From Brouwer theory to the study of homeomorphisms of surfaces

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Abstract. We will state an equivariant foliated version of the classical Brouwer Plane Translation Theorem and will explain how to apply this result to the study of homeomorphisms of surfaces. In particular we will explain why a diffeomorphism of a closed oriented surface of genus ≥ 1 that is the time-one map of a time dependent Hamiltonian vector field has infinitely many periodic orbits. This gives a positive answer in the case of surfaces to a more general question stated by C. Conley. We will give a survey of some recent results on homeomorphisms and diffeomorphisms of surfaces and will explain the links with the improved version of Brouwer's theorem.

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1. Introduction

It is a natural problem to ask if a given dynamical statement about time independent vector fields may be extended to periodic time dependent vector fields. Recall that a complete periodic time dependent smooth vector field on a manifold M defines a family $(F_t)_{t\in\mathbb{R}}$ of diffeomorphisms such that $F_0 = \mathrm{Id}_M$ and $F_{t+T} = F_t \circ F_T$, for every $t \in \mathbb{R}$, if T is the period. To study this system, one usually studies the discrete dynamical system induced by $F = F_T$.

Let us begin with a very simple example. Suppose that M is compact and write $\chi(M)$ for the Euler characteristic of M. If F is a homeomorphism homotopic to the identity with a finite number of fixed points, one knows by the Lefschetz formula that

$$\sum_{F(z)=z} i(F, z) = \chi(M),$$

where $i(F, z)$ is the Lefschetz index. If F is the time-one map of a flow induced by a vector field, the fixed points are necessarily the singularities of ξ, and the previous formula may be deduced from the Poincaré–Hopf formula

$$\sum_{\xi(z)=0} i(\xi, z) = \chi(M),$$

where $i(\xi, z)$ is the Poincaré index.

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Let us now give a more interesting example. Consider a symplectic compact manifold (M, ω) and write n_M (resp. n'_M) for the minimum number of critical points that any smooth function (resp. Morse function) defined on M must have. If F is the time-one map of a family $(F_t)_{t\in\mathbb{R}}$ defined by a 1-periodic time dependent Hamiltonian vector field, we say that a fixed point z is *contractible* if the *trajectory* $\gamma_z : t \mapsto F_t(z)$, defined on $[0, 1]$, is a loop homotopic to a point. Arnold's conjecture [1] states that the number of contractible fixed points is minimized by n_M, and that it is minimized by n'_M if every fixed point of F is non-degenerate (such results are obviously true in the case where the vector field is time independent). Now the the minoration by the sum of the Betti numbers is known to be true in the non-degenerate case (Liu, Tian [49], Fukaya, Ono [31]). See [36] for a history of this problem whose first proven case ($M = \mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$) was solved by Conley and Zehnder [15].

Conley conjectured that the number of contractible periodic points is infinite in the case of a torus $\mathbb{T}^{2n}$. The conjecture is true if F is a diffeomorphism with no degenerate fixed points (Salamon–Zehnder [53]). We will explain in Section 5 why Conley's conjecture is true if $M = \mathbb{T}^2$ and, more generally, if M is a closed surface of genus ≥ 1. The key result is an equivariant foliated version of the Brouwer Plane Translation Theorem, which will be stated in Section 4. Roughly speaking, it asserts that if $(F_t)_{t\in[0,1]}$ is an isotopy from the identity to F on a surface M which has no contractible point, there exists a continuous dynamics on M which is "transverse" to the dynamics of F in the following sense: every orbit is "pushed on its left" by the isotopy. Such a result may be applied in the presence of contractible fixed points if one takes out some of them. Suppose that F is the time-one map of a Hamiltonian flow on a symplectic surface M associated to $H : M \to \mathbb{R}$, then an example of a transverse dynamics is the dynamics of the gradient flow of H if we endow M with a Riemannian structure. We will see that such a transverse gradient-like dynamics may be produced even in the time dependent case.

We will recall the classical Brouwer theory of homeomorphisms of the plane in Section 2 and then state equivariant versions in the case of the annulus in Section 3. We will mention some recent results on dynamics of diffeomorphisms and homeomorphisms of surfaces. In particular, in Section 6 we will recall some new results due to Polterovich and to Franks and Handel about group actions on the space of area preserving diffeomorphisms of surfaces.

2. Brouwer's theory of plane homeomorphisms

The following result is due to Brouwer and recent proofs are given in [12], [16] or [32].

Theorem 2.1 ([10]). *Let f be an orientation preserving homeomorphism of the euclidean plane $\mathbb{R}^2$. If f has a periodic point z of period $q \geq 2$, then there is a simple closed curve Γ, disjoint from the fixed point set* $\mathrm{Fix}(f)$*, such that $i(f, \Gamma) = 1$, where* *the index $i(f, \Gamma)$ is the degree of the map*

$$s \mapsto \frac{f(\Gamma(s)) - \Gamma(s)}{\|f(\Gamma(s)) - \Gamma(s)\|},$$

and $s \mapsto \Gamma(s)$ is a parametrization defined on the unit circle S^1.

A homeomorphism f of $\mathbb{R}^2$ is orientation preserving if and only if it is isotopic to the identity; in the case where f is the time-one map of a flow $(f_t)_{t\in\mathbb{R}}$ whose orbits are tangent to a given continuous vector field ξ, the result is obvious. Indeed, if z is a periodic point of f of period $q \geq 2$, the orbit of z (for the flow) is a simple closed curve Γ, invariant by f, which is the union of periodic points of period q. This clearly implies that $i(f, \Gamma) = 1$.

Theorem 2.1 asserts that any fixed point free and orientation preserving homeomorphism of $\mathbb{R}^2$ is periodic point free. In fact its dynamics has no recurrence at all. More precisely, suppose that f is an orientation preserving homeomorphism of $\mathbb{R}^2$ and that f has a *non-wandering* point z which is not fixed (i.e. every neighborhood of z meets one of its iterate). Let us see why the conclusion of Theorem 2.1 is still satisfied. Choose a *free* topological open disk V containing z (i.e. disjoint from its image by f) and write $q \geq 2$ for the smallest positive integer such that $f^q(V) \cap V \neq \emptyset$. One can compose f with a homeomorphism h supported on V to get a map with a periodic point of period q, so that Theorem 2.1 can be applied to $f \circ h$. The map h being supported on a free set, $f \circ h$ and f have the same fixed points. Moreover, h being isotopic to the identity among the homeomorphisms supported on V, one has $i(f \circ h, \Gamma) = i(f, \Gamma)$ for every simple closed curve $\Gamma \subset \mathbb{R}^2 \setminus \mathrm{Fix}(f)$. The foregoing argument may be generalized to get the useful Franks Lemma:

Proposition 2.2 ([21]). *Let f be an orientation preserving homeomorphism of $\mathbb{R}^2$. If there is a periodic sequence $(V_i)_{i\in\mathbb{Z}/q\mathbb{Z}}$ of pairwise disjoint free topological open disks, and a sequence $(n_i)_{i\in\mathbb{Z}/q\mathbb{Z}}$ of positive integers such that $f^{n_i}(V_i) \cap V_{i+1} \neq \emptyset$, then there is a simple closed curve $\Gamma \subset \mathbb{R}^2 \setminus \mathrm{Fix}(f)$ such that $i(f, \Gamma) = 1$.*

It has been known for a long time that Brouwer theory may be applied to the study of homeomorphisms of surfaces. Let us explain for example why every orientation and area preserving homeomorphism on the sphere S^2 has at least two fixed points. The map f being orientation preserving has at least one fixed point z_1 by the Lefschetz formula. The fact that f preserves the area implies that every point is non-wandering; applying Proposition 2.2 to the map restricted to the topological plane $S^2 \setminus \{z_1\}$, one gets a second fixed point $z_2 \neq z_1$. As noticed by Hamilton [34], then by Brown [11], there is another fixed point result which can be deduced from Proposition 2.2: the Cartwright–Littlewood Fixed Point Theorem [14]. This theorem asserts that any non-separating continuum $K \subset \mathbb{R}^2$ which is invariant by an orientation preserving homeomorphism f of $\mathbb{R}^2$ contains a fixed point. Let us recall Brown's argument. If $K \cap \mathrm{Fix}(f) = \emptyset$, then K is included in a connected component W of $\mathbb{R}^2 \setminus \mathrm{Fix}(f)$. One can choose a lift $\tilde{f}$ of $f|_W$ to the universal covering space $\widetilde{W}$ (homeomorphic to $\mathbb{R}^2$) which fixes a given connected component $\widetilde{K}$ of the preimage of K. The set $\widetilde{K}$ being compact, $\tilde{f}$ should contain a non-wandering point (in fact a recurrent point) while being fixed point free.

Let us state now a much more difficult fixed point theorem, due to Handel, which is very useful in the study of homeomorphisms of surfaces (see [24], [25], [50]):

Theorem 2.3 ([35]). *Let f be an orientation preserving homeomorphism of the closed unit disk D and suppose that*

- *there are $n \geq 3$ points z_i, $1 \leq i \leq n$, in* $\mathrm{Int}(D)$ *such that* $\lim_{k\to-\infty} f^k(z_i) = \alpha_i \in \partial D$ *and* $\lim_{k\to+\infty} f^k(z_i) = \omega_i \in \partial D$*;*
- *the $2n$ points α_i and ω_i are distinct;*
- *there is an oriented convex compact polygon in* $\mathrm{Int}(D)$ *whose i-th side joins α_i to ω_i.*

Then f has a fixed point in $\mathrm{Int}(D)$.

Handel's Fixed Point Theorem is usually applied to a lift f to the universal covering space $\mathrm{Int}(D)$ of a homeomorphism F of a hyperbolic surface (such a lift can always be extended to the closed disk). The core of the proof of Handel is a generalization of the Nielsen–Thurston classification of homeomorphisms of a compact surface M to the case where M is the complement in $\mathbb{R}^2$ of finitely many infinite proper orbits. In fact, it is possible to directly prove Theorem 2.3 by showing the existence of a *periodic free disk chain* of $f|_{\mathrm{Int}(D)}$ (i.e. a family of disks satisfying the assumptions of Proposition 2.2) and thus of a simple closed curve Γ of index 1 (see [44]).

We will go on by recalling Brouwer's Plane Translation Theorem. By Schoenflies' Theorem, any proper topological embedding of the real line $\{0\} \times \mathbb{R}$ may be extended to an orientation preserving homeomorphism h of $\mathbb{R}^2$. The open sets $L(\Gamma) = h(]-\infty, 0[\times\mathbb{R})$ and $R(\Gamma) = h(]0, +\infty[\times\mathbb{R})$ are the two connected components of the complement of the *oriented line* $\Gamma = h(\{0\} \times \mathbb{R})$.

Theorem 2.4 ([10]). *If f is a fixed point free and orientation preserving homeomorphism of $\mathbb{R}^2$, then every point belongs to a Brouwer line, that means an oriented line Γ such that $f(\Gamma) \subset L(\Gamma)$ and $f^{-1}(\Gamma) \subset R(\Gamma)$.*

Such a homeomorphism is usually called a *Brouwer homeomorphism*. Observe that $W = \bigcup_{k\in\mathbb{Z}} f^{k+1}(R(\Gamma)) \setminus f^k(R(\Gamma))$ is an invariant open subset homeomorphic to $\mathbb{R}^2$ and that $f|_W$ is conjugate to a non-trivial translation of $\mathbb{R}^2$. Theorem 2.4 asserts that $\mathbb{R}^2$ can be covered by such invariant subsets. The quotient space $\mathbb{R}^2/f$ of orbits of f is a topological surface which is Hausdorff if and only if f is conjugate to a translation. Brouwer homeomorphisms have been studied for a long time. Among the more recent results one may mention the construction of the *oscillating set* by Béguin and Le Roux [3] which is a new topological invariant of Brouwer homeomorphisms. One may also mention the study of *Reeb components* of Brouwer homeomorphisms by Le Roux [48]. Such objects, which generalize the classical Reeb components of foliations may be defined (in a not easy way) in the framework of Brouwer homeomorphisms and results that are true for foliations may be extended to this discrete case. One may also recall the following results about the topology of the space of Brouwer homeomorphisms when equipped with the compact-open topology: it is arcwise connected and locally contractible [8], more precisely the set of non-trivial affine translations is a strong deformation retract [46].

In the case where $f = f_1$ is the time-one map of a flow $(f_t)_{t\in\mathbb{R}}$ whose orbits are tangent to a continuous vector field ξ, Theorem 2.4 is also obvious. Indeed one may find a complete C^1 vector field η such that $\eta(z)\wedge\xi(z) > 0$ for every $z \in \mathbb{R}^2$ (where $\wedge$ is the usual exterior product on $\mathbb{R}^2$). Every orbit Γ of η is a line by the Poincaré–Bendixson Theorem; it is a Brouwer line because ξ points on Γ from the right to the left. Note that the plane is foliated and not only covered by Brouwer lines. The proof of the Brouwer Plane Translation Theorem is much harder. In all known proofs (see [10], [23], [32], [45]) a non-recurrence lemma, a variation of Proposition 2.2, is needed. We will conclude this section by explaining the ideas of the proofs given in [45] and in Sauzet's thesis [54] as it will be the starting point of the proofs of the foliated versions that we will explain later.

A *brick decomposition* of $\mathbb{R}^2$ is given by a one dimensional stratified set Σ (the *skeleton* of the decomposition) with a zero dimensional submanifold V such that any vertex $v \in V$ is locally the extremity of exactly three edges. A *brick* is the closure of a connected component of $\mathbb{R}^2 \setminus \Sigma$. If f is a fixed point free and orientation preserving homeomorphism of $\mathbb{R}^2$, one can construct a *maximal free decomposition*: it is a brick decomposition with free bricks such that the union of two adjacent bricks is no more free. Moreover if $z \in \mathbb{R}^2$ is a given point, one may suppose that $z \in \Sigma$. Let us write B for the set of bricks. A slightly stronger version of Proposition 2.2, due to Guillou and Le Roux [47] asserts that there is no closed chain of bricks of B. This implies that the relation

$$b\mathcal{R}b' \iff f(b) \cap b' \neq \emptyset$$

generates by transitivity an order $\leq$ on B. The decomposition being maximal, two adjacent bricks are comparable. In fact, it appears that for every brick b, the union of bricks $b' > b$ adjacent to b is non-empty and connected, as is the union of adjacent bricks $b' < b$. This implies that $b_{\geq} = \bigcup_{b'\geq b} b'$ is a connected closed subset satisfying $f(b_{\geq}) \subset \mathrm{Int}(b_{\geq})$. The fact that we are working with bricks implies that the frontier of $b_{\geq}$ is a one dimensional manifold; the inclusion $f(b_{\geq}) \subset \mathrm{Int}(b_{\geq})$ implies that every component of this frontier is a Brouwer line. One may cover the skeleton by Brouwer lines because $\Sigma = \bigcup_{b\in B} \partial b_{\geq}$.

Free cellular decompositions appear already in [32] and are explicitely constructed in a paper of Flucher [20] about a topological version of Conley–Zehnder theorem for $\mathbb{T}^2$. The trick to consider bricks to simplify the proofs was suggested by Guillou. Brick decompositions have been studied in detail in Sauzet's thesis [54] and have been used in some articles ([4], [9], [47]). In [47], Le Roux gives a very precise description of the dynamics of a homeomorphism F of a surface in the neighborhood of an isolated fixed point such that $i(F, z) \neq 1$. In [9], Bonino states the following result about any orientation reversing homeomorphim f of $\mathbb{R}^2$: if f has no periodic point of period 2, it has no periodic point of period ≥ 2 and the complement of the fixed point set may be covered by invariant open subsets, where f is conjugate either to the map $(x, y) \mapsto (x+1, -y)$ or to the map $(x, y) \mapsto \frac{1}{2}(x, -y)$.

3. Equivariant versions of Brouwer's theory

Let us recall the classical Poincaré–Birkhoff Theorem which is the starting point of Arnold's conjecture:

Theorem 3.1 ([6]). *Let F be an area preserving homeomorphism of the annulus $\mathbb{T}^1 \times [0,1]$ isotopic to the identity and let f be a given lift to the universal covering space $\mathbb{R} \times [0,1]$. Denote by $p_1 : \mathbb{R} \times [0,1] \to \mathbb{R}$ the first projection and suppose that $p_1(f(x,0)) < x < p_1(f(x,1))$ for every $x \in \mathbb{R}$. Then f has at least two fixed points which project in different points of $\mathbb{T}^1 \times [0,1]$.*

Soon after the original proof of Birkhoff, it was noticed (see Birkhoff [7], Kerékjártó [37]) that the existence of one fixed point could be deduced by replacing the area preserving assumption with the following *intersection property*: any essential (i.e. not null-homotopic) simple closed curve of $\mathbb{T}^1 \times [0,1]$ meets its image by F. Kerékjártó obtained the result as a consequence of Brouwer's Plane Translation Theorem. Suppose that F and a given lift f satisfy the assumptions of Theorem 3.1 but the area condition, and that f is fixed point free. Then one can extend nicely f to the whole plane in such a way that a Brouwer line may be constructed on $\mathbb{R}\times]0,1[$ that is a lift of a simple closed curve of $\mathbb{T}^1\times]0,1[$ (see [32] for a modern explanation). More recently, Guillou [33] and Sauzet [54] gave a proof of the following "equivariant" version of the Brouwer Plane Translation Theorem:

Theorem 3.2. *Let F be a homeomorphism of $\mathbb{T}^1 \times \mathbb{R}$ isotopic to the identity and let f be a given lift to the universal covering space $\mathbb{R}^2$ that is fixed point free. Then*

- *either there is an essential simple closed curve Γ of $\mathbb{T}^1 \times \mathbb{R}$ such that $F(\Gamma) \cap \Gamma = \emptyset$,*
- *or there is a topological line joining the two ends of $\mathbb{T}^1 \times \mathbb{R}$ that is lifted to $\mathbb{R}^2$ by Brouwer lines of f.*

Franks [21] gave a different method to deduce the Poincaré–Birkhoff Theorem from Brouwer theory by using Proposition 2.2. Suppose that the assumptions of Theorem 3.1 are satisfied and moreover that the number m of fixed points of F which are lifted to fixed points of f is finite. Then one can construct a closed chain of free disks of $f|_{\mathbb{R}\times]0,1[}$, which implies that there is a simple closed curve $\Gamma \subset \mathbb{R}\times]0,1[$ such that $i(f, \Gamma) = 1$. The Lefschetz–Nielsen formula implies that $m \geq 2$. This argument has been used by Franks to give many generalizations of the Poincaré–Birkhoff Theorem, including results on the torus [22]. We will state below such a result which is implicitly contained in Franks' papers and which can also be deduced directly from Theorem 3.2 (see [5]).

Under the hypothesis of the Poincaré–Birkhoff Theorem, F admits periodic points of arbitrarily large period. Indeed, if ρ_- and ρ_+ are the Poincaré rotation numbers defined respectively on $\mathbb{T}^1 \times \{0\}$ and $\mathbb{T}^1 \times \{1\}$, one has $\rho_- < 0 < \rho_+$. For every rational number $p/q \in]\rho_-, \rho_+[$ written in an irreducible way, one may apply again the Poincaré–Birkhoff Theorem to F^q and its lift $T^{-p} \circ f^q$ (where $T(x, y) = (x+1, y)$). One gets a fixed point of $T^{-p} \circ f^q$ which projects onto a periodic point of F of period q. Moreover, the theory of homeomorphisms of the circle gives us such a periodic point if one of the numbers ρ_- or ρ_+ is equal to p/q. Let us give now a more general statement. Let us denote by $\mathbb{A}$ one of the annulus $\mathbb{T}^1 \times [0, 1]$ or $\mathbb{T}^1 \times]0, 1[$ and by $\tilde{\mathbb{A}}$ its universal lift. Write $\pi : \tilde{\mathbb{A}} \to \mathbb{A}$ for the covering projection and $T : (x, y) \mapsto (x + 1, y)$ for the fundamental covering automorphism. Consider a homeomorphism F of $\mathbb{A}$ isotopic to the identity and a lift f to $\tilde{\mathbb{A}}$. Suppose that $z \in \mathbb{A}$ is a positively recurrent point of F and that $\tilde{z} \in \tilde{\mathbb{A}}$ is a preimage of z. For every sequence $(F^{q_k}(z))_{k\geq 0}$ that converges to z, there exists a sequence $(p_k)_{k\geq 0}$ in $\mathbb{Z}$ such that $(T^{-p_k} \circ f^{q_k}(\tilde{z}))_{k\geq 0}$ converges to $\tilde{z}$. The sequence $(p_k)_{k\geq 0}$ is uniquely defined up to a finite number of terms and does not depend on $\tilde{z}$. Let us say that z has a *rotation number* ρ if, for every sequence $(F^{q_k}(z))_{k\geq 0}$ that converges to z, the sequence $(p_k/q_k)_{k\geq 0}$ converges to ρ. Another choice of lift f changes the rotation number by adding an integer.

Proposition 3.3. *Let F be a homeomorphism of $\mathbb{A}$ isotopic to the identity and f a given lift to $\tilde{\mathbb{A}}$. We suppose that*

- *there is a positively recurrent point z_- of rotation number ρ_-;*
- *there is a positively recurrent point z_+ of rotation number $\rho_+ > \rho_-$;*
- *every essential simple closed curve in $\mathbb{A}$ meets its image by F.*

Then for every rational number $p/q \in]\rho_-, \rho_+[$ written in an irreducible way, there is a periodic point z of period q and rotation number p/q.

In the case where F is area preserving, the intersection property is satisfied and Proposition 3.3 may be applied. More can be said in that case ([5], [24], [39]). First, one can prove the existence of a periodic point of period q and rotation number p/q if there is a positively recurrent point of rotation number p/q. Moreover one can prove the existence of a non-trivial interval of rational rotation numbers if there is a positively recurrent point that has no rotation number. As a consequence, the unique case where such a homeomorphism has no periodic point is the case where there exists an irrational number ρ such that every positively recurrent point (thus almost every point) has a rotation number equal to ρ. Such a map is usually called an irrational *pseudo-rotation*.

One may notice the following recent result on irrational pseudo-rotations, previously stated by Kwapisz [38] in the context of the torus, revisited by Béguin, Crovisier, Le Roux, Patou [4] in the case of a closed annulus and extended by Béguin, Crovisier, Le Roux [5] in the case of an open annulus:

For any convergent p/q of ρ, there exists a simple arc γ joining the two ends of the annulus such that the iterates of γ, $F(\gamma), \dots, F^q(\gamma)$ are pairwise disjoint and cyclically ordered as the iterates of a vertical under a rigid rotation of angle ρ.

As a consequence ([4], [5]) one has the following:

The rigid rotation of angle ρ is in the closure of the conjugacy class of the pseudo-rotation.

Note that one does not know if an irrational pseudo-rotation of rotation number ρ is in the closure of the conjugacy class of a rigid rotation of angle ρ.

It has been known for a long time that the dynamics of an irrational pseudo-rotation may be strongly different from the dynamics of a rigid rotation. Anosov and Katok [2] gave an example of a smooth (C^∞) irrational pseudo-rotation on the closed annulus which is weakly mixing (and therefore ergodic) relatively to the Lebesgue measure. Many other pathological examples may be constructed as explained by Fayad and Katok in [17]. All these examples are constructed as a limit of diffeomorphisms which are smoothly conjugate to rigid rotations of rational angle. The rotation number is always a Liouville number. In fact, Fayad and Saprykina [18] proved that every Liouville number is the rotation number of a weakly mixing smooth pseudo-rotation on $\mathbb{T}^1 \times [0, 1]$. Such examples do not exist for Diophantine numbers. Indeed, an unpublished result of Herman states that for a smooth diffeomorphism of $\mathbb{T}^1 \times [0, 1]$, the circle $\mathbb{T}^1 \times \{1\}$ is accumulated by a set of positive measure of invariant curves of F if the rotation number induced on $\mathbb{T}^1 \times \{1\}$ is Diophantine. Let us conclude this section by recalling the following old conjecture of Birkhoff, still unsolved, stating that an irrational pseudo-rotation on the closed annulus which is real analytic must be conjugate to a rigid rotation.

4. Foliated versions of Brouwer's Plane Translation Theorem

As noticed at the end of Section 2, if f is a Brouwer homeomorphism which is the time-one map of a flow whose orbits are tangent to a continuous vector field ξ, then $\mathbb{R}^2$ may be foliated and not only covered by Brouwer lines. Suppose now that G is a discrete group of orientation preserving diffeomorphisms acting freely and properly on $\mathbb{R}^2$ and that ξ is G*-invariant* (that means invariant by every element of G). By considering the surface M/G one can construct a G-invariant C^1 vector field η satisfying $\eta(z) \wedge \xi(z) > 0$. One deduces that there exists a G-invariant foliation by Brouwer lines of f. The following result states that this is a general fact:

Theorem 4.1 ([40], [41]). *If f is a Brouwer homeomorphism, there is an oriented topological foliation $\mathcal{F}$ of $\mathbb{R}^2$ whose leaves are Brouwer lines for f. Moreover, if G is a discrete group of orientation preserving homeomorphisms acting freely and properly on $\mathbb{R}^2$ and if f commutes with every $T \in G$, then $\mathcal{F}$ may be chosen G-invariant.*

Let us give the idea of the proof of the first statement. Consider the maximal free brick decomposition introduced in Section 2. Using Zorn's Lemma one can extend the order $\leq$ to get a weaker one $\leq'$ which is a total order. If $C = (C_{\leftarrow}, C_{\rightarrow})$ is a cut of $\leq'$ the sets $\bigcup_{b\in C_{\leftarrow}}$ and $\bigcup_{b\in C_{\rightarrow}}$ have the same frontier and the (oriented) frontier of $\bigcup_{b\in C_{\rightarrow}}$ is a union of Brouwer lines because $f\left(\bigcup_{b\in C_{\rightarrow}}\right) \subset \mathrm{Int}\left(\bigcup_{b\in C_{\rightarrow}}\right)$. The set $\mathcal{B}$ of such lines covers the skeleton and may be written $\mathcal{B} = \bigcup_{e\in E} \mathcal{B}_e$, where E denotes the set of edges and $\mathcal{B}_e$ the set of lines $\Gamma \in \mathcal{B}$ containing e. One can define a partial order $\preceq$ on the set of oriented lines of $\mathbb{R}^2$: $\Gamma \preceq \Gamma'$ if $R(\Gamma) \subset R(\Gamma')$. The fact that $\leq'$ is a total order implies that two lines of $\mathcal{B}$ do not intersect transversally and consequently that $\preceq$ is a total order when restricted to each $\mathcal{B}_e$. The space $\mathcal{B}$, equipped with the topology generated by the $\mathcal{B}_e$, $e \in E$, is not necessarily Hausdorff but each set $\mathcal{B}_e$ is. In fact each $\mathcal{B}_e$ is compact and the restricted topology coincides with the order topology. As an ordered topological space, $\mathcal{B}$ looks like a *lamination* of $\mathbb{R}^2$, that means a closed subset of leaves of a foliation (in fact it will be isomorphic to a lamination of the foliation that we want to construct). There is a natural (but not unique) way to foliate each brick and then to extend $\mathcal{B}$ by constructing a family of Brouwer lines that cover the plane and that do not intersect transversely. By a desingularization process around each vertex of Σ, one can blow up our extended family to get a foliation by Brouwer lines.

The proof of the second statement is much harder. First one considers a free brick decomposition invariant by every $T \in G$ and maximal for these properties. It is not necessarily maximal among all the free bricks decomposition; however there is a natural G-invariant order $\leq$ on B such that

$$f(b) \cap b' \neq \emptyset \Rightarrow b < b'.$$

Moreover, for every brick b, the union of bricks $b' > b$ adjacent to b is non-empty and connected, as is the union of adjacent bricks $b' < b$. As previously one can cover Σ by a G-invariant family of Brouwer lines. To get our G-invariant foliation, one needs to cover Σ by a G-invariant family of Brouwer lines that do not intersect transversally. If G is abelian (that means if $G = \mathbb{Z}$ or $G = \mathbb{Z}^2$) one knows, by a simple set theory argument, that there is a G-invariant total order $\leq'$ weaker than $\leq$: the previous proof is still valid. If G is not abelian, the existence of such an order does not seem so clear. The construction of $\mathcal{B}$ uses more subtle arguments based on the topology of the surface $\mathbb{R}^2/G$.

If $\mathcal{F}$ is an oriented topological foliation of $\mathbb{R}^2$ whose leaves are Brouwer lines of f, it is easy to prove that for every point z there is an arc $\gamma\colon [0,1] \to \mathbb{R}^2$ joining z to $f(z)$ that is *positively transverse* to $\mathcal{F}$. That means that γ intersects transversely each leaf that it meets, and locally from the right to the left. One deduces immediately (by lifting the isotopy $(F_t)_{t\in[0,1]}$ to an isotopy $(f_t)_{t\in[0,1]}$ from the identity and applying Theorem 4.1 to $f = f_1$):

Corollary 4.2. *Let $(F_t)_{t\in[0,1]}$ be an isotopy from the identity to F on an oriented surface M. Suppose that F has no contractible fixed point. Then there exists a* *topological foliation $\mathcal{F}$ on M that is dynamically transverse to the isotopy: the trajectory $\gamma_z : t \mapsto F_t(z)$ of every point is homotopic, relatively to the extremities, to an arc that is positively transverse to $\mathcal{F}$.*

This result belongs to the category of statements that are obviously true when F is the time-one map of a flow and that can be extended to the case where F is the time-one map of an isotopy from the identity. Let us give now a result that does not. One could ask similarly if there exists a foliation by invariant lines for a Brouwer homeomorphism, as it is true in the case of the time-one map of a flow. The answer is no, there exist Brouwer homeomorphisms without any invariant line (Brown, Slaminka, Transue [13]). Observe that in the case of a flow the foliation by invariant lines was explicitely and uniquely defined. In contrast to this, there are many choices of foliations by Brouwer lines and none of them is canonical.

The only closed surface M where Corollary 4.2 can be applied is the torus $M = \mathbb{T}^2$. Indeed, the Lefschetz–Nielsen formula implies the existence of a contractible fixed point for any homeomorphism isotopic to the identity on an oriented closed surface of genus $\neq 1$. In the case of a torus, a stronger hypothesis on the isotopy will imply additional properties of the foliation. Write f for the natural lift of F to $\mathbb{R}^2$ defined by the isotopy and recall the definition of the *rotation set* $R(f)$ whose origin goes back to Schwartzman [55]. The map $f - \mathrm{Id}_{\mathbb{R}^2}$ is invariant by the integer translations and lifts a continuous function $\psi : \mathbb{T}^2 \to \mathbb{R}^2$. For every Borel probability measure which is invariant by F, one may define the *rotation vector* $\rho(\mu) = \int_{\mathbb{T}^2} \psi \, d\mu \in \mathbb{R}^2 \approx H_1(\mathbb{T}^2, \mathbb{R})$ and the set $R(f)$ of rotation vectors of all invariant probability measures. The set $R(f)$ is a non-empty convex compact subset of $H_1(\mathbb{T}^2, \mathbb{R})$. If one supposes that $0 \notin R(f)$ (which of course implies that f is fixed point free) one can find cohomology classes $\kappa \in H^1(\mathbb{T}^2, \mathbb{R})$ that are positive on $R(f)$. One has the following:

Theorem 4.3 ([43]). *Let F be a homeomorphism of $\mathbb{T}^2$ isotopic to the identity and let f be a lift of F to $\mathbb{R}^2$. Suppose that $\kappa \in H^1(\mathbb{T}^2, \mathbb{R})$ is positive on the rotation set $R(f)$. Then there is a non-vanishing smooth closed 1-form ω whose cohomology class is κ, and such that $H(f(z)) - H(z) > 0$ if H is a primitive of the lifted form on $\mathbb{R}^2$.*

The level curves of H define a foliation of $\mathbb{R}^2$ by Brouwer lines of f. It projects onto a foliation diffeomorphic to a linear one, the leaves are closed if κ is a rational class, they are dense if not. One may ask if a similar statement occurs in higher dimension. Let $(F_t)_{t\in[0,1]}$ be an isotopy from the identity on a compact manifold M and write $\gamma_z : t \to F_t(z)$ for the trajectory of any point z. Let μ be a Borel probability measure invariant by F. If ω is a smooth closed 1-form, the integral $\int_M \left(\int_{\gamma_z} \omega\right) d\mu(z)$ is well defined and vanishes when ω is exact. As it depends linearly on the cohomology class $[\omega]$ of ω, one may find $\rho(\mu) \in H_1(M, \mathbb{R})$ such that

$$\int_M \left(\int_{\gamma_z} \omega\right) d\mu(z) = \langle [\omega], \rho(\mu) \rangle.$$

The rotation set of the isotopy is the set of rotation vectors $\rho(\mu)$ of invariant probability measures. Here again it is a non-empty convex compact subset of $H_1(M, \mathbb{R})$. Suppose now that $\kappa \in H^1(M, \mathbb{R})$ is positive on the rotation set of the isotopy:

Does there exist a non-vanishing smooth 1*-form* ω *such that* $[\omega] = \kappa$ *and* $\int_{\gamma_z} \omega > 0$ *for every* $z \in M$?

The answer is yes if F is the time-one map of a flow $(F_t)_{t\in\mathbb{R}}$ induced by a smooth vector field ξ (see Fried [30] or Schwartzman [55]). More precisely, ω may be chosen such that $\langle \omega(z), \xi(z)\rangle > 0$ for every $z \in M$. Fried's proof may be adapted in the discrete case to find a smooth closed 1-form ω such that $[\omega] = \kappa$ and $\int_{\gamma_z} \omega > 0$ for every $z \in M$. The problem is that ω can vanish. In the case of a time-one map of a flow, if the rotation set does not contain zero, κ may be chosen in $H^1(M, \mathbb{Z})$ and ω will be written $\omega = dH$ where $H : M \to \mathbb{T}^1$ is a submersion. Consequently M fibers over $\mathbb{T}^1$. Therefore one may naturally ask:

Suppose that on a given compact manifold M one may find an isotopy from the identity whose rotation set does not contain 0*, does the manifold necessarily fiber over* $\mathbb{T}^1$?

Theorem 4.3 gives us an example where a dynamical assumption on an isotopy implies dynamical properties of some foliation dynamically transverse to the isotopy. In many situations such an assumption will imply dynamical properties of every foliation dynamically transverse. This is the fundamental fact that will permit us to apply Theorem 4.1 and its corollary to the study of homeomorphisms of surfaces. We will conclude this section by an example: a short proof of Proposition 3.3. We will give first two useful statements which illustrate how conservative assumptions satisfied by a homeomorphism can be transposed to dissipative properties of a dynamically transverse foliation. Suppose that $(F_t)_{t\in[0,1]}$ is an isotopy from the identity to F without contractible fixed point on a surface M and that $\mathcal{F}$ is a foliation dynamically transverse. This implies that for every point $z' \in M$ and for every $k \geq 1$ one may find an arc joining z' to $F^k(z')$ that is positively transverse to $\mathcal{F}$. It is easy to prove that this arc may be perturbed into a loop positively transverse to $\mathcal{F}$ if the extremities z' and $F^k(z')$ are sufficiently close to a previously given point z. Hence the following holds:

For every non-wandering point z, there is a loop based on z that is positively transverse to $\mathcal{F}$.

Fix now a point z and define the set W of points $z' \in M$ which can be joined by an arc from z that is positively transverse to $\mathcal{F}$. It may be noticed that $F(\overline{W}) \subset \mathrm{Int}(W)$. Hence the next assertion is true:

If every point is non-wandering, then for every points z and z'*, there is an arc joining z to* z' *that is positively transverse to* $\mathcal{F}$.

Let us now prove Proposition 3.3. Suppose for example that $\rho_- < 0 < \rho_+$ and that $\mathbb{A}$ is open. We want to prove that the intersection property is not satisfied if f has no fixed point. In this case we can construct an oriented foliation $\mathcal{F}$ on $\mathbb{A}$ which is lifted to $\tilde{\mathbb{A}}$ into a foliation by Brouwer lines of f. The points $z_-$and z_+ being recurrent, there are loops Γ_- and Γ_+ based respectively on z_- and z_+ that are positively transverse to $\mathcal{F}$. Write $[\Gamma] \in H_1(\mathbb{A}, \mathbb{Z})$ for the homology class of any loop Γ and consider the generator $[\Gamma_0]$ of the loop $\Gamma_0 : t \mapsto (t + \mathbb{Z}, 1/2)$ defined on $[0, 1]$. The fact that $\rho_- < 0$ implies that Γ_- may be chosen such that $[\Gamma_-] = n_-[\Gamma_0]$ where $n_- < 0$. Similarly one may suppose that $[\Gamma_+] = n_+[\Gamma_0]$ where $n_+ > 0$. Using the fact that $\mathcal{F}$ is a non-singular foliation, it is straightforward to prove the following:

– the loops Γ_- and Γ_+ are disjoint;

– there is a unique relatively compact annular component U of $\mathbb{A} \setminus (\Gamma_- \cup \Gamma_+)$;

– the frontier of U is the union of two simple essential loops positively transverse to $\mathcal{F}$;

– the leaves on ∂U are all leaving U or all entering in U.

The Poincaré–Bendixson Theorem implies the existence of a closed leaf inside U. This leaf does not meet its image by F because it is lifted into a Brouwer line of f.

5. Hamiltonian homeomorphisms of surfaces

Let us say that a homeomorphism F on an oriented closed surface, time-one map of an isotopy from the identity $(F_t)_{t\in[0,1]}$, is *Hamiltonian* if it preserves a probability measure μ whose support is M and whose rotation vector is 0. The classical example is obtained when M is endowed with a symplectic structure ω and when the isotopy is defined by a time dependent Hamiltonian vector field. The measure is nothing but the normalized measure induced by the volume form ω. Let us give another example. Consider an irrational pseudo-rotation F on $\mathbb{T}^1 \times]0, 1[$ and extend F to the end compactification of the annulus. One gets a Hamiltonian homeomorphism on the sphere that has no periodic points but the two fixed ends. As we will see in this section, extended irrational pseudo-rotations are the only examples, up to conjugacy, of Hamiltonian homeomorphisms having finitely many periodic points.

It was shown by Franks [25] that a Hamiltonian homeomorphism on S^2 which has at least three fixed points admits infinitely many periodic points. More recently Franks and Handel [26] proved that a non-trivial Hamiltonian diffeomorphism of a surface of positive genus admits periodic points of arbitrarily large periods (and that this is also the case on a sphere if F has at least three fixed points). Their arguments are mainly of topological nature. The differentiability condition prevents the dynamics to be too wild in a neighborhood of a non-isolated fixed point. For each connected component U of the complement of the fixed point set, they construct a normal form of the restriction map $F|_U$ in the sense of Thurston–Nielsen's theory of homeomorphisms of surfaces like it is usually done for a surface of finite type. There are three cases to look at and in each case periodic orbits may be found for different reasons, the case where there exists at least one pseudo-Anosov component, the case where there is a twist condition in a reducing annulus, the case where the map is isotopic to the identity. The last case is the most difficult one and subtle geometric arguments that already appeared in [35] are needed.

We will state now a more general result, which gives a positive answer to Conley's conjecture in the case of surfaces:

Theorem 5.1 ([41], [42]). *Suppose that F is a Hamiltonian time-one map of an isotopy from the identity $(F_t)_{t\in[0,1]}$ on a compact oriented surface M of genus $g \geq 1$.*

i) *If $F \neq \mathrm{Id}_M$, there are periodic points of arbitrarily large period.*

ii) *If the set of contractible fixed points is contained in a disk of M, there are contractible periodic points of arbitrarily large period.*

Moreover we have a similar result in the case where M is a sphere if we suppose that F has at least three fixed points.

Let us explain first what happens when F is the time-one map of a time independent Hamiltonian flow associated to a function $H: M \to \mathbb{R}$ on a surface of genus ≥ 1. Let us suppose that there are finitely many critical points of H (there are at least three). The minimum of H corresponds to a contractible fixed point z_0. This point is surrounded by invariant curves which are level curves of H. The map F is conjugate to a rotation on each curve. Thus one gets a foliated open annulus with one end corresponding to z_0 and one "critical" end which does not correspond to a point (because M is not a sphere) but to a degenerate curve containing a critical point of H. The rotation number of F on each curve (which is a well defined real number) depends continuously of the curve, never vanishes and tends to zero when the curve tends to the critical level. This implies that the rotation numbers take their values onto a non-trivial interval. One concludes that there are contractible periodic points of arbitrarily large period.

In the case where F is the time-one map of a time dependent Hamiltonian flow, Floer [19] and Sikorav [56] proved that F has at least three contractible fixed points, giving a positive answer to Arnold's conjecture for surfaces. In symplectic geometry contractible fixed points of Hamiltonian isotopies are usually found by studying the dynamics of the gradient flow of a function $\mathcal{H}$ defined on an infinite dimensional space (space of loops) or on a high dimensional space (if one uses generating functions) whose critical points are in bijection with contractible fixed points. Franks [24] gave a purely topological proof of the existence of three contractible fixed points for a Hamiltonian diffeomorphism making use of Handel's Fixed Point Theorem, the proof of which was extended by Matsumoto [50] to the case of Hamiltonian homeomorphisms. The fundamental idea in the proof of Theorem 5.1 is to make a link between the symplectic and the topological methods by producing a "singular" dynamically transverse foliation and by proving that its dynamics is "gradient-like". This will permit us first to find again Matsumoto's result, then to produce a topological "twist property". Such a property is easy to prove if F is a diffeomorphism with no degenerate fixed points. We will give here some ideas of the proof of assertion ii) of Theorem 5.1. We will begin by the simplest case where the set $\mathrm{Fix}(F)_{\mathrm{cont}}$ of contractible fixed point is finite.

Case where $M = S^2$ and $\sharp\mathrm{Fix}(F)_{\mathrm{cont}} < +\infty$. Here $\mathrm{Fix}(F)_{\mathrm{cont}}$ coincides with the set $\mathrm{Fix}(F)$ of fixed points. We suppose that F preserves a probability measure μ with total support and that $3 \leq \sharp\mathrm{Fix}(F) < +\infty$. We want to prove that F has periodic points of arbitrarily large period. Let us say that $Z \subset \mathrm{Fix}(F)$ is *unlinked* if F is isotopic to the identity relatively to Z. This is always the case if $\sharp Z \leq 3$. As $\mathrm{Fix}(F)$ is supposed to be finite, one can find a *maximal* (for the inclusion) unlinked set Z and one knows that $\sharp Z \geq 3$. Fix an isotopy $(F_t)_{t\in[0,1]}$ such that $F_t(z) = z$ for every $z \in Z$ and every $t \in [0, 1]$, and look at the restricted isotopy to $N = S^2 \setminus Z$. It is standard to prove that $(F_t|_N)_{t\in[0,1]}$ has no contractible point, by maximality of Z. By Corollary 4.2, one may construct a foliation $\mathcal{F}$ on N which is dynamically transverse to the isotopy. As we suppose that F preserves μ we know that every point is non-wandering, which implies that every point belongs to a loop that is positively transverse to $\mathcal{F}$. This clearly implies that $\mathcal{F}$ has no closed leaf and more generally has only wandering leaves. In fact the dynamics of $\mathcal{F}$ is easy to understand:

- *any leaf λ joins a point $\alpha(\lambda) \in Z$ to a different point $\omega(\lambda) \in Z$;*
- *there is no sequence of leaves $(\lambda_i)_{i\in\mathbb{Z}/p\mathbb{Z}}$ such that $\omega(\lambda_i) = \alpha(\lambda_{i+1})$ for any $i \in \mathbb{Z}/p\mathbb{Z}$.*

Fix a leaf λ and consider the annulus $A = S^2 \setminus (\alpha(\lambda) \cup \omega(\lambda))$. The isotopy $(F_t|_A)_{t\in[0,1]}$ may be lifted to the universal covering space $\tilde{A}$ of A into an isotopy $(f_t)_{t\in[0,1]}$ from the identity. We will apply Proposition 3.3 by finding two positively recurrent points with different rotation numbers. The map $f = f_1$ clearly fixes every point of the preimage of $Z \setminus (\alpha(\lambda) \cup \omega(\lambda))$, which implies that the rotation vector of any point of $Z \setminus (\alpha(\lambda) \cup \omega(\lambda))$ is 0. The foliation $\mathcal{F}$ is lifted to a foliation on the preimage $\widetilde{N}$ of N which is dynamically transverse to the isotopy $(f_t|_{\widetilde{N}})_{t\in[0,1]}$. Any lift of λ is a Brouwer line of f because λ joins the two ends of the annulus. It is not difficult, using classical arguments of Ergodic Theory (and in particular the Birkhoff Ergodic Theorem), to prove that F has positively recurrent points whose rotation number is $\neq 0$ (this is the case for almost every point that has a preimage between a given lift $\tilde{\lambda}$ and its image by f).

Case where $g \geq 1$ and $\sharp\mathrm{Fix}(F)_{\mathrm{cont}} < +\infty$. Here again suppose that the set $\mathrm{Fix}_{\mathrm{cont}}(F)$ of contractible fixed points is finite and say that $Z \subset \mathrm{Fix}_{\mathrm{cont}}(F)$ is unlinked if there is an isotopy $(F_t)_{t\in[0,1]}$ (homotopic to the one given by hypothesis) such that $F_t(z) = z$ for every $z \in Z$ and every $t \in [0, 1]$. Fix a maximal unlinked set Z. Again, there exists a foliation $\mathcal{F}$ on $N = M \setminus Z$ which is dynamically transverse to the isotopy $(F_t|_N)_{t\in[0,1]}$, and we would like to understand the dynamics of $\mathcal{F}$. As we suppose that F preserves μ we already know that every point belongs to a loop positively transverse to $\mathcal{F}$. The fact that the rotation of μ is zero implies a stronger result:

For every $\upsilon \in H_1(M, \mathbb{Z})$ and every $z \in M$ there is a loop $\Gamma \subset N$ positively transverse to $\mathcal{F}$ and based in z such that $[\Gamma] = \upsilon$.

One must prove that the set $C(z) \subset H_1(M, \mathbb{Z})$ of homology classes of loops in N based in z and positively transverse to $\mathcal{F}$, which is stable by addition, is the whole group $H_1(M,\mathbb{Z})$. The nullity of the rotation vector of μ implies that every class $\kappa \in H^1(M,\mathbb{R})$ takes different signs on $C(z)$ and therefore that the convex hull in $H_1(M,\mathbb{R})$ of $C(z)$ contains a neighborhood of 0. It becomes easy to prove that $C(z)$ is a subgroup and therefore a lattice of $H_1(M,\mathbb{Z})$. If one now applies the transverse transitivity condition stated in the previous section to a natural finite covering of M, one obtains that $C(z) = H_1(M,\mathbb{Z})$.

It is easy to deduce that there is no closed leaf and more precisely that every leaf is wandering. In fact one can prove that the dynamics of $\mathcal{F}$ is gradient-like. Note first that any loop $\Gamma \subset N$ homologous to zero induces naturally by duality a function $\Lambda_\Gamma\colon M\setminus\Gamma \to \mathbb{Z}$ defined up to a constant, where $\Lambda_\Gamma(z')-\Lambda_\Gamma(z)$ denotes the algebraic intersection number $\Gamma \wedge \Gamma'$ between Γ and any arc Γ' joining z to z'. Observe now that Λ_Γ decreases along the oriented leaves if Γ is positively transverse to $\mathcal{F}$. In other words, the sub-level surfaces of Λ_Γ define a filtration of $\mathcal{F}$. The property stated above permits us to construct a loop Γ homologous to zero and positively transverse to $\mathcal{F}$ which sufficiently "fills" the surface in the following sense:

– *every connected component U of $M \setminus \Gamma$ is the interior of a closed disk of M and contains at most one point of Z;*
– *if there exists a leaf of $\mathcal{F}$ which joins $z \in Z$ to $z' \in Z$, then $\Lambda_\Gamma(z') < \Lambda_\Gamma(z)$.*

Using the Poincaré–Bendixson Theorem, one may deduce first that every leaf meets Γ and then that it joins a point $z \in Z$ to a point $z' \in Z$. In fact, the dynamics of F is trivial inside a component U with no singularity and well understood inside a component that contains a singularity. Such a singularity is necessarily a sink, a source or a generalized saddle point (with $p \geq 1$ attracting sectors alternating with $p \geq 1$ repelling sectors).

An easy consequence of the previous results is the fact that $\sharp Z \geq 3$. Existence of contractible periodic points of arbitrarily large period is much more difficult to get. One wants to generalize the case where F is the time-one map of a Hamiltonian flow associated to a function $H\colon M \to \mathbb{R}$. If M is equipped with a Riemannian metric, the foliation by orbits of the gradient flow of H on the complement of the set Z of critical points is dynamically transverse to the isotopy and the point z_0 where H reaches its minimum is a sink of the foliation. In our more general situation, one will choose a sink of $\mathcal{F}$ and then will prove that there exists periodic points inside the basin of attraction W (for the foliation). The set W has no reason to be invariant by F. However the two following facts

– there exists at least one contractible fixed point in the frontier of W,
– there is a radial foliation on W which is pushed along the isotopy,

give us a weak twist condition. Some plane topology arguments and the use of the discrete Conley index permit us to find periodic points inside W.

Case where $\sharp\mathrm{Fix}(F)_{\mathrm{cont}} = +\infty$. The case where the set of contractible fixed points is infinite is much harder to deal with because it does not seem so easy to find maximal unlinked sets, which are necessary to construct a dynamically transverse foliation. Under the hypothesis ii) of Theorem 5.1, there is a unique component N of $M \setminus \mathrm{Fix}_{\mathrm{cont}}(F)$ such that the inclusion $i: N \to M$ induces an isomorphism between the first groups of homology and this component is fixed. If there is a lift f of $F|_N$ to the universal covering space of N which commutes with every covering transformation, then by Corollary 4.2 a dynamically transverse foliation $\mathcal{F}$ may be constructed. Of course there is no decomposition of the dynamics of $\mathcal{F}$ in elementary pieces as in the finite case. However, the previous arguments may be generalized, even if they are not so easy to get. In the case where such a lift does not exist, we will get contractible periodic orbits of arbitrarily large periods for different reasons, that will be explained in the next section.

There are natural reasons to study carefully homeomorphisms of surfaces of infinite type. Consider a volume form on S^2 and write $\mathrm{Diff}^k_\omega(S^2)$ for the set of C^k diffeomorphisms that preserve ω. Consider $F \in \mathrm{Diff}^k_\omega(S^2)$ and fix a connected component U of $S \setminus \overline{\mathrm{Per}(F)}$. There is an integer q such that $F^q(U) = U$. By Theorem 2.1, one knows that there would be a fixed point of F^q in U if U were a disk, which is not the case. By Franks result stated above [25] it cannot be a hyperbolic surface of finite type. Therefore it is an annulus (and in that case the restricted map is an irrational pseudo-rotation) or a surface of infinite type. One may ask the following:

Can U be a surface of infinite type or should it be necessarily an annulus?

The interest in this question comes from the following: it is not difficult to prove that there is a residual set $\mathcal{G} \subset \mathrm{Diff}^k_\omega(S^2)$ (for the C^k-topology) such that for every $F \in \mathcal{G}$ there are no annulus among the connected components of $S^2 \setminus \overline{\mathrm{Per}(F)}$ (see [29]). A positive answer to the previous question would imply that the periodic orbits are generically dense. What is known is that the union of the stable manifolds of the hyperbolic periodic points is dense [29], a result extended by Xia [57] to any compact surface.

6. On the group of diffeomorphisms of surfaces

Consider a compact Riemannian manifold M. If F is a C^1 diffeomorphism one can define its *growth sequence* $(\Gamma_n(F))_{n\geq 0}$ where

$$\Gamma_n(F) = \max\Big(\max_{z\in M} \|T_z F^n\|,\ \max_{z\in M} \|T_z F^{-n}\|\Big).$$

The growth sequence of a non-trivial diffeomorphism may be bounded. This is the case for a periodic map, a translation on a torus or a rigid rotation on S^2. Even when it is not bounded it may tend to $+\infty$ not very quickly (see Polterovitch, Sodin [52]). The situation is different in the case of area preserving diffeomorphisms of surfaces. More precisely:

Theorem 6.1. *If F is a non-trivial Hamiltonian diffeomorphism of a closed oriented surface of genus ≥ 1, there exists $C > 0$ such that $\Gamma_n(F) \geq Cn$ for every $n \geq 0$.*

Proved by Sikorav and Polterovitch in the special case of the torus, the result was generalized to other surfaces by Polterovich [51] applying a result of Schwarz related to Floer homology. Note that in the case of a surface of genus ≥ 2, the result is still true for any area preserving diffeomorphism isotopic to the identity. Indeed, by the Lefschetz–Nielsen formula, such a map has at least one contractible fixed point. Therefore the diffeomorphism admits two probability measures with different rotation vectors if it is not Hamiltonian. It is not difficult to see that such a property implies that the conclusion of Theorem 6.1 is necessarily true.

Applications of the previous result to actions of higher rank lattices in simple Lie groups on compact manifolds were given in [51], yielding a positive answer, in the special case of surfaces, to a more general conjecture of Zimmer:

Theorem 6.2. *Fix a volume form ω on a closed oriented surface M of genus $g \geq 2$. Then any morphism ψ of $\mathrm{SL}(n, \mathbb{Z})$ in the group $\mathrm{Diff}^{\infty}_{\omega}(M)$ of diffeomorphisms which preserves ω has a finite image if $n \geq 3$.*

Franks and Handel in [27] gave an alternative proof which works in the C^1 case and includes the case $g \leq 1$. The smoothness of F is used in a much weaker way, mainly to construct a Thurston–Nielsen normal form on the complement of the fixed point set. The two important properties satisfied by the group $\mathrm{SL}(n, \mathbb{Z})$, $n \geq 3$, and by any normal subgroup of finite order are the following (the first one is due to Margulis):

- it is almost simple (every normal subgroup is finite or has a finite index);
- it contains a subgroup isomorphic to the group of upper triangular integer valued matrices of order 3 with 1 on the diagonal (the integer Heisenberg group).

More precisely, using algebraic properties of the mapping class group, it is sufficient to study the case where ψ takes its values in the subgroup $\mathrm{Diff}^1_{\omega,*}(M)$ of diffeomorphisms of $\mathrm{Diff}^1_{\omega}(M)$ which are isotopic to the identity. Using the second property, there exist three elements F, G, H in $\mathrm{Im}(\psi)$ such that $[G, H] = F$, $[F, G] = [F, H] = \mathrm{Id}_M$ and such that F is the image of an element of infinite order. To get the theorem it is sufficient to prove that $F = \mathrm{Id}_M$, because this would imply that $\mathrm{Ker}(\psi)$, being an infinite normal subgroup, has a finite index. Note that F is Hamiltonian because it is a commutator and that $F^{n^2} = [G^n, H^n]$. The fact that $F = \mathrm{Id}_M$ will follow from the next result (and the fact that F has periodic orbits if it is not trivial):

Theorem 6.3 ([28]). *Suppose that F is a diffeomorphism of a closed surface M of genus g which satisfies the following distorsion property: it belongs to a finitely generated subgroup of diffeomorphisms isotopic to the identity and there are two sequences n_k and p_k with $p_k = o(n_k)$ and $n_k \to +\infty$ such that F^{n_k} can be written as the product of p_k elements chosen in the (finite) set of generators. Then F is isotopic to the identity relatively to the fixed point set and has no periodic points except the fixed points if $g \geq 2$, if $g = 1$ and $\mathrm{Fix}\, F \neq \emptyset$, or if $g = 0$ and $\sharp\,\mathrm{Fix}\, F \geq 3$.*

The proof uses the Thurston–Nielsen normal form on the complement of the fixed point set explained in Section 5. The distorsion property implies that F is isotopic to the identity relatively to the fixed point set. Every iterate F^k will also satisfy the distorsion property and should be isotopic to the identity relatively to its fixed point set. But this situation cannot occur in case $\mathrm{Fix}(F^k) \neq \mathrm{Fix}(F)$ if $g \geq 2$, if $g = 1$ and $\mathrm{Fix}\, F \neq \emptyset$, or if $g = 0$ and $\sharp\,\mathrm{Fix}\, F \geq 3$.

Let us conclude this article by explaining how to get another interpretation of Theorem 6.1 and Theorem 6.3 with the use of the foliated version of Brouwer's Plane Translation Theorem. We will look at the case of a surface of genus $g \geq 1$ by using the notion of *linking number*. The case of the sphere may be studied in a similar way by using an appropriate notion of linking number.

Suppose that F is the time-one map of an isotopy from the identity $(F_t)_{t\in[0,1]}$ on a closed surface M of genus $g \geq 1$ and lift the isotopy to an isotopy from the identity $(f_t)_{t\in[0,1]}$ on the universal covering space $\widetilde{M}$. One may identify the universal lift $\widetilde{M}$ of M with the complex plane if $g = 1$ or with the Poincaré disk if $g \geq 2$. If z and z' are two fixed points of f, the degree of the map $\xi : S^1 \to S^1$ defined by

$$\xi(e^{2i\pi t}) = \frac{f_t(z) - f_t(z')}{|f_t(z) - f_t(z')|}$$

is called the linking number $I(z, z')$ of z and z'. One course $I(z, z') = 0$ if z' is the image of z by a covering automorphism. There exits a "natural lift" of $f|_{\widetilde{M}\setminus\{z\}}$ to the universal covering space of the annulus $\widetilde{M} \setminus \{z\}$ which fixes the preimages of every image of z by a covering automorphism of $\widetilde{M}$. The linking number $I(z, z')$ is nothing but the rotation number (up to the sign) of the fixed point z' of $f|_{\widetilde{M}\setminus\{z\}}$ for this natural lift. Note that for every integer $n \geq 1$, the linking number of z and z' for f^n is equal to $nI(z, z')$. In the case where $I(z, z') \neq 0$ it is not difficult to deduce that there exists $C > 0$ such that $\Gamma_n(F) \geq Cn$ for every $n \geq 0$ and also that F does not satisfy the distorsion property in the group of diffeomorphisms isotopic to the identity. Observe that if F preserves a probability measure with total support, then $f|_{\widetilde{M}\setminus\{z\}}$ satisfies the intersection property. Therefore, in this case, if f has two fixed points z and z' such that $I(z, z') \neq 0$, it has periodic points with arbitrarily large period which project onto contractible periodic points of F. The next statement permits us to understand why, in the proof of Theorem 5.1, it is sufficient to study the case where the map $F_{|N}$ has a lift to the universal covering space that commutes with the covering transformations.

Proposition 6.4 ([42]). *Let $(F_t)_{t\in[0,1]}$ be an isotopy from the identity to F on a closed surface M of genus $g \geq 1$ and $(f_t)_{t\in[0,1]}$ the lifted isotopy to the universal covering space $\widetilde{M}$ starting from the identity. Suppose that there is a connected component N of $M \setminus \mathrm{Fix}_{\mathrm{cont}}(F)$ such that the inclusion $i : N \to M$ induces an isomorphism between the first groups of homology and that there is no lift of $F|_N$ to the universal covering space of N that commutes with the covering automorphisms. Then there are two fixed points z and z' of $f = f_1$ such that $I(z, z') \neq 0$.*

Let us give the ideas of the proof. By an approximation argument it is sufficient to study the case where $\mathrm{Fix}_{\mathrm{cont}}(F)$ is finite. One considers a maximal unlinked set $Z \subset \mathrm{Fix}_{\mathrm{cont}}(F)$. By hypothesis one knows that $Z \neq \mathrm{Fix}_{\mathrm{cont}}(F)$. One may suppose that our isotopy $(F_t)_{t\in[0,1]}$ fixes every point of Z. We consider a foliation $\mathcal{F}$ on $N' = M \setminus Z$ dynamically transverse to $(F_t|_{N'})_{t\in[0,1]}$ and lift it to a foliation $\widetilde{\mathcal{F}}$ onto the preimage $\widetilde{N}'$ of N' in $\widetilde{M}$. Fix a point $z' \in \mathrm{Fix}(f) \cap \widetilde{N}'$. There is a loop $\Gamma_0 \subset \widetilde{N}'$ based in z' that is positively transverse to $\widetilde{\mathcal{F}}$ and homotopic in $\widetilde{N}'$ to the trajectory Γ_1 of z'. The dual function $\Lambda_{\Gamma_0} : \widetilde{M} \setminus \Gamma_0 \to \mathbb{Z}$ assigning to z the index of Γ_0 relatively to z is zero outside a compact set and takes finitely many values. One may suppose for example that the maximum l^+ of Λ_{Γ_0} is different from zero. The loop Γ_0 being positively transverse to the foliation, it is easy to prove that every component of $\widetilde{M} \setminus \Gamma_0$ where Λ_{Γ_0} takes the value l^+ is the interior of a closed disk whose boundary is a simple loop transverse to the foliation. Therefore, there exists a singularity z inside this component. The loop Γ_1 being homotopic to Γ_0 in $\widetilde{N}$, the index of Γ_1 relatively to z is equal to l^+. This number is nothing but the linking number $I(z, z')$.

The linking number $I(z, z')$ between a fixed point z and a periodic point z' of f may be defined similarly. The previous proof may be adapted to get:

Proposition 6.5 ([42]). *Let $(F_t)_{t\in[0,1]}$ be an isotopy from the identity to F on a closed surface M of genus $g \geq 1$ and $(f_t)_{t\in[0,1]}$ the lifted isotopy to the universal covering space $\widetilde{M}$ starting from the identity. For every periodic point z' of $f = f_1$ that is not fixed, there is a fixed point z of f such that $I(z, z') \neq 0$.*

Let us explain how to deduce Theorem 6.1 from Proposition 6.5. Suppose that F is a non-trivial Hamiltonian diffeomorphism of a closed surface M of genus $g \geq 1$. One can choose a periodic point z' of period ≥ 2. If z' is contractible, then one gets the conclusion of Theorem 6.1 by applying Proposition 6.5. If the rotation vector of z' is non-zero, the conclusion follows easily. It will follow also in the missing case where z' is not contractible but has a rotation vector equal to zero. This is possible only if $g \geq 2$. Identify $\widetilde{M}$ with the Poincaré disk. If $\tilde{z}'$ is a preimage of z' in $\widetilde{M}$, it is not difficult to prove that there exits $C > 0$ such that for every $n \geq 0$ the hyperbolic distance between z' and $f^n(z')$ is minimized by Cn, which implies the validity of the conclusion of Theorem 6.1.

The previous arguments imply that a diffeomorphism which satisfies the assumptions of Theorem 6.3 has no periodic points of period ≥ 2 if $g \geq 2$ or if $g = 1$ and F has a contractible fixed point, and that every fixed point is contractible. One may adapt the arguments of Proposition 6.4 to prove that for every connected component U of $M \setminus \mathrm{Fix}_{\mathrm{cont}}(F)$, the map $F|_U$ has necessarily a lift to the universal covering space which commutes with the covering automorphisms.

The analogs of Theorem 6.2 and Theorem 6.3 for homeomorphisms are unknown. They should be deduced from the (positive) answer to the following open question:

Suppose that f is the lift to the universal covering space of a homeomorphism F isotopic to the identity defined by an isotopy $(F_t)_{t\in[0,1]}$ and that f has two fixed *point z and z' such that $I(z, z') \neq 0$. Does this imply that F does not satisfy the distorsion property in the group of homeomorphisms isotopic to the identity*?

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