}\in\mathcal{O}_X(U_{ij})$. Let $\beta_{ij}=\dfrac{\rho_{ij}}{\theta_{ij}^{(0)}}$. The cocycle relation for (θ'_{ij}) is equivalent to

$$\beta_{ik} \;=\; \beta_{ij}+(\alpha_{ij}^{(0)})^{n-1}\beta_{jk} \ , \tag{16}$$

i.e. (β_{ij}) represents an element β of $H^1(X,L^{n-1})$.

The line bundles $\mathbb{D}$, $\mathbb{D}'$ are extensions to X_n of a line bundle $\mathbb{D}_{n-1}$ on X_{n-1}. Let $D=\mathbb{D}_{n-1|X}$. We have canonical exact sequences

$$0\longrightarrow D\otimes L^{n-1}\longrightarrow\mathbb{D}\longrightarrow\mathbb{D}_{n-1}\longrightarrow 0\ ,\quad 0\longrightarrow D\otimes L^{n-1}\longrightarrow\mathbb{D}'\longrightarrow\mathbb{D}_{n-1}\longrightarrow 0\ ,$$

corresponding to $\sigma,\sigma'\in\operatorname{Ext}^1_{\mathcal{O}_{X_n}}(\mathbb{D}_{n-1},D\otimes L^{n-1})$ respectively. From 4.6.3 we have an exact sequence

$$0\longrightarrow H^1(X,L^{n-1})\xrightarrow{\ \Psi\ }\operatorname{Ext}^1_{\mathcal{O}_{X_n}}(\mathbb{D}_{n-1},D\otimes L^{n-1})\longrightarrow\mathbb{C}\ ,$$

and $\sigma'-\sigma=\Psi(\beta)$.

Now $\nabla_0(\mathbb{D}') - \nabla_0(\mathbb{D})$ is represented by the cocycle $\left(\frac{d\theta'_{ij}}{\theta'_{ij}} - \frac{d\theta_{ij}}{\theta_{ij}}\right)$ (cf. 5). We have

$$\frac{d\theta'_{ij}}{\theta'_{ij}} - \frac{d\theta_{ij}}{\theta_{ij}} \;=\; (n-1)\beta_{ij}t^{n-2}dt + t^{n-1}d\beta_{ij}\ .$$

Let $\mathbb{L} = \mathcal{I}_{X,X_{n+1}|X_2}$, which is a line bundle on X_2. Recall that $\delta^1_{\mathbb{L}^{n-1}}$ denotes the connecting morphism $H^1(X, L^{n-1}) \to H^2(X, L^n)$ coming from the exact sequence $0 \to L^n \to \mathbb{L}^{n-1} \to L^{n-1} \to 0$ on X_2.

7.2.1. Theorem: *We have* $\overline{\sigma}_{X_{n+1}}(\nabla_0(\mathbb{D}') - \nabla_0(\mathbb{D})) = n\delta^1_{\mathbb{L}^{n-1}}(\beta)$.

Proof. The line bundle $\mathbb{L}^{n-1}$ is defined by the family $\big((\alpha^{(0)}_{ij} + \alpha^{(1)}_{ij}t)^{n-1}\big)$. We have

$$(\alpha^{(0)}_{ij} + \alpha^{(1)}_{ij}t)^{n-1} \;=\; \big(\alpha^{(0)}_{ij}\big)^{n-1} + (n-1)\big(\alpha^{(0)}_{ij}\big)^{n-2}\alpha^{(1)}_{ij}t\ .$$

Hence from Lemma 4.3.1, $n\delta^1_{\mathbb{L}^{n-1}}(\beta)$ is represented by (μ_{ijk}), $\mu_{ijk} \in \mathcal{O}_X(U_{ijk})[t]/(t^n)$, with

$$\mu_{ijk} \;=\; n\big(\alpha^{(0)}_{ij}\big)^{n-2}\big((n-1)\alpha^{(1)}_{ij}\beta_{jk} + \alpha^{(0)}_{ij}D_{ij}(\beta_{jk})\big)\ .$$

We consider the exact sequence

$$0 \longrightarrow L^n \longrightarrow \Omega_{X_{n+1}|X_n} \longrightarrow \Omega_{X_n} \longrightarrow 0\ ,$$

and the associated map

$$\delta : H^1(X_n, \Omega_{X_n}) \longrightarrow H^2(X, L^n)\ .$$

We construct $\Omega_{X_{n+1}}$ as in 4.4. Let

$$A_{ij} \;=\; (n-1)\beta_{ij}t^{n-2}dt + t^{n-1}d\beta_{ij}\ .$$

Then we have

$$\overline{\sigma}_{X_{n+1}}(\nabla_0(\mathbb{D}') - \nabla_0(\mathbb{D})) \;=\; \delta(\nabla_0(\mathbb{D}') - \nabla_0(\mathbb{D}))\ ,$$

and it is represented by the family $(\nu_{ijk|X_n})$, $\nu_{ijk} \in H^0(\Omega_{U_{ijk}\times\mathbf{Z}_{n+1}})$, where

$$\nu_{ijk} \;=\; A_{ij} - A_{ik} + \mathbf{d}_{ij}(A_{jk})\ .$$

A long computation, using the equalities (16), $D_{ik} = D_{ij} + \alpha^{(0)}_{ij}D_{jk}$ and

$$\alpha^{(1)}_{ik} \;=\; \alpha^{(0)}_{ij}D_{ij}(\alpha^{(0)}_{jk}) + \big(\alpha^{(0)}_{ij}\big)^2\alpha^{(1)}_{jk} + \alpha^{(0)}_{jk}\alpha^{(1)}_{ij}\ ,$$

shows that $\nu_{ijk} = n\big(\alpha^{(0)}_{ij}\big)^{n-2}\big((n-1)\alpha^{(1)}_{ij}\beta_{jk} + \alpha^{(0)}_{ij}D_{ij}(\beta_{jk})\big)t^{n-1}dt$, which, by Lemma 4.3.1 and 2.1.5, implies Theorem 7.2.1. $\square$

Let $\mathbf{P}$ be an irreducible component of $\mathrm{Pic}(X)$. Recall that $\Gamma^{\mathbf{P}}(X_{n+1})$ is the smooth subvariety of $\mathrm{Pic}^{\mathbf{P}}(X_n)$ of line bundles which can be extended to X_{n+1}. The following result is an easy consequence of Theorem 7.2.1:

7.2.2. Proposition: *Let Z be the image of he restriction morphism $\Gamma^{\mathbf{P}}(X_{n+1}) \to \Gamma^{\mathbf{P}}(X_n)$. Then Z is smooth and $\Gamma^{\mathbf{P}}(X_{n+1})$ is an affine bundle over Z with associated vector bundle $\mathcal{O}_Z \otimes \ker(\delta^1_{\mathbb{L}^{n-1}})$.*

8. Products of curves

Let C and D be smooth projective irreducible curves, $X = C \times D$ and $\pi_C : X \to C$, $\pi_D : X \to D$ the projections. Let g_C (resp. g_D) be the genus of C (resp. D). Suppose that $g_C \geq 2$, $g_D \geq 2$.

8.1. Basic properties

8.1.1. – Let L_C (resp. L_D) be a line bundle on C (resp. D) and $L = \pi_C^*(L_C) \otimes \pi_D^*(L_D)$. Then we have canonical isomorphisms

$$\begin{array}{lrcl} \text{(i)} & H^0(X,L) & \simeq & H^0(C,L_C) \otimes H^0(D,L_D), \\ \text{(ii)} & H^1(X,L) & \simeq & \big(H^0(X,L_C) \otimes H^1(D,L_D)\big) \oplus \big(H^1(C,L_C) \otimes H^0(D,L_D)\big), \\ \text{(iii)} & H^2(X,L) & \simeq & H^1(C,L_C) \otimes H^1(D,L_D). \end{array}$$

8.1.2. – Let M_C (resp. M_D) be a line bundle on C (resp. D) and $M = \pi_C^*(M_C) \otimes \pi_D^*(M_D)$. Using 2.1.5, 3.1, and the direct sum decompositions of (ii) and (iii), it is easy to see that the only components of the canonical map $H^1(X,L) \otimes H^1(X,M) \to H^2(X, L \otimes M)$ that could not vanish are the two canonical maps

$$\big(H^1(L_C) \otimes H^0(L_D)\big) \otimes \big(H^0(M_C) \otimes H^1(M_D)\big) \longrightarrow H^1(L_C \otimes M_C) \otimes H^1(L_D \otimes M_D)\ ,$$
$$\big(H^0(L_C) \otimes H^1(L_D)\big) \otimes \big(H^1(M_C) \otimes H^0(M_D)\big) \longrightarrow H^1(L_C \otimes M_C) \otimes H^1(L_D \otimes M_D)\ ,$$

induced by

$$H^1(L_C) \otimes H^0(M_C) \to H^1(L_C \otimes M_C)\ , \quad H^0(L_D) \otimes H^1(M_D) \to H^1(L_D \otimes M_D)\ ,$$

and

$$H^0(L_C) \otimes H^1(M_C) \to H^1(L_C \otimes M_C)\ , \quad H^1(L_D) \otimes H^0(M_D) \to H^1(L_D \otimes M_D)$$

respectively.

8.1.3. – Similarly, the canonical map

$$H^1(X, \pi_C^*(L_C \otimes M_C) \otimes \pi_D^*(L_D)) \otimes H^1(C, \pi_C^*(M_C^*)) \to H^2(X,L)$$

comes from the two canonical maps

$$\big(H^0(L_C \otimes M_C) \otimes H^1(L_D)\big) \otimes H^1(M_C^*) \longrightarrow H^1(L_C) \otimes H^1(L_D)\ ,$$
$$\big(H^1(L_C \otimes M_C) \otimes H^0(L_D)\big) \otimes \big(H^0(M_C^*) \otimes H^1(\mathcal{O}_D)\big) \longrightarrow H^1(L_C) \otimes H^1(L_D)$$

induced by $H^0(C, L_C \otimes M_C) \otimes H^1(C, M_C^*) \to H^1(C, L_C)$ and

$$H^1(L_C \otimes M_C) \otimes H^0(M_C^*) \longrightarrow H^1(L_C)\ , \quad H^0(L_D) \otimes H^1(\mathcal{O}_D) \longrightarrow H^1(L_D)$$

respectively.

8.1.4. – We have $\Omega_X \simeq \pi_C^*(\omega_C) \oplus \pi_D^*(\omega_D)$, hence $\omega_X \simeq \pi_C^*(\omega_C) \otimes \pi_D^*(\omega_D)$ and

$$H^1(X,\Omega_X) \ = \ \big(H^0(\omega_C)\otimes H^1(\mathcal{O}_D)\big) \oplus H^1(\omega_C) \oplus \big(H^1(\mathcal{O}_C)\otimes H^0(\omega_D)\big) \oplus H^1(\omega_D)\ .$$

With respect to this decomposition, the canonical class of L is $\nabla_0(L) = \nabla_0(L_C) + \nabla_0(L_D)$.

We have $T_X \otimes L = (\pi_C^*(T_C)\otimes L) \oplus (\pi_D^*(T_D)\otimes L)$, and

$$\text{(17)}\qquad H^1(X, T_X\otimes L) \ = \ \big(H^0(T_C\otimes L_C)\otimes H^1(L_D)\big) \oplus \big(H^1(T_C\otimes L_C)\otimes H^0(L_D)\big) \oplus \big(H^0(L_C)\otimes H^1(T_D\otimes L_D)\big) \oplus \big(H^1(L_C)\otimes H^0(T_D\otimes L_D)\big)\ .$$

8.2. Primitive double schemes

8.2.1. *Line bundles on X_2* – We use the notations of 8.1. The non trivial primitive double schemes with associated smooth variety X and associated line bundle L are parametrized by $\mathbb{P}(H^1(X,T_X\otimes L))$. Let $\eta \in H^1(X,T_X\otimes L)$ and X_2 the associated double scheme. Using the decomposition (17), we see that η has four components: $\eta = (\eta_1,\eta_2,\eta_3,\eta_4)$.

We denote by ϕ_{L_C}, ϕ_{L_D} the canonical maps $H^0(T_C\otimes L_C)\otimes H^1(\omega_C) \to H^1(L_C)$ and $H^0(T_D\otimes L_D)\otimes H^1(\omega_D)\to H^1(L_D)$ respectively.

Let $\Psi : H^1(X,T_X\otimes L)\otimes H^1(X,\Omega_X) \to H^2(X,L)$ be the canonical morphism. Recall that M can be extended to a line bundle on X_2 if and only if $\Delta(M)=0$, where $\Delta(M) = \Psi(\eta\otimes\nabla_0(M))$ (cf. [16], Theorem 7.1.2). From 8.1.4, we have

$$\text{(18)}\qquad \Delta(M) \ = \ (\phi_{L_C}\otimes I_{H^1(L_D)})(\eta_1\otimes\nabla_0(M_C)) + (I_{H^1(L_C)}\otimes\phi_{L_D})(\eta_4\otimes\nabla_0(M_D))\ .$$

The canonical class of a line bundle on a smooth projective curve is in fact an integer. For every line bundle $\mathcal{L}$ on C we have $\nabla_0(\mathcal{L}) = \deg(\mathcal{L})c$, where $c = \nabla_0(\mathcal{O}_C(P))$, for any $P\in C$.

8.3. The case $\omega_{X_2} \simeq \mathcal{O}_{X_2}$

Let X_2 be a primitive double scheme such that $(X_2)_{red} = X$, with associated line bundle L on X. According to Corollary 5.3.2, we have $\omega_{X_2}\simeq\mathcal{O}_{X_2}$ if and only if $L\simeq\omega_X$.

Suppose that $L\simeq\omega_X$. We have an isomorphism $H^1(\omega_C)\simeq\mathbb{C}$ such that if $\mathcal{L}$ is a line bundle of degree d on C, we have $\nabla_0(\mathcal{L}) = d$ (and similarly on D). Let

$$\eta \ = \ (\eta_1,\eta_2,\eta_3,\eta_4) \ \in H^1(X,T_X\otimes\omega_X)$$

(using the decomposition (17)). Here we have

$$\eta_1\in\mathbb{C}\ ,\quad \eta_2\in H^1(C,\mathcal{O}_C)\otimes H^0(D,\omega_D)\ ,\quad \eta_3\in H^0(C,\omega_C)\otimes H^1(D,\mathcal{O}_D)\ ,\quad \eta_4\in\mathbb{C}\ .$$

If $M = \pi_C^*(M_C)\otimes\pi_D^*(M_D)$,

$$\Delta(M) \ = \ \eta_1\nabla_0(M_C) + \eta_4\nabla_0(M_D)\ ,$$

hence M can be extended to a line bundle on X_2 if and only if $\eta_1\deg(M_C) + \eta_4\deg(M_D) = 0$.

8.3.1. Corollary: *The scheme X_2 is projective if and only if $\eta_1=\eta_4=0$ or $\eta_1\eta_4<0$ and $\dfrac{\eta_1}{\eta_4}$ is rational.*

We have by 8.1.1

$$H^2(T_X\otimes L^2) \ \simeq\ \big(H^1(C,\omega_C)\otimes H^1(D,\omega_D^2)\big)\oplus\big(H^1(C,\omega_C^2)\otimes H^1(D,\omega_D)\big) \ =\ 0\ .$$

Hence by 4.1.3, X_2 can be extended to a primitive multiple scheme of multiplicity 3 if and only ω_X can be extended to a line bundle on X_2. This is the case if and only $(g_C-1)\eta_1+(g_D-1)\eta_4=0$, and then X_2 is projective.

8.4. An example of extensions of double primitive schemes

Let Θ be a theta characteristic on C such that $h^0(C,\Theta)>0$. We take $L_C=\Theta$, $L_D=\omega_D$. So we have

$$H^1(L) \ =\ \big(H^1(\Theta)\otimes H^0(\omega_D)\big)\oplus\big(H^0(\Theta)\otimes H^1(\omega_D)\big)\ ,\quad H^2(L)=H^1(\Theta)\otimes H^1(\omega_D)\ .$$

In the decomposition (17) the first summand is 0, and

$$H^1(T_X\otimes L) \ =\ \big(H^1(\Theta^{-1})\otimes H^0(\omega_D)\big)\oplus\big(H^0(\Theta)\otimes H^1(\mathcal{O}_D)\big)\oplus H^1(\Theta)\ .$$

Let X_2 be a non trivial primitive double scheme, corresponding to $\eta=(\eta_1,\eta_2\eta_3,\eta_4)\in\mathbb{P}(H^1(T_X\times L))$. We have $\eta_1=0$, and we will also suppose that $\eta_4=0$. Such double schemes are parametrized by $\mathbb{P}((H^1(\Theta^{-1})\otimes H^0(\omega_D))\oplus(H^0(\Theta)\otimes H^1(\mathcal{O}_D)))$.

We have, from (18), $\Delta(M)=0$. Hence every line bundle on X can be extended to a line bundle on X_2, and X_2 is projective. Such extensions of M are parametrized by an affine space, with underlying vector space $H^1(L)$.

8.4.1. Proposition: *Suppose that $\eta_4=0$ and $\eta_3\neq 0$. Then there exists an extension of X_2 to a primitive multiple scheme X_3 of multiplicity 3, and X_3 is projective.*

Proof. By Theorem 4.5.1, to prove the existence of X_3 it suffices to show that if

$$\chi : H^1(T_X\otimes L)\otimes H^1(L)\longrightarrow H^2(T_X\otimes L^2)$$

is the canonical map, then $\chi(\{\eta\}\otimes H^1(L))=H^2(T_X\otimes L^2)$. We have

$$\begin{aligned} H^2(T_X\otimes L^2) &= H^2(p_D^*(\omega_D^2))\oplus H^2(p_C^*(\omega_C)\otimes p_D^*(\omega_D)) \ =\ H^2(p_C^*(\omega_C)\otimes p_D^*(\omega_D))\\ &= H^1(\omega_C)\otimes H^1(\omega_D)\ \simeq\ \mathbb{C}\ ,\end{aligned}$$

and χ is equivalent to the canonical map

$$\chi' : \big(H^0(\Theta)\otimes H^1(\mathcal{O}_D)\big)\otimes\big(H^1(\Theta)\otimes H^0(\omega_D)\big)\longrightarrow H^1(\omega_C)\otimes H^1(\omega_D)\ ,$$

and we have to prove that $\chi'(\{\eta_3\}\otimes(H^1(\Theta)\otimes H^0(\omega_D))=H^1(\omega_C)\otimes H^1(\omega_D)$. Let $(\sigma_i)_{1\leq i\leq g_D}$ be a basis of $H^1(\mathcal{O}_D)$, and $(\sigma_i^*)_{1\leq i\leq g_D}$ the dual basis of $H^0(\omega_D)$. We can write $\eta_3=\displaystyle\sum_{i=1}^{g}\mu_i\otimes\sigma_i$, with $\mu_i\in H^0(\Theta)$. If $1\leq j\leq g_D$ and $\rho\in H^1(\Theta)$, we have $\chi'(\eta_3\otimes(\rho\otimes\sigma_j^*))=\rho\mu_j$. Since $\eta_3\neq 0$, we can choose j such that $\mu_j\neq 0$. We have an exact sequence on C

$$0\longrightarrow\Theta\xrightarrow{\ \times\mu_j\ }\Theta^2=\omega_C\longrightarrow\mathbb{T}\longrightarrow 0\ ,$$

where $\mathbb{T}$ is a torsion sheaf. It follows that $H^1(\times\mu_j):H^1(\Theta)\to H^1(\omega_C)$ is surjective. Since $H^2(L^2)=\{0\}$, every line bundle on X_2 can be extended to X_3. This proves Proposition 8.4.1. □

8.5. Picard groups

We use the notations of 8.2.1. Let $M_C \in \mathrm{Pic}(C)$, $M_D \in \mathrm{Pic}(D)$, $M = \pi_C^*(M_C) \otimes \pi_D^*(M_D)$ be such that $\Delta(M) = 0$, so that M can be extended to a line bundle on X_2. Let $\mathbf{P}$ be the irreducible component of $\mathrm{Pic}(C \times D)$ that contains $\pi_C^*(M_C) \otimes \pi_D^*(M_D)$. Let $\mathrm{Pic}^{\mathbf{P}}(X_2) \subset \mathrm{Pic}(X_2)$ be the irreducible component of line bundles $\mathcal{L}$ such that $\mathcal{L}_{|X} \in \mathbf{P}$. From 7, it is an affine bundle over $\mathbf{P}$, with associated vector bundle $\mathcal{O}_{\mathbf{P}} \otimes H^1(X, L)$.

Let $C' = C$, $X = C \times D$, and $p_C : X \times C' \to C$, $p_{C'} : X \times C' \to C'$, $p_D : X \times C' \to D$, $p_X : X \times C' \to X$, $p_{C\times C} : X \times C' \to C \times C'$ the projections.

Let $n = \deg(M_C)$ and $\Delta_C \subset C \times C$ the diagonal. Then $\mathbb{L} = p_{C\times C}^*(\mathcal{O}(n\Delta_C)) \otimes p_D^*(M_D)$ is a family of lines bundles on X parametrized by C'. These line bundles belong to $\mathbf{P}$, and $\mathbb{L}$ defines an immersion $C' \to \mathrm{Pic}^{\mathbf{P}}(X)$.

We have
$$\Delta(\mathbb{L}) \ = \ \Delta(p_{C\times C}^*(\mathcal{O}(n\Delta_C))) + \Delta(p_D^*(M_D)) \ ,$$
Let $q_{C\times C} : p_{C\times C}^*(\Omega_{C\times C}) \to \Omega_{X\times C'}$, $q_D : p_D^*(\omega_D) \to \Omega_{X\times C'}$ be the injective canonical morphisms. We have
$$\nabla_0(p_{C\times C}^*(\mathcal{O}(n\Delta_C))) \ = \ H^1(q_{C\times C})(\nabla_0(\mathcal{O}(n\Delta_C))) \ , \quad \nabla_0(p_D^*(M_D)) \ = \ H^1(q_D)(\nabla_0(M_D)) \ .$$
We have $H^1(X \times C', \Omega_{X\times C'}) = A_C \oplus A_{C'} \oplus A_D$, with
$$\begin{aligned} A_C &= H^1(\omega_C) \oplus \big(H^0(\omega_C) \otimes (H^1(\mathcal{O}_D) \oplus H^1(\mathcal{O}_{C'}))\big) \ , \\ A_{C'} &= H^1(\omega_{C'}) \oplus \big(H^0(\omega_{C'}) \otimes (H^1(\mathcal{O}_D) \oplus H^1(\mathcal{O}_C))\big) \ , \\ A_D &= H^1(\omega_D) \oplus \big(H^0(\omega_D) \otimes (H^1(\mathcal{O}_C) \oplus H^1(\mathcal{O}_{C'}))\big) \ , \end{aligned}$$

and $\mathrm{im}(H^1(q_D)) \ = \ H^1(\omega_D)$,
$$\mathrm{im}(H^1(q_{C\times C})) \ = H^1(\omega_C) \oplus (H^0(\omega_C) \otimes H^1(\mathcal{O}_{C'})) \oplus H^1(\omega_{C'}) \oplus (H^0(\omega_{C'}) \otimes H^1(\mathcal{O}_C)) \ .$$
We have $\nabla_0(p_D^*(M_D)) = \nabla_0(M_D)$, and by 9.4, $q_{C\times C}(\nabla_0(\mathcal{O}(n\Delta_C))) = n(1, I_{H^0(\omega_C)}, -1, -I_{H^0(\omega_C)})$.

We see $X_2 \times C'$ as a primitive multiple scheme, with underlying smooth scheme $X \times C'$ and associated line bundle $p_X^*(L)$. We have
$$H^1(T_{X\times C'} \otimes p_X^*(L)) \ \simeq \ H^1(T_X \otimes L) \oplus (H^0(T_X \otimes L) \otimes H^1(\mathcal{O}_{C'})) \oplus B \ ,$$
with
$$\begin{aligned} B \ = \ & (H^1(L_C) \otimes H^0(L_D) \otimes H^0(\omega_{C'}))\oplus \\ & (H^0(L_C) \otimes H^1(L_D) \otimes H^0(\omega_{C'})) \oplus (H^0(L_C) \otimes H^0(L_D) \otimes H^1(\omega_{C'})), \end{aligned}$$
The element η' of $H^1(T_{X\times C'} \otimes p_X^*(L))$ associated to $X_2 \times C'$ is η (in the factor $H^1(T_X \otimes L)$).

We have
$$\begin{aligned} H^2(p_X^*(L)) \ = \ & (H^1(L_C) \otimes H^1(L_D))\oplus \\ & (H^1(L_C) \otimes H^0(L_D) \otimes H^1(\mathcal{O}_{C'})) \oplus (H^0(L_C) \otimes H^1(L_D) \otimes H^1(\mathcal{O}_{C'})) \ . \end{aligned}$$

Let $\beta_{L_C} : H^0(\omega_C) \otimes H^1(T_C \otimes L_C) \to H^1(L_C)$ be the canonical map. As in 8.2.1 we can compute $\Delta(\mathbb{L})$ and find that it has the component $\Delta(M) = 0$ in the factor $H^1(L_C) \otimes H^1(L_D)$, and the component $(\beta_{L_C} \otimes I_{H^1(\omega_C)\otimes H^0(L_D)})(\eta_2 \otimes I_{H^0(\omega_C)})$ in the factor $H^1(L_C) \otimes H^0(L_D) \otimes H^1(\mathcal{O}_{C'})$. It follows that

8.5.1. Theorem: *If* $n(\beta_{L_C}\otimes I_{H^1(\omega_C)\otimes H^0(L_D)})(\eta_2\otimes I_{H^0(\omega_C)})\neq 0$, *then the affine bundle* $\mathrm{Pic}^{\mathbf{P}}(X_2)$ *is not a vector bundle over* $\mathbf{P}$.

8.5.2. *The case* $L=\omega_{C\times D}$ – We have $\eta_1,\eta_4\in\mathbb{C}$, $\eta_2\in H^1(C,\mathcal{O}_C)\otimes H^0(D,\omega_D)$, and $\Delta(M)=0$ is equivalent to $\eta_1\deg(M_C)+\eta_4\deg(M_D)=0$,. Theorem 8.5.1 implies that if $\deg(M_C)\neq 0$ and $\eta_2\neq 0$, then $\mathrm{Pic}^{\mathbf{P}}(X_2)$ is not a vector bundle over $\mathbf{P}$.

9. Moduli spaces of vector bundles on primitive multiple curves

Let C be a smooth irreducible projective curve of genus $g\geq 2$, and $Y=C_2$ a *ribbon* over C, i.e. a primitive multiple curve of multiplicity 2 such that $Y_{red}=C$. Let $L=\mathcal{I}_{C,Y}$ be the associated line bundle on C.

9.1. Construction of the moduli spaces

We suppose that $\deg(L)<0$. Let r, d be coprime integers such that $r>0$. Let $M_C(r,d)$ be the moduli space of stable sheaves on C of rank r and degree d. It is a smooth projective variety of dimension $r^2(g-1)+1$. On $M_C(r,d)\times C$ there is an universal bundle E. For every closed point $m\in M_C(r,d)$, E_m is the stable vector bundle corresponding to m.

Let $p_M : M_C(r,d)\times C\to M_C(r,d)$, $p_C : M_C(r,d)\times C\to C$ be the projections. The vector bundle of 6.2.9,

$$\mathbb{A} \;=\; R_1p_{M*}\big(\mathcal{H}om(E,E\otimes p_C^*(L))\big)$$

is of rank $r^2(\deg(L)+g-1)$. For every $m\in M_C(r,d)$, $\mathbb{A}_m=\mathrm{Ext}^1_{\mathcal{O}_C}(E_m,E_m\otimes L)$.

The conditions of 6.4 are satisfied. Hence we obtain a fine moduli space $\mathbf{M}_Y(r,d)$ for the vector bundles $\mathbb{E}$ on Y such that $\mathbb{E}_{|C}\in M_C(r,d)$. We have a canonical morphism

$$\tau_{r,d} : \mathbf{M}_Y(r,d)\longrightarrow M(r,d)$$

associating $\mathbb{E}_{|C}$ to $\mathbb{E}$, and $\mathbf{M}_Y(r,d)$ is an affine bundle on $M(r,d)$ with associated vector bundle $\mathbb{A}$.

The same kind of construction can be applied in higher multiplicity to obtain Theorems 1.3.1 and 1.3.3.

9.2. Picard groups

We don't suppose here that $\deg(L)<0$. We will use the results of 7. For $r=1$, we have $M(r,d)=\mathrm{Pic}^d(C)$, the variety of line bundles of degree d on C. As on C we have a decomposition of the Picard group of Y:

$$\mathrm{Pic}(Y) \;=\; \bigcup_{d\in\mathbb{Z}}\mathrm{Pic}^d(Y)\ ,$$

where $\mathrm{Pic}^d(Y) = \mathbf{M}_Y(1,d)$ is the variety of line bundles on Y whose restriction to C has degree d. With the notations of 7, $\mathrm{Pic}^d(Y) = \mathrm{Pic}^{\mathbf{P}}(Y)$, where $\mathbf{P}$ is the set of line bundles of degree d on C. Consider the restriction morphism

$$\tau_{1,d} : \mathrm{Pic}^d(Y) \longrightarrow \mathrm{Pic}^d(C)$$

which is an affine bundle over $\mathrm{Pic}^d(C)$, with associated line bundle $\mathcal{O}_{\mathrm{Pic}^d(C)} \otimes H^1(C,L)$.

Let $0 \to L \to \Omega_{Y|C} \to \omega_C \to 0$ be the canonical exact sequence, associated to $\sigma_Y \in H^1(C, T_C \otimes L)$. Let $p_Y : Y \times C \to Y$ be the projection. Let

$$\Sigma_{Y\times C} : \qquad 0 \longrightarrow p_Y^*(L) \longrightarrow \Omega_{Y\times C|C\times C} \longrightarrow \Omega_{C\times C} \longrightarrow 0 \ ,$$

be the exact sequence (12) corresponding to $Y \times C$, associated with

$$\sigma_{Y\times C} \ \in \ \mathrm{Ext}^1_{\mathcal{O}_{C\times C}}(p_1^*(\omega_C), p_1^*(L)) \oplus \mathrm{Ext}^1_{\mathcal{O}_{C\times C}}(p_2^*(\omega_C), p_1^*(L)) \ .$$

Then $\Sigma_{Y\times C}$ is $p_1^*(\Sigma_Y) \oplus p_2^*(\Sigma)$, where Σ is the exact sequence

$$0 \longrightarrow 0 \longrightarrow \omega_C = \omega_S \longrightarrow 0 \ .$$

It follows that the component of $\sigma_{Y\times C}$ in $\mathrm{Ext}^1_{\mathcal{O}_{C\times C}}(p_2^*(\omega_C), p_1^*(L))$ vanishes, and that

$$\sigma_{Y\times C} \ = \ \sigma_Y \in H^1(C, T_C \otimes L) \subset \mathrm{Ext}^1_{\mathcal{O}_{C\times C}}(p_1^*(\omega_C), p_1^*(L))$$

(cf. (19)).

According to Theorem 6.2.10 it is equivalent to say that $\mathrm{Pic}^d(Y)$ is not a vector bundle over $\mathrm{Pic}^d(C)$, and that a Poincaré bundle $\mathbb{D}$ on $C \times \mathrm{Pic}^d(C)$ cannot be extended to a line bundle of $Y \times \mathrm{Pic}^d(C)$. To see this we can suppose that $d = -1$. We have then an embedding

$$\begin{aligned} C &\hookrightarrow \mathrm{Pic}^{-1}(C) \\ P &\longmapsto \mathcal{O}_C(-P) \ . \end{aligned}$$

We will show that with suitable hypotheses, $\mathbb{D}_{|C\times C}$ cannot be extended to $Y \times C$. Let $p_1 : C \times C \to C$ (resp. $p_2 : C \times C \to C$) be the first (resp. second) projection. It suffices to prove that for some $N \in \mathrm{Pic}(C)$, $\mathbb{D}_{|C\times C} \otimes p_2^*(N)$ cannot be extended to $Y \times C$. So we can assume that $\mathbb{D}_{|C\times C} = \mathcal{O}_{C\times C}(-\Gamma)$, where $\Gamma \subset C \times C$ is the diagonal.

We see $Y \times C$ as a primitive multiple scheme, with associated line bundle $p_1^*(L)$ on $C \times C$. We will show that $\Delta(\mathcal{O}_{C\times C}(-\Gamma)) \in H^2(C \times C, p_1^*(L))$ is non zero. This implies that $\mathcal{O}_{C\times C}(-\Gamma)$ cannot be extended to $Y \times C$ (cf. 4.6.1).

We have

$$\Delta(\mathcal{O}_{C\times C}(-\Gamma)) \ \in \ H^2(C \times C, p_1^*(L)) \ = \ L(H^0(C,\omega_C), H^1(C,L)) \ ,$$

and $\Delta(\mathcal{O}_{C\times C}(-\Gamma)) = \sigma_{Y\times C}.\nabla_0(\mathcal{O}_{C\times C}(-\Gamma))$ (from [16], Theorem 7.1.2). It follows easily from 8.1.2 that $\Delta(\mathcal{O}_{C\times C}(-\Gamma))$ is the image of σ_Y by the canonical map

$$\eta : H^1(C, T_C \otimes L) \longrightarrow L(H^0(C,\omega_C), H^1(C,L)) \ .$$

It follows that if $\eta(\sigma_Y) \neq 0$, then $\mathrm{Pic}^d(Y)$ is not a vector bundle over $\mathrm{Pic}^d(C)$.

We will prove

9.2.1. Theorem : $\mathrm{Pic}^d(Y)$ *is not a vector bundle over* $\mathrm{Pic}^d(C)$ *if* Y *is not trivial and either* C *is not hyperelliptic and* $\deg(L) \leq 2 - 2g$, *or* $L = \omega_C$.

9.3. The duality morphism

We will use the results of 8.1.

Let Z be smooth irreducible projective curve. Let $p_Z : Z \times C \to Z$ and $p_C : Z \times C \to C$ be the projections. Let N be a vector bundle on C. Then we have (from 8.1)

$$H^1(Z \times C, p_C^*(N)) \ \simeq \ H^1(C, N) \oplus \big(H^0(C, N) \otimes H^1(Z, \mathcal{O}_Z)\big) \ . \tag{19}$$

We have also $H^2(Z \times C, p_C^*(N)) \simeq H^1(C, N) \otimes H^1(Z, \mathcal{O}_Z)$.

9.3.1. *Canonical bundles* – We have $\Omega_{Z\times C} = p_Z^*(\omega_Z) \oplus p_C^*(\omega_C)$, hence

$$\begin{aligned} H^1(Z \times C, \Omega_{Z\times C}) \ &\simeq \ H^1(Z, \omega_Z) \oplus \big(H^0(Z, \omega_Z) \otimes H^1(C, \mathcal{O}_C)\big) \oplus \\ &\qquad H^1(C, \omega_C) \oplus \big(H^0(Z, \mathcal{O}_Z) \otimes H^1(C, \omega_C)\big) \\ &= \ \mathbb{C} \oplus L(H^0(C, \omega_C), H^0(Z, \omega_Z)) \oplus \mathbb{C} \oplus L(H^0(Z, \omega_Z), H^0(C, \omega_C)) \ . \end{aligned} \tag{20}$$

We have $H^2(Z \times C, \Omega_{Z\times C}) \simeq H^1(Z, \omega_Z) \otimes H^1(C, \omega_C) = \mathbb{C}$.

9.3.2. Lemma: *The duality morphism*

$$H^1(Z \times C, \Omega_{Z\times C}) \times H^1(Z \times C, \Omega_{Z\times C}) \longrightarrow H^2(Z \times C, \omega_{Z\times C}) = \mathbb{C}$$

is

$$\big((\alpha_1, \phi_1, \alpha_2, \phi_2), (a_1, f_1, a_2, f_2)\big) \longmapsto \alpha_1 a_2 - \alpha_2 a_1 + tr(f_2\phi_1) - tr(f_1\phi_2) \ .$$

Proof. This follows from 8.1. For the minus signs, see 2.1.5. ☐

9.4. Proof of Theorem 9.2.1

9.4.1. *The canonical class of* $\mathcal{O}_{C\times C}(-\Gamma)$ – We have $\nabla_0(\mathcal{O}_{C\times C}(-\Gamma)) = -\mathrm{cl}_\Gamma$, where cl_Γ is the cohomology class of Γ. It is the element of $H^1(C \times C, \Omega_{C\times C})$ corresponding to the canonical linear form

$$\tau : H^1(C \times C, \Omega_{C\times C}) \longrightarrow H^1(\Gamma, \omega_\Gamma) = H^1(C, \omega_C) = \mathbb{C}$$

using the duality described in 9.3.1. For every

$$(a_1, f_1, a_2, f_2) \ \in \ \mathbb{C} \oplus \mathrm{End}(H^0(C, \omega_C)) \oplus \mathbb{C} \oplus \mathrm{End}(H^0(C, \omega_C)) \ .$$

we have

$$\tau(a_1, f_1, a_2, f_2) \ = \ a_1 + a_2 + tr(f_1) + tr(f_2) \ .$$

It follows that

$$\mathrm{cl}_\Gamma \ = \ (1, I_{H^0(C,\omega_C)}, -1, -I_{H^0(C,\omega_C)}) \ .$$

9.4.2. *Proof of Theorem 9.2.1* – The case $L = \omega_C$ is obvious. So we suppose that C is not hyperelliptic and $\deg(L) \leq 2 - 2g$. The transpose of the evaluation morphism $\mathcal{O}_C \otimes H^0(C, \omega_C) \to \omega_C$ is injective and its cokernel F is stable (from [26], Corollary 3.5). From the exact sequence

$$0 \longrightarrow T_C \otimes L \longrightarrow L \otimes H^0(C, \omega_C)^* \longrightarrow F \otimes L \longrightarrow 0 \ ,$$

we have an exact sequence

$$0 \longrightarrow H^0(C, F \otimes L) \longrightarrow H^1(C, T_C \otimes L) \xrightarrow{\eta} L(H^0(C, \omega_C), H^1(C, L)) \ .$$

Since $\deg(F \otimes L) \leq 0$ and $F \otimes L$ is stable, we have $H^0(C, F^* \otimes L) = \{0\}$, and η is injective. Since $\sigma_Y \neq 0$ (Y is not trivial), we have $\Delta(\mathcal{O}_{C\times C}(-\Gamma)) \neq 0$. This proves Theorem 9.2.1.

9.5. Moduli spaces of vector bundles of rank $r \geq 2$ and degree -1

9.5.1. Theorem : *Suppose that C is not hyperelliptic, Y not trivial and $\deg(L) \leq 2-2g$. Then $\mathbf{M}_Y(r,-1)$ is not a vector bundle over $M(r,-1)$.*

Proof. Let E be a stable vector bundle on C of rank r and degree 0 and $x \in C$. Then for every non zero linear form $\varphi : E_x \to \mathbb{C}$, the kernel of the induced morphism $\tilde{\varphi} : E \to \mathcal{O}_x$ is a stable vector bundle: let $F \subset \ker(\tilde{\varphi})$ be a subbundle such that $0 < \mathrm{rk}(F) < r$; then since $F \subset E$ and E is stable, we have $\deg(F) < 0$. Hence $\dfrac{\deg(F)}{\mathrm{rk}(F)} < -\dfrac{1}{r}$.

Let $\mathbb{P} = \mathbb{P}(E^*)$ and $\pi : \mathbb{P} \to C$, $p_{\mathbb{P}} : \mathbb{P} \times C \to C$, $p_C : \mathbb{P} \times C \to C$ the projections. Let $\Delta \subset C \times C$ be the diagonal. We have a canonical obvious surjective morphism on sheaves on $\mathbb{P} \times C$

$$\mathbf{p} : p_C^*(E) \otimes p_{\mathbb{P}}^*(\mathcal{O}_{\mathbb{P}}(-1)) \longrightarrow \mathcal{O}_{(\pi\times p_C)^{-1}(\Delta)} \ .$$

For every $x \in C$ and non zero $\varphi \in E_x^*$, $\mathbf{p}_{\mathbb{C}\varphi}$ is the morphism $E \otimes \mathbb{C}\varphi \to \mathcal{O}_x$ induced by φ. The kernel of $\mathbf{p}$ is a vector bundle $\mathbb{E}$ on $\mathbb{P} \times C$ that is a family of stable vector bundles on C, of rank r and degree -1, parametrized by $\mathbb{P}$.

Let $\mathcal{E}$ be a universal bundle on $M(r,-1) \times C$. Since $r \geq 2$, E^* has a rank 1 subbundle D, inducing a section s of π. Then $\mathbb{F} = s^*(\mathbb{E})$ is a family of stable vector bundles on C, of rank r and degree -1, parametrized by C. Hence there exists a morphism $\tau : C \to M(r,-1)$ and a line bundle H on C such that $\mathbb{F} \simeq (\tau \times I_C)^*(\mathcal{E}) \otimes q^*(H)$ (where $q : C \times C \to C$ is the first projection).

We have a canonical immersion $i : C \to \mathrm{Pic}^{-1}(C)$, sending x to $\mathcal{O}_C(-x) \otimes \det(E)$. Let $\mathbf{det} : M(r,-1) \to \mathrm{Pic}^{-1}(C)$ be the determinant morphism. Then $\mathbf{det} \circ \tau$ is the identity morphism (from C to $i(C) = C$). It follows that τ is an immersion.

Now suppose that $\mathbf{M}_Y(r,-1)$ is a vector bundle over $M(r,-1)$. By Theorem 6.2.10, $\mathcal{E}$ can be extended to a vector bundle on $\mathbf{M}_Y(r,-1) \times Y$, hence $\mathcal{E}_{|\tau(C)\times C}$ can be extended to a vector bundle on $\tau(C) \times Y$. Hence $\det(\mathcal{E}_{|\tau(C)\times C})$ can be extended to a line bundle on $i(C) \times C$. But this is impossible by the proof of Theorem 9.2.1. □

9.6. Picard groups of primitive multiple schemes of multiplicity 3

Let C_3 be an extension of C_2 of multiplicity 3. We use the notations of 7. Let $\mathbf{P} = \mathrm{Pic}^d(C)$. Every line bundle on C_2 can be extended to C_3, since the obstructions lie in H^2 of vector bundles on C. Hence we have $\Gamma^{\mathbf{P}}(C_3) = \mathrm{Pic}^d(C_2)$.

Let $\mathrm{Pic}^d(C_3)$ denote the component of $\mathrm{Pic}(C_3)$ of line bundles whose restriction to C is of degree d. From 7 we see that $\mathrm{Pic}^d(C_3)$ is a smooth irreducible variety, and an affine bundle over $\mathrm{Pic}^d(C_2)$ with associated vector bundle $\mathcal{O}_{\mathrm{Pic}^d(C_2)} \otimes \big(H^1(C,L^2)/\operatorname{im}(\delta^0)\big)$, where $\delta^0 : H^0(\mathcal{O}_{C_2}) \to H^1(C,L^2)$ is the map coming from the exact sequence $0 \to L^2 \to \mathcal{O}_{C_3} \to \mathcal{O}_{C_2} \to 0$.

We have an exact sequence

$$0 \longrightarrow H^0(C,L) \longrightarrow H^0(\mathcal{O}_{C_2}) \longrightarrow H^0(\mathcal{O}_C) = \mathbb{C} \longrightarrow 0 \ ,$$

so that $\operatorname{im}(\delta^0)$ is the same as the image of the restriction of δ^0 to $H^0(C,L)$.

Now suppose that we have another extension C'_3 of C_2 to multiplicity 3. Then $\mathcal{I}_{C,C_3}$ and $\mathcal{I}_{C,C'_3}$ are two extensions of L to a line bundle on C_2, corresponding to $\sigma, \sigma' \in \mathrm{Ext}^1_{\mathcal{O}_{C_2}}(\mathcal{O}_C, L)$ respectively. We have also two maps $\delta^0, \delta'^0 : H^0(C,L) \to H^1(C,L^2)$.

According to 4.6.3, we have $\sigma' - \sigma = \Psi(\mu)$, for some $\mu \in H^1(C,L)$. Let $\chi : H^0(C,L) \otimes H^1(C,L) \to H^1(C,L^2)$ be the canonical product.

From Lemma 4.3.2 follows easily

9.6.1. Proposition: *We have, for every* $\eta \in H^0(C,L)$, $(\delta'^0 - \delta^0)(\eta) = \chi(\eta \otimes \mu)$.

This implies that it is possible that $\operatorname{im}(\delta^0) \neq 0$. If C_2 is the trivial ribbon (cf. 4.1), then $\delta^0 = 0$, so we obtain a formula for δ'^0.

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