COHERENT SHEAVES ON PRIMITIVE MULTIPLE SCHEMES

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RESUME. A primitive multiple scheme is a Cohen-Macaulay scheme Y such that the associated reduced scheme $X = Y_{red}$ is smooth, irreducible, and that Y can be locally embedded in a smooth variety of dimension $\dim(X) + 1$. If n is the multiplicity of Y, there is a canonical filtration $X = X_1 \subset X_2 \subset \cdots \subset X_n = Y$, such that X_i is a primitive multiple scheme of multiplicity i. The ideal sheaf $\mathcal{I}_X$ of X is a line bundle on X_{n-1} and $L = \mathcal{I}_X/\mathcal{I}_X^2$ is a line bundle on X, called the *associated line bundle* of Y.

Even if X is projective, Y needs not to be quasi projective. We define in every case the *reduced Hilbert polynomial* $P_{red,\mathcal{O}_X(1)}(E)$ of a coherent sheaf E on Y, depending on the choice of an ample line bundle $\mathcal{O}_X(1)$ on X. If $\mathcal{E}$ is a flat family of sheaves on Y parameterized by a smooth curve C, then $P_{red,\mathcal{O}_X(1)}(\mathcal{E}_c)$ does not depend on $c \in C$. We study flat families of sheaves in two important cases: the families of *quasi locally free sheaves*, and if $n = 2$ those of *balanced sheaves*. Balanced sheaves are generalizations of vector bundles on Y, and could be used to expand already known moduli spaces of vector bundles on Y.

When X is a smooth projective surface, and Y is of multiplicity 2 we study the simplest examples of balanced sheaves: the sheaves $\mathcal{E}$ such that there is an exact sequence

$$0 \longrightarrow \mathcal{I}_P \otimes L \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_P = \mathcal{E}_{|X} \longrightarrow 0 ,$$

where $\mathcal{I}_P \subset \mathcal{O}_X$ is the ideal sheaf of a point $P \in X$. They can also be described as the ideal sheaves $\mathcal{E}$ of subschemes of Y concentrated on P, and such that $\mathcal{E}_P$ is generated by two elements whose images in $\mathcal{O}_{X,P}$ generate the maximal ideal. There is a moduli space for such sheaves, which is an affine bundle on X with associated vector bundle $T_X \otimes L$ (where T_X is the tangent bundle of X). The associated class in $H^1(X, T_X \otimes L)$ can be determined.

SUMMARY

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1. Introduction

A *primitive multiple scheme* is a Cohen-Macaulay scheme Y over $\mathbb{C}$ such that:

- $Y_{red} = X$ is a smooth connected variety,
- for every closed point $x \in X$, there exists a neighborhood U of x in Y, and a smooth variety S of dimension $\dim(X) + 1$ such that U is isomorphic to a closed subscheme of S.

We call X the *support* of Y. It may happen that Y is quasi-projective, and in this case it is projective if X is.

For every closed subscheme $Z \subset Y$, let $\mathcal{I}_Z$ (or $\mathcal{I}_{Z,Y}$) denote the ideal sheaf of Z in Y. For every positive integer i, let X_i be the closed subscheme of Y corresponding to the ideal sheaf $\mathcal{I}_X^i$. The smallest integer n such that $X_n = Y$ is called the *multiplicity* of Y. For $1 \leq i \leq n$, X_i is a primitive multiple scheme of multiplicity i, $L = \mathcal{I}_X / \mathcal{I}_{X_2}$ is a line bundle on X, and we have $\mathcal{I}_{X_i} / \mathcal{I}_{X_{i+1}} = L^i$. We call L the line bundle on X *associated* to Y. The ideal sheaf $\mathcal{I}_X$ can be viewed as a line bundle on X_{n-1}. If $n = 2$, Y is called a *primitive double scheme.*

The simplest case is when Y is contained in a smooth variety S of dimension $\dim(X) + 1$. Suppose that Y has multiplicity n. Let $P \in X$ and $f \in \mathcal{O}_{S,P}$ a local equation of X. Then we have $\mathcal{I}_{X_i,P} = (f^i)$ for $1 < j \leq n$ in S, in particular $\mathcal{I}_{Y,P} = (f^n)$ (where $\mathcal{I}_{X_i}, \mathcal{I}_Y$ are the ideal sheaves of $X_i \cap S$ and $Y \cap S$ in S respectively), and $L = \mathcal{O}_X(-X)$.

For any $L \in \mathrm{Pic}(X)$, the *trivial primitive scheme* of multiplicity n, with induced smooth variety X and associated line bundle L on X is the n-th infinitesimal neighborhood of X, embedded by the zero section in the dual bundle L^*, seen as a smooth variety.

The primitive multiple curves where defined in [17], [2]. Primitive double curves were parameterized and studied in [3] and [16]. More results on primitive multiple curves can be found in [5], [6], [7], [8], [9], [10], [11], [12], [4], [22], [23], [24]. Some primitive double schemes are studied in [1], [18] and [19]. The case of varieties of any dimension is studied in [14], where the following subjects were treated:

- construction and parameterization of primitive multiple schemes,
- obstructions to the extension of a vector bundle on X_m to X_{m+1},
- obstructions to the extension of a primitive multiple scheme of multiplicity n to one of multiplicity $n + 1$.

In [15], the construction and properties of fine moduli spaces of vector bundles on primitive multiple schemes are described. Suppose that $Y = X_n$ is of multiplicity n, and can be extended to X_{n+1} of multiplicity $n + 1$, and let M_n be a fine moduli space of vector bundles on X_n. With suitable hypotheses, a fine moduli space M_{n+1} for the vector bundles on X_{n+1} whose restriction to X_n belongs to M_n can be constructed. This applies in particular to Picard groups.

The subject of this paper is the study of more general coherent sheaves on primitive multiple schemes. More precisely:

- We will define an invariant of coherent sheaves on Y depending on the choice of an ample line bundle on X, the *reduced Hilbert polynomial*, which is the same as the usual Hilbert polynomial when Y is projective. But it is defined even when Y is not quasi-projective.
- We define and study *primitive multiple rings*. Among them are the local rings of closed points in a primitive multiple scheme. They make the presentations more readable.

– We give some properties of *quasi locally free sheaves* already studied on primitive multiple curves, and of *balanced sheaves*, which are natural generalizations of vector bundles. The main subjects are flat families of such sheaves.
– We study the simplest balanced sheaves on X_2 which are not locally free, i.e. sheaves whose restriction to X is the ideal sheaf of a point.

1.1. Reduced Hilbert polynomials

It may happen that Y is not quasi-projective. For example, there are exactly two non trivial primitive multiple schemes such that X is a projective space of dimension ≥ 2: the two with $X = \mathbb{P}_2$, one of multiplicity 2, the other of multiplicity 4 (cf. [14]). On these two schemes the only line bundle is the trivial one, so they are not quasi-projective, and there is no notion of Hilbert polynomial for sheaves on such schemes.

If Z is a projective scheme, $\mathcal{O}_Z(1)$ an ample line bundle and E a coherent sheaf on Z, let $P_{\mathcal{O}_Z(1)}(E)$ be the Hilbert polynomial of E with respect to $\mathcal{O}_Z(1)$.

Let $Y = X_n$ be a primitive multiple scheme of multiplicity n and associated smooth scheme X. Suppose that X is projective. Let $\mathcal{O}_X(1)$ be an ample line bundle on X. Let $\mathcal{E}$ be a coherent sheaf on Y and

$$\mathcal{F}_m = 0 \subset \mathcal{F}_{m-1} \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathcal{E} \tag{1}$$

a filtration such that, for $0 \leq i < m$, $\mathcal{F}_i/\mathcal{F}_{i+1}$ is concentrated on X. We will see (proposition 5.1.2) that

$$P_{red,\mathcal{O}_X(1)}(\mathcal{E}) = \sum_{i=0}^{m-1} P_{\mathcal{O}_X(1)}(\mathcal{F}_i/\mathcal{F}_{i+1})$$

does not depend on the filtration (1). It is called the *reduced Hilbert polynomial* of $\mathcal{E}$ with respect to $\mathcal{O}_X(1)$. This expands to higher dimensions the notions of generalized rank and degree defined in [5], [7] on primitive multiple curves. A similar definition can be made for *reduced Chern classes*. These invariants behave correctly with exact sequences of sheaves.

If $\mathcal{O}_X(1)$ can be extended to a line bundle $\mathcal{O}_Y(1)$ on Y, then $\mathcal{O}_Y(1)$ is ample and
$P_{red,\mathcal{O}_X(1)} = P_{\mathcal{O}_Y(1)}$.

The reduced Hilbert polynomial of a coherent sheaf is invariant by flat deformations of the sheaf: let C be an irreducible smooth curve, and $\mathcal{E}$ a coherent sheaf on $X_n \times C$, flat on C. Then the map

$$C \longrightarrow \mathbb{Q}[T]$$

$$c \longmapsto P_{red,\mathcal{O}_X(1)}(\mathcal{E}_c)$$

is constant (corollary 5.2.2). To prove this we need a preliminary result that was pointed out by János Kollár, which has its own interest: suppose that we have a filtration

$$0 = \mathcal{G}_m \subset \mathcal{G}_{m-1} \subset \cdots \subset \mathcal{G}_1 \subset \mathcal{G}_0 = \mathcal{E}$$

such that for $0 \leq i < m$, $\mathcal{G}_i/\mathcal{G}_{i+1}$ is concentrated on $X \times C$. Then there exists a filtration

$$0 = \mathcal{F}_m \subset \mathcal{F}_{m-1} \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathcal{E}$$

such that the $\mathcal{F}_i$ are flat on C, as well as the quotients $\mathcal{F}_i/\mathcal{F}_{i+1}$, which are concentrated on $X \times C$, and that there is a finite subset $\Sigma \subset C$ such that the two filtrations coincide on $X_n \times (C\backslash\Sigma)$ (theorem 5.2.1).

1.2. Quasi locally free sheaves

1.2.1. *Canonical filtrations* – Let $\mathcal{E}$ be a coherent sheaf on Y. The *first canonical filtration* of $\mathcal{E}$ is

$$\mathcal{E}_n = 0 \subset \mathcal{E}_{n-1} \subset \cdots \subset \mathcal{E}_1 \subset \mathcal{E}_0 = \mathcal{E} \ ,$$

where for $0 \leq i < n$, $\mathcal{E}_{i+1}$ is the kernel of the canonical surjective morphism $\mathcal{E}_i \to \mathcal{E}_{i|X}$. For $0 \leq i < n$, let $G_i(\mathcal{E}) = \mathcal{E}_i/\mathcal{E}_{i+1} = \mathcal{E}_{i|X}$. The *second canonical filtration of* $\mathcal{E}$ is

$$\mathcal{E}^{(0)} = \{0\} \subset \mathcal{E}^{(1)} \subset \cdots \subset \mathcal{E}^{(n-1)} \subset \mathcal{E}^{(n)} = \mathcal{E} \ ,$$

where for $1 \leq i \leq n-1$, $\mathcal{E}^{(i)}$ is the maximal subsheaf annihilated by $\mathcal{I}_X^i$. For $0 < i \leq n$, let $G^{(i)}(\mathcal{E}) = \mathcal{E}^{(i)}/\mathcal{E}^{(i-1)}$. The sheaves $G^{(i)}(\mathcal{F})$ and $G_i(\mathcal{F})$ are concentrated on X.

Let $\mathcal{E}$ be a coherent sheaf on Y and $x \in X$ a closed point. We say that $\mathcal{E}$ is *quasi locally free at* x if there exist integers $m_i \geq 0$, $1 \leq i \leq n$, such that

$$\mathcal{E}_x \ \simeq \ \bigoplus_{1\leq i\leq n} m_i \mathcal{O}_{X_i,x} \ .$$

The set U of such points x is nonempty and open, the integers m_i are uniquely determined and do not depend on x. If $U = X$, $\mathcal{E}$ is called *quasi locally free*. The sequence $(m_1, \ldots, m_n)$ is called the *type* of $\mathcal{E}$. If $\mathcal{F}$ is a coherent sheaf on Y, $\mathcal{F}$ is torsion free (resp. quasi locally free) if and only if all the $G^{(i)}(\mathcal{F})$ (resp. $G_i(\mathcal{F})$) are torsion free (resp. locally free).

We will see (in theorem 4.5.1) that if we have a flat family of quasi locally free sheaves $\mathcal{E}$ on Y, all of the same type, then the $\mathcal{E}_i$ and $\mathcal{E}^{(i)}$ form flat families of sheaves. Flat families of such sheaves having this property were called *good families* in [13], 3.2.

1.3. Balanced sheaves

Let $\mathbb{E}$ be a vector bundle on $Y = X_n$ and $E = \mathbb{E}_{|X}$. Then for $0 \leq i \leq n$, $\mathbb{E}_i$ is locally free on X_{n-i}, $\mathbb{E}_i = \mathbb{E}^{(n-i)}$ and $G_i(\mathbb{E}) = G^{(n-i)}(\mathbb{E}) = E \otimes L^i$.

Let $\mathcal{E}$ be a coherent sheaf on X_n. We say that $\mathcal{E}$ is *balanced* if $\mathcal{E}_i = \mathcal{E}^{(n-i)}$ for $1 \leq i \leq n$ (see 6.1 for an equivalent definition). Here also $G_i(\mathcal{E}) = G^{(n-i)}(\mathcal{E}) = \mathcal{E}_{|X} \otimes L^i$.

Balanced sheaves are the most natural sheaves that could be extensions of torsion free sheaves on X to X_n. To build moduli spaces of these sheaves, the work of [15] on vector bundles should be extended to torsion free sheaves.

1.3.1. *The case of primitive double schemes* – Suppose that $n = 2$, i.e. Y is a primitive double scheme. We prove that the fact that a sheaf is balanced is an *open property*: let C be a smooth curve and $\mathcal{E}$ a family of coherent sheaves on Y, parameterized by C, flat on C. Suppose that for some closed point $c_0 \in C$, $\mathcal{E}_{c_0}$ is balanced. Then there exists an open neighborhood U of c_0 such that for every $c \in U$, $\mathcal{E}_c$ is balanced (cf. theorem 6.2.1).

Let $\mathcal{E}$ a flat family of balanced sheaves on X_2 parameterized by a smooth connected curve C. We prove that the restrictions $\mathcal{E}_{c|X}$ form a flat family of sheaves on X parameterized cy C (corollary 6.2.2).

1.3.2. *Extension of sheaves on a smooth surface to multiplicity 2* – Suppose that $n = 2$ and $\dim(X) = 2$. Let S be a smooth variety and $\boldsymbol{F}$ a coherent sheaf on $X \times S$ such that

– $\boldsymbol{F}$ is flat on S.
– For every $s \in S$, $\boldsymbol{F}_s$ is torsion free and simple.
– $\dim(\mathrm{Ext}^1_{\mathcal{O}_X}(\mathbf{F}_s, \mathbf{F}_s \otimes L))$ is independent of $s \in S$.

We prove in 6.3 that the set of points $s \in S$ such that $\boldsymbol{F}_s$ can be extended to a balanced sheaf on X_2 is closed in S. This could allow to extend some results on moduli spaces of vector bundles on X_2 (cf. [15]) to moduli spaces of balanced sheaves.

1.4. Ideal sheaves

Let $\mathfrak{I}$ be the ideal sheaf of a zero dimensional subscheme Z of X_n. Suppose that for every closed point x of Z, the $\mathcal{O}_{X_n,x}$-module $\mathfrak{I}_x$ is generated by a regular sequence. This is equivalent to the fact that the image of $\mathfrak{I}_x$ in $\mathcal{O}_{X,x}$ is generated by a regular sequence. Then $\mathfrak{I}$ is balanced (cf. propositions 3.7.1 and 3.7.2).

1.4.1. *Ideal sheaves on primitive double surfaces* – We suppose now that $\dim(X) = 2$ and $n = 2$. Let $P \in X$ and $\mathcal{I}$ an ideal sheaf on X_2 of a subscheme whose support is $\{P\}$. Suppose that $\mathcal{I}_P$ is generated by two elements x, y whose images in $\mathcal{O}_{X,P}$ are generators of the maximal ideal. We show that

– The set of such ideal sheaves, with fixed P, has a natural structure of affine space with associated vector space $T_{X,P} \otimes L_P$, where T_X is the tangent bundle of X (cf. proposition 7.3.1).
– The set $\mathbb{I}$ of all these sheaves, for all P, has a natural structure of an affine bundle on X, with associated vector bundle $T_X \otimes L$.
– We build in 7.5 a *universal family of ideal sheaves* $\boldsymbol{\mathcal{J}}$ parameterized by $\mathbb{I}$, such that for every ideal sheaf $I \in \mathbb{I}$, $\boldsymbol{\mathcal{J}}_I \simeq I$.

We have a canonical exact sequence of vector bundles on X:

$$0 \longrightarrow L = \mathcal{I}_X \longrightarrow \Omega_{X_2|X} \longrightarrow \Omega_X \longrightarrow 0 \ ,$$

associated with $\sigma \in \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X, L) = H^1(X, T_X \otimes L)$. To the affine bundle $\mathbb{I}$ is also associated $\eta \in H^1(X, T_X \otimes L)$. We prove that $\mathbb{C}\sigma = \mathbb{C}\eta$.

To obtain a moduli space of sheaves we consider the set $\boldsymbol{I}(X_2) = \mathbb{I} \times \mathrm{Pic}(X_2)$ of sheaves of the form $I \otimes D$, where $I \in \mathbb{I}$ and $D \in \mathrm{Pic}(X_2)$. Let $p_1 : \boldsymbol{I}(X_2) \to \mathbb{I}$, $p_1 : \boldsymbol{I}(X_2) \to \mathrm{Pic}(X_2)$ be the projections. The exists an open cover $(P_i)_{i \in I}$ of $\mathrm{Pic}(X_2)$ such that for every $i \in I$, there is a Poincaré bundle $\mathcal{D}_i$ on $X_2 \times P_i$. This defines $\mathrm{Pic}(X_2)$ as a *fine moduli space* of vector bundles (cf. [15], 2.6, 4.6).

Let $\mathcal{H}_i = p_1^\sharp(\mathcal{J}) \otimes p_2^\sharp(\mathcal{D}_i)$, which is a sheaf on $\mathbb{I} \times P_i$. For every ideal $\mathcal{I}$ in $\mathbb{I}$ and $D \in P_i$, $\mathcal{H}_{\mathcal{I},D} = \mathcal{I} \otimes \mathcal{D}_{i,D} \simeq \mathcal{I} \otimes D$. Then $\boldsymbol{I}(X_2)$, with the open cover $(\mathbb{I} \times P_i)_{i\in I}$ and the sheaves $\mathcal{H}_i$ is a fine moduli space in the sense of 2.6 (proposition 7.5.3).

1.5. Outline of the paper

In Chapter 2, we give several preliminary definitions and technical results that will be used in the next chapters.

In Chapter 3, we define and study the primitive multiple rings. Some results obtained here are used in the next chapters.

In Chapter 4 we recall the definitions and properties of primitive multiple schemes, and we prove the results obtained on quasi locally free sheaves.

In Chapter 5 we define and give some properties of reduced Hilbert polynomials of coherent sheaves on primitive multiple schemes.

The Chapter 6 is devoted to balanced sheaves on primitive multiple schemes.

In Chapter 7 we study the balanced sheaves on a primitive double surface X_2, whose restriction to X are ideal sheaves of a point.

Notations and terminology: – Let $x_0, \ldots, x_k \in \mathbb{C}$, not all zero. We will also denote by $(x_0, \ldots, x_k)$ the element $\mathbb{C}.(x_0, \ldots, x_k)$ of $\mathbb{P}_k$.

– A *scheme* is a noetherian separated scheme over $\mathbb{C}$.

– If X is a scheme and $Y \subset X$ a subscheme, $\mathcal{I}_{Y,X}$ (or $\mathcal{I}_Y$) denotes the ideal sheaf of Y in X.

– Let X, Y, Z be schemes, $\mathcal{E}$ a coherent sheaf on $X \times Z$, and $f : Y \to Z$ a morphism. Then $f^\sharp(\mathcal{E}) = (I_X \times f)^*(\mathcal{E})$.

2. Preliminaries

2.1. Canonical class of a line bundle

Let Z be a scheme over $\mathbb{C}$ and L a line bundle on Z. To L one associates an element $\nabla_0(L)$ of $H^1(Z, \Omega_Z)$, called the *canonical class of* L. If Z is smooth and projective, and $L = \mathcal{O}_Z(Y)$, where $Y \subset Z$ is a smooth hypersurface, then $\nabla_0(L)$ is the cohomology class of Y.

Let $(Z_i)_{i\in I}$ be an open cover of Z such that L is defined by a cocycle (θ_{ij}), $\theta_{ij} \in \mathcal{O}_Z(Z_{ij})^*$. Then $\left(\dfrac{d\theta_{ij}}{\theta_{ij}}\right)$ is a cocycle which represents $\nabla_0(L)$.

2.2. Extensions of modules

Let R be a commutative ring and M a R-module. Suppose that we have a free resolution

$$\cdots \mathbb{F}_2 \xrightarrow{\phi_2} \mathbb{F}_1 \xrightarrow{\phi_1} \mathbb{F}_0 \twoheadrightarrow M \ .$$

Let N be a R-module. We have exact sequences

$$\mathrm{Hom}(\mathbb{F}_0, N) \longrightarrow \mathrm{Hom}(\mathrm{im}(\phi_1), N) \xrightarrow{\ \gamma\ } \mathrm{Ext}^1_R(M,N) \longrightarrow 0 \ ,$$
$$0 \longrightarrow \mathrm{Hom}(\mathrm{im}(\phi_1), N) \xrightarrow{\ \beta\ } \mathrm{Hom}(\mathbb{F}_1, N) \longrightarrow \mathrm{Hom}(\mathbb{F}_2, N) \ .$$

Let $0 \to N \to P \to M \to 0$ be an exact sequence, corresponding to $\sigma \in \mathrm{Ext}^1_R(M,N)$. Suppose that $\sigma = \gamma(f)$, $f : \mathrm{im}(\phi_1) \to N$, and $f_1 = \beta(f)$. Then P is constructed as follows: let $\eta = f_1 \oplus \phi_1 : \mathbb{F}_1 \to N \oplus \mathbb{F}_0$. Then $P = \mathrm{coker}(\eta)$. The inclusion $N \hookrightarrow P$ is induced by $N \hookrightarrow N \oplus \mathbb{F}_0$, and $P \twoheadrightarrow M$ by $\mathbb{F}_0 \twoheadrightarrow M$.

2.3. Refinements of filtrations

Let Z be a scheme, $\mathcal{E}$ a sheaf of $\mathcal{O}_Z$-modules, and two filtrations of $\mathcal{E}$

$$\mathcal{D} : \ \mathcal{D}_n = 0 \subset \mathcal{D}_{n-1} \subset \cdots \subset \mathcal{D}_1 \subset \mathcal{D}_0 = \mathcal{E} \ ,$$
$$\mathcal{F} : \ \mathcal{F}_m = 0 \subset \mathcal{E}_{m-1} \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathcal{E} \ .$$

We say that $\mathcal{D}$ is a *refinement* of $\mathcal{F}$ if for every j, $0 \le j \le m$, there exists i, $0 \le i \le n$, such that $\mathcal{E}_i = \mathcal{F}_j$. In this case we write $\mathcal{E} \to \mathcal{F}$.

Two filtrations $(\mathcal{E}_i)_{0\le i\le n}$, $(\mathcal{F}_j)_{0\le j\le m}$ of $\mathcal{E}$ are called *similar* if $m = n$, and if there exists a permutation σ of $\{0,\ldots,n\}$ such that for every $i \in \{0,\ldots,n\}$ we have $\mathcal{E}_i/\mathcal{E}_{i+1} \simeq \mathcal{F}_{\sigma(i)}/\mathcal{F}_{\sigma(i)+1}$. In this case we write $\mathcal{E} \simeq \mathcal{F}$.

2.3.1. Proposition: *Let*

$$\mathcal{D} \ : \ \mathcal{D}_n = 0 \subset \mathcal{D}_{n-1} \subset \cdots \subset \mathcal{D}_1 \subset \mathcal{D}_0 = \mathcal{E} \ ,$$
$$\mathcal{F} \ : \ \mathcal{F}_m = 0 \subset \mathcal{E}_{m-1} \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathcal{E}$$

be two filtrations of $\mathcal{E}$. Then there exist refinements $\mathcal{D}'$ of $\mathcal{D}$, $\mathcal{F}'$ of $\mathcal{F}$, which are similar.

Proof. **Step 1 –** We first suppose that $n = 2$. For $(\mathcal{D}'_k)_{0\le k\le p}$ we take

$$0 = \mathcal{D}_1 \cap \mathcal{F}_m \subset \mathcal{D}_1 \cap \mathcal{F}_{m-1} \subset \cdots \mathcal{D}_1 \cap \mathcal{F}_1 \subset \mathcal{D}_1 \subset \mathcal{D}_1 + \mathcal{F}_{m-1} \subset \cdots \subset \mathcal{D}_1 + \mathcal{F}_1 \subset \mathcal{D}_1 + \mathcal{F}_0 = \mathcal{E} \ ,$$

and we build $(\mathcal{F}'_k)_{0\le k\le p}$ by inserting $\mathcal{F}_{j+1} + \mathcal{D}_1 \cap \mathcal{F}_j$ between $\mathcal{F}_{j+1}$ and $\mathcal{F}_j$. We have

$$(\mathcal{F}_{j+1} + \mathcal{D}_1 \cap \mathcal{F}_j)/\mathcal{F}_{j+1} \ \simeq \ (\mathcal{D}_1 \cap \mathcal{F}_j)/(\mathcal{D}_1 \cap \mathcal{F}_{j+1}) \ ,$$
$$\mathcal{F}_j/(\mathcal{F}_{j+1} + \mathcal{D}_1 \cap \mathcal{F}_j) \ \simeq \ (\mathcal{D}_1 + \mathcal{F}_j)/(\mathcal{D}_1 + \mathcal{F}_{j+1}) \ .$$

Step 2 – Now suppose that we have two similar filtrations of $\mathcal{E}$

$$\mathcal{G} \ : \ \mathcal{G}_p = 0 \subset \mathcal{G}_{p-1} \subset \cdots \subset \mathcal{G}_1 \subset \mathcal{G}_0 = \mathcal{E} \ ,$$
$$\mathcal{H} : \ \mathcal{H}_p = 0 \subset \mathcal{H}_{p-1} \subset \cdots \subset \mathcal{H}_1 \subset \mathcal{H}_0 = \mathcal{E} \ ,$$

and that we have a refinement $\mathcal{G}'$ of $\mathcal{G}$. Then there exists a refinement $\mathcal{H}'$ of $\mathcal{H}$ which is similar to $\mathcal{G}'$: let σ be a permutation of $\{0,\ldots,p\}$ such that $\mathcal{G}_i/\mathcal{G}_{i+1} \simeq \mathcal{H}_{\sigma(i)}/\mathcal{H}_{\sigma(i)+1}$. Then to each

$\mathcal{G}'_k$ such that $\mathcal{G}_{i+1} \subset \mathcal{G}'_k \subset \mathcal{G}_i$ corresponds a sheaf $\mathcal{K}$ such that $\mathcal{H}_{\sigma(i)+1} \subset \mathcal{K} \subset \mathcal{H}_{\sigma(i)}$. All these sheaves make the filtration $\mathcal{H}'$.

Step 3 – Now we prove proposition 2.3.1 by induction on n. It is already proved for $n = 2$. Suppose that it is true for $n - 1 \geq 2$. Then we consider the filtrations

$$\mathcal{D}^0 \ : \ \mathcal{D}^0_{n-1} = 0 \subset \mathcal{D}^0_{n-2} = \mathcal{D}_{n-2} \subset \cdots \subset \mathcal{D}^0_1 = \mathcal{D}_1 \subset \mathcal{D}^0_0 = \mathcal{E}$$

and $\mathcal{F}$. There are refinements $\mathcal{D}'' = (\mathcal{D}''_k)_{0\leq k\leq p}$ of $\mathcal{D}^0$ and $\mathcal{F}''$ of $\mathcal{F}$ which are similar. There exists an integer q such that $\mathcal{D}''_q = \mathcal{D}_{n-2}$. We have filtrations of $\mathcal{D}_{n-2}$

$$0 \subset \mathcal{D}_{n-1} \subset \mathcal{D}_{n-2} \ ,$$

$$0 = \mathcal{D}''_p \subset \mathcal{D}''_{p-1} \subset \cdots \subset \mathcal{D}''_q = \mathcal{D}_{n-2} \ .$$

From step 1, we can find refinements $(\mathcal{D}^1_k)_{0\leq k\leq r}$ of the first, $(\mathcal{D}^2_k)_{0\leq k\leq r}$ of the second, which are similar. Now we build two refinements $\mathcal{D}^3$, $\mathcal{D}^4$, of $\mathcal{D}^0$, the first by replacing in $\mathcal{D}''$, $(\mathcal{D}''_p, \ldots, \mathcal{D}''_q)$ with $(\mathcal{D}^1_r, \ldots, \mathcal{D}^1_0)$, and the second by replacing in $\mathcal{D}''$, $(\mathcal{D}''_p, \ldots, \mathcal{D}''_q)$ with $(\mathcal{D}^2_r, \ldots, \mathcal{D}^2_0)$. We have $\mathcal{D}^3 \simeq \mathcal{D}^4$, $\mathcal{D}^3 \to \mathcal{D}$, $\mathcal{D}^4 \to \mathcal{D}''$ and $\mathcal{D}'' \simeq \mathcal{F}''$.

From step 2 we can find a refinement $\mathcal{F}^4$ of $\mathcal{F}''$ such that $\mathcal{F}^4 \simeq \mathcal{D}^4$. So we can take $\mathcal{D}' = \mathcal{D}^3$, $\mathcal{F}' = \mathcal{F}^4$. □

2.4. Affine bundles

Let $f : \mathcal{A} \to S$ be a morphism of schemes, and $r \geq 0$ an integer. We say that f (or $\mathcal{A}$) is an *affine bundle* of rank r over S if there exists an open cover $(S_i)_{i\in I}$ of S such that for every $i \in I$ there is an isomorphism $\tau_i : f^{-1}(S_i) \to S_i \times \mathbb{C}^r$ over S_i such that for every distinct $i, j \in I$, $f_j \circ f_i^{-1} : S_{ij} \times \mathbb{C}^r \to S_{ij} \times \mathbb{C}^r$ is of the form

$$(x, u) \longmapsto (x, A_{ij}(x)u + b_{ij}(x)) \ ,$$

where A_{ij} is an $r \times r$-matrix of elements of $\mathcal{O}_S(S_{ij})$ and b_{ij} is a morphism from S_{ij} to $\mathbb{C}^r$. We have then the cocycle relations

$$A_{ij}A_{jk} \ = \ A_{ik} \ , \quad b_{ik} \ = \ A_{ij}b_{jk} + b_{ij} \ .$$

The first relation shows that the family (A_{ij}) defines a vector bundle $\mathbb{A}$ on S, and the second that (b_{ij}) defines $\lambda \in H^1(S, \mathbb{A})$ (according to [14], 2.1.1).

The vector bundle $\mathbb{A}$ is uniquely defined, as well as $\eta(\mathcal{A}) = \mathbb{C}\lambda \in \big(\mathbb{P}(H^1(S, \mathbb{A})) \cup \{0\}\big)/\operatorname{Aut}(\mathbb{A})$.

We say that $\mathcal{A}$ *is a vector bundle* if f has a section. This is the case if and only $\lambda = 0$, and then a section of f induces an isomorphism $\mathcal{A} \simeq \mathbb{A}$ over S.

For every closed point $s \in S$ there is a canonical action of the additive group $\mathbb{A}_s$ on $\mathcal{A}_s$

$$\begin{aligned} \mathbb{A}_s \times \mathcal{A}_s &\longrightarrow \mathcal{A}_s \\ (u, a) &\longmapsto a + u \end{aligned}$$

such that for every $a \in \mathcal{A}_s$, $u \mapsto a + u$ is an isomorphism $\mathbb{A}_s \simeq \mathcal{A}_s$.

2.5. Locally free resolutions

Let X, S be smooth algebraic varieties, with X projective. Let $\mathcal{E}$ be a flat family of torsion free sheaves on X parameterized by S, and

$$\cdots\mathbb{E}_m \xrightarrow{f_m} \mathbb{E}_{m-1} \xrightarrow{f_{m-1}} \cdots \longrightarrow \mathbb{E}_0 \xrightarrow{f_0} \mathcal{E} \longrightarrow 0$$

an exact sequence, where the $\mathbb{E}_i$ are flat families of torsion free sheaves on X parameterized by S.

2.5.1. Lemma: *The sheaf $\mathcal{E}$ is torsion free.*

Proof. Let $(x,s)\in X\times S$, $e\in\mathcal{E}_{(x,s)}$, $\alpha\in\mathcal{O}_{X\times S,(x,s)}$ be such that $\alpha e=0$. Let U be an open neighborhood of (x,s) such that $e\in\mathcal{E}(U)$, $\alpha\in\mathcal{O}_{X\times S}(U)$. Suppose that $\alpha\neq 0$. If $(x',s')\in U$, and $\mathcal{E}_{s',x'}$ is a free $\mathcal{O}_{X\times\{s'\},x'}$-module , then $\mathcal{E}_{(x',s')}$ is a free $\mathcal{O}_{X\times S,(x',s')}$-module. Hence $e=0$ on a neighborhood of (x',s'). If $s'\in S$ is such that $U_{s'}=U\cap(X\times\{s'\})\neq\emptyset$, then $\mathcal{E}_{|U_{s'}}$ is locally free on a nonempty open subset of $U_{s'}$. Hence $e_{|U_{s'}}=0$. Since this is true for every such s', we have $e=0$. □

2.5.2. Lemma: *Let $H\subset S$ be a hypersurface, and $\mathcal{F}$ a torsion free sheaf on $X\times S$. Then we have*

$$\mathrm{Tor}^1_{\mathcal{O}_{X\times S}}(\mathcal{F},\mathcal{O}_{X\times H}) \ = \ 0\ .$$

Proof. We have a locally free resolution

$$0\longrightarrow \mathcal{O}_{X\times S}(-(X\times H)) \xrightarrow{\gamma} \mathcal{O}_{X\times S}\longrightarrow \mathcal{O}_{X\times H}\longrightarrow 0$$

that we use to compute $\mathrm{Tor}^1_{\mathcal{O}_{X\times S}}(\mathcal{E},\mathcal{O}_{X\times H})$. It is the kernel of

$$I_{\mathcal{F}}\otimes\gamma:\mathcal{F}(-(X\times H))\longrightarrow\mathcal{F}\ ,$$

which is injective since $\mathcal{F}$ is torsion free. □

2.5.3. Proposition: *For every closed point $s\in S$, the sequence*

$$\cdots\mathbb{E}_{m,s} \xrightarrow{f_{m,s}} \mathbb{E}_{m-1,s} \xrightarrow{f_{m-1,s}} \cdots \longrightarrow \mathbb{E}_{0,s} \xrightarrow{f_{0,s}} \mathcal{E}_s \longrightarrow 0$$

is exact.

Proof. For $0\leq i<m$, let $N_i=\ker(f_i)$, so that we have exact sequences

$$0\longrightarrow N_i\longrightarrow\mathbb{E}_i\longrightarrow N_{i-1}\longrightarrow 0 \qquad \text{if} \quad i>0\ ,$$

$$0\longrightarrow N_0\longrightarrow\mathbb{E}_0\longrightarrow\mathcal{E}\longrightarrow 0\ .$$

Let $d=\dim(S)$, and $S_1,\ldots,S_d$ hypersurfaces containing s, smooth at s and that intersect transversally at s. By replacing S with an open neighborhood of s we can assume the $S_1\cap\cdots\cap S_d=\{s\}$.

From lemma 2.5.2, since the N_i are torsion free, we have exact sequences

$$0\longrightarrow N_{i|X\times S_1}\longrightarrow\mathbb{E}_{i|X\times S_1}\longrightarrow N_{i-1|X\times S_1}\longrightarrow 0 \qquad \text{if} \quad i>0\ ,$$

$$0\longrightarrow N_{0|X\times S_1}\longrightarrow\mathbb{E}_{0|X\times S_1}\longrightarrow\mathcal{E}_{|X\times S_1}\longrightarrow 0\ ,$$

i.e. we have a locally free resolution of $\mathcal{E}_{|X\times S_1}$

$$\cdots\mathbb{E}_{m|X\times S_1}\xrightarrow{f_m}\mathbb{E}_{m-1|X\times S_1}\xrightarrow{f_{m-1}}\cdots\longrightarrow\mathbb{E}_{0|X\times S_1}\xrightarrow{f_0}\mathcal{E}_{|X\times S_1}\longrightarrow 0\ ,$$

and by lemma 2.5.1, $\mathcal{E}_{|X\times S_1}$ and the $N_{i|X\times S_1}$ are torsion free. So we can continue this process, and by induction on d we finally arrive at the resolution of proposition 2.5.3. □

2.6. Fine moduli spaces of sheaves

Let χ be a set of isomorphism classes of coherent sheaves on a scheme Z. Suppose that χ is *open*, i.e. for every scheme V and coherent sheaf $\mathbb{E}$ on $Z\times V$, flat on V, if $v\in V$ is a closed point such that $\mathbb{E}_v\in\chi$, then there exists an open neighborhood U of v such that $\mathbb{E}_u\in\chi$ for every closed point $u\in U$. A *fine moduli space for* χ is the data of a scheme M and of

– a bijection

$$M^0\longrightarrow\chi$$

$$m\longmapsto E_m$$

(where M^0 denotes the set of closed points of M),

– an open cover $(M_i)_{i\in I}$ of M, and for every $i\in I$, a coherent sheaf $\mathcal{E}_i$ on $Z\times M_i$ such that for every $m\in M_i$, $\mathcal{E}_{i,m}\simeq E_m$,

such that: for any scheme S, any coherent sheaf $\mathcal{F}$ on $Z\times S$, flat on S, such that for any closed point $s\in S$, $\mathcal{F}_s\in\chi$, there is a morphism $f_{\mathcal{F}}:S\to M$ such that for every $s\in S$, if $m=f_{\mathcal{F}}(s)$ then $\mathcal{F}_s\simeq E_m$, and if $m\in M_i$, then there exists an open neighborhood U of s such that $f_{\mathcal{F}}(U)\subset M_i$ and $(I_X\times f_{\mathcal{F}|U})^*(\mathcal{E}_i)\simeq\mathcal{F}_{|Z\times U}$.

3. Primitive multiple rings

Let R be an entire noetherian ring and K its fraction field.

Let n be a positive integer and $R[n]=R[t]/(t^n)$, which we call the *primitive multiple ring of multiplicity* n associated to R.

We will also consider rings A,A_n isomorphic to $R,R[n]$ respectively, with the ideal $\mathcal{I}\subset A_n$ corresponding to (t), and $A=A_n/\mathcal{I}$. Sometimes it is useful not to specify a generator of $\mathcal{I}$, in particular when considering families of rings.

An element u of $R[n]$ can be written in an unique way as

$$u\ =\ \sum_{i=0}^{n-1}u_it^i\ ,$$

with $u_i\in R$ for $0\leq i\leq n$. Then u is a zero divisor if and only if $u_0=0$, i.e. if u is a multiple of t. Let S_n be the multiplicative system of non zero divisors of $R[n]$, i.e. $S_n=\{u\in R[n];u_0\neq 0\}$. It is easy to see that $S_n^{-1}R[n]=K[t]/(t^n)=K[n]$ (the primitive multiple ring of multiplicity n associated to K).

3.1. Canonical filtrations

Let M be a $R[n]$-module. Then it is easy to see that we have a canonical isomorphism $S_n^{-1}M \simeq M \otimes_{R[n]} K[n]$, and a natural morphism of $R[n]$-modules $i_M : M \to M \otimes_{R[n]} K[n]$.

For $0 \leq i \leq n$, let $M_i = t^i M$, which is a submodule of M. The filtration

$$0 = M_n \subset M_{n-1} \subset \cdots \subset M_1 \subset M_0 = M$$

is called the *first canonical filtration* of M. For $0 \leq i < n$, let

$$G_i(M) \ = \ M_i/M_{i+1} \ \simeq \ M_i \otimes_{R[n]} R \ .$$

For $0 \leq i \leq n$, let $M^{(i)} = \{m \in M; t^i m = 0\}$, which is a submodule of M that contains M_{n-i}. The filtration

$$M^{(0)} = \{0\} \subset M^{(1)} \subset \cdots \subset M^{(n-1)} \subset M^{(n)} = M$$

is called the *second canonical filtration* of M. For $0 < i \leq n$, let

$$G^{(i)}(M) \ = \ M^{(i)}/M^{(i-1)} \ ,$$

which is a R-module.

For example we have, for $M = R[n]^p$ and $0 \leq i < n$,

$$M_i \ = \ M^{(n-i)} \ \simeq \ (R[n-i])^p \ , \qquad G_i(M) \ \simeq \ R^p \ . \tag{2}$$

For $1 \leq i \leq n-1$, the multiplication by t, $M \to M$, induces an injective morphism of R-modules

$$\lambda_i : G^{(i+1)}(M) \longrightarrow G^{(i)}(M)$$

(for a A-module M : $\lambda_i : G^{(i+1)}(M) \otimes_A \mathcal{J} \to G^{(i)}(M)$). Let $\Gamma^{(i-1)}(M) = \operatorname{coker}(\lambda_i)$. Similarly we have for $0 \leq i \leq n-2$ a surjective morphism of R-modules

$$\mu_i : G_i(M) \longrightarrow G_{i+1}(M)$$

(resp. $\mu_i : G_i(M) \otimes_A \mathcal{J} \longrightarrow G_{i+1}(M)$). Let $\Gamma_i(M) = \ker(\mu_i)$.

We have then a canonical isomorphism $\Gamma_i(M) \simeq \Gamma^{(i)}(M)$ (resp. $\Gamma_i(M) \simeq \Gamma^{(i)}(M) \otimes_A \mathcal{J}^{i+1}$, cf. prop. 4.3.4).

3.2. Some canonical extensions

We can view $R[i]$, $0 \leq i < n$, as a $R[n]$-module. We have an exact sequence

$$0 \longrightarrow R[i] = tR[i+1] \hookrightarrow R[i+1] \longrightarrow R \longrightarrow 0$$

(resp. $0 \to \mathcal{J}A_{i+1} \simeq A_i \to A_{i+1} \to A \to 0$).

3.2.1. Lemma: *We have* $\operatorname{Ext}^1_{R[n]}(R, R[i]) \simeq R$, *and given an extension* $0 \to R[i] \to P \to R \to 0$ *of* $R[n]$*-modules, corresponding to* $\sigma \in R$, *then* $P \simeq R[i+1]$ *if and only if* σ *is invertible.*

Proof. We use 2.2 and the following free resolution of R as a $R[n]$-module:

$$\cdots R[n] \xrightarrow{Xt^{n-1}} R[n] \xrightarrow{Xt} R[n] \twoheadrightarrow R\ .$$

Let $\iota : R[n-1] = tR[n] \to R[n]$ be the inclusion, and $\eta = \sigma \oplus \iota : R[n-1] \to R[i] \oplus R[n]$. Then $P = \mathrm{coker}(\eta)$. Lemma 3.2.1 follows easily from this description. □

3.3. Quasi free modules

For $1 \leq i \leq n$, $R[i]$ is a $R[n]$-module. Let M be a $R[n]$-module. We say that M is *quasi free* if there exist integers $n_i \geq 0$, $1 \leq i \leq n$, such that $M \simeq \bigoplus_{i=1}^{n} R[i]^{n_i}$.

The proof of the following result is similar to that of [5], théorème 5.1.3.

3.3.1. Theorem: *Let M be a $R[n]$-module of finite type. Then M is quasi free if and only $G_i(M)$ is a free R-module for $0 < i \leq n$.*

If R is a field then the $G_i(M)$ are R-vector spaces, hence they are free. It follows that

3.3.2. Corollary: *Every finitely generated $K[n]$-module is quasi locally free (as a $K[n]$-module).*

3.4. Torsion

Let M be a $R[n]$-module of finite type. The set of elements $m \in M$ such that there exists $\alpha \in S_n$ such that $\alpha m = 0$ is a submodule $T(M)$ of M, called the *torsion submodule* of M. We say that M is a *torsion module* if $M = T(M)$, and that M is *torsion free* if $T(M) = 0$, or equivalently if for every non zero $m \in M$ and $u \in R[n]$, we have $um \neq 0$. The module $M/T(M)$ is torsion free. It is easy to see that $T(M) = \ker(i_M)$ (cf. 3.1). Hence M is torsion free if and only if i_M is injective.

3.4.1. Proposition: *M is torsion free if and only if it is isomorphic to a submodule of a free $R[n]$-module.*

Proof. A free module is torsion free, so is M if it is a submodule of a free module. Conversely, suppose that M is torsion free. Then $i_M : M \to M \otimes_{R[n]} K[n]$ is injective. By corollary 3.3.2, $M \otimes_{R[n]} K[n]$ is quasi locally free as a $K[n]$-module, i.e. there are integers $n_i \geq 0$, $1 \leq i \leq n$, such that we have an isomorphism

$$M \otimes_{R[n]} K[n] \ \simeq\ \bigoplus_{i=1}^{n} K[i]^{n_i}\ .$$

Let $m_1, \ldots, m_q$ be generators of M. We can write $i_M(m_j) = \sum_{i=1}^{n} \lambda_{j,i}$, with $\lambda_{j,i} \in K[i]^{n_i}$. There exists $\mu \in R[n]$, such that $\mu_0 \neq 0$, and $\mu\lambda_{j,i} \in R[i]^{n_i}$ for $1 \leq j \leq q$, $1 \leq i \leq n$. We have then

$$i_M(M) \subset \bigoplus_{i=1}^{n} \left(\frac{1}{\mu}R[i]\right)^{n_i} ,$$

and since $\left(\frac{1}{\mu}R[i]\right)^{n_i} \simeq R[i]^{n_i}$, we obtain an inclusion $M \subset \bigoplus_{i=1}^{n} R[i]^{n_i}$. But $R[i] \simeq (t^{n-i}) \subset R[n]$, and finally $M \subset R[n]^m$, with $m = \sum_{i=1}^{n} n_i$. □

3.4.2. Proposition: $\mathrm{Ext}^1_{R[n]}(M, R[n])$ *is a torsion module.*

Proof. Using the exact sequence $0 \to T(M) \to M \to M/T(M) \to 0$, it suffices to treat the cases of torsion modules, and of torsion free modules.

Suppose that $M = T(M)$. Since $T(M)$ is finitely generated, there exists $x = \sum_{i=0}^{n-1} u_i t^i \in R[n]$ with $u_0 \neq 0$, such that $xT(M) = \{0\}$. The morphism $\mathrm{Ext}^1_{R[n]}(M, R[n]) \to \mathrm{Ext}^1_{R[n]}(M, R[n])$ induced by the multiplication by $x : M \to M$, is also the multiplication by x, and it is zero, i.e. $x\,\mathrm{Ext}^1_{R[n]}(M, R[n]) = \{0\}$.

Suppose that M is torsion free. Let $\sigma \in \mathrm{Ext}^1_{R[n]}(M, R[n])$, corresponding to an extension

$$0 \longrightarrow R[n] \xrightarrow{\alpha} N \xrightarrow{p} M \longrightarrow 0 .$$

The module N is finitely generated and torsion free, hence by proposition 3.4.1, it is a submodule of a free module: $N \subset R[n]^m$. Let $\alpha(1) = (r_1, \dots, r_m) \in R[n]^m$. Since α is injective, some r_i is not a multiple of t. Let $p_i : N \to R[n]$ be the restriction to N of the ith projection $R[n]^m \to R[n]$, so that $p_i \circ \alpha : R[n] \to R[n]$ is the multiplication by r_i. We have then a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R[n] & \xrightarrow{\alpha} & N & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\times r_i} & & \downarrow{p_i \oplus p} & & \| & & \\
0 & \longrightarrow & R[n] & \longrightarrow & R[n] \oplus M & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

where the lower sequence is the trivial exact sequence. It is associated to $r_i\sigma$, hence $r_i\sigma = 0$. □

3.4.3. Proposition: *M is torsion free if and only $M^{(1)}$ is. In this case $G^{(i)}(M)$ and $M/M^{(i)}$ are torsion free for $1 \leq i \leq n$.*

Proof. If M is torsion free then so is $M^{(1)} \subset M$. It M is not torsion free, let $m \in M$, $m \neq 0$, be such that there exists $\alpha \in R[n]$, not a zero divisor, such that $\alpha m = 0$. Let k be the biggest integer such that $z^k m \neq 0$. Then $z^k m \in M^{(1)}$ and $\alpha z^k m = 0$. Hence $M^{(1)}$ is not torsion free.

Suppose that M is torsion free. Let $\overline{m} \in M/M^{(1)}$, α a non zero divisor in $R[n]$, be such that $\alpha\overline{m} = 0$. If $m \in M$ is over $\overline{m}$, we have $\alpha m \in M^{(1)}$, i.e. $z\alpha m = 0$. Since M is torsion free, we have $zm = 0$, $m \in M^{(1)}$ and $\overline{m} = 0$. This proves that $M/M^{(1)}$ is torsion free.

The fact that $G^{(i)}(M)$ and $M/M^{(i)}$ are torsion free for $1 \leq i \leq n$ is easily proved by induction on i. □

3.5. Duality

Let M be a $R[n]$-module of finite type, and $M^\vee = \mathrm{Hom}(M, R[n])$ the *dual* of M. If $1 \leq i \leq n$ and M is a $R[i]$-module, we have

$$M^\vee \ = \ \mathrm{Hom}(M, (t^{n-i})) \ = \ \mathrm{Hom}(M, R[i]) \ ,$$

So the dual of M as a $R[i]$-module is also $M^\vee$. In particular $(R[i])^\vee \simeq R[i]$. The dual of a $R[n]$-module is torsion free. If M is a torsion module, then $M^\vee = \{0\}$.

The following is immediate

3.5.1. Lemma: *For $1 \leq i < n$, we have $(M^\vee)^{(i)} \simeq (M/M_i)^\vee$.*

Let $t_M : M \to M^{\vee\vee}$ be the canonical morphism. If $m \in M$, $t_M(m)$ is the linear form

$$\begin{array}{ccc} M^\vee & \longrightarrow & R[n] \\ \phi & \longmapsto & \phi(m) \end{array} .$$

If M is a free module, t_M is an isomorphism.

3.5.2. Proposition: *We have* $\ker(t_M) = T(M)$.

Proof. First we prove that t_M is injective if M is torsion free. By proposition 3.4.1, M is then a submodule of a free module E. Let $m \in M$, $m \neq 0$. Then there exists a linear form $\psi : E \to R[n]$ such that $\psi(m) \neq 0$. If $\phi \in M^\vee$ is the restriction of ψ, we have $\phi(m) \neq 0$.

Now for a general M, the exact sequence $0 \to T(M) \to M \to M/T(M) \to 0$ implies that $(M/T(M))^\vee \simeq M^\vee$. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(M) & \longrightarrow & M & \longrightarrow & M/T(M) & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle t_M} & & \downarrow{\scriptstyle t_{M/T(M)}} & & \\ & & & & M^{\vee\vee} & \xrightarrow{\simeq} & (M/T(M))^{\vee\vee} & & \end{array}$$

and the result follows from the injectivity of $t_{M/T(M)}$. □

3.5.3. Proposition: $\mathrm{coker}(t_M)$ *is a torsion module.*

Proof. We first prove the following result: suppose that we have an exact sequence $0 \to K \to M \xrightarrow{\ \pi\ } Q \to 0$ of $R[n]$-modules, and that $\mathrm{coker}(t_K)$ and $\mathrm{coker}(t_Q)$ are torsion modules. Then $\mathrm{coker}(t_M)$ is a torsion module. We have an exact sequence

$$0 \to Q^\vee \xrightarrow{\ {}^t\pi\ } M^\vee \to K^\vee \xrightarrow{\ \delta\ } \mathrm{Ext}^1_{R[n]}(Q, R[n]) \ .$$

By proposition 3.4.2, $\mathrm{Ext}^1_{R[n]}(Q,R[n])$ is a torsion module. Let $U=\ker(\delta)$, $T=\mathrm{im}(\delta)$. We have exact sequences

$$0\to Q^\vee \xrightarrow{\ {}^t\pi\ } M^\vee \to U\to 0\ ,\quad 0\to U\to K^\vee\to T\to 0\ ,$$

whence exact sequences

$$0\to U^\vee\to M^{\vee\vee} \xrightarrow{\ {}^{tt}\pi\ } Q^{\vee\vee}\to T_2\to 0\ ,\quad 0\to K^{\vee\vee}\xrightarrow{\ \mu\ } U^\vee\to T_3\to 0\ ,$$

for some torsion modules T_2, T_3. Let $V=\mathrm{im}({}^{tt}\pi)$. Then we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & Q & \longrightarrow & 0\\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle t_M} & & \downarrow{\scriptstyle\beta} & & \\ 0 & \longrightarrow & U^\vee & \longrightarrow & M^{\vee\vee} & \longrightarrow & V & \longrightarrow & 0 \end{array}$$

hence an exact sequence

$$\mathrm{coker}(\alpha)\longrightarrow \mathrm{coker}(t_M)\longrightarrow \mathrm{coker}(\beta)\longrightarrow 0\ .$$

Now α is the composition $K\xrightarrow{\ t_K\ } K^{\vee\vee}\xrightarrow{\ \mu\ } U^\vee$, so $\mathrm{coker}(\alpha)$ is a torsion module. and the composition $Q\xrightarrow{\ \beta\ } V\subset Q^{\vee\vee}$ is t_Q, so $\mathrm{coker}(\beta)$ is a torsion module. It follows that $\mathrm{coker}(t_M)$ is also a torsion module.

Proposition 3.5.3 is then easily proved by induction on the minimal number of generators of M (or on n, using the canonical filtrations of M). □

3.6. Balanced modules

Let M be a $R[n]$-module of finite type. Let

$$\boldsymbol{\lambda}(M)\ =\ \lambda_1\circ\cdots\circ\lambda_{n-1}:G^{(n)}(M)\longrightarrow G^{(1)}(M)\ ,$$
$$\boldsymbol{\mu}(M)\ =\ \mu_{n-2}\circ\cdots\circ\mu_0:G_0(M)\longrightarrow G_{n-1}(M)$$

(resp.

$$\boldsymbol{\lambda}(M)\ =\ \lambda_1\circ\cdots\circ\lambda_{n-1}:G^{(n)}(M)\otimes_{A_n}\mathcal{J}^{n-1}\longrightarrow G^{(1)}(M)\ ,$$
$$\boldsymbol{\mu}(M)\ =\ \mu_{n-2}\circ\cdots\circ\mu_0:G_0(M)\otimes_{A_n}\mathcal{J}^{n-1}\longrightarrow G_{n-1}(M)\ )\ .$$

(cf. 3.1).

3.6.1. Definition: *We say that M is* balanced *if $\boldsymbol{\lambda}(M)$ is surjective.*

3.6.2. Proposition: *The following properties are equivalent*

(i) *M is balanced.*
(ii) *$\lambda_1,\ldots,\lambda_{n-1}$ are surjective.*
(iii) $\Gamma_1(M)=\cdots=\Gamma_{n-1}(M)=0$.
(iv) $\Gamma^{(1)}(M)=\cdots=\Gamma^{(n-1)}(M)=0$.
(v) *$\mu_0,\ldots,\mu_{n-2}$ are injective.*
(vi) *$\boldsymbol{\mu}(M)$ is injective.*

(cf. prop. 6.1.3).

3.6.3. Proposition: *The $R[n]$-module M is balanced if and only if $M_i = M^{(n-i)}$ for $1 \leq i \leq n$.*

(cf. prop. 6.1.4).

In particular a $R[2]$-module is balanced if and only if $M^{(1)} = M_1$.

3.6.4. Proposition: *Let M be a balanced $R[n]$-module.*

1– *Let*

$$0 = N_n \subset N_{n-1} \subset \cdots \subset N_1 \subset N_0 = N$$

be a filtration such that, for $0 \leq i < n$, $N_i/N_{i+1} \neq \{0\}$ and $t.(N_i/N_{i+1}) = \{0\}$. Then we have $N_i = M_i$ for $1 \leq i < n$.

2– *Let $M' \subset M$ be a submodule. Suppose that $t^{n-1}.M' = \{0\}$ and $t.(M/M') = \{0\}$. Then we have $M' = M_1$.*

Proof. We first prove **1**. Let $u \in N_1$. Let $p : N_1 \to N_1/N_2$ be the projection. Then $p(tu) = tp(u) = 0$, hence $tu \in N_2$, and similarly $t^{i-1}u \in N_i$ for $2 \leq i < n$. It follows that $N_1 \subset M^{(n-1)}$. Since $M_1 \subset N_1$ and $M_1 = M^{(n-1)}$, we have $N_1 = M_1$. Then **1** is easily proved by induction on n.

2 is a consequence of **1** (take the first canonical filtration of M'). □

3.7. Balanced ideals and regular sequences

3.7.1. Proposition: *Let p be a positive integer, and for $1 \leq i \leq p$, $x_i = \sum_{j=0}^{n-1} x_{i,j}t^j \in R[n]$, with $x_{i,j} \in R$ for $0 \leq j < n$. Let I (resp. I_0) be the ideal of $R[n]$ (resp. R) generated by $x_1, \ldots, x_p$ (resp. $x_{1,0}, \ldots, x_{p,0}$). Then*

(i) *$(x_1, \ldots, x_p)$ is a regular sequence in $R[n]$ if and only if $(x_{1,0}, \ldots, x_{p,0})$ is a regular sequence in R.*

(ii) *If $(x_1, \ldots, x_p)$ is a regular sequence in $R[n]$, then for every $y \in R$, $t^{n-1}y \in I$ if and only if $y \in I_0$.*

Proof. By induction on n. The case $n = 1$ is obvious. Suppose that the theorem is true for $n - 1 \geq 1$. We will prove that it is true for n by induction on p, the case $p = 1$ being obvious. Suppose that it is true for $p - 1 \geq 1$.

Suppose that $(x_1, \ldots, x_p)$ is a regular sequence in $R[n]$. We will prove that $(x_{1,0}, \ldots, x_{p,0})$ is a regular sequence in R. By the induction hypothesis, $(x_{1,0}, \ldots, x_{p-1,0})$ is a regular sequence in R, and we have to show that $x_{p,0}$ is not a zero divisor in $R/(x_{1,0}, \ldots, x_{p-1,0})$. Let $a \in R$ be such that $ax_{p,0} \in (x_{1,0}, \ldots, x_{p-1,0})$. Hence $t^{n-1}ax_p \in (x_1, \ldots, x_{p-1})$. By (ii) and the induction hypothesis, we have $a \in (x_{1,0}, \ldots, x_{p-1,0})$, i.e. $a = 0$ in $R/(x_{1,0}, \ldots, x_{p-1,0})$. Hence $(x_{1,0}, \ldots, x_{p,0})$ is a regular sequence in R.

Suppose now that $(x_{1,0}, \ldots, x_{p,0})$ is a regular sequence in R. We will prove that $(x_1, \ldots, x_p)$ is a regular sequence in $R[n]$. By the induction hypothesis, $(x_1, \ldots, x_{p-1})$ is a regular sequence in

$R[n]$, and we must show that x_p is not a zero divisor in $R[n]/(x_1,\ldots,x_{p-1})$. Let $a\in R[n]$ be such that $ax_p\in(x_1,\ldots,x_{p-1})$. By the induction hypothesis (the result is true for $n-1$), we can write $a=b+t^{n-1}c$, with $b\in(x_1,\ldots,x_{p-1})$ and $c\in R$. We have then $t^{n-1}cx_p\in(x_1,\ldots,x_{p-1})$. By (ii) and the induction hypothesis, we have $cx_{p,0}\in(x_{1,0},\ldots,x_{p-1,0})$. Since $(x_{1,0},\ldots,x_{p-1,0})$ is a regular sequence, we have $c\in(x_{1,0},\ldots,x_{p-1,0})$. It follows that $a\in(x_1,\ldots,x_{p-1})$, hence $(x_1,\ldots,x_p)$ is a regular sequence in $R[n]$.

Now we prove (ii). If $y\in I_0$, it is clear that $t^{n-1}y\in I$. Conversely, suppose that $t^{n-1}y\in I$. We can write $t^{n-1}y=t^k\sum_{i=1}^p\alpha_ix_i$, with $k\in\{0,\ldots,n-1\}$ maximal, and $\alpha_1,\ldots,\alpha_p\in R[n]$. We must show that $k=n-1$. Suppose that $k<n-1$. We can write $t^{n-1-k}y=\sum_{i=1}^p\alpha_ix_i+t^{n-k}z$, for some $z\in R[n]$, i.e. $t^{n-1-k}(y-tz)=\sum_{i=1}^p\alpha_ix_i$. By the induction hypothesis (the result is true for $n-k$), the term of degree 0 of $y-tz$ belongs to I_0, i.e $y\in I_0$. □

3.7.2. Proposition: *Let $(x_1,\ldots,x_p)$ be a regular sequence in $R[n]$, and I the ideal of $R[n]$ generated by $x_1,\ldots,x_p$. Then I is a balanced $R[n]$-module.*

Proof. Let $I'\subset R[n-1]$ be the ideal generated by the images $x'_1,\ldots,x'_p$ of $x_1,\ldots,x_p$ in $R[n-1]$. By proposition 3.7.1 (i), $(x'_1,\ldots,x'_p)$ is a regular sequence in $R[n-1]$. Let $\phi:I\to I'$ be the restriction of the canonical morphism $R[n]\to R[n-1]$. Then we have $\ker(\phi)=I\cap(t^{n-1})$, and by proposition 3.7.1 (ii), $I\cap(t^{n-1})=t^{n-1}I$. It follows that ϕ induces an isomorphism $I/t^{n-1}I\simeq I'$. Hence, using an induction on n, we see that for $1\leq<n$, I_k is canonically isomorphic to the ideal of $R[k]$ generated by the images of $x_1,\ldots,x_p$. Proposition 3.7.2 follows easily. □

3.7.3. *Other examples* – Suppose that $R=\mathbb{C}[X,Y]$, and let $I=(X^2,Y^2,XY)$, $J=(X^2,Y^2+t,XY)$ in $R[2]$. Then we have $I_1=I^{(1)}=t.(X^2,Y^2,XY)$, so I is balanced. But $J_1=t.(X^2,Y^2,XY)$, and $tX\in J^{(1)}\backslash J_1$ (because $tX=X(Y^2+t)-Y.XY$). Hence J is not balanced.

4. Primitive multiple schemes

4.1. Definition and construction

Let X be a smooth connected variety, and $d=\dim(X)$. A *multiple scheme with support* X is a Cohen-Macaulay scheme Y such that $Y_{red}=X$. If Y is quasi-projective we say that it is a *multiple variety with support* X. In this case Y is projective if X is.

Let n be the smallest integer such that $Y=X^{(n-1)}$, $X^{(k-1)}$ being the k-th infinitesimal neighborhood of X, i.e. $\mathcal{I}_{X^{(k-1)}}=\mathcal{I}_X^k$. We have a filtration $X=X_1\subset X_2\subset\cdots\subset X_n=Y$ where

X_i is the biggest Cohen-Macaulay subscheme contained in $Y \cap X^{(i-1)}$. We call n the *multiplicity* of Y.

We say that Y is *primitive* if, for every closed point x of X, there exists a smooth variety S of dimension $d+1$, containing a neighborhood of x in Y as a locally closed subvariety. In this case, $L = \mathcal{I}_X/\mathcal{I}_{X_2}$ is a line bundle on X, X_j is a primitive multiple scheme of multiplicity j and we have $\mathcal{I}_{X_j} = \mathcal{I}_X^j$, $\mathcal{I}_{X_j}/\mathcal{I}_{X_{j+1}} = L^j$ for $1 \leq j < n$. We call L the line bundle on X *associated* to Y. The ideal sheaf $\mathcal{I}_{X,Y}$ can be viewed as a line bundle on X_{n-1}.

Let $P \in X$. Then there exist elements $y_1, \ldots, y_d$, t of $m_{S,P}$ whose images in $m_{S,P}/m_{S,P}^2$ form a basis, and such that for $1 \leq i < n$ we have $\mathcal{I}_{X_i,P} = (t^i)$. In this case the images of $y_1, \ldots, y_d$ in $m_{X,P}/m_{X,P}^2$ form a basis of this vector space.

Even if X is projective, we do not assume that Y is projective.

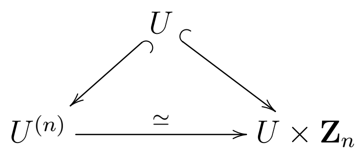
The simplest case is when Y is contained in a smooth variety S of dimension $d+1$. Suppose that Y has multiplicity n. Let $P \in X$ and $f \in \mathcal{O}_{S,P}$ a local equation of X. Then we have $\mathcal{I}_{X_i,P} = (f^i)$ for $1 < i \leq n$ in S, in particular $\mathcal{I}_{Y,P} = (f^n)$, and $L = \mathcal{O}_X(-X)$.

For any $L \in \mathrm{Pic}(X)$, the *trivial primitive variety* of multiplicity n, with induced smooth variety X and associated line bundle L on X is the n-th infinitesimal neighborhood of X, embedded by the zero section in the dual bundle L^*, seen as a smooth variety.

4.1.1. *Construction of primitive multiple schemes* – Let Y be a primitive multiple scheme of multiplicity n, $X = Y_{red}$. Let $\mathbf{Z}_n = \mathrm{spec}(\mathbb{C}[t]/(t^n))$. Then for every closed point $P \in X$, there exists an open neighborhood U of P in X, and an open neighborhood $U^{(n)}$ of P in Y such that

- $U^{(n)} \cap X = U$,
- There exists a section $\mathcal{O}_X(U) \to \mathcal{O}_Y(U^{(n)})$ of the restriction map $\mathcal{O}_Y(U^{(n)}) \to \mathcal{O}_X(U)$,
- $L_{|U}$ is trivial,

and there exists a commutative diagram



i.e. Y is locally trivial ([6], théorème 5.2.1, corollaire 5.2.2). For every open subset V of X, $V^{(n)}$ will denote the corresponding open subset of Y.

It follows that we can construct a primitive multiple scheme of multiplicity n by taking an open cover $(U_i)_{i \in I}$ of X and gluing the varieties $U_i \times \mathbf{Z}_n$ (with automorphisms of the $U_{ij} \times \mathbf{Z}_n$ leaving U_{ij} invariant).

Let $(U_i)_{i \in I}$ be an affine open cover of X such that we have trivializations

$$\delta_i : U_i^{(n)} \xrightarrow{\simeq} U_i \times \mathbf{Z}_n,$$

and $\delta_i^* : \mathcal{O}_{U_i \times \mathbf{Z}_n} \to \mathcal{O}_{U_i^{(n)}}$ the corresponding isomorphisms. Let

$$\delta_{ij} = \delta_j \delta_i^{-1} : U_{ij} \times \mathbf{Z}_n \xrightarrow{\simeq} U_{ij} \times \mathbf{Z}_n \ .$$

Then $\delta_{ij}^* = \delta_i^{*-1}\delta_j^*$ is an automorphism of $\mathcal{O}_{U_i\times Z_n} = \mathcal{O}_X(U_{ij})[t]/(t^n)$, such that for every $\phi \in \mathcal{O}_X(U_{ij})$, seen as a polynomial in t with coefficients in $\mathcal{O}_X(U_{ij})$, the term of degree zero of $\delta_{ij}^*(\phi)$ is the same as the term of degree zero of ϕ.

4.1.2. *The ideal sheaf of X* – There exists $\alpha_{ij} \in \mathcal{O}_X(U_{ij}) \otimes_{\mathbb{C}} \mathbb{C}[t]/(t^{n-1})$ such that $\delta_{ij}^*(t) = \alpha_{ij}t$. Let $\alpha_{ij}^{(0)} = \alpha_{ij|X} \in \mathcal{O}_X(U_i)$. For every $i \in I$, $\delta_i^*(t)$ is a generator of $\mathcal{I}_{X,Y|U^{(n)}}$. So we have local trivializations

$$\lambda_i : \mathcal{I}_{X,Y|U_i^{(n-1)}} \longrightarrow \mathcal{O}_{U_i^{(n-1)}}$$

$$\delta_i^*(t) \longmapsto 1$$

Hence $\lambda_{ij} = \lambda_i\lambda_j^{-1} : \mathcal{O}_{U_{ij}^{(n-1)}} \to \mathcal{O}_{U_{ij}^{(n-1)}}$ is the multiplication by $\delta_j^*(\alpha_{ij})$. It follows that $(\delta_j^*(\alpha_{ij}))$ (resp. $(\alpha_{ij}^{(0)})$) is a cocycle representing the line bundle $\mathcal{I}_{X,Y}$ (resp. L) on X_{n-1} (resp. X).

4.2. The case of double schemes

We suppose that $n = 2$. Let $\alpha_i : L_{|U_i} \to \mathcal{O}_{U_i}$ be isomorphisms such that $\alpha_{ij} = \alpha_i \circ \alpha_j^{-1}$ on U_{ij}. Then we have the following description of δ_{ij}^*:

- There are derivations D_{ij} of $\mathcal{O}_X(U_{ij})$ such that $\delta_{ij}^*(\beta) = \beta + D_{ij}(\beta)t$ for every $\beta \in \mathcal{O}_X(U_{ij})$.
- $\delta_{ij}^*(t) = \alpha_{ij}t$.

The relation $\delta_{ij}^*\delta_{jk}^* = \delta_{ik}^*$ is equivalent to $D_{ij} + \alpha_{ij}D_{jk} = D_{ik}$. We can view D_{ij} as section of $T_{X|U_{ij}}$, the family $(\alpha_i^{-1} \otimes D_{ij})$ represents an element λ of $H^1(X, T_X \otimes L)$, and $\mathbb{C}\lambda$ is independent of the choice of the δ_{ij}^* and α_i. We will note $\mathbb{C}\lambda = \zeta(X_2)$.

According to [14] (and [3] for curves), two primitive double schemes X_2, X_2', with underlying smooth variety X and associated line bundle L are isomorphic (over X) if and only if $\zeta(X_2) = \zeta(X_2')$. And X_2 is the trivial primitive double scheme if and only $\zeta(X_2) = 0$.

4.3. Canonical filtrations, quasi locally free sheaves

Let X be a smooth and irreducible variety. Let $Y = X_n$ be a primitive multiple scheme of multiplicity n, with underlying smooth variety X, and associated line bundle L on X.

Let $P \in X$ be a closed point.

The two *canonical filtrations* are useful tools to study the coherent sheaves on primitive multiple schemes. They have been defined for $\mathcal{O}_{X_n,P}$-modules in 3.1.

4.3.1. The *first canonical filtration of* $\mathcal{E}$ is

$$\mathcal{E}_n = 0 \subset \mathcal{E}_{n-1} \subset \cdots \subset \mathcal{E}_1 \subset \mathcal{E}_0 = \mathcal{E}$$

where for $0 \leq i < n$, $\mathcal{E}_{i+1}$ is the kernel of the canonical surjective morphism $\mathcal{E}_i \to \mathcal{E}_{i|X}$. For $0 \leq i < n$, let

$$G_i(\mathcal{E}) \ = \ \mathcal{E}_i/\mathcal{E}_{i+1} \ = \ \mathcal{E}_{i|X} \ .$$

We have $\mathcal{E}/\mathcal{E}_i = \mathcal{E}_{|X_i}$.

4.3.2. One defines similarly the *second canonical filtration of* $\mathcal{E}$:

$$\mathcal{E}^{(0)} = \{0\} \subset \mathcal{E}^{(1)} \subset \cdots \subset \mathcal{E}^{(n-1)} \subset \mathcal{E}^{(n)} = \mathcal{E},$$

where, for $1 \leq i \leq n-1$, $\mathcal{E}^{(i)}$ is the maximal subsheaf $\mathcal{F}$ of $\mathcal{E}$ such that $\mathcal{I}^i_{X,X_n}\mathcal{F} = 0$. For $0 < i \leq n$, let

$$G^{(i)}(\mathcal{E}) \ = \ \mathcal{E}^{(i)}/\mathcal{E}^{(i-1)} \ .$$

4.3.3. *Relationship between the two filtrations* – Let $\mathcal{E}$ be a coherent sheaf on X_n. For $1 \leq i \leq n-1$, the canonical morphism $\mathcal{E} \otimes \mathcal{I}_{X,X_n} \to \mathcal{E}$ induces an injective morphism of sheaves on X

$$\lambda_i : G^{(i+1)}(\mathcal{E}) \otimes L \longrightarrow G^{(i)}(\mathcal{E}) \ .$$

Let $\Gamma^{(i-1)}(\mathcal{E}) = \mathrm{coker}(\lambda_i)$. Similarly we have for $0 \leq i \leq n-2$ a surjective morphism of sheaves on X

$$\mu_i : G_i(\mathcal{E}) \otimes L \longrightarrow G_{i+1}(\mathcal{E}) \ .$$

Let $\Gamma_i(\mathcal{E}) = \ker(\mu_i)$.

4.3.4. Proposition: *For* $0 \leq i < n$ *we have* $\Gamma_i(\mathcal{E}) \simeq \Gamma^{(i)}(\mathcal{E}) \otimes L^{i+1}$.

Proof. We suppose that $\mathcal{I}_{X,X_n}$, which is a line bundle on X_{n-1}, can be extended to a line bundle $\mathbb{L}$ on X_n. Let $\phi_i : \mathcal{E}_i \otimes \mathbb{L} \to \mathcal{E}_{i+1}$ be the canonical surjective morphism. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}^{(i)} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}_i \otimes \mathbb{L}^{-i} & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow {\scriptstyle \phi_i \otimes I_{\mathbb{L}^{-i-1}}} & & \\
0 & \longrightarrow & \mathcal{E}^{(i+1)} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}_{i+1} \otimes \mathbb{L}^{-i-1} & \longrightarrow & 0
\end{array}$$

It follows that $\ker(\phi_i) \simeq G^{(i+1)}(\mathcal{E}) \otimes L^{i+1}$.

The result follows from the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^{(i+2)}(\mathcal{E})\otimes L^{i+2} & \longrightarrow & \mathcal{E}_{i+1}\otimes\mathbb{L} & \longrightarrow & \mathcal{E}_{i+2} & \longrightarrow & 0\\
 & & \big\downarrow{\scriptstyle \lambda_{i+1}\otimes I_{L^{i+1}}} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^{(i+1)}(\mathcal{E})\otimes L^{i+1} & \longrightarrow & \mathcal{E}_{i}\otimes\mathbb{L} & \longrightarrow & \mathcal{E}_{i+1} & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G_i(\mathcal{E})\otimes L & \xrightarrow{\mu_i} & G_{i+1}(\mathcal{E}) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

In the general case, X can be covered by open subsets U such that, if U_n is the open subset of X_n corresponding to U, then $\mathcal{I}_{X,X_n|U_n}$ can be extended to a line bundle on U_n. We obtain isomorphisms $\Gamma_i(\mathcal{E}_{|U_n}) \simeq \Gamma^{(i)}(\mathcal{E}_{|U_n}) \otimes L^{i+1}$, and it is easy to see that they coincide on the intersections of any two of these open subsets U, and thus define the global isomorphism of proposition 4.3.4. □

4.3.5. *Quasi locally free sheaves* – Let P be a closed point of X. Let M be a $\mathcal{O}_{X_n,P}$-module of finite type. Then M is called *quasi free* if there exist non negative integers $m_1, \ldots, m_n$ and an isomorphism $M \simeq \oplus_{i=1}^n m_i \mathcal{O}_{X_i,P}$. The integers $m_1, \ldots, m_n$ are uniquely determined: it is easy to recover them from the first canonical filtration of M. We say that $(m_1, \ldots, m_n)$ is the *type* of M.

Let $\mathcal{E}$ be a coherent sheaf on X_n. We say that $\mathcal{E}$ is *quasi free at* P if $\mathcal{E}_P$ is quasi free, and that $\mathcal{E}$ is *quasi locally free* on a nonempty open subset $U \subset X$ if it is quasi free at every point of U. If $U = X$ we say that $\mathcal{E}$ is quasi locally free. In this case the types of the modules $\mathcal{E}_x$, $x \in X$, are the same (the *type* of $\mathcal{E}$), and for every $x \in X$ there exists a neighborhood $V \subset X_n$ of x and an isomorphism

$$\mathcal{E}_{|V} \ \simeq \ \bigoplus_{n=1}^{n} m_i \mathcal{O}_{X_i \cap V} \ . \tag{3}$$

For every coherent sheaf $\mathcal{E}$ on X_n there exists a nonempty open subset $U \subset X$ such that $\mathcal{E}$ is quasi locally free on U (cf. [15], 4.7).

4.3.6. Proposition: *Let $\mathcal{E}$ be a quasi locally free sheaf on X_n. Then*

(i) *$\mathcal{E}$ is reflexive.*

(ii) *For every positive integer i we have $\mathcal{E}xt^i_{\mathcal{O}_{X_n}}(\mathcal{E}, \mathcal{O}_{X_n}) = 0$.*

(iii) *For every vector bundle $\mathbb{F}$ on X_n and every integer i we have*
$\mathrm{Ext}^i_{\mathcal{O}_{X_n}}(\mathcal{E}, \mathbb{F}) \simeq H^i(X_n, \mathcal{E}^\vee \otimes \mathbb{F})$.

(iv) *Let* $Y \subset X$ *be a closed subvariety of codimension* ≥ 2, $U = X\backslash Y$. *Let* $\mathcal{E}$ *be a coherent sheaf on* X_n, *such that* $\mathcal{E}_P$ *is quasi free for every* $P \in Y$. *Then the restriction map* $H^0(X_n, \mathcal{E}) \to H^0(U^{(n)}, \mathcal{E})$ *is an isomorphism.*

Proof. The assertion (iii) follows easily from (i), with the Ext spectral sequence, and (iv) is immediate. Assertions (i) and (ii) are local, so we can replace X_n with $U \times \mathbf{Z}_n$, where U is an open subset of X_n. Let $U_i = U \times \mathbf{Z}_i$, for $1 \leq i \leq n$. The results follow from (3) and the free resolutions

$$\cdots\cdot \longrightarrow \mathcal{O}_{U_n} \xrightarrow{\times t^i} \mathcal{O}_{U_n} \xrightarrow{\times t^{n-i}} \mathcal{O}_{U_n} \xrightarrow{\times t^i} \mathcal{O}_{U_i}$$

□

4.4. Morphisms of quasi locally free sheaves

Let $r, r_1, \ldots, r_n$ be integers, with $r_i \geq 0$ and $r \geq \sum_{i-1}^n r_i$. Let $\mathcal{E} = \bigoplus_{1\leq i\leq n}(\mathcal{O}_{X_i} \otimes \mathbb{C}^{r_i})$ and $\phi : \mathcal{O}_{X_n} \otimes \mathbb{C}^r \to \mathcal{E}$ a morphism.

Let $k = (\sum_{i-1}^n r_i) - r$. Chose a direct sum decomposition $\mathbb{C}^r = \mathbb{C}^k \oplus \big(\bigoplus_{1\leq i\leq n} \mathbb{C}^{r_i}\big)$. The *canonical morphism* $\phi_0 : \mathcal{O}_{X_n} \otimes \mathbb{C}^r \to \mathcal{E}$ is defined as follows: $\phi_0 = 0$ on $\mathcal{O}_{X_n} \otimes \mathbb{C}^k$, and on $\mathcal{O}_{X_n} \otimes \mathbb{C}^{r_i}$, $\phi = p_i \otimes I_{\mathbb{C}^{r_i}}$, where $p_i : \mathcal{O}_{X_n} \to \mathcal{O}_{X_i}$ is the restriction.

4.4.1. Lemma: **1** − *Let* $x \in X$ *be a closed point. If* ϕ *is surjective at* x *then there exists an open neighborhood* U *of* x, *and automorphisms* α *of* $\mathcal{O}_{U^{(n)}} \otimes \mathbb{C}^r$, β *of* $\mathcal{E}_{|U^{(n)}}$ *such that* $\phi_{|U^{(n)}} = \beta \circ \phi_{0|U^{(n)}} \circ \alpha$.

2 − ϕ *is surjective if and only of* $\phi_{|X}$ *is. In this case* $\ker(\phi)$ *is quasi locally free.*

Proof. We will prove **1** by induction on n, and **2** is an easy consequence of **1**. The result is obvious if $n = 1$. Suppose that $n > 1$ and that the lemma is true on X_{n-1}. The component of ϕ, $\phi_n : \mathcal{O}_{X_n} \otimes \mathbb{C}^r \to \mathcal{O}_{X_n} \otimes \mathbb{C}^{r_n}$, is surjective, so we have a direct sum decomposition $\mathbb{C}^r = \mathbb{C}^{r_n} \oplus \mathbb{C}^{r-r_n}$ such that there exist U, α, β such that $\phi' = \beta^{-1} \circ \phi_{|U^{(n)}} \circ \alpha^{-1}$ is the identity on $\mathcal{O}_{U^{(n)}} \otimes \mathbb{C}^{r_n}$ and zero on $\mathcal{O}_{U^{(n)}} \otimes \mathbb{C}^{r-r_n}$. We can even choose α, β such that $\phi'(\mathcal{O}_{U^{(n)}} \otimes \mathbb{C}^{r_n})$ is the summand $\mathcal{O}_{U^{(n)}} \otimes \mathbb{C}^{r_n}$ of $\mathcal{E}_{|U^{(n)}}$. On the summand $\mathcal{O}_{U^{(n)}} \otimes \mathbb{C}^{r-r_n}$ of $\mathcal{O}_{X_n} \otimes \mathbb{C}^r$, ϕ' is a morphism $\mathcal{O}_{U^{(n)}} \otimes \mathbb{C}^{r-r_n} \to \bigoplus_{1\leq i\leq n-1}(\mathcal{O}_{X_i} \otimes \mathbb{C}^{r_i})$, which vanish on $L^{n-1} \otimes \mathbb{C}^{r-r_n}$, so that ϕ' is a surjective morphism $\mathcal{O}_{U^{(n-1)}} \otimes \mathbb{C}^{r-r_n} \to \bigoplus_{1\leq i\leq n-1}(\mathcal{O}_{X_i} \otimes \mathbb{C}^{r_i})$. The result is easily obtained by applying the induction hypothesis on ϕ'. □

4.4.2. Corollary: *Let* $\mathcal{N}$, $\mathcal{E}$ *coherent sheaves on* X_n, *with* $\mathcal{N}$ *quasi locally free. Let*

$$0 \longrightarrow \mathcal{N} \xrightarrow{\psi} \mathcal{O}_{X_n} \otimes \mathbb{C}^r \longrightarrow \mathcal{E} \longrightarrow 0$$

an exact sequence. The $\mathcal{E}$ *is quasi locally free if and only if*

$${}^t\psi : \mathcal{O}_{X_n} \otimes \mathbb{C}^r \longrightarrow \mathcal{N}^*$$

is surjective.

Proof. If $\mathcal{E}$ is quasi locally free, then $\mathcal{E}xt^1_{\mathcal{O}_{X_n}}(\mathcal{E},\mathcal{O}_{X_n})=0$ by proposition 4.3.6, hence ${}^t\psi$ is surjective.

Conversely, suppose that ${}^t\psi$ is surjective. We have then an exact sequence

$$0\longrightarrow\mathcal{E}^*\longrightarrow\mathcal{O}_{X_n}\otimes\mathbb{C}^r\xrightarrow{\ {}^t\psi\ }\mathcal{N}^*\longrightarrow 0\ ,$$

hence $\mathcal{E}^*$ is quasi locally free by lemma 4.4.1. Since $\mathcal{E}xt^1_{\mathcal{O}_{X_n}}(\mathcal{N}^*,\mathcal{O}_{X_n})=0$ by proposition 4.3.6, we have an exact sequence

$$0\longrightarrow\mathcal{N}^{**}=\mathcal{N}\xrightarrow{\ \psi\ }\mathcal{O}_{X_n}\otimes\mathbb{C}^r\longrightarrow\mathcal{E}^{**}\longrightarrow 0\ .$$

It follows that $\mathcal{E}^{**}=\mathcal{E}$, and $\mathcal{E}$ is quasi locally free. □

4.5. Flat families of quasi locally free sheaves

Let C be an irreducible smooth curve.

4.5.1. Theorem: *Let $\mathcal{E}$ be a coherent sheaf on $X_n\times C$, flat on C. Suppose that for every closed point $c\in C$, $\mathcal{E}_c$ is quasi locally free, and that the type of $\mathcal{E}_c$ is independent of c. Then*

1 – *$\mathcal{E}$ is quasi locally free.*

2 – *For $1\leq i\leq n-1$, $\mathcal{E}_i$ and $\mathcal{E}_{i-1}/\mathcal{E}_i$ are flat on C, and for every $c\in C$, the morphism $(\mathcal{E}_i)_c\to(\mathcal{E}_{i-1})_c$ induced by $\mathcal{E}_i\subset\mathcal{E}_{i-1}$ is injective, and*

$$(\mathcal{E}_i)_c\ =\ (\mathcal{E}_c)_i\ ,\qquad(\mathcal{E}_{i-1}/\mathcal{E}_i)_c\ =(\mathcal{E}_{i-1})_c/(\mathcal{E}_i)_c\ .$$

3 – *For $1\leq i\leq n-1$, $\mathcal{E}^{(i)}$ and $\mathcal{E}^{(i+1)}/\mathcal{E}^{(i)}$ are flat on C, and for every $c\in C$,the morphism $(\mathcal{E}^{(i)})_c\to(\mathcal{E}^{(i+1)})_c$ induced by $\mathcal{E}^{(i)}\subset\mathcal{E}^{(i+1)}$ is injective, and*

$$(\mathcal{E}^{(i)})_c\ =(\mathcal{E}_c)^{(i)}\ ,\quad(\mathcal{E}^{(i+1)}/\mathcal{E}^{(i)})_c\ =(\mathcal{E}^{(i+1)})_c/(\mathcal{E}^{(i)})_c\ .$$

Proof. We will prove the same statement with a nonempty subset of $X\times C$ instead of $X\times C$. The proof is by induction on n. The result is obvious if $n=1$. Suppose that $n>1$ and that it is true for $n-1$.

Let $x\in X$, $c_0\in C$ be closed points. Since $\mathcal{E}_{c_0}$ is quasi locally free, there exists an open neighborhood $V\subset X$ of x such that $\mathcal{E}_{c_0|V}$ is free: $\mathcal{E}_{c_0|V}\simeq\mathcal{O}_V\otimes\mathbb{C}^r$. It follows that there exists an open neighborhood $U\subset X\times C$ of (x,c_0) and a surjective morphism

$$\phi:\mathcal{O}_{U^{(n)}}\otimes\mathbb{C}^r\longrightarrow\mathcal{E}_{U^{(n)}}\ .$$

Let $\mathcal{N}=\ker(\phi)$. From the exact sequence

$$0\longrightarrow\mathcal{N}\xrightarrow{\ \psi\ }\mathcal{O}_{U^{(n)}}\otimes\mathbb{C}^r\xrightarrow{\ \phi\ }\mathcal{E}_{|U^{(n)}}\longrightarrow 0$$

and [20], Exposé IV, proposition 1.1, $\mathcal{N}$ is flat on C. For every $c\in C$ we have an exact sequence

$$0\longrightarrow\mathcal{N}_c\longrightarrow\mathcal{O}_{U_c^{(n)}}\otimes\mathbb{C}^r\xrightarrow{\ \phi_c\ }\mathcal{E}_{c|U_c^{(n)}}\longrightarrow 0$$

(with $U_c=(X\times\{c\})\cap U$). Since the type of $\mathcal{E}_c$ is constant, ϕ_c induces an isomorphism $\mathcal{O}_{U_c}\otimes\mathbb{C}^r\simeq\mathcal{E}_{c|U_c}$. It follows that $\mathcal{N}_c$ is a quasi locally free sheaf on $U_c^{(n-1)}$, and its type is independent of c (if U_c is non empty).

For every $c \in C$ such that U_c is non empty, let $\mathcal{I}_c$ be the ideal sheaf of $X \times \{c\} \subset X \times C$. For every infinite subset $\Sigma \subset C$, we have

$$\bigcap_{c\in\Sigma} \mathcal{I}_c.(\mathcal{O}_{U^{(n)}} \otimes \mathbb{C}^r) \ = \ 0 \ .$$

We have, if $U_c \neq \emptyset$,

$$L^{n-1}.\mathcal{N} \ \subset \ \mathcal{I}_c.\mathcal{N} \ \subset \ \mathcal{I}_c.(\mathcal{O}_{U^{(n)}} \otimes \mathbb{C}^r)$$

(recall that L^{n-1} is the ideal sheaf of X_{n-1} in X_n). It follows that $L^{n-1}.\mathcal{N} = 0$, i.e. $\mathcal{N}$ is a sheaf on $U^{(n-1)}$. Hence from the induction hypothesis, $\mathcal{N}$ is quasi locally free. For every $(x, c) \in U$,

$${}^t\psi_{c,x} : \mathbb{C}^r \longrightarrow \mathcal{N}^*_{c,x}$$

is surjective. It follows from corollary 4.4.2 that $\mathcal{E}_{|U^{(n)}}$ is quasi locally free. This proves **1**, and **2**, **3** are immediate consequences of **1**. □

There is an easy converse of theorem 4.5.1:

4.5.2. Proposition: *Let $\mathcal{E}$ be a coherent sheaf on $X_n \times C$, $c \in C$, $x \in X$ closed points, and U a neighborhood of (x, c) in $X_n \times C$ such that $\mathcal{E}_{|U}$ is quasi locally free. Then $\mathcal{E}_{|U}$ is flat on C.*

4.6. The Picard group of a primitive double scheme

(cf. [15], 7.)

Let $\boldsymbol{P} \subset \mathrm{Pic}(X)$ be an irreducible component, such that some line bundle in $\boldsymbol{P}$ can be extended to X_2. Let $\mathrm{Pic}^{\boldsymbol{P}}(X_2)$ be the set of line bundles on X_2 whose restriction to X belongs to $\boldsymbol{P}$. If $\boldsymbol{P} = \mathrm{Pic}^0(X)$ we will note $\mathrm{Pic}^{\boldsymbol{P}}(X_2) = \mathrm{Pic}^0(X_2)$. Let $\Gamma^0(X_2) \subset \mathrm{Pic}^0(X)$ (resp. $\Gamma^{\boldsymbol{P}}(X_2) \subset \mathrm{Pic}^{\boldsymbol{P}}(X)$) be the set of line bundles that can be extended to X_2. Then $\Gamma^{\boldsymbol{P}}(X_2)$ is a smooth variety, isomorphic to $\Gamma^0(X_2) \subset \mathrm{Pic}^0(X)$, and $\Gamma^0(X_2)$ is a subgroup of $\mathrm{Pic}^0(X)$.

The variety $\mathrm{Pic}^0(X_2)$ is an algebraic group, the $\mathrm{Pic}^{\boldsymbol{P}}(X_2)$ have a natural structure of smooth varieties, and the choice of an element $\mathrm{Pic}^{\boldsymbol{P}}(X_2)$ defines in an obvious way an isomorphism $\mathrm{Pic}^0(X_2) \simeq \mathrm{Pic}^{\boldsymbol{P}}(X_2)$. The $\mathrm{Pic}^{\boldsymbol{P}}(X_2)$ are the irreducible components of $\mathrm{Pic}(X_2)$. Each $\mathrm{Pic}^{\boldsymbol{P}}(X_2)$ has a natural structure of affine bundle on $\Gamma^{\boldsymbol{P}}(X_2)$, with associated vector bundle $\mathcal{O}_X \otimes H^1(X, L)$. In general this affine bundle is not banal (cf. [15], theorem 9.2.1 when X is a curve).

There is an open cover $(P_i)_{i\in I}$ of $\mathrm{Pic}^{\boldsymbol{P}}(X_2)$, and for every $i \in I$, a *Poincaré bundle* $\mathbb{L}_i$ on $X_2 \times P_i$ such that, with these data, $\mathrm{Pic}^{\boldsymbol{P}}(X_2)$ is a *fine moduli space* in the sense of 2.6.

5. Reduced Chern classes and reduced Hilbert polynomials

5.1. Definitions

If Z is a projective scheme, $\mathcal{O}_Z(1)$ an ample line bundle and E a coherent sheaf on Z, let $P_{\mathcal{O}_Z(1)}(E)$ be the Hilbert polynomial of E with respect to $\mathcal{O}_Z(1)$.

Let X be a smooth, projective and irreducible variety. Let X_n be a primitive multiple scheme of multiplicity n, with underlying smooth variety X, and associated line bundle L on X.

Let $\mathcal{O}_X(1)$ be an ample line bundle on X. For every coherent sheaf E on X, let $c(E) \in A^*(X)$ (or $H^*(X,\mathbb{Z})$) be the total Chern class of E.

Let $\mathcal{E}$ be a coherent sheaf on X_n, and

$$\mathcal{E}_n = 0 \subset \mathcal{E}_{n-1} \subset \cdots \subset \mathcal{E}_1 \subset \mathcal{E}_0 = \mathcal{E}$$

the first canonical filtration of $\mathcal{E}$. Let

$$c_{red}(\mathcal{E}) \ = \ \prod_{i=0}^{n-1} c(\mathcal{E}_i/\mathcal{E}_{i+1}) \ \in \ A^*(X) \ (\text{or } H^*(X,\mathbb{Z})) \ , \qquad P_{red,\mathcal{O}_X(1)}(\mathcal{E}) = \sum_{i=0}^{n-1} P_{\mathcal{O}_X(1)}(\mathcal{E}_i/\mathcal{E}_{i+1}) \ .$$

We call $c_{red}(\mathcal{E})$ the *total reduced Chern class of* $\mathcal{E}$, and $P_{red,\mathcal{O}_X(1)}(\mathcal{E})$ the *reduced Hilbert polynomial of* $\mathcal{E}$ (with respect to $\mathcal{O}_X(1)$).

5.1.1. *Relations with the usual Hilbert polynomial* – If $\mathcal{O}_X(1)$ can be extended to a line bundle $\mathcal{O}_{X_n}(1)$ on Y, then $\mathcal{O}_{X_n}(1)$ is ample (prop. 4.6.1 of [14]), and $P_{red,\mathcal{O}_X(1)}(\mathcal{E}) = P_{\mathcal{O}_{X_n}(1)}(\mathcal{E})$.

5.1.2. Proposition: 1 – *Let* $\mathcal{F}_m = 0 \subset \mathcal{F}_{m-1} \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathcal{E}$ *be a filtration such that, for* $0 \leq i < m$, $\mathcal{F}_i/\mathcal{F}_{i+1}$ *is concentrated on* X. *Then we have*

$$c_{red}(\mathcal{E}) \ = \ \prod_{i=0}^{m-1} c(\mathcal{F}_i/\mathcal{F}_{i+1}) \ , \qquad P_{red,\mathcal{O}_X(1)}(\mathcal{E}) = \sum_{i=0}^{m-1} P_{\mathcal{O}_X(1)}(\mathcal{F}_i/\mathcal{F}_{i+1}) \ .$$

2 – *Let* $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ *be an exact sequence of coherent sheaves on* X_n. *Then we have*

$$c(\mathcal{E}) \ = c(\mathcal{E}')c(\mathcal{E}'') \quad \textit{and} \quad P_{red,\mathcal{O}_X(1)}(\mathcal{E}) \ = \ P_{red,\mathcal{O}_X(1)}(\mathcal{E}') + P_{red,\mathcal{O}_X(1)}(\mathcal{E}'') \ .$$

Proof. **1-** is an immediate consequence of proposition 2.3.1. We now prove **2-**. According to proposition 2.3.1 there exist similar refinements $(\mathcal{E}'_j)_{0\leq j\leq p}$ of the filtration $\mathcal{E}' \subset \mathcal{E}$ and $(\mathcal{F}'_j)_{0\leq j\leq p}$ of the first canonical filtration of $\mathcal{E}$. There exists an integer j_0 such that $0 \leq j_0 \leq p$ and $\mathcal{F}'_{j_0} = \mathcal{E}'$. We have

$$c_{red}(\mathcal{E}) \ = \ \prod_{j=0}^{p-1} c(\mathcal{F}'_j/\mathcal{F}'_{j+1}) \ ,$$

and from **2-**

$$c_{red}(\mathcal{E}') \ = \ \prod_{j=j_0}^{p-1} c(\mathcal{F}'_j/\mathcal{F}'_{j+1}) \ , \qquad c_{red}(\mathcal{E}'') \ = \ \prod_{j=0}^{j_0-1} c(\mathcal{F}'_j/\mathcal{F}'_{j+1}) \ .$$

Hence $c_{red}(\mathcal{E}) = c_{red}(\mathcal{E}')c_{red}(\mathcal{E})$, and similarly $P_{red,\mathcal{O}_X(1)}(\mathcal{E}) = P_{red,\mathcal{O}_X(1)}(\mathcal{E}') + P_{red,\mathcal{O}_X(1)}(\mathcal{E}'')$. □

5.2. Invariance by deformation

Let C be an irreducible smooth curve and $\mathcal{E}$ a coherent sheaf on $X_n \times C$, flat on C. The following result was pointed out by János Kollár:

5.2.1. Theorem: *Suppose that we have a filtration*

$$0 = \mathcal{G}_m \subset \mathcal{G}_{m-1} \subset \cdots \subset \mathcal{G}_1 \subset \mathcal{G}_0 = \mathcal{E}$$

such that for $0 \leq i < m$, $\mathcal{G}_i/\mathcal{G}_{i+1}$ *is concentrated on* $X \times C$. *Then there exists a filtration*

$$0 = \mathcal{F}_m \subset \mathcal{F}_{m-1} \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathcal{E} \tag{4}$$

such that

- $\mathcal{F}_1, \ldots, \mathcal{F}_{m-1}$ *and* $\mathcal{F}_0/\mathcal{F}_1, \ldots, \mathcal{F}_{m-2}/\mathcal{F}_{m-1}$ *are flat on* C.
- $\mathcal{F}_0/\mathcal{F}_1, \ldots, \mathcal{F}_{m-1}/\mathcal{F}_m$ *are concentrated on* $X \times C$.
- *There exists a finite subset* $\Sigma \subset C$ *such that the two preceding filtrations restricted to* $X_n \times (C \backslash \Sigma)$ *are the same.*

Proof. Let $c_0 \in C$ and $\mathcal{I}_{c_0}$ its ideal sheaf. Let z be a generator of the maximal ideal of $\mathcal{O}_{C,c_0}$. Let $F = \mathcal{G}_0/\mathcal{G}_1$, and $G \subset F$ be the subsheaf union of the subsheaves annihilated by the powers of $\mathcal{I}_{c_0}$.

Then for every $x \in X$, $(F/G)_{(x,c_0)}$ is a flat $\mathcal{O}_{C,c_0}$-module: Let $\overline{u} \in (F/G)_{(x,c_0)}$ and suppose that $z\overline{u} = 0$. Let $u \in F_{(x,c_0)}$ be over $\overline{u}$. Then $zu \in G_{(x,c_0)}$, so there exists an integer $k \geq 0$ such that $z^k.zu = z^{k+1}u = 0$. Hence $u \in G_{(x,c_0)}$ and $\overline{u} = 0$.

Let $H \subset \mathcal{E}$ be the inverse image of G. From [20], Exposé IV, proposition 1.1, $H_{(x,c_0)}$ is a flat $\mathcal{O}_{C,c_0}$-module.

Since X is projective, there is a neighborhood C_0 of c_0 such that $(F/G)_{|X \times C_0}$ and $H_{|X_n \times C_0}$ are flat on C_0. Note that H coincide with $\mathcal{G}_1$ on $X_n \times (C \backslash \{c_0\})$. By taking othen c_0 we can cover C with a finite number of such open subsets C_0 and finally obtain a subsheaf $\mathcal{G}'_1 \subset \mathcal{E}$ such that:

- $\mathcal{G}_1 \subset \mathcal{G}'_1$ and $\mathcal{G}'_1$ is flat on C.
- $\mathcal{E}/\mathcal{G}'_1$ is flat on C and concentrated on $X \times C$.
- There is an open subset $\Sigma_1 \subset C$ such that $\mathcal{G}'_{1|X_n \times (C \backslash \Sigma_1)} = \mathcal{G}_{1|X_n \times (C \backslash \Sigma_1)}$.

We can continue this process with $\mathcal{G}'_1$ instead of $\mathcal{E}$, with the filtration
$0 = \mathcal{G}_m \subset \mathcal{G}_{m-1} \subset \cdots \subset \mathcal{G}_2 \subset \mathcal{G}'_1$, and so on. We finally obtain the filtration

$$0 \subset \mathcal{F}_m \subset \mathcal{F}_{m-1} \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathcal{E}$$

by subsheaves flat on C, as well as the quotients $\mathcal{F}_i/\mathcal{F}_{i+1}$, which are concentrated on $X \times C$, and this filtration coincide with the original one on $X_n \times (C \backslash \Sigma)$, where $\Sigma \subset C$ is finite. But we have $\mathcal{F}_m = 0$, since it is flat on C and 0 on $X_n \times (C \backslash \Sigma)$. □

5.2.2. Corollary: *The map*

$$C \longrightarrow \mathbb{Q}[T]$$

$$c \longmapsto P_{red,\mathcal{O}_X(1)}(\mathcal{E}_c)$$

is constant.

Proof. We can view $X_n \times C$ as a primitive multiple scheme, and $(X_n \times C)_{red} = X \times C$. Now, in theorem 5.2.1, for the first filtration we take the first canonical filtration of $\mathcal{E}$. We obtain the filtration (4). For every $c \in C$, we get a filtration of $\mathcal{E}_c$

$$0 = \mathcal{F}_{m,c} \subset \mathcal{F}_{m-1,c} \subset \cdots \subset \mathcal{F}_{1,c} \subset \mathcal{F}_{0,c} = \mathcal{E}_c \ ,$$

such that for $0 \leq i < m$, $\mathcal{F}_{i,c}/\mathcal{F}_{i-1,c}$ is concentrated on $X \times C$ and isomorphic to $(\mathcal{F}_i/\mathcal{F}_{i-1})_c$. Hence we have

$$P_{red,\mathcal{O}_X(1)}(\mathcal{E}_c) \ = \ \sum_{i=0}^{m-1} P_{\mathcal{O}_X(1)}\big((\mathcal{F}_i/\mathcal{F}_{i-1})_c\big) \ .$$

Since $\mathcal{F}_i/\mathcal{F}_{i-1}$ is flat on C, $c \mapsto P_{\mathcal{O}_X(1)}\big((\mathcal{F}_i/\mathcal{F}_{i-1})_c\big)$ is constant, and so is $c \mapsto P_{red,\mathcal{O}_X(1)}(\mathcal{E}_c)$. □

6. Balanced sheaves on primitive multiple schemes

6.1. Balanced sheaves

Let X be a smooth and irreducible variety. Let X_n be a primitive multiple scheme of multiplicity n, with underlying smooth variety X, and associated line bundle L on X.Let $\mathcal{E}$ be a coherent sheaf on X_n. From 4.3.3 we have canonical morphisms

$$\lambda_{i-1} \otimes I_{L^{i-2}} : G^{(i)}(\mathcal{E}) \otimes L^{i-1} \longrightarrow G^{(i-1)}(\mathcal{E}) \otimes L^{i-2} \qquad \text{for} \ \ 2 \leq i \leq n \ ,$$

which is injective, and

$$\mu_i \otimes I_{L^{n-2-i}} : G_i(\mathcal{E}) \otimes L^{n-1-i} \longrightarrow G_{i+1}(\mathcal{E}) \otimes L^{n-2-i} \qquad \text{for} \ \ 0 \leq i \leq n-2 \ ,$$

which is surjective. Let

$$\boldsymbol{\lambda}(\mathcal{E}) \ = \lambda_1 \circ (\lambda_2 \otimes I_L) \circ \cdots \circ (\lambda_{n-1} \otimes I_{L^{n-2}}) : G^{(n)}(\mathcal{E}) \otimes L^{n-1} \longrightarrow G^{(1)}(\mathcal{E}) \ ,$$
$$\boldsymbol{\mu}(\mathcal{E}) \ = \ \mu_{n-2} \circ (\mu_{n-3} \otimes I_L) \circ \cdots \circ (\mu_0 \otimes I_{L^{n-2}}) : G_0(\mathcal{E}) \otimes L^{n-1} \longrightarrow G_{n-1}(\mathcal{E}) \ .$$

We will use the easy following lemma:

6.1.1. Lemma: *Let A be a commutative ring, $k \geq 3$ an integer, $M_1, \ldots, M_k$ A-modules, and $f_i : M_i \to M_{i+1}$, $1 \leq i < k$, injective (resp. surjective) morphisms. Then $f_{k-1} \circ \cdots f_1 : M_1 \to M_k$ is surjective (resp. injective) if and only $f_1, \ldots, f_k$ are surjective (resp. injective).*

6.1.2. Definition: *We say that $\mathcal{E}$ is* balanced *if $\boldsymbol{\lambda}(\mathcal{E})$ is surjective.*

From lemma 6.1.1 and proposition 4.3.4, we have

6.1.3. Proposition: *The following properties are equivalent*

(i) $\mathcal{E}$ *is balanced.*
(ii) $\lambda_1, \ldots, \lambda_{n-1}$ *are surjective*

(iii) $\Gamma_1(\mathcal{E}) = \cdots = \Gamma_{n-1}(\mathcal{E}) = 0$.
(iv) $\Gamma^{(1)}(\mathcal{E}) = \cdots = \Gamma^{(n-1)}(\mathcal{E}) = 0$.
(v) $\mu_0, \ldots, \mu_{n-2}$ *are injective.*
(vi) $\boldsymbol{\mu}(\mathcal{E})$ *is injective.*

If $\mathcal{E}$ *is balanced, then* $\lambda_1, \ldots, \lambda_{n-1}$, $\mu_0, \ldots, \mu_{n-2}$, $\boldsymbol{\lambda}$ *and* $\boldsymbol{\mu}$ *are isomorphisms.*

6.1.4. Proposition: *Let* $\mathcal{E}$ *be a coherent sheaf on* X_n*. Then* $\mathcal{E}$ *is balanced if and only if* $\mathcal{E}_i = \mathcal{E}^{(n-i)}$ *for* $1 \leq i \leq n$*.*

Proof. If $\mathcal{E}_i = \mathcal{E}^{(n-i)}$ for $1 \leq i \leq n$, we have $\lambda_i = \mu_{n-i}$, hence λ_i and μ_{n-i} are isomorphisms, and $\mathcal{E}$ is balanced.

Conversely, suppose that $\mathcal{E}$ is balanced. For $1 \leq j \leq n$, let $\beta_j : \mathcal{E}_{n-j}/\mathcal{E}_{n-j+1} \to \mathcal{E}^{(j)}/\mathcal{E}^{(j-1)}$ be the morphism induced by the inclusion $\mathcal{E}_{n-j} \subset \mathcal{E}^{(j)}$.

We have a commutative diagram

$$\begin{array}{ccc} \mathcal{E}/\mathcal{E}_1 \otimes L^{n-1} & \xrightarrow{\boldsymbol{\mu}} & \mathcal{E}_{n-1} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta_1} \\ \mathcal{E}/\mathcal{E}^{(n-1)} & \xrightarrow{\boldsymbol{\lambda}} & \mathcal{E}^{(1)} \end{array}$$

where the surjective morphism α is induced by the inclusion $\mathcal{E}_1 \subset \mathcal{E}^{(n-1)}$ and β is the inclusion. Since $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are isomorphisms, so are α and β_1. For $2 \leq i \leq n-2$ we have a commutative diagram

$$\begin{array}{ccc} (\mathcal{E}_{n-i}/\mathcal{E}_{n-i+1}) \otimes L & \xrightarrow{\mu_{n-i}} & \mathcal{E}_{n-i+1}/\mathcal{E}_{n-i+2} \\ \downarrow{\scriptstyle \beta_i} & & \downarrow{\scriptstyle \beta_{i-1}} \\ (\mathcal{E}^{(i)}/\mathcal{E}^{(i-1)}) \otimes L & \xrightarrow{\lambda_{i-1}} & \mathcal{E}^{(i-1)}/\mathcal{E}^{(i-2)} \end{array}$$

Using the fact that λ_{i-1}, μ_{n-i}, β_1 are isomorphisms we see by induction on i that β_i is an isomorphism. Again by induction on i it is then easy to see that $\mathcal{E}_i = \mathcal{E}^{(n-i)}$. □

6.1.5. Examples: – Vector bundles on X_n are balanced sheaves.

– Recall the a sheaf of ideals $\mathcal{I}$ on X_n is called *regular* if for every closed point $x \in X_n$, $\mathcal{I}_x$ is generated by the elements of a regular sequence in $\mathcal{O}_{X_n,x}$. It follows from proposition 3.7.2 that a regular ideal is a balanced sheaf.

6.1.6. *The case of double schemes* – We suppose that $n = 2$. Let $\mathcal{E}$ be a coherent sheaf of X_2. Then we have $\mathcal{E}_1 \subset \mathcal{E}^{(1)}$. Let $F \subset \mathcal{E}$ be a subsheaf and $E = \mathcal{E}/F$. Then F is concentrated on X if and only if $F \subset \mathcal{E}^{(1)}$, and in this case E is concentrated on X if and only if $\mathcal{E}_1 \subset F$.

Suppose that $\mathcal{E}_1 \subset F \subset \mathcal{E}^{(1)}$. The canonical morphism $\mathcal{E} \otimes L \to \mathcal{E}$ induces $\phi_F : E \otimes L \to F$, and ϕ_F is injective (resp. surjective) if and only if $F = \mathcal{E}^{(1)}$ (resp. $F = \mathcal{E}_1$). The sheaf $\mathcal{E}$ is concentrated on X if and only if $\phi_F = 0$.

If $\mathcal{E}$ is balanced then $\mathcal{E} \otimes L \to \mathcal{E}$ induces a canonical isomorphism $\mathcal{E}_1 \simeq \mathcal{E}_{|X} \otimes L$.

Let $F \subset \mathcal{E}$ be a subsheaf concentrated on X. If ϕ_F is an isomorphism, then $\mathcal{E}$ is balanced, and $F = \mathcal{E}_1$. It follows that

6.1.7. Lemma: *Let E, $\mathcal{E}$ be coherent sheaves on X, X_2 respectively. Suppose that E is simple, and that we have an exact sequence $0 \to E \otimes L \to \mathcal{E} \to E \to 0$. If $\mathcal{E}$ is not concentrated on X, then $\mathcal{E}$ is a balanced sheaf and $\mathcal{E}_{|X} = E$.*

6.2. Flat families of balanced sheaves on primitive double schemes

We suppose that $n = 2$ and that X is projective. Let C be an irreducible smooth curve. We can view $X_2 \times C$ as a primitive double scheme.

Let Z be a scheme over $\mathbb{C}$. We can extend the definitions of 4.3 and of balanced sheaves to $X_2 \times Z$.

6.2.1. Theorem: **1** − *Let $\mathcal{E}$ be a coherent sheaf on $X_2 \times Z$, flat on Z. Suppose that for some closed point $z \in Z$, $\mathcal{E}_z$ is balanced. Then there exists a neighborhood V of z such that $\mathcal{E}_{|X_2 \times V}$ is balanced, $\mathcal{E}_{1|X_2 \times V}$, $\mathcal{E}_{|X \times V}$ are flat on V and $\mathcal{E}_{1,v} = \mathcal{E}_{v,1}$, $(\mathcal{E}_{|X \times V})_v = \mathcal{E}_{v|X}$ for every closed point $v \in V$.*

2 − *Let $\mathcal{E}$ be a coherent sheaf on $X_2 \times C$, flat on C. Suppose that $\mathcal{E}$ is balanced. Then for every $c \in C$, $\mathcal{E}_c$ is balanced.*

Proof. We first prove **1**. Let $x_0 \in X$, and U a neighborhood of x_0 such $L_{|U}$ can be extended to a line bundle $\mathbb{L}$ on $U^{(2)}$. Let $p : U^{(2)} \times Z \to U^{(2)}$ be the projection. We have canonical exact sequences

$$0 \longrightarrow (\mathcal{E}^{(1)})_{|U^{(2)} \times Z} \otimes p^*(\mathbb{L}) \longrightarrow \mathcal{E}_{|U^{(2)} \times Z} \otimes p^*(\mathbb{L}) \longrightarrow \mathcal{E}_{1|U^{(2)} \times Z} \longrightarrow 0 \ ,$$

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_{|X \times Z} \longrightarrow 0 \ .$$

From the first one we deduce the exact sequence

$$0 \longrightarrow \mathrm{Tor}^1_{\mathcal{O}_{X_2 \times Z}}(\mathcal{E}_1, \mathcal{O}_{X_2 \times \{z\}}) \longrightarrow (\mathcal{E}^{(1)})_z \otimes \mathbb{L} \longrightarrow \mathcal{E}_z \otimes \mathbb{L} \longrightarrow \mathcal{E}_{1,z} \longrightarrow 0 \ .$$

It follows from proposition 3.6.4 that $\mathcal{E}_{1,z} = \mathcal{E}_{z,1}$ on U. In particular the morphism $\mathcal{E}_{1,z} \to \mathcal{E}_z$ induced by the inclusion $\mathcal{E}_1 \subset \mathcal{E}$ is injective on $U^{(2)}$. From the second canonical exact sequence and[20], Exposé IV, corollaire 5.7, $(\mathcal{E}_{|X \times Z})_{(x,z)}$ is a flat $\mathcal{O}_{Z,z}$-module, for every $x \in U$. By [20], Exposé IV, Theorem 6.10, there is an open neighborhood W_0 of z such that $\mathcal{E}_{|U \times W_0}$ is flat on W_0, and from [20], Exposé IV, proposition 1.1, $\mathcal{E}_{1|U \times W_0}$ is flat on W_0. We can then take a finite number of points $x_0 \in X$ such that the union of corresponding open subsets $U \subset X$ is X. Taking the intersection W of the corresponding open subsets $W_0 \subset Z$, we see that $\mathcal{E}_{1|X \times W}$ $\mathcal{E}_{|X \times W}$ are flat on W.

From proposition 3.6.4 we have $(\mathcal{E}_{|X \times Z})_z = \mathcal{E}_{z|X}$. Hence the canonical surjective morphism $\Phi : \mathcal{E}_{|X \times Z} \to \mathcal{E}_1 \otimes L^*$ is an isomorphism on $X \times \{z\}$. Let $\mathcal{N} = \ker(\Phi)$. Then $\mathcal{N}_{|X \times W}$ is flat on W. Let $\mathfrak{m}_z \subset \mathcal{O}_{Z,z}$ be the maximal ideal, and $x \in X$. Then since $\mathcal{N}_z = 0$, we have $\mathcal{N}_{(x,z)} = \mathfrak{m}_z \mathcal{N}_{(x,z)}$, and $\mathcal{N}_{(x,z)} = 0$. To prove **1** we take $V \subset W$ such that $X \times V$ does not meet the support of $\mathcal{N}$.

Now we prove **2**. Let $c \in C$, $x \in X$, t a local section of L, defined at x and that generates L around x, and z a generator of $\mathfrak{m}_c$. We must prove that $(\mathcal{E}_c)_{1,x} = (\mathcal{E}_c)^{(1)}_x$. Let $e_c \in (\mathcal{E}_c)^{(1)}_x$, i.e. $te_c = 0$. Let $e \in \mathcal{E}_{(x,c)}$ over e_c. Then there exists $u \in \mathcal{E}_{(x,c)}$ such that $te = zu$. We have $z.tu = t.zu = t^2e = 0$, hence, since $\mathcal{E}_{(x,c)}$ is a flat $\mathcal{O}_{C,c}$-module, $tu = 0$. Since $\mathcal{E}$ is balanced, we can write $u = tv$, with $v \in \mathcal{E}_{(x,c)}$. We have $t(e - zv) = 0$, and since $\mathcal{E}$ is balanced, there exists $f \in \mathcal{E}_{(x,c)}$ such that $e - zv = tf$. If f_c is the image of f in $\mathcal{E}_{c,x}$, we have $e_c = tf_c$, and $e_c \in (\mathcal{E}_c)_{1,x}$. Hence $(\mathcal{E}_c)^{(1)}_x \subset (\mathcal{E}_c)_{1,x}$. This proves **2**. □

6.2.2. Corollary: *Let $\mathcal{E}$ be a coherent sheaf on $X_2 \times C$, flat on C. Suppose that for some $c \in C$, $\mathcal{E}_c$ is balanced. Then there exists a neighborhood V of c such that for every $v \in V$, $\mathcal{E}_v$ is balanced, $\mathcal{E}_{1|X_2\times V}$, $\mathcal{E}_{|X\times V}$ are flat on V and $\mathcal{E}_{1,v} = \mathcal{E}_{v,1}$, $(\mathcal{E}_{|X\times V})_v = \mathcal{E}_{v|X}$ for every $v \in V$.*

6.3. Flat families of balanced sheaves on primitive double surfaces

We suppose that $n = 2$, X is projective, and $\dim(X) = 2$. Let S be a smooth variety and $\mathbf{E}, \mathbf{F}$ families of torsion free sheaves on X, parameterized by S and flat on S. Let

$$f_0 : \mathbb{F}_0 \longrightarrow \mathbf{F}$$

be a surjective morphism, where $\mathbb{F}_0$ is a vector bundle on $X \times S$ such that for every $s \in S$ and positive integer i we have $h^i(X, \mathbb{F}^*_{0s} \otimes \mathbf{E}_s) = 0$. Let $X_0 = \ker(f_0)$. By 2.5, X_0 is a flat family of torsion free sheaves, and by [21], prop. 1.1, for every $s \in S$, $X_{0,s}$ is a reflexive sheaf, and since $\dim(X) = 2$, $X_{0,s}$ is locally free. Hence X_0 is locally free.

Let $s \in S$. From the exact sequence $0 \to X_{0,s} \to \mathbb{F}_{0,s} \to \mathbf{F}_s \to 0$, we have an exact sequences

$$0 \to \mathrm{Hom}(\mathbf{F}_s, \mathbf{E}_s) \longrightarrow \mathrm{Hom}(\mathbb{F}_{0,s}, \mathbf{E}_s) \longrightarrow \mathrm{Hom}(X_{0,s}, \mathbf{E}_s) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathbf{F}_s, \mathbf{E}_s) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathbb{F}_{0,s}, \mathbf{E}_s) = \{0\},$$

and

$$0 \to \mathrm{Hom}(\mathbf{F}_s, \mathbf{E}_s) \longrightarrow \mathrm{Hom}(\mathbb{F}_{0,s}, \mathbf{E}_s) \longrightarrow \mathrm{Hom}(X_{0,s}, \mathbf{E}_s) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{X_2}}(\mathbf{F}_s, \mathbf{E}_s) \xrightarrow{\delta_s} \mathrm{Ext}^1_{\mathcal{O}_{X_2}}(\mathbb{F}_{0,s}, \mathbf{E}_s) \xrightarrow{\theta} \mathrm{Ext}^1_{\mathcal{O}_{X_2}}(X_{0,s}, \mathbf{E}_s) .$$

Hence we have an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathbf{F}_s, \mathbf{E}_s) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{X_2}}(\mathbf{F}_s, \mathbf{E}_s) \xrightarrow{\delta_s} \mathrm{Ext}^1_{\mathcal{O}_{X_2}}(\mathbb{F}_{0,s}, \mathbf{E}_s) \xrightarrow{\theta} \mathrm{Ext}^1_{\mathcal{O}_{X_2}}(X_{0,s}, \mathbf{E}_s) .$$

6.3.1. Lemma: *Let A (resp. B) be a vector bundle (resp. a coherent sheaf) on X. Then there is a canonical functorial isomorphism*

$$\mathcal{E}xt^1_{\mathcal{O}_{X_2}}(A, B) \simeq \mathcal{H}om(A \otimes L, B) .$$

The proof is similar to that of lemma 4.6.2 of [15].

By the Ext spectral sequence and the fact that $H^i(X, \mathbb{F}^*_{0,s} \otimes \mathbf{E}_s) = \{0\}$ for $i = 1, 2$, the canonical map

$$\mathrm{Ext}^1_{\mathcal{O}_{X_2}}(\mathbb{F}_{0,s}, \mathbf{E}_s) \longrightarrow H^0(\mathcal{E}xt^1_{\mathcal{O}_{X_2}}(\mathbb{F}_{0,s}, \mathbf{E}_s)) = \mathrm{Hom}(\mathbb{F}_{0,s} \otimes L, \mathbf{E}_s)$$

is an isomorphism.

Hence we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathcal{O}_{X_2}}(\mathbb{F}_{0,s},\mathbf{E}_s) & \xrightarrow{\theta} & \mathrm{Ext}^1_{\mathcal{O}_{X_2}}(X_{0,s},\mathbf{E}_s) \\ \Vert & & \downarrow \\ \mathrm{Hom}(\mathbb{F}_{0,s}\otimes L,\mathbf{E}_s) & \xrightarrow{\alpha} & \mathrm{Hom}(X_{0,s}\otimes L,\mathbf{E}_s) \end{array}$$

(where α is induced by $X_{0s}\subset\mathbb{F}_{0s}$). We have $\ker(\alpha)\simeq\mathrm{Hom}(\mathbf{F}_s\otimes L,\mathbf{E}_s)$, hence $\mathrm{im}(\delta_s)=\ker(\theta)\subset\mathrm{Hom}(\mathbf{F}_s\otimes L,\mathbf{E}_s)$, and we have an exact sequence

$$0\longrightarrow\mathrm{Ext}^1_{\mathcal{O}_X}(\mathbf{F}_s,\mathbf{E}_s)\longrightarrow\mathrm{Ext}^1_{\mathcal{O}_{X_2}}(\mathbf{F}_s,\mathbf{E}_s)\xrightarrow{\ \delta_s\ }\mathrm{Hom}(\mathbf{F}_s\otimes L,\mathbf{E}_s)\ .$$

We suppose now that

– $\mathbf{E}=\mathbf{F}\otimes p_X^*(L)$, where $p_X:X\times S\to X$ is the projection.
– For every $s\in S$, $\mathbf{F}_s$ is simple.
– $\dim(\mathrm{Ext}^1_{\mathcal{O}_X}(\mathbf{F}_s,\mathbf{F}_s\otimes L))$ is independent of $s\in S$.

It follows that we have an exact sequence

$$0\longrightarrow\mathrm{Ext}^1_{\mathcal{O}_X}(\mathbf{F}_s,\mathbf{F}_s\otimes L)\longrightarrow\mathrm{Ext}^1_{\mathcal{O}_{X_2}}(\mathbf{F}_s,\mathbf{F}_s\otimes L)\xrightarrow{\ \delta_s\ }\mathbb{C}\ .$$

6.3.2. Proposition: *Let $s\in S$, $\sigma\in\mathrm{Ext}^1_{\mathcal{O}_{X_2}}(\mathbf{F}_s,\mathbf{F}_s\otimes L)$, and $0\to\mathbf{F}_s\otimes L\to\mathcal{E}\to\mathbf{F}_s\to 0$ the corresponding extension. Then $\mathcal{E}$ is a balanced sheaf if and only if $\delta_s(\sigma)\neq 0$.*

Proof. If $\delta_s(\sigma)=0$ then $\sigma\in\mathrm{Ext}^1_{\mathcal{O}_X}(\mathbf{F}_s,\mathbf{F}_s\otimes L)$, hence $\mathcal{E}$ is concentrated on X and cannot be balanced. Conversely assume that $\delta_s(\sigma)\neq 0$. By lemma 6.1.7 it suffices to prove that $\mathcal{E}$ is not concentrated on X, which is obvious since otherwise we would have $\sigma\in\mathrm{Ext}^1_{\mathcal{O}_X}(\mathbf{F}_s,\mathbf{F}_s\otimes L)$ and $\delta_s(\sigma)=0$. □

Let $n=\dim(\mathrm{Ext}^1_{\mathcal{O}_X}(\mathbf{F}_s,\mathbf{F}_s\otimes L))$. It follows that $\mathbf{F}_s$ can be extended to a balanced sheaf on X_2 if and only if $\dim(\mathrm{Ext}^1_{\mathcal{O}_{X_2}}(\mathbf{F}_s,\mathbf{F}_s\otimes L))=n+1$.

6.3.3. Corollary: *The set $\{s\in S;\mathbf{F}_s$ can be extended to a balanced sheaf$\}$ is closed is S.*

Proof. This follows from the upper semicontinuity of the map $s\mapsto\dim(\mathrm{Ext}^1_{\mathcal{O}_{X_2}}(\mathbf{F}_s,\mathbf{F}_s\otimes L))$. □

7. Extensions of ideal sheaves on primitive double surfaces

Let X be a complex smooth projective surface, L a line bundle on X and X_2 a primitive double scheme, with underlying smooth variety X and associated line bundle L. Let $P\in X$ be a closed point.

Notations – Let $\mathcal{O}_1=\mathcal{O}_{X,P}$, $\mathcal{O}_2=\mathcal{O}_{X_2,P}$, $\mathcal{I}=\mathcal{I}_{X,X_2,P}$, and $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_1$. We have $\mathcal{I}\simeq\mathcal{O}_1$ and $\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I}\simeq\mathfrak{m}\mathcal{I}$. Let $\bar{t}$ be a generator of $\mathcal{I}$.

– If R is a ring and n is a positive integer, nR will denote the direct sum of n copies of R.
– elements of $\mathcal{O}_1$ will be denoted by greek letters: $\alpha, \beta, \gamma, \dots$
– elements of $\mathcal{O}_2$ will be denoted by latin letters: $a, b, c, \dots$
– for the restrictions to X we will use the index $_0$: $a_0, b_0, c_0, \dots$
– elements of the ideal $\mathcal{J}$ will be denoted by overscored letters: $\overline{a}, \overline{\alpha}, \dots$

7.1. Ideals of $\mathcal{O}_2$

Let $\mathbb{J}_P$ the set of ideals $\boldsymbol{J}$ of $\mathcal{O}_2$ such that there exist generators x, y of $\boldsymbol{J}$ such that $\mathfrak{m} = (x_0, y_0)$.
Let $x, y \in \mathcal{O}_2$ be such that x_0, y_0 are generators of $\mathfrak{m}$. Then the elements of $\mathbb{J}_P$ are the ideals of the form $\boldsymbol{J} = (x + \overline{A}, y + \overline{B})$, with $\overline{A}, \overline{B} \in \mathcal{J}$.

7.1.1. Proposition: *Let $\overline{A}, \overline{B}, \overline{A'}, \overline{B'} \in \mathcal{J}$, and τ (resp. τ') the image of $-y_0\overline{A} + x_0\overline{B}$ (resp. $-y_0\overline{A'} + x_0\overline{B'}$) in $(\mathfrak{m}/\mathfrak{m}^2) \otimes_{\mathcal{O}_1} \mathcal{J}$. The following properties are equivalent:*

(i) $(x + \overline{A}, y + \overline{B}) = (x + \overline{A'}, y + \overline{B'})$.
(ii) $\tau = \tau'$.
(iii) $\overline{A'} - \overline{A} \in \mathfrak{m}\mathcal{J}$ *and* $\overline{B'} - \overline{B} \in \mathfrak{m}\mathcal{J}$.
(iv) *the* $\mathcal{O}_2$*-modules* $(x + \overline{A}, y + \overline{B})$ *and* $(x + \overline{A'}, y + \overline{B'})$ *are isomorphic.*

Proof. Suppose that (iii) is true. Then

$$y_0(\overline{A'} - \overline{A}) \; = \; x_0(\overline{B'} - \overline{B}) \; = \; 0 \quad (\text{mod. } \mathfrak{m}^2\mathcal{J}) \ , \tag{5}$$

hence $\tau = \tau'$ and (ii) is true. Conversely, if (ii) is true, we have
$y_0(\overline{A'} - \overline{A}) = x_0(\overline{B'} - \overline{B})$ (mod. $\mathfrak{m}^2\mathcal{J}$). Hence we can write

$$y_0(\overline{A'} - \overline{A}) \; = \; x_0(\overline{B'} - \overline{B}) + \overline{t}\phi \ ,$$

with $\phi \in \mathfrak{m}^2$. Suppose that $\overline{A'} - \overline{A} = a + \alpha$, $\overline{B'} - \overline{B} = b + \beta$, with $a, b \in \mathbb{C}$, $\alpha, \beta \in \mathfrak{m}$. We have

$$\big(y_0 a - x_0 b + [y_0\alpha - x_0\beta - \phi]\big)\overline{t} \; = \; 0 \ ,$$

hence $y_0 a - x_0 b + [y_0\alpha - x_0\beta - \phi] = 0$. Since $y_0\alpha - x_0\beta - \phi \in \mathfrak{m}^2$, we have $a = b = 0$, and (iii) is true.

Now we prove that (i) implies (ii). Suppose that (i) is true. There exist $\lambda, \mu, \epsilon, \rho \in \mathcal{O}_2$ such that

$$x + \overline{A'} \; = \; \lambda(x + \overline{A}) + \mu(y + \overline{B}) \ , \quad y + \overline{B'} \; = \; \epsilon(x + \overline{A}) + \rho(y + \overline{B}) \ . \tag{6}$$

We have $\lambda_0 x_0 + \mu_0 y_0 = x_0$, hence there is some $\Psi \in \mathcal{O}_2$ such that $\lambda_0 = 1 - \Psi y_0$ and $\mu_0 = \Psi x_0$. Similarly there exists $\theta \in \mathcal{O}_2$ such that $\epsilon_0 = \theta y_0$ and $\rho_0 = 1 - \theta x_0$. Hence we can write

$$\lambda = 1 - \psi y + \overline{w} \ , \quad \mu = \psi x + \overline{z} \ , \quad \epsilon = \theta y + \overline{u} \ , \quad \rho = 1 - \theta x + \overline{v} \ ,$$

with $\overline{w}, \overline{z}, \overline{u}, \overline{v} \in \mathcal{J}$. It follows that, with $\overline{\tau} = -y_0\overline{A} + x_0\overline{B}$, $\overline{\tau'} = -y_0\overline{A'} + x_0\overline{B'}$,

$$\overline{\tau'} - \overline{\tau} \; = \; -x_0y_0\overline{w} - y_0^2\overline{z} - y_0\psi_0\overline{\tau} + x_0^2\overline{u} + x_0y_0\overline{v} - x_0\theta_0\overline{\tau} \ ,$$

hence $\tau = \tau'$ and (ii) is true.

Now we prove that (iii) implies (i). Suppose that $\overline{A'} - \overline{A} \in \mathfrak{m}\mathcal{I}$ and $\overline{B'} - \overline{B} \in \mathfrak{m}\mathcal{I}$. So we can write $\overline{A'} = \overline{A} + x_0\overline{u} + y_0\overline{v}$, with $\overline{u}, \overline{v} \in \mathcal{I}$. Hence

$$x + \overline{A'} \ = \ x + \overline{A} + x_0\overline{u} + y_0\overline{v} \ = \ (1+\overline{u})(x+\overline{A}) + (y+\overline{B})\overline{v} \ ,$$

(since $\overline{A}\overline{u} = \overline{B}\overline{v} = 0$) so $x + \overline{A'} \in (x+\overline{A}, y+\overline{B})$. Similarly $y + \overline{B'} \in (x+\overline{A}, y+\overline{B})$. Hence $(x+\overline{A'}, y+\overline{B'}) \subset (x+\overline{A}, y+\overline{B})$. In the same way $(x+\overline{A}, y+\overline{B}) \subset (x+\overline{A'}, y+\overline{B'})$ and (i) is true.

Suppose now that (iii) is true. Then we have proved that (i) is true, and so is (iv). Conversely, suppose that (iv) is true. Let $\sigma : (x+\overline{A}, y+\overline{B}) \to (x+\overline{A'}, y+\overline{B'})$ be an isomorphism. It induces an automorphism of the $\mathcal{O}_1$-module (x_0, y_0), which is the multiplication by some invertible element of $\mathcal{O}_1$. Hence with respect to the generators $x+\overline{A}$, $y+\overline{B}$ of $(x+\overline{A}, y+\overline{B})$, and $x+\overline{A'}$, $y+\overline{B'}$ of $(x+\overline{A'}, y+\overline{B'})$, σ is represented by a matrix $\begin{pmatrix}\alpha+\overline{u} & \overline{v}\\ \overline{w} & \alpha+\overline{z}\end{pmatrix}$, with $\alpha \in \mathcal{O}_2$ invertible, $\overline{u}, \overline{v}, \overline{w}, \overline{z} \in \mathcal{I}$. Replacing σ with $\alpha^{-1}\sigma$, we can assume that $\alpha = 1$. The equation

$$(y+\overline{B'})\sigma(x+\overline{A'}) \ = \ (x+\overline{A'})\sigma(y+\overline{B'})$$

gives

$$x_0[y_0(\overline{u}-\overline{z}) + \overline{B'} - \overline{B} + x_0\overline{w}] \ = \ y_0[y_0\overline{v} + \overline{A'} - \overline{A}]$$

in $\mathcal{I}$. It follows that $\overline{A'} - \overline{A} \in \mathfrak{m}\mathcal{I}$, and similarly $\overline{B'} - \overline{B} \in \mathfrak{m}\mathcal{I}$. So (iii) is true. □

7.2. Extensions of ideals of $\mathcal{O}_{X,P}$

7.2.1. Theorem: *The balanced $\mathcal{O}_2$-modules M such that $M \otimes_{\mathcal{O}_2} \mathcal{O}_1 \simeq \mathfrak{m}$ are the ideals of $\mathbb{J}_P$.*

Proof. The ideals of $\mathbb{J}_P$ are balanced modules by proposition 3.7.2, and if $\mathcal{J} \in \mathbb{J}_P$, then $\mathcal{J} \otimes_{\mathcal{O}_2} \mathcal{O}_1 \simeq \mathfrak{m}$. Conversely we will study the extensions of $\mathcal{O}_2$-modules

$$0 \longrightarrow \mathfrak{m} \otimes_{\mathcal{O}_1} \mathcal{I} \to M \longrightarrow \mathfrak{m} \longrightarrow 0 \ ,$$

and prove that when M is balanced, then it is an ideal of $\mathbb{J}_P$.

We have the following free resolution of $\mathfrak{m}$ as a $\mathcal{O}_2$-module:

$$3\mathcal{O}_2 \xrightarrow{\phi_2} 3\mathcal{O}_2 \xrightarrow{\phi_1} 2\mathcal{O}_2 \xrightarrow{\phi_0} \mathfrak{m} \longrightarrow 0 \ , \qquad (7)$$

where ϕ_0, ϕ_1, ϕ_2 are represented by the matrices

$$\begin{pmatrix} x & y \end{pmatrix} \ , \quad \begin{pmatrix} y & \overline{t} & 0\\ -x & 0 & \overline{t}\end{pmatrix} \ , \quad \begin{pmatrix} \overline{t} & 0 & 0\\ -y & \overline{t} & 0\\ x & 0 & \overline{t}\end{pmatrix}$$

respectively. We have

$$K \ = \ker(\phi_0) \ = \{(ey+\overline{\gamma}, -ex+\overline{\delta});\ e \in \mathcal{O}_2,\ \overline{\gamma}, \overline{\delta} \in \mathcal{I}\} \ .$$

Step 1. *Morphisms from K to $\mathfrak{m} \otimes_{\mathcal{O}_1} \mathcal{I} = \mathfrak{m}\mathcal{I}$* – Recall that $\mathcal{I}$ is isomorphic to $\mathcal{O}_1$, so $\mathfrak{m} \otimes_{\mathcal{O}_1} \mathcal{I} \simeq \mathfrak{m}$. These morphisms are the $\Theta_{\overline{\tau},\rho}$, $\overline{\tau} \in \mathfrak{m} \otimes_{\mathcal{O}_1} \mathcal{I}$, $\rho \in \mathcal{O}_1$, with

$$\Theta_{\overline{\tau},\rho}(ey+\overline{\gamma}, -ex+\overline{\delta}) \ = \ e_0\overline{\tau} + \rho(x_0\overline{\gamma} + y_0\overline{\delta}) \ .$$

Step 2. *Description of* $\mathrm{Ext}^1_{\mathcal{O}_2}(\mathfrak{m},\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I})$ – From (7) follows the complex

$$\begin{array}{ccccc} 2(\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I}) & \xrightarrow{\Psi_1} & 3(\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I}) & \xrightarrow{\Psi_2} & 3(\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I}) \\ \| & & \| & & \| \\ \mathrm{Hom}(2\mathcal{O}_2,\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I}) & & \mathrm{Hom}(3\mathcal{O}_2,\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I}) & & \mathrm{Hom}(3\mathcal{O}_2,\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I}) \end{array} ,$$

(whence $\mathrm{Ext}^1_{\mathcal{O}_2}(\mathfrak{m},\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I})\simeq\ker(\Psi_2)/\mathrm{im}(\Psi_1)$) where Ψ_1, Ψ_2 are represented by the matrices

$$\begin{pmatrix} y_0 & -x_0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} , \quad \begin{pmatrix} 0 & y_0 & -x_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

respectively. It follows easily that we have

$$\ker(\Psi_2) \ = \ \{(\alpha, x_0\otimes\overline{\epsilon}, y_0\otimes\overline{\epsilon});\ \alpha\in\mathfrak{m},\ \overline{\epsilon}\in\mathcal{I}\} \ = \ (\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I})\oplus W ,$$

with $W=(x_0,y_0)\otimes\mathcal{I}\subset 2(\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I})$,

$$\mathrm{im}(\Psi_1) \ = \ \mathfrak{m}^2\otimes_{\mathcal{O}_1}\mathcal{I} \ \subset \ \mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I} \quad (\text{the first factor in } 3(\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I})) .$$

Hence $\mathrm{Ext}^1_{\mathcal{O}_2}(\mathfrak{m},\mathfrak{m}\otimes\mathcal{I}_X)\simeq\big((\mathfrak{m}/\mathfrak{m}^2)\otimes_{\mathcal{O}_1}\mathcal{I}\big)\oplus\mathcal{O}_1$.

Step 3. *Description of the extensions* – The morphisms $f:3\mathcal{O}_2\to\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I}$ such that $f(\mathrm{im}(\phi_2))=\{0\}$ (i.e. the elements of $\ker(\Psi_2)$) are the $f_{\overline{\tau},\rho}$, with $\overline{\tau}\in\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I}$, $\rho\in\mathcal{O}_1$ and

$$f_{\overline{\tau},\rho}(a,b,c) \ = \ a_0\overline{\tau}+\rho(x_0b_0\overline{t}+y_0c_0\overline{t}) ,$$

inducing $\Theta_{\overline{\tau},\rho}:K\to\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I}$. Let $\tau\in(\mathfrak{m}/\mathfrak{m}^2)\otimes_{\mathcal{O}_1}\mathcal{I}$ be the image of $\overline{\tau}$. Then the element of $\mathrm{Ext}^1_{\mathcal{O}_2}(\mathfrak{m},\mathfrak{m}\otimes\mathcal{I}_X)$ corresponding to $f_{\overline{\tau},\rho}$ is (τ,ρ).

Let

(8) $$0\longrightarrow\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I}\xrightarrow{\ i\ }M\xrightarrow{\ p\ }\mathfrak{m}\longrightarrow 0$$

be the extension associated to the image of $f_{\overline{\tau},\rho}$ in $\mathrm{Ext}^1_{\mathcal{O}_2}(\mathfrak{m},\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I})$. We have an exact sequence

(9) $$0\longrightarrow K\xrightarrow{g=(\iota,\Theta_{\overline{\tau},\rho})}2\mathcal{O}_2\oplus(\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I})\xrightarrow{\ q\ }M\longrightarrow 0 ,$$

where ι is the inclusion. Hence we have

(10) $$\mathrm{im}(g) \ = \ \{(ey+\overline{\gamma},-ex+\overline{\delta},e_0\overline{\tau}+\rho(x_0\overline{\gamma}+y_0\overline{\delta}));\ e\in\mathcal{O}_2,\ \overline{\gamma},\overline{\delta}\in\mathcal{I}\} .$$

In the exact sequence 8, $i:\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I}\to M$ comes from the inclusion
$\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I}\hookrightarrow 2\mathcal{O}_2\oplus(\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I})$, and $q:M\to\mathfrak{m}$ from ϕ_0.

It follows easily from (9) that M is a $\mathcal{O}_1$-module (i.e. $\mathcal{I}M=\{0\}$) if and only if $\rho=0$.

Step 4. *Properties of the extensions* – We will describe the morphism $h:\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I}\to\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I}$ induced by the canonical morphism $M\otimes_{\mathcal{O}_2}\mathcal{I}\to M$ and (8) (M is balanced if and only if h is an isomorphism). Let $\alpha\in\mathfrak{m}$, that we can write as $\alpha=\phi_0(a,b)=a_0x_0+b_0y_0$, with $a,b\in\mathcal{O}_2$. Then from step 3, $h(\alpha\otimes\overline{t})$ is the unique $\beta\otimes\overline{t}\in\mathfrak{m}\otimes_{\mathcal{O}_1}\mathcal{I}$ such that $(0,0,\beta\otimes\overline{t})-(\overline{t}a,\overline{t}b,0)\in g(K)$. Using (9), we see that $\beta=-\rho\alpha$, so that h is the multiplication by $-\rho$.

Now suppose that $\rho \neq 0$. Then is is easy to see that $M^{(1)} = \mathrm{im}(i) = \mathfrak{m} \otimes_{\mathcal{O}_1} \mathcal{I}$, and that $M^{(1)}/M_1 \simeq \mathfrak{m}/\rho\mathfrak{m}$. It follows that M is balanced if and only if ρ is invertible, and that (8) is the canonical exact sequence $0 \to M^{(1)} \to M \to M^{(0)} \to 0$ if and only id $\rho = -1$.

Step 5. *Description of the morphisms $M \to \mathcal{O}_2$* – We suppose that $\rho = -1$. Such a morphism comes from $\psi : 2\mathcal{O}_2 \oplus \mathfrak{m}\mathcal{I} \to \mathcal{O}_2$ such that $\psi \circ g = 0$. There exist $a, b \in \mathcal{O}_2$, $\alpha \in \mathcal{O}_1$, such that $\psi(u, v, \overline{w}) = au + bv + \alpha\overline{w}$ for every $u, v \in \mathcal{O}_2$, $\overline{w} \in \mathfrak{m}\mathcal{I}$. Using (10), we see that $\psi \circ g = 0$ is equivalent to

$$ax - by + \alpha\overline{\tau} \;=\; 0\;, \quad a_0 = \alpha x_0 \quad \text{and} \quad b_0 = \alpha y_0\;.$$

Suppose that $\alpha = 1$. We have then

$$ay - bx \;=\; -\overline{\tau}\;, a - 0 \;=\; x_0\;, b_0 \;= y_0\;.$$

So we can write $a = x + \overline{A}$, $b = y + \overline{B}$, with $\overline{A}, \overline{B} \in \mathcal{I}$, and $\overline{\tau} = -y_0\overline{A} + x_0\overline{B}$.

Now we show that ψ is injective. Let $(u, v, \lambda) \in 2\mathcal{O}_2 \oplus \mathfrak{m}\mathcal{I}$ be such that $\psi(u, v, \lambda) = 0$, i.e. $u(x + \overline{A}) + v(y + \overline{B}) + \lambda = 0$. Then we have $u_0x_0 + v_0y_0 = 0$, hence we can write $u = ey + \overline{\gamma}$, $v = -ex + \overline{\delta}$, with $e \in \mathcal{O}_2$, $\gamma, \delta \in \mathcal{I}$, and we obtain $-e_0\overline{\tau} + x\overline{\gamma} + y\overline{\delta} + \lambda = 0$, whence $(u, v, \lambda) = g(ey + \overline{\gamma}, -ex + \overline{\delta})$, and ψ is injective.

Since ψ is injective, M is isomorphic to $\mathrm{im}(\psi) \subset \mathcal{O}_2$, and $\mathrm{im}(\psi) = (x + \overline{A}, y + \overline{B})$. This proves theorem 7.2.1. □

7.3. Parameterization of $\mathbb{J}_P$

Let $x, y \in \mathcal{O}_2$ be such that x_0, y_0 are generators of $\mathfrak{m}$. If $\overline{A}, \overline{B} \in \mathcal{I}$, let $\boldsymbol{\tau}_{x,y}(\overline{A}, \overline{B})$ be the image of $-y_0\overline{A} + x_0\overline{B}$ in $\mathfrak{m}\mathcal{I}/\mathfrak{m}^2\mathcal{I}$. By proposition 7.1.1,

$$\begin{array}{ccc} \mathbb{J}_P & \longrightarrow & \mathfrak{m}\mathcal{I}/\mathfrak{m}^2\mathcal{I} \\ (x + \overline{A}, y + \overline{B}) & \longmapsto & \boldsymbol{\tau}_{x,y}(\overline{A}, \overline{B}) \end{array}$$

is a bijection.

Let $x', y' \in \mathcal{O}_2$ be such that x'_0, y'_0 are generators of $\mathfrak{m}$. Then there exist $\overline{u}, \overline{v} \in \mathfrak{m}\mathcal{I}$ such that $(x', y') = (x + \overline{u}, y + \overline{v})$. There exists an invertible matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with coefficients in $\mathcal{O}_2$, such that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \;=\; \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x + \overline{u} \\ y + \overline{v} \end{pmatrix}\;.$$

Let $\Delta = \alpha\delta - \beta\gamma$. If $\overline{A'}, \overline{B'} \in \mathcal{I}$, then $(x' + \overline{A'}, y + \overline{B'}) = (x + \overline{A}, y + \overline{B})$ with

$$\overline{A} \;=\; \frac{\delta_0}{\Delta_0}\overline{A'} - \frac{\beta_0}{\Delta_0}\overline{B'} + \overline{u}\;, \quad \overline{B} \;= -\frac{\gamma_0}{\Delta_0}\overline{A'} + \frac{\alpha_0}{\Delta_0}\overline{B'} + \overline{v}\;.$$

It follows that

$$\boldsymbol{\tau}_{x,y}(\overline{A}, \overline{B}) \;=\; \frac{1}{\Delta_0}\boldsymbol{\tau}_{x',y'}(\overline{A'}, \overline{B'}) - y_0\overline{u} + x_0\overline{v}\;,$$

We have $\mathfrak{m}\mathcal{I}/\mathfrak{m}^2\mathcal{I} \simeq \Omega_{X,P} \otimes L_P$. Let $\Delta(x, y)$ be the image of $x_0 \wedge y_0$ in $\omega_{X,P}$, and denote by $\Delta(x, y)^{-1}$ the corresponding element of $\omega^*_{X,P}$. For $\overline{A}, \overline{B} \in \mathcal{I}$ and $J = (x + \overline{A}, y + \overline{B})$, let

$$\boldsymbol{\lambda}_{P,x,y}(J) \;=\; \Delta(x, y)^{-1} \otimes \boldsymbol{\tau}_{x,y}(\overline{A}, \overline{B}) \;\in\; \omega^*_{X,P} \otimes \Omega_{X,P} \otimes L_P \;\simeq\; T_{X,P} \otimes L\;.$$

Then $\boldsymbol{\lambda}_{P,x,y} : \mathbb{J}_P \to T_{X,P} \otimes L$ is a bijection. We have

$$\boldsymbol{\lambda}_{P,x',y'}(J) \; = \; \boldsymbol{\lambda}_{P,x,y}(J) + \Delta(x,y)^{-1} \otimes (-y_0\overline{u} + x_0\overline{v}) \; . \tag{11}$$

Hence for every $J_1, J_2 \in \mathbb{J}$ we have

$$\boldsymbol{\lambda}_{P,x,y}(J_1) - \boldsymbol{\lambda}_{P,x,y}(J_2) \; = \boldsymbol{\lambda}_{P,x',y'}(J_1) - \boldsymbol{\lambda}_{P,x',y'}(J_2) \; .$$

It follows that

7.3.1. Proposition: $\mathbb{J}_P$ *has a natural structure of affine space, with underlying* $\mathbb{C}$*-vector space* $T_{X,P} \otimes L$*.*

7.4. Ideal sheaves on X_2

7.4.1. *Extension of ideals* – Let $J \in \boldsymbol{J}$. Then J extends naturally to an ideal sheaf J_2 on X_2, P being the unique point $Q \in X$ such that $J_{2,Q} \neq \mathcal{O}_{X_2,Q}$. More generally, let Z be a smooth variety, U an open subset of X_2, and $U_0 = U \cap X$. Let $\alpha : Z \to U_0$ be a morphism. Then $Z_\alpha = \{(z, \alpha(z));\ z \in Z\}$ is a closed subvariety of $Z \times X$. Let $\mathcal{I}$ be a sheaf of ideals on $Z \times U$ such that the support of $\mathcal{O}_{Z\times U}/\mathcal{I}$ (i.e. the set of points where $\mathcal{I} \neq \mathcal{O}_{Z\times U}$) is Z_α. The $\mathcal{I}$ extends naturally to a sheaf of ideals on $Z \times X_2$, such that the support of $\mathcal{O}_{Z\times X_2}/\mathcal{I}'$ is Z_α.

7.4.2. *Families of ideals* – Suppose that

- The open subset U_0 is affine.
- There exists a section $\sigma : \mathcal{O}_X(U_0) \to \mathcal{O}_{X_2}(U)$ of the restriction $\mathcal{O}_{X_2}(U) \to \mathcal{O}_X(U_0)$, so we can view $\mathcal{O}_X(U_0)$ as a subring of $\mathcal{O}_{X_2}(U)$.
- there exist $x, y \in \mathcal{O}_X(U_0)$ such that for every $Q \in U_0$, $\mathfrak{m}_Q = (x - x(Q), y - y(Q))$ (where $\mathfrak{m}_Q$ is the maximal ideal of $\mathcal{O}_{X,Q}$), hence $\Omega_X(U_0)$ is generated by dx and dy.
- $L_{|U} = \mathcal{I}_{X|U}$ is trivial. Let $\overline{t}$ be a generator of $L_{|U}$.

So we have a local description of $\Omega_{X|U_0}$:

- with differentials: $\Omega_X(U_0)$ is generated by dx and dy.
- for every $Q \in U_0$, $\Omega_{X,Q} \simeq \mathfrak{m}_Q/\mathfrak{m}_Q^2$, and for every $\alpha, \beta \in \mathcal{O}_{X,Q}$, $\alpha dx + \beta dy$ corresponds to the image of $\alpha(Q)(x_0 - x_0(Q)) + \beta(Q)(y_0 - y_0(Q))$ in $\mathfrak{m}_Q/\mathfrak{m}_Q^2$.

Let $Y = U \times (U_0 \times \mathbb{C}^2)$, $p : Y \to U$, $p_0 : Y \to U_0$, $q_1, q_2 : Y \to \mathbb{C}$ be the projections. Then $(p^*(\sigma(x)) - p_0^*(x_0) + q_1\overline{t}, p^*(\sigma(y)) - p_0^*(y_0) + q_2\overline{t})$ is a regular sequence at each point of Y. Let

$$\mathbf{I} \; = \; (p^*(\sigma(x)) - p_0^*(x_0) + q_1\overline{t}, p^*(\sigma(y)) - p_0^*(y_0) + q_2\overline{t}) \; ,$$

which is an ideal sheaf on Y, of a subscheme $Z \subset Y$. One of the irreducible and connected components of Z, Z_0, contains the points (P, P, u), $P \in U_0, u \in \mathbb{C}^2$. Let $\boldsymbol{\mathfrak{J}}^{\sigma,\overline{t},x,y} = \mathcal{I}_{Z_0}$, which by 7.4.1 can be seen as a family of ideal sheaves on X_2 parameterized by $U_0 \times \mathbb{C}_2$ and flat on it. For every $(P, a, b) \in U_0 \times \mathbb{C}^2$, we have

$$\left(\boldsymbol{\mathfrak{J}}^{\sigma,\overline{t},x,y}\right)_{P,a,b} \; = \; (\sigma(x) - x(P) + a\overline{t}, \sigma(y) - y(P) + b\overline{t}) \; .$$

We have an isomorphism

$$\Theta^{x,y,\overline{t}} : \mathcal{O}_{U_0} \otimes \mathbb{C}^2 \longrightarrow \left(\Omega_X \otimes \omega_X^* \otimes L\right)_{|U_0} \; = \; (T_X \otimes L)_{|U_0} \; ,$$

defined by

$$\Theta^{x,y,\bar{t}}(a,b) \;=\; (-a.dy + b.dx) \otimes (dx \wedge dy)^{-1} \otimes \bar{t}\ ,$$

for every $a, b \in \mathcal{O}_X(U_0)$. Let

$$\mathfrak{J}^{\sigma,\bar{t},x,y} \;=\; ((\Theta^{x,y,\bar{t}})^{-1})^{\sharp}(\mathfrak{I}^{\sigma,\bar{t},x,y})\ ,$$

so that $\left(\mathfrak{J}^{\sigma,\bar{t},x,y}\right)_{P,\Theta^{x,y,\bar{t}}(a,b)} = \left(\mathfrak{I}^{\sigma,\bar{t},x,y}\right)_{P,a,b}$.

7.4.3. Proposition: $\mathfrak{J}^{\sigma,\bar{t},x,y}$ *is independent of x,y and* $\bar{t}$.

Proof. For every $Q \in U_0$ we have

$$\left(\Theta^{x,y,\bar{t}}(a,b)\right)_Q \;=\; \boldsymbol{\lambda}_{Q,\sigma(x),\sigma(y)}\left((\mathfrak{J}^{\sigma,\bar{t},x,y})_{Q,\Theta^{x,y,\bar{t}}(a,b)}\right)\ .$$

In other words, for every $\eta \in \left(\Omega_X \otimes \omega_X^* \otimes L\right)_Q$,

$$\boldsymbol{\lambda}_{Q,\sigma(x),\sigma(y)}\left((\mathfrak{J}^{\sigma,\bar{t},x,y})_{Q,\eta}\right) \;=\; \eta\ .$$

If x, y are replaced with $x', y' \in \mathcal{O}_X(U_0)$, we have, for every $P \in U_0$

$$(\sigma(x) - x(P), \sigma(y) - y(P)) \;=\; (\sigma(x') - x'(P), \sigma(y') - y'(P))$$

(it is the ideal of $\mathcal{O}_{X_2,P}$ generated by $\sigma(\mathfrak{m}_Q)$). Hence, by formula (11), the maps $\boldsymbol{\lambda}_{Q,\sigma(x),\sigma(y)}$ and $\boldsymbol{\lambda}_{Q,\sigma(x'),\sigma(y')}$ are the same, and since they are isomorphisms, $\mathfrak{J}^{\sigma,\bar{t},x,y}$ is independent of the choice of x, y and $\bar{t}$. □

We will note $\mathfrak{J}^{\sigma} = \mathfrak{J}^{\sigma,\bar{t},x,y}$. It is a flat family of ideal sheaves on X_2, parameterized by $(T_X \otimes L)_{|U_0}$.

The map

$$f_Q : T_{X,Q} \otimes L_Q = \left(\Omega_X \otimes \omega_X^* \otimes L\right)_Q \longrightarrow \mathbb{J}_Q$$

$$\eta \longmapsto (\mathfrak{J}^{\sigma})_{Q,\eta}$$

is a bijection, i.e. $\left(\Omega_X \otimes \omega_X^* \otimes L\right)_Q$ parameterizes the set of ideal sheaves of $\mathcal{O}_{X_2}$, which are balanced and whose restriction to X is the ideal sheaf of Q.

Let $\mathbb{J}_{U_0} = \bigcup_{Q \in U_0} \mathbb{J}_Q$. Then $\mathbb{J}_{U_0}$ parameterizes the set of ideal sheaves of $\mathcal{O}_{X_2}$, which are balanced and whose restriction to X is the ideal sheaf of a point of U_0. We have a bijection

$$\mu_\sigma : (T_X \otimes L)_{|U_0} \longrightarrow \mathbb{J}_{U_0}$$

over U_0 defined by the maps f_Q.

7.4.4. – *Change of parameters* – Suppose that we have another section $\sigma' : \mathcal{O}_X(U_0) \to \mathcal{O}_{X_2}(U)$ of the restriction $\mathcal{O}_{X_2}(U) \to \mathcal{O}_X(U_0)$. Then there is a derivation D of $\mathcal{O}_X(U_0)$such that

$$\sigma'(\lambda) \;=\; \sigma(\lambda) + D(\lambda)\bar{t}\ ,$$

for every $\lambda \in \mathcal{O}_X(U_0)$. Let

$$x' \;=\; \sigma'(x) \;=\; \sigma(x) + D(x)\bar{t}\ , \qquad y' \;=\; \sigma'(y) \;=\; \sigma(y) + D(y)\bar{t}\ .$$

We can view $D \otimes \bar{t}$ as a section of $(T_X \otimes L)_{|U_0}$.

7.4.5. Proposition: *For every $Q \in U_0$, $u \in T_{X,Q} \otimes L_Q$,*
$$\mu_{\sigma'}(u) \ = \ \mu_\sigma\big(u + (D \otimes \overline{t})(Q)\big) \ .$$

Proof. From the definitions, using x', y' instead of x, y, we have
$$\begin{aligned}\big(\mathcal{J}^{\sigma'}\big)_{P,\Theta^{x',y',\overline{t}}(a,b)} \ &= \ (\sigma(x) - x(P) + (a + D(P)(x))\overline{t}, \sigma(y) - y(P) + (b + D(P)(y))\overline{t}) \ , \\ &= \ \big(\mathcal{J}^{\sigma}\big)_{P,\Theta^{x,y,\overline{t}}(a+D(P)(x),b+D(P)(y))} \ ,\end{aligned}$$
which is equivalent to
$$\mu_{\sigma',\overline{t}}((-a.dy + b.dx) \otimes (dx \wedge dy)^{-1} \otimes \overline{t}) \ = \ \mu_{\sigma,\overline{t}}\big(- (a + D(x))dy + (b + D(y))dx\big) \otimes (dx \wedge dy)^{-1} \otimes \overline{t} \ .$$
Using the canonical isomorphism $\omega^*_{X,P} \otimes \Omega_{X,P} \simeq T_{X,P}$, one obtains proposition 7.4.5. □

7.5. Moduli spaces of ideals and universal sheaves

7.5.1. *Ideal sheaves on X* – Let $\Delta \subset X \times X$ be the diagonal. Then $\mathcal{I}_\Delta$ can be seen as a family of ideal sheaves on X, parameterized by X. For every $x \in X$
$$\mathcal{I}_{\Delta,x} \ = \ \mathcal{I}_{\Delta|X\times\{x\}} \ = \ \mathcal{I}_x \ .$$

7.5.2. *Construction of the moduli space and the universal sheaf* – Let $(V_i)_{1\leq i\leq k}$ be an open affine cover of X such that for each V_i the conditions of 7.4.2 are satisfied: we have a section $\sigma_i : \mathcal{O}_X(V_i) \to \mathcal{O}_{X_2}(V_i)$ of the restriction $\mathcal{O}_{X_2}(V_i) \to \mathcal{O}_X(V_i)$, and $\overline{t}_i$ is a generator of $L_{|V_i}$. We obtain the family $\mathcal{J}^{\sigma_i}$ of ideal sheaves on X_2 parameterized by $(T_X \otimes L)_{|V_i}$. If $1 \leq i < j \leq k$ we have
$$\sigma_{j|V_{ij}} \ = \ \sigma_{i|V_{ij}} + D_{ij}.\overline{t}_i \ ,$$
where D_{ij} is a derivation of $\mathcal{O}_X(V_{ij})$. With the notations of 4, $(\alpha_i^{-1} \otimes D_{ij})_{1\leq i<j\leq k}$ represents an element λ of $H^1(X, T_X \otimes L)$. From 4.2, $\mathbb{P}(H^1(X, T_X \otimes L)) \cup \{0\}$ parameterizes the set of primitive double schemes Y such that $Y_{red} = X$ and associated line bundle L, and $\mathbb{C}\lambda = \zeta(X_2)$ is the element corresponding to X_2.

We have an affine isomorphism
$$\begin{aligned}\rho_{ij} : (T_X \otimes L)_{|V_{ij}} &\longrightarrow (T_X \otimes L)_{|V_{ij}} \\ u &\longmapsto u + D_{ij}(u) \otimes \overline{t}_i\end{aligned}$$
and
$$\rho_{ij}^\sharp(\mathcal{J}^{\sigma_i}) \ \simeq \ \mathcal{J}^{\sigma_j} \ .$$
The ρ_{ij} define an affine bundle $\mathbb{I}$ on X, with associated vector bundle $T_X \otimes L$. According to 2.4 we have

7.5.3. Proposition: *The element $\eta(\mathbb{I})$ of $\big(\mathbb{P}(H^1(X, T_X \otimes L)) \cup \{0\}\big)/\operatorname{Aut}(T_X \otimes L)$ associated to $\mathbb{I}$ is the image of $\zeta(X_2)$.*

The closed points of $\mathbb{I}$ are the ideal sheaves $\mathcal{J}$ on X_2 of subschemes Z such that

– Z contains only one closed point P of X.
– there exist $x, y \in \mathcal{O}_{X_2,P}$ such that their images in $\mathcal{O}_{X,P}$ generate the maximal ideal, and $\mathcal{J}_P = (x, y)$.

The sheaves $\boldsymbol{\mathfrak{J}}^{\sigma_i}$ can be glued, using 7.4.4, to obtain a flat family of ideal sheaves $\boldsymbol{\mathfrak{J}}$ parameterized by $\mathbb{I}$. It is a balanced sheaf on $X_2 \times \mathbb{I}$. If $p_X : \mathbb{I} \to X$ is the projection, we have

$$\boldsymbol{\mathfrak{J}}_{|X\times\mathbb{I}} \simeq p_X^\sharp(\mathcal{J}_\Delta) .$$

7.6. Moduli spaces of sheaves

The ideal sheaves $\mathcal{J} \in \mathbb{I}$ can also be deformed by tensoring with a line bundle. This is why, to obtain a moduli space of sheaves, one must consider the sheaves $\mathcal{J} \otimes D$, $\mathcal{J} \in \mathbb{I}$, $D \in \mathrm{Pic}(X_2)$.

7.6.1. *Moduli spaces on rank 1 sheaves on X* – Let $\mathbb{D}$ be a Poincaré bundle on $\mathrm{Pic}(X)$ and $\mathbb{M}_X = X \times \mathrm{Pic}(X)$. Let $p_1 : \mathbb{M}_X \to X$, $p_2 : \mathbb{M}_X \to \mathrm{Pic}(X)$ be the projections, and

$$\mathfrak{D} = p_1^\sharp(\mathcal{J}_\Delta) \otimes p_2^\sharp(\mathbb{D}) .$$

It is a flat family of sheaves on X parameterized by $\mathbb{M}_X$. For every $(x, D) \in \mathbb{M}_X$, $\mathfrak{D}_{(x,D)} = \mathcal{J}_x \otimes D$.

The following properties are well known:

– Let $(x, D), (x', D') \in \mathbb{M}_X$. Suppose that $\mathfrak{D}_{(x,D)} \simeq \mathfrak{D}_{(x',D')}$. Then $(x, D) = (x', D')$. So we can see $\mathbb{M}_X$ as the family of sheaves $\mathfrak{D}_{(x,D)}$.
– Let $\mathcal{F}$ be a flat family of sheaves on X parameterized by a scheme Y. Let $y \in Y$ be a closed point such that $\mathcal{F}_y \in \mathbb{M}_X$. Then there exists an open neighborhood V of y such that for every closed point $v \in V$, $\mathcal{F}_v \in \mathbb{M}_X$.
– Let $\mathcal{F}$ be a flat family of sheaves on X parameterized by a scheme Y. Suppose that for every closed point $y \in Y$, $\mathcal{F}_y \in \mathbb{M}_X$. Then there exists a unique morphism $f_\mathcal{F} : Y \to \mathbb{M}_X$, and a line bundle Γ on Y such that $\mathcal{F} \simeq f^\sharp(\mathfrak{D}) \otimes p_Y^*(\Gamma)$ (where $p_Y = X \times Y \to Y$ is the projection).

Thus $\mathbb{M}_X$ is a moduli space of rank 1 sheaves on X (cf. 2.6).

7.6.2. *Moduli spaces on rank 1 sheaves on X_2* – We can do the same work on X_2, using $\mathbb{I}$ and $\mathrm{Pic}(X_2)$.

7.6.3. Lemma: *Let $\mathcal{J}$, $\mathcal{J}'$ be ideals in $\mathbb{I}$, and $\mathbb{D}$ a line bundle on X_2. If $\mathcal{J}' \simeq \mathcal{J} \otimes \mathbb{D}$ then $\mathbb{D} = \mathcal{O}_{X_2}$ and $\mathcal{J}' = \mathcal{J}$.*

Proof. This follows easily from the fact that $\mathcal{J}^* = \mathcal{J}'^* = \mathcal{O}_{X_2}$. □

Let $\boldsymbol{I}(X_2) = \mathbb{I} \times \mathrm{Pic}(X_2)$. Let $q_1 : \boldsymbol{I}(X_2) \to \mathbb{I}$, $q_2 : \boldsymbol{I}(X_2) \to \mathrm{Pic}(X_2)$ be the projections. Let $(P_i)_{i\in I}$ be an open cover of $\mathrm{Pic}(X_2)$ such that for every $i \in I$, there is a Poincaré bundle $\mathcal{D}_i$ on $X_2 \times P_i$, which define $\mathrm{Pic}(X_2)$ as a *fine moduli space* of sheaves (cf. 2.6, 4.6).

Let $\mathcal{H}_{\boldsymbol{i}} = q_1^\sharp(\boldsymbol{\mathcal{J}}) \otimes q_2^\sharp(\mathcal{D}_i)$, which is a sheaf on $\mathbb{I} \times P_i$. For every ideal $\mathcal{J}$ in $\mathbb{I}$ and $D \in P_i$, $\mathcal{H}_{\mathcal{J},D} = \mathcal{J} \otimes \mathcal{D}_{i,D}$.

7.6.4. Proposition: *Let Y be a scheme and $\mathcal{E}$ a coherent sheaf on $X_2 \times Y$, flat over Y. Let $y \in Y$ be a closed point such that $\mathcal{E}_y \in \boldsymbol{I}(X_2)$. Then there exists an neighborhood $V \subset Y$ of y such that*

– *$\mathcal{E}_{1|X\times V}$ and $\mathcal{E}_{|X\times V}$ are flat on V.*
– *For every closed point $v \in V$, $\mathcal{E}_v \in \boldsymbol{I}(X_2)$, and $(\mathcal{E}_1)_v = (\mathcal{E}_v)_1$, $(\mathcal{E}_{|X\times V})_v = (\mathcal{E}_v)_{|X}$.*

Proof. This follows from theorem 6.2.1, 1-. □

The proof of the following result is similar of that of theorem 1.1.3 of [15], and shows that $\boldsymbol{I}(X_2)$ is a moduli space of sheaves in the sense of 2.6.

7.6.5. Theorem: *Let Y be a scheme and $\mathcal{E}$ a coherent sheaf on $X_2 \times Y$, flat over Y, such that for every closed point $y \in Y$, $\mathcal{E}_y \in \boldsymbol{I}(X_2)$. Then there exists a unique morphism $f_{\mathcal{E}} : Y \to \boldsymbol{I}(X_2)$ and an open cover $(Y_i)_{i\in I}$ such that for every $i \in I$, $f_{\mathcal{E}}^\sharp(\mathcal{H})_{|X_2\times Y_i} \simeq \mathcal{E}_{|X_2\times Y_i}$.*

Let $\Gamma(X_2) \subset \mathrm{Pic}(X)$ be the algebraic subgroup of vector bundles that can be extended to X_2. According to 4.6, $\mathrm{Pic}(X_2)$ is an affine bundle on $\Gamma(X_2)$ with associated vector bundle $\mathcal{O}_{\Gamma(X_2)} \otimes H^1(X,L)$. Let $p_X : X \times \Gamma(X_2) \to X$, be the projection. Then $\boldsymbol{I}(X_2)$ is an affine bundle on $X \times \Gamma(X_2)$, with associated vector bundle $(\mathcal{O}_{X\times\Gamma(X_2)} \otimes H^1(X,L)) \oplus p_X^*(T_X \otimes L)$.

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