The work of Artur Avila

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Abstract. Artur Avila is awarded a Fields Medal for his profound contributions to dynamical systems theory, which have changed the face of the field, using the powerful idea of renormalization as a unifying principle.

1. Introduction

The citation for Avila's award states:

"Avila leads and shapes the field of dynamical systems. With his collaborators, he has made essential progress in many areas, including real and complex one-dimensional dynamics, spectral theory of the one-frequency Schrödinger operator, flat billiards and partially hyperbolic dynamics. Avila's work on real one-dimensional dynamics brought completion to the subject, with full understanding of the probabilistic point of view, accompanied by a complete renormalization theory. His work in complex dynamics led to a thorough understanding of the fractal geometry of Feigenbaum Julia sets. In the spectral theory of one-frequency difference Schrödinger operators, Avila came up with a global description of the phase transitions between discrete and absolutely continuous spectra, establishing surprising stratified analyticity of the Lyapunov exponent. In the theory of flat billiards, Avila proved several long-standing conjectures on the ergodic behavior of interval-exchange maps. He made deep advances in our understanding of the stable ergodicity of typical partially hyperbolic systems. Avila's collaborative approach is an inspiration for a new generation of mathematicians."

Avila has published a huge number of papers, many of them solving long standing conjectures, with many collaborators. It is impossible to give an overview of his contribution in a small number of pages, even in rough outlines.

Fortunately, on the rather recent occasion of the Brin prize for Avila, two detailed papers were published, giving an excellent presentation of his work, at least in two of his main areas of research: one dimensional dynamics and the billiards dynamics [2, 3]. The interested reader is strongly encouraged to read these reviews.

I chose the option of following very closely the oral "laudation" that I presented during ICM Seoul. I had to select a very small number of results among many other possibilities.

It is intended for the general mathematician, certainly not for the expert, and its only purpose is to catch a glimpse of Avila's work.

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2. The setting

If I had to give a summary of four centuries of research in dynamics, in a few sentences, I would write, following a joke by Yulij Ilyashenko, that there are three main stages in this history.

The *first stage* was initiated by Newton:

"You are given an ordinary differential equation and your task is to find its solutions".

Differential calculus has been indeed remarkably successful.

The *second stage* was initiated by Poincaré at the turn of the twentieth century, when he realized that in most cases it is simply impossible to find a formula for solutions. This corresponds for instance to the birth of chaos theory.

"You are given an ordinary differential equation and your task is to say something about its solutions."

If possible something useful, for instance something describing the qualitative behavior when time goes to infinity.

The *third stage* began when mathematicians realized that, in practice, physicists never know exactly the differential equation they want to solve. There are always unknown quantities, which may be small, but which do have some influence on the motion, some tiny friction for instance. One could say that this period began in the 1960's with Smale and Thom:

"You are NOT given an ordinary differential equation and your task is to say something about its solutions."

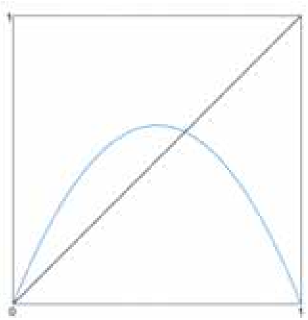
This is the field of research of Artur Avila. Most of his results turn around the question:

"What does a typical dynamical system look like?"

3. One dimensional dynamics and renormalization

Let us start with a basic example.

Consider a *unimodal map* f from an interval to itself, that is, a map having a single maximum. Assume that the second derivative is negative at the maximum.



Pick a point x in the interval, take its image by f and iterate the process. One gets the *orbit* of x, denoted $\{f^n(x)\}$. The main question is to describe the sequence $f^n(x)$. Where does it go? Where does it accumulate? According to Smale-Thom's message one should not try to answer this question for *every* f, but for a *typical* f.

Here is one of the very first great results of Artur, jointly with Misha Lyubich and Wellington de Melo, right after his PhD, improved a bit later in a joint work with Moreira.

In a non-trivial real analytic family $f_\lambda(\lambda \in \Lambda)$ *of unimodal maps (where* Λ *is some finite dimensional parameter space), there is a dichotomy: for Lebesgue almost every* λ*, the map* f_λ *is either* Regular *or* Stochastic.

Of course, one should be more precise about the words used in this statement.

In the *regular* case, Lebesgue almost every orbit converges to some attracting cycle. After some time, the dynamics becomes essentially periodic: no chaos appears. This is the easy situation. The set of values of the parameter λ for which this regular case holds is typically an open and dense set in the parameter space Λ (but not of full Lebesgue measure).

The second case, *stochastic*, is *chaotic*. But chaos should not be understood as a negative word. It does not mean that one cannot describe the motion. There is some *absolutely continuous* measure on the interval such that for Lebesgue almost every initial condition x the sequence $f^n(x)$ is *asymptotically distributed* according to this measure, unless it converges to a periodic cycle. So, this chaotic mode is still well understood since a single good measure describes the dynamics. The set of values of the parameter λ for which this happens has *positive* Lebesgue measure.

The theorem is that the union of regular and stochastic dynamics has *full Lebesgue measure* in the parameter space.

This result has a very long history and it is not possible to mention here all preliminary steps. The reader is referred to Misha Lyubich survey paper [3]. This "Regular or Stochastic dichotomy" was the first occasion confirming the general Palis conjecture on the behavior of almost all orbits for typical dissipative dynamical systems.

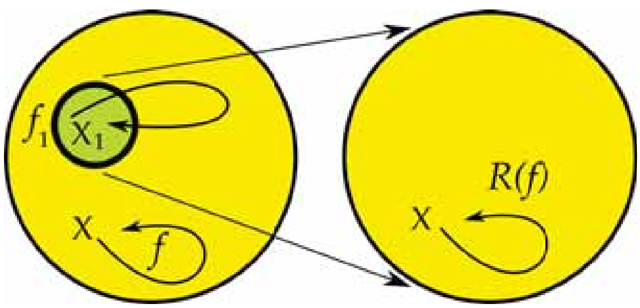
As Lyubich writes "we have reached a full probabilistic understanding of real analytic unimodal dynamics, and Artur Avila has been the key player in the final stage of the story".

Of course, I cannot give any description of the proof of such a difficult theorem but I would like at least to explain one of the key tools. The so called *renormalization operator* has certainly not been invented by Artur but he knows better than anybody else how to use it! It quickly became his magic stick: he uses it in most of his papers. That was the topic of his plenary lecture in the previous Congress, in Hyderabad [1].

Start from a dynamical system, say a map f from a space X to itself.

Choose some small part X_1 of X and assume that the orbit of every point in X_1 comes back in X_1, maybe after many iterates.

Let us consider the map f_1 from X_1 to X_1 which maps every point of X_1 to its first return in X_1 under the iterates of f.

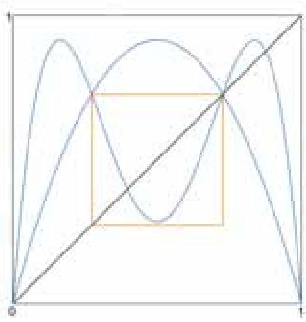


In many cases, the small X_1 is somehow similar to the big X and there is a zooming out, from X_1 to X, so that one can "renormalize" f_1 as a map from X to X. Let us denote this new map by $R(f)$. Therefore, one can think of R as an operator sending a dynamical system f from X to X to some other dynamical system from X to X. This is called the *renormalization operator*. The magical fact is that there is a strong correspondence between the dynamics of R, acting on the space of maps f, and the dynamics of a typical element f.

As Adrien Douady used to say:

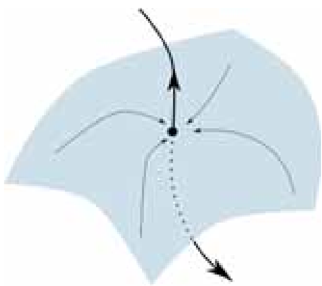
"We first plough in the dynamical plane and then harvest in parameter plane".

Let us have a look at the first historical example. Consider a unimodal map f from the interval to itself. The graph of f, together with the graph of its square f^2 (i.e. $f \circ f$), may look like in the following picture.



In this case there is a subinterval invariant by f^2. Restricting f^2 to this interval, zooming out and fliping, one gets back to the initial interval equipped with another unimodal map $R(f)$.

The general picture for this renormalization operator in this very special case has been a conjecture for many years. This figure illustrates the dynamics of the operator R on the infinite dimensional space of unimodal maps.



Coulet-Tresser and Feigenbaum, in the the late 70's, had the intuition, based on numerical evidence, that there is a fixed point for the renormalization operator R, the so-called *Feigenbaum map*. Moreover the linearization at this fixed point has a one-dimensional expanding direction and is contracting on some hypersurface. It was a wonderful joint venture of many mathematicians to transform this intuition into a theorem. Among them, Lanford, Sullivan and McMullen. Avila and Lyubich could eventually achieve Sullivan's dream: instead of a computer assisted proof, they produced a "brain assisted proof" using some sophisticated technical preparation and then, just the standard Schwarz Lemma. A "proof from the book" as Erdös would have said.

4. Billiard tables

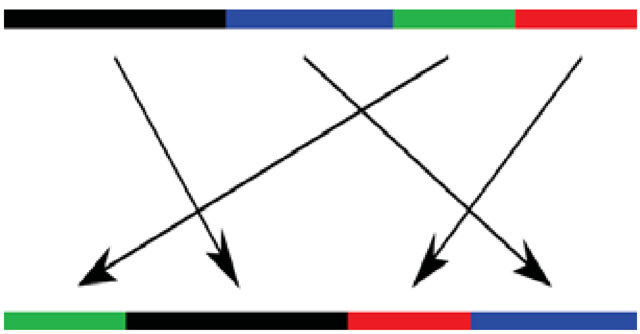
Imagine a box containing some perfect ideal gas, a huge number of bouncing molecules. For simplicity, let us make the assumption, not very realistic physically, that the gas is so dilute that the molecules don't collide between themselves.

Each individual molecule travels along straight lines in the box and bounces from time to time on the boundary. Molecules follow the orbits of a classical billiard ball game.

Let us make things simple and suppose that the box is actually 2 dimensional: a polygon in the plane. Choose a point x on the boundary of the polygon, which is a finite union of segments, and choose an initial velocity v, say of norm 1. Hit a ball there in that direction and wait until the ball bounces again on the boundary in x' and gets off in some other direction v'. This defines a dynamical system T which maps (x, v) to (x', v'). Let us make the even stronger assumption that the angles of this polygon are rational multiple of π. The rationality of the angles implies that the directions of the travelling ball can only take a finite number of values.

On a rectangle for instance, the velocity vector takes only four values. Therefore one can reduce the dynamics from dimension 2 to dimension 1. Now the configuration space will be a finite union of intervals. Each side of the polygon defines a finite number of intervals, one for each direction.

This kind of map is called an *interval exchange transformation*. Formally, the definition is the following. Take the unit interval $[0, 1]$ and split it into k subintervals. Now reorganize the intervals according to some permutation. This defines a bijection from $[0, 1]$ to itself. Don't worry about the endpoints. It is somehow like a generalized cards shuffling: you split your deck into several intervals and you permute them. Therefore, the dynamics of a rational polygonal billiard table is reduced to the dynamics of interval exchanges maps. Note that the space of interval exchange maps, with a given number k of subintervals, is parameterized by the product of the permutation group on k objects and a simplex, describing the lengths of the subintervals. In particular in this case, the space of dynamical systems under consideration is finite dimensional.



Now, let me state a theorem, due to Avila and Forni, again in the spirit of Smale-Thom.

Almost all interval exchange transformations are weakly mixing (except for trivial situations).

I should explain the words and say at least something about the proof.

"Almost all" should be clear since the space of interval exchange maps is finite dimensional so that one has the Lebesgue measure at our disposal.

Let me define "mixing" first. Let f from X to X be a transformation preserving a probability measure μ. One says that f is *mixing* if, for every pair A, B of measurable subsets of X we have $\lim_{n\to\infty} \mu(A \cap f^n(B)) = \mu(A)\mu(B)$. This means that when times goes to infinity the dynamics somehow forgets the past: the events A and $f^n(B)$ have a tendency to become independent. So a mixing dynamical system is a good approximation to randomness. Katok showed, however, that an interval exchange map is never mixing.

"Weak mixing" is, of course, a weakening of the concept of mixing. It simply means that $\mu(A \cap f^n(B))$ converges to $\mu(A)\mu(B)$ in a weaker sense: restricting n to some subset E of the integers, of density 1. Almost as good as mixing.

Avila-Forni's theorem is a major progress in the understanding of the dynamics of billards. The main tool to prove this theorem is again renormalization. The renormalization operator in this context acts on the space of interval exchange maps, which is a finite union of finite dimensional simplices. The important fact is that, even though each interval exchange is a rather simple dynamical system, this renormalization operator turns out to be very chaotic. This chaoticity in parameter space is the key to the understanding of a typical interval exchange map. For many more details, see the survey paper by Giovani Forni [2].

5. Schrödinger operators

This is a topic in which the dynamical insight of Artur radically changed the landscape.

Imagine a 1-dimensional discrete quantum particle. Its state is described by some l^2 function ψ on $\mathbf{Z}$ with complex values. One can think that the probability that the particle is located at a point n is the square of the modulus of $\psi(n)$.

The time evolution of ψ, as usual, is described by the Schrödinger equation: the time derivative of ψ is $iH\psi$ where H is the Schrödinger operator:

$$H(\psi)(n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n).$$

The first two terms give a discrete version of the Laplace operator and $V(n)$ is some bounded potential describing the environment of the particle.

Note that H is a bounded self adjoint operator on l^2. Everything depends on the *spectrum* of H and the *spectral measure*.

Let me recall that the *spectrum* is the set of energies E such that $H - E.Id$ is not invertible. It is a compact set $\sigma(H)$ in $\mathbf{R}$.

The *spectral measure* associated to some ψ is the measure μ_ψ (supported on $\sigma(H)$)such that for every continuous real valued function g, one has $\langle \psi, g(H)\psi \rangle = \int g \, d\mu_\psi$.

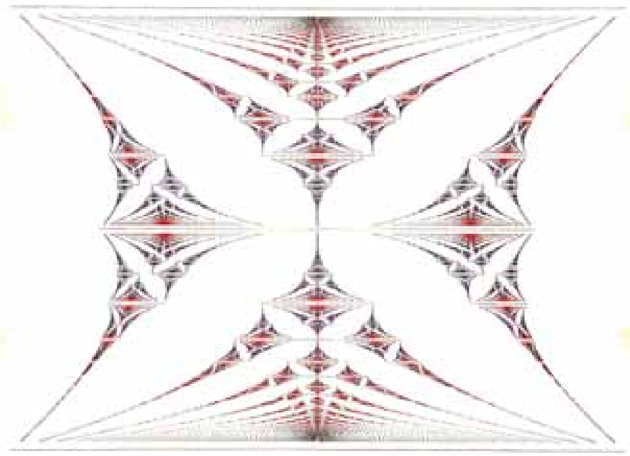
The spectral measures provide a key to the understanding of the dynamics of the quantum particle. To say things in a non precise way:

- The particle "travels freely" if μ_ψ is absolutely continuous: the medium is conductor.
- The particle "travels a little bit" if μ_ψ is singular continuous.
- The particle "does not travel" if μ_ψ is pure point. The medium is insulator.

The most interesting case occurs when V is quasi-periodic. One can think for instance of a quasicrystal. The special case of $V(n) = 2\lambda\cos(2\pi n\alpha)$ arises in this context as the simplest example. This is called the *almost Mathieu operator*.

Based on numerics, the shape of the spectrum was conjectured to be a Cantor set when α is irrational. In 1981, Mark Kac offered ten Martinis for a proof of this fact. Barry Simon coined the term *Ten Martini Problem*.

The following picture is the famous *Hofstadter butterfly*.



Slicing this butterfly by a vertical line with first coordinate α, one gets the spectrum for the critical case $\lambda = 1$. Many papers were devoted to the ten Martini problem and other conjectures in the 1980's and 1990's. It is probably fair to say that spectral theorists had exhausted their toolboxes. New ideas and approaches were needed. Artur introduced new dynamical methods in the problem and could solve the most difficult conjectures. Here is a sample of some results:

Theorem (Avila-Jitomirskaya 2009). *For all $\lambda \neq 0$, and all irrational α, the spectrum $\sigma_{\lambda,\alpha}$ is a Cantor set.*[1]

Theorem (Avila-Krikorian 2006). $Leb(\sigma_{\lambda,\alpha}) = 4|1 - |\lambda||$.

This was already known by Jitomirskaya and Krasovsky in the non critical case, when λ is not equal to 1.

Theorem (Avila, Damanik, 2008). *For all irrational α and $|\lambda| < 1$, the spectrum is purely absolutely continuous.*

The key tool in the proofs of these difficult theorem is again *renormalization*.

Let me also mention, without giving any explanation that Artur created recently a global theory of one frequency Schrödinger operators, describing in detail what he calls the *stratified analyticity* of the Lyapounov exponent and the boundary of non uniform hyperbolicity.

Artur started his career by solving a number of long standing problems and conjectures but he is also an exceptional theory builder. The whole theory was developed by Artur and this required outstanding insight and exceptional technical abilities.

6. A gem

Let me finish by mentioning a puzzling theorem of Artur, which is somehow isolated in his work. This is not directly related to dynamics: this is a pure partial differential equations

[1]They could not get the ten Martinis since meanwhile Mark Kac had unfortunately passed away.

result. It is easy to state and Artur told me that almost every mathematician listening to this theorem for the first time is immediately convinced that this is very easy and that he can provide a short simple proof. But, this is not so...

Let f be a diffeomorphism of class C^1 of some compact manifold of class C^∞. It is well known, and easy to prove, that you can approximate f by C^∞ diffeomorphisms in the C^1 topology.

Artur's theorem is that *if the manifold is equipped with a C^∞ volume form and if f preserves the volume, it can approximated in the C^1 topology by C^∞ diffeomorphisms* which are volume preserving.

Artur's proof starts with a triangulation and does the approximation by induction on the skeleton. It reminds me of the wonderful proofs by Gromov of his h-principles in PDE.

Avila's contributions are amazing: I convinced that this is just a beginning.

References

[1] Avila, Artur, *Dynamics of renormalization operators*, Proceedings of the International Congress of Mathematicians, Volume I, 154–175, Hindustan Book Agency, New Delhi, 2010.

[2] Forni, Giovanni, *On the Brin Prize work of Artur Avila in Teichmüller dynamics and interval-exchange transformations*, J. Mod. Dyn. **6** (2012), no. 2, 139–182.

[3] Lyubich, Mikhail, *Forty years of unimodal dynamics: on the occasion of Artur Avila winning the Brin Prize*, J. Mod. Dyn. **6** (2012), no. 2, 183–203.

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