

Geometric Quantization on Kähler and Symplectic Manifolds

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Abstract

We explain various results on the asymptotic expansion of the Bergman kernel on Kähler manifolds and also on symplectic manifolds. We also review the “quantization commutes with reduction” phenomenon for a compact Lie group action, and its relation to the Bergman kernel.

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0. Introduction

In the theory of quantization, one attempts to associate to a symplectic manifold (X, ω) a Hilbert space H and a mapping from the space of functions on X into the space of operators on H , and this in a canonical way. The mapping should give some reasonable relationship between the Poisson bracket on the function side and the commutator on the operator side. It is generally acknowledged that there is no canonical way to construct a quantization of X without making use of certain additional structures.

In the theory of the geometric quantization of Kostant and Souriau, (X, ω) is assumed to be prequantizable, that is, there exists a prequantum line bundle (L, h^L, ∇^L) on X (i.e., ω is the first Chern form of L associated with the Hermitian connection ∇^L). Given a compatible almost complex structure J and a Riemannian metric g^{TX} , we can define canonically a Dirac operator D^L acting on $\Omega^{0, \bullet}(X, L)$, the smooth $(0, \bullet)$ -forms on X with coefficients in L .

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Assume that X is compact. Following an observation by Bott, we take, as a quantization of X , $\text{Ind}(D_+^L) = \text{Ker}(D_+^L) - \text{Coker}(D_+^L)$ of $D_+^L := D^L|_{\Omega^{0,\text{even}}}$, which is a formal difference of finite dimensional Hilbert spaces. The virtual dimension of $\text{Ind}(D_+^L)$, which can be computed by the Atiyah-Singer index theorem, does not depend on the choice of the connection and of the metric on L .

For $p \gg 1$, $\text{Ind}(D_+^{L^p}) = \text{Ker}(D_+^{L^p})$ is an ordinary finite dimensional Hilbert space. The Bergman kernel is defined as the integral kernel $P_p(x, x')$ associated with the orthogonal projection P_p from $\Omega^{0,\bullet}(X, L^p)$ onto $\text{Ker}(D_+^{L^p})$. We will show that when $p \rightarrow +\infty$, the Bergman kernel $P_p(x, x')$ has an asymptotic expansion whose coefficients contain interesting geometric informations about X and L . The kind of expansion obtained for the kernel $P_p(x, x')$ also characterizes the Berezin-Toeplitz operators. Their semi-classical limit provides a precise way to relate the classical and quantum observables.

Assume that a compact connected Lie group G acts on X , and that the action lifts to (L, h^L, ∇^L) . Then the quantization of X is a G -virtual representation, and it is interesting to determine the multiplicity of the irreducible representations of G . The Guillemin-Sternberg conjecture “quantization commutes with reduction” gives a precise geometric answer to this problem by using the associated moment map. Here we explain the behavior of the G -invariant part of $P_p(x, x')$ as $p \rightarrow +\infty$, and we relate this behavior to the Guillemin-Sternberg conjecture.

New difficulties appear when the manifold X is no longer supposed to be compact, since in this case $\text{Ind}(D_+^L)$ is not well defined. In her ICM 2006 plenary lecture, Michèle Vergne proposed to replace $\text{Ind}(D_+^L)$ by a certain transversal index introduced by Atiyah, under the natural hypothesis that the moment map is proper, and that the zero-set of the vector field induced by the moment map is compact. She conjectured that “quantization commutes with reduction” still holds in this case.

If (X, ω, J) is a compact Kähler manifold and if L is holomorphic, then for $p \gg 1$, $\text{Ker}(D_+^{L^p})$ is the space of holomorphic sections $H^0(X, L^p)$ of L^p on X . This leads to many applications of the asymptotic expansion of the Bergman kernel in Kähler geometry.

We refer the reader to our book with Marinescu [41] for a comprehensive study of the Bergman kernel and applications, and to the survey by Michèle Vergne [68] on the Guillemin-Sternberg conjecture. One can find more comments, references and motivations in [41] and [68].

This paper is organized as follows. The first two sections are based on our work with Dai, Liu and Marinescu, the last two sections are based on our work with Zhang. In Section 1, we review the definition of Bergman kernel and Berezin-Toeplitz quantization.

In Section 2, we discuss the asymptotic expansion of the Bergman kernel, and also Toeplitz operators.

In Section 3, we examine the corresponding results when a compact Lie group G acts on X and the action lifts to L .

In Section 4, we outline Ma-Zhang’s solution of the Vergne conjecture.

1. Quantization on Symplectic Manifolds

In Section 1.1, we review the basic definitions, and the spectral gap property of the Dirac operator. Then we explain the model example \mathbb{C}^n in Section 1.2.

1.1. Dirac operators and quantization. Let (X, ω) be a compact symplectic manifold of real dimension $2n$ with compatible almost complex structure J , i.e., $\omega(\cdot, J\cdot) > 0$, $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$. We endow X with a Riemannian metric g^{TX} compatible with J , i.e., $g^{TX}(J\cdot, J\cdot) = g^{TX}(\cdot, \cdot)$. Let (E, h^E) be a Hermitian vector bundle on X with Hermitian connection ∇^E and curvature $R^E = (\nabla^E)^2$.

The almost complex structure J induces a splitting of the complexification of the tangent bundle, $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Let $T^{*(0,1)}X$ be the dual space of $T^{(0,1)}X$. For any $v \in T^{(1,0)}X$, let $\bar{v}^* \in T^{*(0,1)}X$ be the metric dual of v , then

$$\mathbf{c}(v) = \sqrt{2}\bar{v}^* \wedge, \quad \mathbf{c}(\bar{v}) = -\sqrt{2}i_{\bar{v}}, \tag{1.1}$$

define the Clifford actions of v, \bar{v} on $\Lambda^{0,\bullet} := \Lambda^\bullet(T^{*(0,1)}X)$, where \wedge and i denote the exterior and interior multiplications respectively.

Consider the Levi-Civita connection ∇^{TX} of (TX, g^{TX}) with associated curvature R^{TX} . Let $\nabla^{T^{(1,0)}X}$ be the connection on $T^{(1,0)}X$ induced by projecting ∇^{TX} ; $\nabla^{T^{(1,0)}X}$ induces the connection ∇^{\det} on $\det(T^{(1,0)}X)$. The Clifford connection ∇^{Cl} on $\Lambda^{0,\bullet}$ is induced canonically by ∇^{TX} and ∇^{\det} (cf. [41, §1.3]). Finally, let $\nabla^{\Lambda^{0,\bullet} \otimes E}$ be the connection on $\Lambda^{0,\bullet} \otimes E$ induced by ∇^{Cl} and ∇^E .

Let dv_X be the Riemannian volume form of (TX, g^{TX}) and $\Omega^{0,\bullet}(X, E)$ be the space of smooth sections of $\Lambda^{0,\bullet} \otimes E$ endowed with the L^2 -norm $\|\cdot\|_{L^2}$ induced by h^E, g^{TX} . Let $\{e_j\}_{j=1}^{2n}$ be an orthonormal frame of (TX, g^{TX}) .

Definition 1.1. The *spin^c Dirac operator* D^E is defined by

$$D^E := \sum_j \mathbf{c}(e_j) \nabla_{e_j}^{\Lambda^{0,\bullet} \otimes E} : \Omega^{0,\bullet}(X, E) \longrightarrow \Omega^{0,\bullet}(X, E), \quad D_{\pm}^E := D^E|_{\Omega^{0,\text{even}}}. \tag{1.2}$$

The operator D^E is a formally self-adjoint, first order elliptic differential operator on $\Omega^{0,\bullet}(X, E)$, which interchanges $\Omega^{0,\text{even}}(X, E)$ and $\Omega^{0,\text{odd}}(X, E)$ (cf. [41, §1.3]).

Thus $\text{Ker}(D_+^E), \text{Ker}(D_-^E)$ are finite dimensional Hilbert spaces and the *quantization space* of E is defined as their formal difference

$$Q(E) := \text{Ind}(D_+^E) := \text{Ker}(D_+^E) - \text{Ker}(D_-^E). \tag{1.3}$$

The Atiyah-Singer index theorem [3, §4.1], [41, Th. 1.3.9] allows us to compute the virtual dimension of $Q(E)$ by using characteristic numbers:

$$\dim Q(E) = \int_X \text{Td}(T^{(1,0)}X) \text{ch}(E), \tag{1.4}$$

where $\text{ch}(\cdot), \text{Td}(\cdot)$ are the Chern character and the Todd class of the corresponding complex vector bundles. In particular, the virtual dimension of $Q(E)$ does not depend on the choice of J, g^{TX} or the metric and connection on E . If $\text{Ker}(D_-^E) = 0$, then the quantization space $Q(E)$ is an ordinary vector space.

We explain now the idea of the geometric quantization introduced by Kostant [33] and Souriau [62]. Let (L, h^L) be a Hermitian line bundle over X endowed with a Hermitian connection ∇^L with curvature $R^L = (\nabla^L)^2$. We assume that (L, h^L, ∇^L) satisfies the *prequantization condition*, that is

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L. \tag{1.5}$$

For $p \in \mathbb{N}$, we denote by $D^{L^p \otimes E}$ the Dirac operator associated to $L^p \otimes E$ with $L^p := L^{\otimes p}$, and set

$$E_p := \Lambda^{0,\bullet} \otimes L^p \otimes E, \quad D_p := D^{L^p \otimes E}, \quad D_{\pm,p} := D_p|_{\Omega^{0,\frac{\text{even}}{\text{odd}}}}. \tag{1.6}$$

Let $L^2(X, E_p)$ be the L^2 -completion of $(\Omega^{0,\bullet}(X, L^p \otimes E), \|\cdot\|_{L^2})$.

The following result is the starting point of the asymptotic expansion results for the Bergman kernel which we describe in the sequel. The proof is based on a direct application of the Lichnerowicz formula for D_p^2 .

Theorem 1.2 (Ma-Marinescu [37, Th. 1.1, 2.5], [41, Th. 1.5.5]). *There exists $C > 0$ such that for any $p \in \mathbb{N}$, the spectrum of D_p^2 satisfies*

$$\text{Spec}(D_p^2) \subset \{0\} \cup [2p\nu_0 - C, +\infty[, \tag{1.7a}$$

$$\text{Ker}(D_{-,p}) = 0 \quad \text{for } p \gg 1, \tag{1.7b}$$

where $\nu_0 = \inf\{R_x^L(u, \bar{u}) : u \in T_x^{(1,0)}X, |u|^2 = 1, x \in X\} > 0$.

Thus for $p \gg 1$, $Q(L^p \otimes E) = \text{Ker}(D_p^2)$ is an ordinary vector space and its dimension is a polynomial in p of degree n given by (1.4). The analogue of Theorem 1.2 in the holomorphic setting was first obtained by Bismut and Vasserot [8, Th. 1.1] by using Demailly’s version of the Bochner-Kodaira-Nakano formula (cf. [41, Th. 1.4.12]). Formula (1.7b) was first established by Borthwick-Urbe [10, Th. 2.3] and Braverman [14, Th. 2.6] by using Melin’s inequality. Mathai-Zhang [46, Th. 1.3] obtained a version of (1.7b) for the proper cocompact group action case by applying the method in [37].

Definition 1.3. The orthogonal projection $P_p : L^2(X, E_p) \rightarrow \text{Ker}(D_p)$ is called the *Bergman projection*. The *Bergman kernel* of D_p is the smooth kernel $P_p(x, x') \in E_{p,x} \otimes E_{p,x'}^*$, $(x, x' \in X)$, of P_p with respect to $dv_X(x')$, i.e., for any $s \in L^2(X, E_p)$, we have

$$(P_p s)(x) = \int_X P_p(x, x')s(x') dv_X(x'). \tag{1.8}$$

For $f \in \mathcal{C}^\infty(X, \text{End}(E))$, set

$$T_{f,p} : L^2(X, E_p) \rightarrow L^2(X, E_p), \quad T_{f,p} = P_p f P_p. \tag{1.9}$$

Here the action of f is the pointwise multiplication by f . The map which associates to $f \in \mathcal{C}^\infty(X, \text{End}(E))$ the family of bounded operators $\{T_{f,p}\}_p$ on $L^2(X, E_p)$ is called the *Berezin-Toeplitz quantization*.

Definition 1.4. A *Toeplitz operator* is a sequence $\{T_p\}_{p \in \mathbb{N}}$ of linear operators $T_p : L^2(X, E_p) \rightarrow L^2(X, E_p)$ satisfying $T_p = P_p T_p P_p$, such that there exists a sequence $g_l \in \mathcal{C}^\infty(X, \text{End}(E))$ such that for all $k \geq 0$, there exists $C_k > 0$ with

$$\left\| T_p - \sum_{l=0}^k T_{g_l,p} p^{-l} \right\| \leq C_k p^{-k-1} \quad \text{for any } p \in \mathbb{N}^*, \tag{1.10}$$

where $\|\cdot\|$ denotes the operator norm on the space of bounded operators. The section g_0 is called the *principal symbol* of $\{T_p\}$.

We express (1.10) symbolically by

$$T_p = \sum_{l=0}^k T_{g_l,p} p^{-l} + \mathcal{O}(p^{-k-1}). \tag{1.11}$$

If (1.10) holds for any $k \in \mathbb{N}$, then we write (1.11) with $k = +\infty$.

The Poisson bracket $\{\cdot, \cdot\}$ on (X, ω) is defined as follows. For $f, g \in \mathcal{C}^\infty(X)$, let $\xi_f \in \mathcal{C}^\infty(X, TX)$ be defined by $2\pi i_{\xi_f} \omega = df$. Then $\{f, g\} := \xi_f(dg)$.

In the spirit of the geometric quantization, (X, ω) represents the classical phase space and the Poisson algebra $(\mathcal{C}^\infty(X), \{\cdot, \cdot\})$ represents the classical observables, while $\text{Ker}(D_p)$ is the quantum space and the linear operators on $\text{Ker}(D_p)$ are the quantum observables. The process $p \rightarrow +\infty$ is called the semi-classical limit, which is a way to relate the classical and quantum observables.

1.2. Bergman kernel on \mathbb{C}^n . Let us consider the canonical real coordinates (Z_1, \dots, Z_{2n}) on \mathbb{R}^{2n} and the complex coordinates (z_1, \dots, z_n) on \mathbb{C}^n . The two sets of coordinates are linked by the relation $z_j = Z_{2j-1} + \sqrt{-1}Z_{2j}$, $j = 1, \dots, n$. We consider the L^2 -norm $\|\cdot\|_{L^2} = \left(\int_{\mathbb{R}^{2n}} |\cdot|^2 dZ\right)^{1/2}$ on the obvious L^2 -space on \mathbb{R}^{2n} , with $dZ = dZ_1 \cdots dZ_{2n}$ the Lebesgue measure. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $z \in \mathbb{C}^n$, put $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

Let $L = \mathbb{C}$ be the trivial holomorphic line bundle on \mathbb{C}^n with the canonical section $\mathbf{1} : \mathbb{C}^n \rightarrow L, z \mapsto (z, 1)$. Let h^L be the metric on L defined by

$$|\mathbf{1}|_{h^L}(z) := e^{-\frac{1}{4} \sum_{j=1}^n a_j |z_j|^2} = \rho(Z) \quad \text{for } z \in \mathbb{C}^n, \tag{1.12}$$

with $a_j > 0$ for $j \in \{1, \dots, n\}$. The space of L^2 -integrable holomorphic sections of L with respect to h^L and dZ is the classical Segal-Bargmann space of L^2 -integrable holomorphic functions with respect to the volume form ρdZ . It is well-known that $\{z^\beta : \beta \in \mathbb{N}^n\}$ forms an orthogonal basis of this space.

To introduce the model operator \mathcal{L} we set:

$$b_i = -2 \frac{\partial}{\partial z_i} + \frac{1}{2} a_i \bar{z}_i, \quad b_i^+ = 2 \frac{\partial}{\partial \bar{z}_i} + \frac{1}{2} a_i z_i, \quad \mathcal{L} = \sum_i b_i b_i^+. \tag{1.13}$$

We can interpret the operator \mathcal{L} in terms of complex geometry. Let $\bar{\partial}^{L^*}$ be the adjoint of the Dolbeault operator $\bar{\partial}^L$ on (L, h^L) over $(\mathbb{C}^n, \frac{\sqrt{-1}}{2} \sum_j dz_j \wedge d\bar{z}_j)$. We have the isometry $\Omega^{0,\bullet}(\mathbb{C}^n, \mathbb{C}) \rightarrow \Omega^{0,\bullet}(\mathbb{C}^n, L)$ given by $\alpha \mapsto \rho^{-1} \alpha$. If $\square^L = \bar{\partial}^{L^*} \bar{\partial}^L + \bar{\partial}^L \bar{\partial}^{L^*}$ denotes the Kodaira Laplacian acting on $\Omega^{0,\bullet}(\mathbb{C}^n, L)$, then $\rho \square^L \rho^{-1} : \Omega^{0,\bullet}(\mathbb{C}^n, \mathbb{C}) \rightarrow \Omega^{0,\bullet}(\mathbb{C}^n, \mathbb{C})$ is given by $\frac{1}{2} \mathcal{L} + \sum_j a_j d\bar{z}^j \wedge i \frac{\partial}{\partial \bar{z}_j}$, and its restriction on functions is $\frac{1}{2} \mathcal{L}$.

The operator \mathcal{L} is the complex analogue of the harmonic oscillator, the operators b, b^+ are creation and annihilation operators respectively. Each eigenspace of \mathcal{L} has infinite dimension, but we can still give an explicit description.

Theorem 1.5 (Ma-Marinescu [38, Th. 1.15], [41, Th. 4.1.20]). *The spectrum of \mathcal{L} on $L^2(\mathbb{R}^{2n})$ is given by*

$$\text{Spec}(\mathcal{L}) = \left\{ 2 \sum_{i=1}^n \alpha_i a_i : \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n \right\} \tag{1.14}$$

and an orthogonal basis of the eigenspace of $\lambda \in \text{Spec}(\mathcal{L})$ is given by

$$B_\lambda = \left\{ b^\alpha \left(z^\beta \exp \left(-\frac{1}{4} \sum_i a_i |z_i|^2 \right) \right) : 2 \sum_i \alpha_i a_i = \lambda, \text{ with } \alpha, \beta \in \mathbb{N}^n \right\} \tag{1.15}$$

where $b^\alpha := b_1^{\alpha_1} \dots b_n^{\alpha_n}$. Moreover, $\cup_\lambda \{B_\lambda : \lambda \in \text{Spec}(\mathcal{L})\}$ forms a complete orthogonal basis of $L^2(\mathbb{R}^{2n})$.

Let $\mathcal{P}(Z, Z')$ be the smooth kernel of \mathcal{P} , which is the orthogonal projection from $(L^2(\mathbb{R}^{2n}), \|\cdot\|_{L^2})$ onto $\text{Ker}(\mathcal{L})$, with respect to dZ' . Then $\mathcal{P}(Z, Z')$ is the classical Bergman kernel on \mathbb{C}^n given by

$$\mathcal{P}(Z, Z') = \prod_{i=1}^n \frac{a_i}{2\pi} \exp \left(-\frac{1}{4} \sum_i a_i (|z_i|^2 + |z'_i|^2 - 2z_i \bar{z}'_i) \right). \tag{1.16}$$

2. Asymptotic Expansion of Toeplitz Operators

The starting point for our work on the asymptotic expansion of the Bergman kernel has been the heat equation proof by Bismut [6] of Demailly’s holomorphic Morse inequalities [21]. For a unified treatment of these two questions, we refer to the book [41]. Here, we give various results on expansions of Bergman kernels, and also on Toeplitz operators.

This Section is organized as follows. In Section 2.1, we give the asymptotic expansion of the Bergman kernel.

In Section 2.2, we describe a characterization of the Toeplitz operators in terms of their asymptotic expansion.

In Section 2.3, we specify the results to the Kähler case.

We will use the notation and assumptions of Section 1.1.

2.1. Asymptotic expansion of Bergman kernel. Let $d^X(x, x')$ be the Riemannian distance between $x, x' \in X$. Let a^X be the injectivity radius of (X, g^{TX}) . We denote by $B^X(x, \varepsilon)$ and $B^{T_x X}(0, \varepsilon)$ the open balls in X and $T_x X$ with centers x and 0 and radius ε , respectively. Then the exponential map $T_x X \ni Z \rightarrow \exp_x^X(Z) \in X$ is a diffeomorphism from $B^{T_x X}(0, \varepsilon)$ onto $B^X(x, \varepsilon)$ for $\varepsilon \leq a^X$. From now on, we identify $B^{T_x X}(0, \varepsilon)$ with $B^X(x, \varepsilon)$ via the exponential map for $\varepsilon \leq a^X$. When a function is calculated using normal coordinates based at x , we will add a subscript x .

We fix $x_0 \in X$. For $Z \in B^{T_{x_0} X}(0, \varepsilon)$, we identify $E_{p,Z}$ with E_{p,x_0} by parallel transport with respect to the connection $\nabla^{E_p} := \nabla^{\Lambda^{0,\bullet} \otimes L^p \otimes E}$ along the curve $\gamma_Z : [0, 1] \ni u \rightarrow uZ$.

Let dv_{TX} be the Riemannian volume form on $(T_{x_0} X, g^{T_{x_0} X})$. There exists a smooth positive function κ_{x_0} on $B^{T_{x_0} X}(0, \varepsilon)$ defined by

$$dv_X(Z) = \kappa_{x_0}(Z)dv_{TX}(Z), \quad \kappa_{x_0}(0) = 1. \tag{2.1}$$

We will identify the 2-form R^L with the Hermitian matrix $\dot{R}^L \in \text{End}(T^{(1,0)}X)$ such that for $W, Y \in T^{(1,0)}X$, $R^L(W, \bar{Y}) = \langle \dot{R}^L W, \bar{Y} \rangle$. We choose an orthonormal basis $\{w_i\}_{i=1}^n$ of $T_{x_0}^{(1,0)}X$ such that

$$\dot{R}^L(x_0) = \text{diag}(a_1(x_0), \dots, a_n(x_0)) \in \text{End}(T_{x_0}^{(1,0)}X) \quad \text{with } a_j(x_0) > 0. \tag{2.2}$$

Then $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \bar{w}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \bar{w}_j)$, $j = 1, \dots, n$, form an orthonormal basis of $T_{x_0} X$. We use the identification $(Z_1, \dots, Z_{2n}) \in \mathbb{R}^{2n} \rightarrow \sum_i Z_i e_i \in T_{x_0} X$. In what follows, we also use the corresponding complex coordinates $z = (z_1, \dots, z_n)$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

Let $\pi : TX \times_X TX \rightarrow X$ be the obvious projection. Let $\{\Theta_p\}_{p \in \mathbb{N}}$ be a sequence of linear operators $\Theta_p : L^2(X, E_p) \rightarrow L^2(X, E_p)$ with smooth kernels $\Theta_p(x, y)$ with respect to $dv_X(y)$. In terms of our trivialization, $\Theta_p(x, y)$ induce smooth sections $\Theta_{p, x_0}(Z, Z')$ of $\pi^*(\text{End}(\Lambda^{0,\bullet} \otimes E))$ over $TX \times_X TX$, with $Z, Z' \in T_{x_0} X$. Recall that $\mathcal{P}_{x_0} = \mathcal{P}$ was defined in (1.16).

Notation 2.1. Let $\{Q_{r, x_0}\}_{0 \leq r \leq k, x_0 \in X}$ be a family $Q_{r, x_0} \in \text{End}(\Lambda^{0, \bullet} \otimes E)_{x_0}[Z, Z']$ of polynomials in Z, Z' , smooth with respect to the parameter $x_0 \in X$. We will write

$$p^{-n} \Theta_{p, x_0}(Z, Z') \cong \sum_{r=0}^k (Q_{r, x_0} \mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}), \tag{2.3}$$

if there exist $\varepsilon' \in]0, a^X[$, $C_0 > 0$ with the following property: for any $l \in \mathbb{N}$, there exist $C_{k, l} > 0$, $M > 0$ such that for any $x_0 \in X$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon'$ and $p \in \mathbb{N}^*$, the following estimate holds:

$$\begin{aligned} & \left| p^{-n} \Theta_{p, x_0}(Z, Z') \kappa_{x_0}^{\frac{1}{2}}(Z) \kappa_{x_0}^{\frac{1}{2}}(Z') - \sum_{r=0}^k (Q_{r, x_0} \mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} \right|_{\mathcal{C}^l(X)} \\ & \leq C_{k, l} p^{-\frac{k+1}{2}} (1 + \sqrt{p}|Z| + \sqrt{p}|Z'|)^M \exp(-\sqrt{C_0 p}|Z - Z'|) + \mathcal{O}(p^{-\infty}). \end{aligned} \tag{2.4}$$

Here $|\cdot|_{\mathcal{C}^l(X)}$ is the \mathcal{C}^l norm with respect to the parameter $x_0 \in X$.

If $K \subset X \times X$ is compact, we will write that as $p \rightarrow +\infty$, $P_p(x, x') = \mathcal{O}(p^{-\infty})$ for $x, x' \in K$ if for any $k, l \in \mathbb{N}$, the \mathcal{C}^l norm of $P_p(x, x')$ for $x, x' \in K$ with respect to the connections ∇^L, ∇^E and the metrics h^L, h^E, g^{TX} is dominated by Cp^{-k} .

We denote by $I_{\mathbb{C} \otimes E}$ the projection from $\Lambda^{0, \bullet} \otimes E$ onto $\mathbb{C} \otimes E$ relative to the decomposition $\Lambda^{0, \bullet} = \mathbb{C} \oplus \Lambda^{0, >0}$.

We have the following full asymptotic expansion of the Bergman kernel.

Theorem 2.2 (Dai-Liu-Ma [20, Prop. 4.1 and Th. 4.18'], [41, Th. 8.1.4]). *For any $x_0 \in X$ and $r \in \mathbb{N}$, there exist polynomials $J_{r, x_0}(Z, Z') \in \text{End}(\Lambda^{0, \bullet} \otimes E)_{x_0}$ in Z, Z' with the same parity as r and with $\deg J_{r, x_0} \leq 3r$, whose coefficients are functions of the curvatures and their derivatives, such that for any $k \in \mathbb{N}$, in the sense of Notation 2.1,*

$$p^{-n} P_{p, x_0}(Z, Z') \cong \sum_{r=0}^k (J_{r, x_0} \mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}), \tag{2.5}$$

with $J_{0, x_0} = I_{\mathbb{C} \otimes E}$. Moreover, for any $\varepsilon > 0$, we have

$$P_p(x, x') = \mathcal{O}(p^{-\infty}) \quad \text{if } d^X(x, x') \geq \varepsilon. \tag{2.6}$$

Idea of the proof. Using the spectral gap property in Theorem 1.2, and finite propagation speed of solutions of hyperbolic equations, we get (2.6). Also we can localize the asymptotics of $P_p(x_0, x')$ in the neighborhood of x_0 . The second step consists in working on \mathbb{R}^{2n} . To conclude the proof, we combine the spectral gap property, the rescaling of the coordinates and functional analytic techniques inspired by Bismut-Lebeau [7, §11]. □

By taking $\mathbf{b}_r(x_0) = (J_{2r, x_0} \mathcal{P}_{x_0})(0, 0)$, we get from (2.5) that for any $k, l \in \mathbb{N}$, there exists $C_{k,l} > 0$ such that for any $p \in \mathbb{N}^*$,

$$\left| P_p(x, x) - \sum_{r=0}^k \mathbf{b}_r(x) p^{n-r} \right|_{\mathcal{C}^l(X)} \leq C_{k,l} p^{n-k-1}. \tag{2.7}$$

We will give an algorithm to compute the coefficients J_{r, x_0} in the expansion, by using a formal power series trick.

For $s \in \mathcal{C}^\infty(\mathbb{R}^{2n}, (\Lambda^{0,\bullet} \otimes E)_{x_0})$, $Z \in \mathbb{R}^{2n}$, $|Z| \leq \varepsilon$, and for $t = \frac{1}{\sqrt{p}}$, set

$$(S_t s)(Z) := s(Z/t), \quad \mathcal{L}_t := S_t^{-1} \kappa^{1/2} t^2 D_p^2 \kappa^{-1/2} S_t. \tag{2.8}$$

By [20, Th. 4.6] (cf. [41, Th. 4.1.7]), there exist second order differential operators \mathcal{O}_r such that for any $m \in \mathbb{N}$, we have an asymptotic expansion when $t \rightarrow 0$,

$$\mathcal{L}_t = \mathcal{L}_0 + \sum_{r=1}^m t^r \mathcal{O}_r + \mathcal{O}(t^{m+1}), \quad \text{with } \mathcal{L}_0 = \mathcal{L} + 2 \sum_j a_j \bar{w}^j \wedge i_{\bar{w}_j}. \tag{2.9}$$

Then $P^N = I_{\mathbb{C} \otimes E} \mathcal{P}$ is the orthogonal projection of $(L^2(\mathbb{R}^{2n}, (\Lambda^{0,\bullet} \otimes E)_{x_0}), \|\cdot\|_{L^2})$ onto $N = \text{Ker}(\mathcal{L}_0)$. Set $P^{N^\perp} = \text{Id} - P^N$. We define by recursion $f_r(\lambda) \in \text{End}(L^2(\mathbb{R}^{2n}, (\Lambda^{0,\bullet} \otimes E)_{x_0}))$ by

$$f_0(\lambda) = (\lambda - \mathcal{L}_0)^{-1}, \quad f_r(\lambda) = (\lambda - \mathcal{L}_0)^{-1} \sum_{j=1}^r \mathcal{O}_j f_{r-j}(\lambda). \tag{2.10}$$

Let δ be the counterclockwise oriented circle in \mathbb{C} of center 0 and radius $\nu_0/2$.

We denote by \mathcal{F}_{r, x_0} the operator with smooth kernel

$$\mathcal{F}_{r, x_0}(Z, Z') = J_{r, x_0}(Z, Z') \mathcal{P}(Z, Z') \tag{2.11}$$

with respect to dZ' . Then by [38, (1.110)] (cf. also [41, (4.1.91)])

$$\mathcal{F}_{r, x_0} = \frac{1}{2\pi\sqrt{-1}} \int_\delta f_r(\lambda) d\lambda. \tag{2.12}$$

By Theorem 1.5, (2.10), (2.12) and by the residue formula, we can express \mathcal{F}_{r, x_0} in terms of \mathcal{L}_0^{-1} , P^N , P^{N^\perp} , \mathcal{O}_k (with $k \leq r$). This gives a direct method to compute \mathcal{F}_{r, x_0} . In [39, §2], we find an explicit computation for \mathcal{F}_{2, x_0} when $\omega(\cdot, \cdot) = g^{TX}(J\cdot, \cdot)$ (i.e., $\dot{R}^L = 2\pi \text{Id}$). We have in particular:

Theorem 2.3 (Ma-Marinescu [39, Th. 2.1]). *If $\omega(\cdot, \cdot) = g^{TX}(J\cdot, \cdot)$, we have*

$$\text{Tr}_{|\Lambda(T^{*(0,1)}X)}[\mathbf{b}_1(x)] = \frac{1}{8\pi} \left[r^X + \frac{1}{4} |\nabla^X J|^2 + 4 \sum_j R^E(w_j, \bar{w}_j) \right]. \tag{2.13}$$

Here $\nabla^X J$ is the covariant derivative of J with respect to ∇^{TX} , and r^X is the scalar curvature of (X, g^{TX}) . In Donaldson [22], the term $r^X + \frac{1}{4}|\nabla^X J|^2$ in (2.13) is called the Hermitian scalar curvature. It is a natural substitute for the Riemannian scalar curvature in the almost-Kähler case. It was used by Donaldson to define the moment map on the space of compatible almost-complex structures.

Ma-Zhang [44] obtained a family version of Theorem 2.2.

2.2. Asymptotic expansion of Toeplitz operators. Here is a useful characterization of the Toeplitz operators in terms of their kernel.

Theorem 2.4. (Ma-Marinescu [40, Th. 4.9, Rem. 4.10], [41, Lemmas 7.2.2, 7.2.4, Th. 7.3.1]) *Let $\{T_p : L^2(X, E_p) \rightarrow L^2(X, E_p)\}$ be a family of bounded linear operators. Then $\{T_p\}$ is a Toeplitz operator if and only if it satisfies the following three conditions:*

- (i) For any $p \in \mathbb{N}$, $P_p T_p P_p = T_p$.
- (ii) For any $\varepsilon_0 > 0$, $T_p(x, x') = \mathcal{O}(p^{-\infty})$ if $d^X(x, x') \geq \varepsilon_0$.
- (iii) There exists a family of polynomials $\{Q_{r, x_0} \in \text{End}(\Lambda^{0, \bullet} \otimes E)_{x_0}[Z, Z']\}_{x_0 \in X}$ which has the same parity as r , such that for any $k \in \mathbb{N}$, we have in the sense of (2.3) and (2.4),

$$p^{-n} T_{p, x_0}(Z, Z') \cong \sum_{r=0}^k (Q_{r, x_0} \mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}). \tag{2.14}$$

In this case, its principal symbol is $g_0(x_0) = Q_{0, x_0}(0, 0)|_{\mathbb{C} \otimes E} \in \text{End}(E_{x_0})$.

Remark 2.5. For $f \in \mathcal{C}^\infty(X, \text{End}(E))$, conditions (i), (ii), (iii) of Theorem 2.4 for $\{T_{f, p}\}$ are consequences of Theorem 2.2 and of the Taylor expansion of f at x_0 . The coefficients Q_{r, x_0} in (2.14) corresponding to the Toeplitz operator $\{T_{f, p}\}$ are denoted by $Q_{r, x_0}(f)$, and $Q_{0, x_0}(f) = f(x_0)I_{\mathbb{C} \otimes E}$.

By taking $\mathbf{b}_{r, f}(x_0) = (Q_{2r, x_0}(f) \mathcal{P}_{x_0})(0, 0)$, we get from (2.14) that for any $k, l \in \mathbb{N}$, there exists $C_{k, l} > 0$ such that for any $p \in \mathbb{N}^*$, we have

$$\left| T_{f, p}(x, x) - \sum_{r=0}^k \mathbf{b}_{r, f}(x) p^{n-r} \right|_{\mathcal{C}^l(X)} \leq C_{k, l} p^{n-k-1}. \tag{2.15}$$

In [40, (4.15)] (cf. also [41, (7.2.16)]), we find a precise formula for $Q_{r, x_0}(f)$ by using the Taylor expansion of f at x_0 , $J_{j, x_0}(j \leq r)$ and \mathcal{P}_{x_0} in (2.5), from which the computation $\mathbf{b}_{r, f}(x_0)$ can be derived.

Theorem 2.6 (Ma-Marinescu [40, Th. 1.1], [41, Th. 7.4.1]). *The product of the Toeplitz operators $T_{f, p}$ and $T_{g, p}$, with $f, g \in \mathcal{C}^\infty(X, \text{End}(E))$, is a Toeplitz*

operator, i.e., it admits the asymptotic expansion in the sense of (1.11):

$$T_{f,p} T_{g,p} = \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),p} + \mathcal{O}(p^{-\infty}), \tag{2.16}$$

where C_r are bidifferential operators, $C_0(f,g) = fg$ and $C_r(f,g) \in \mathcal{C}^\infty(X, \text{End}(E))$.

If $f, g \in (\mathcal{C}^\infty(X), \{\cdot, \cdot\})$ with the Poisson bracket defined in Section 1.1, we have

$$[T_{f,p}, T_{g,p}] = \frac{\sqrt{-1}}{p} T_{\{f,g\},p} + \mathcal{O}(p^{-2}). \tag{2.17}$$

Theorem 2.6 implies that the set of Toeplitz operators is closed under the composition of operators, and so it forms an associative algebra.

For $E = \mathbb{C}$, Theorem 2.6 shows that we can associate to $f, g \in \mathcal{C}^\infty(X)$ a formal power series $\sum_{l=0}^{\infty} \hbar^l C_l(f,g) \in \mathcal{C}^\infty(X)[[\hbar]]$, where C_l are bidifferential operators. Therefore, we have constructed in a canonical way an associative star-product $f * g = \sum_{l=0}^{\infty} \hbar^l C_l(f,g)$, called the *Berezin-Toeplitz star-product*. Note that the existence of formal star product on symplectic manifolds was established by De Wilde and Lecomte in 1983. We refer to Fedosov’s book [24] for more information on the theory of deformation quantization. In Theorem 2.6, we gave a geometric realization of the associative star-product.

2.3. The Kähler case. In this subsection, we assume that (X, ω, J) is a compact Kähler manifold, (L, h^L) is a holomorphic Hermitian line bundle with Chern connection ∇^L verifying (1.5), and (E, h^E) is a holomorphic Hermitian vector bundle with Chern connection ∇^E . We assume also that $\omega = \frac{\sqrt{-1}}{2\pi} R^L$ is the Kähler form of (X, g^{TX}) . Let $\bar{\partial}^{L^p \otimes E, *}$ be the adjoint of the Dolbeault operator $\bar{\partial}^{L^p \otimes E}$ on $\Omega^{0,\bullet}(X, L^p \otimes E)$. In this case, D_p in (1.6) is given by

$$D_p = \sqrt{2}(\bar{\partial}^{L^p \otimes E} + \bar{\partial}^{L^p \otimes E, *}). \tag{2.18}$$

Thus D_p^2 preserves the \mathbb{Z} -grading on $\Omega^{0,\bullet}(X, L^p \otimes E)$. By Hodge theory and the Kodaira vanishing theorem, we have

$$\text{Ker}(D_p) = H^0(X, L^p \otimes E) \quad \text{for } p \gg 1. \tag{2.19}$$

The Bergman projection P_p reduces to a projection from $\mathcal{C}^\infty(X, L^p \otimes E)$ onto $H^0(X, L^p \otimes E)$, a Toeplitz operator $\{T_p\}$ is now a sequence of linear operators acting on $\mathcal{C}^\infty(X, L^p \otimes E)$. Thus we don’t need to introduce differential forms, and we can work on $\mathcal{C}^\infty(X, L^p \otimes E)$. In this situation, J_{r,x_0} , $\mathbf{b}_r(x_0)$, $Q_{r,x_0}(f)$, $\mathbf{b}_{r,f}(x_0)$ introduced in (2.5), (2.7), Remark 2.5 and (2.15) take values in $\text{End}(E)_{x_0}$.

Let $\mathbb{P}(H^0(X, L^p)^*)$ be the projective space associated to the dual of $H^0(X, L^p)$, and let ω_{FS} be the Fubini–Study (1, 1)-form. The Kodaira map

$\phi_p : X \rightarrow \mathbb{P}(H^0(X, L^p)^*)$ is defined by $\phi_p(x) = \{H^0(X, L^p) \ni s \rightarrow s(x) \in L_x^p\}$ for $x \in X$. The Kodaira embedding theorem asserts that for $p \gg 1$, ϕ_p is a holomorphic embedding and $\phi_p^* \mathcal{O}(1) = L^p$. Let $h_p^{\phi_p^* \mathcal{O}(1)}$ be the metric on $\phi_p^* \mathcal{O}(1)$ induced by the metric $h^{\mathcal{O}(1)}$ on $\mathcal{O}(1)$. Then for $E = \mathbb{C}$, we have (cf. [41, Th. 5.1.3])

$$h_p^{\phi_p^* \mathcal{O}(1)}(x) = P_p(x, x)^{-1} h^{L^p}(x). \quad (2.20)$$

The question of the convergence as $p \rightarrow +\infty$ of $\frac{1}{p} \phi_p^*(\omega_{FS})$ was raised by Yau [71, §6.1]. By (2.7) for $E = \mathbb{C}$, and (2.20), as $p \rightarrow +\infty$, $\frac{1}{p} \phi_p^*(\omega_{FS})$ converges to ω in the \mathcal{C}^∞ topology: for any $l \geq 0$, there exists $C_l > 0$ such that

$$\left| \frac{1}{p} \phi_p^*(\omega_{FS}) - \omega \right|_{\mathcal{C}^l(X)} \leq C_l / p^2. \quad (2.21)$$

When $l = 2$, the estimate of the type (2.21) was obtained by Tian [64] with p^2 replaced by \sqrt{p} , by using the Bergman kernel on the diagonal, $P_p(x, x)$. Ruan [59] obtained (2.21) with p instead of p^2 . Bouche [11] proved that $\lim_{p \rightarrow +\infty} p^{-n} P_p(x, x) = 1$ in the \mathcal{C}^0 topology. The expansion (2.7) was first established by Catlin [17] and Zelditch [72].

Lu [36] calculated more coefficients \mathbf{b}_r via R^{TX} . Let $\text{Ric} = \text{Ric}_g(J \cdot, \cdot)$ be the $(1, 1)$ -form associated to the Ricci curvature Ric_g of g^{TX} . Let Δ be the positive Laplacian acting on functions on X ; set $|R^{TX}|^2 = \sum_{ijkl} |\langle R^{TX}(w_i, \bar{w}_j) w_k, \bar{w}_l \rangle|^2$.

Theorem 2.7 (Lu [36, Th. 1.1]). *When $E = \mathbb{C}$, we have*

$$\mathbf{b}_1 = \frac{r^X}{8\pi}, \quad \pi^2 \mathbf{b}_2 = -\frac{\Delta r^X}{48} + \frac{1}{96} |R^{TX}|^2 - \frac{1}{24} |\text{Ric}|^2 + \frac{1}{128} (r^X)^2. \quad (2.22)$$

Wang [70] also computed \mathbf{b}_1 in (2.7) for general E . When $E = \mathbb{C}$, the existence of an asymptotic expansion similar to (2.5) for $|Z|, |Z'| \leq C/\sqrt{p}$ was also obtained in [61, Th. 1]. For other versions of the asymptotic expansion see [17], [31], [18], [4]. The main tool in [17], [72], [18], [31], and [61] is the Boutet de Monvel-Sjöstrand parametrix for the Szegő kernel [13], [25]. The coefficients were computed in [64], [36], [70] by constructing appropriate peak sections, using Hörmander's L^2 $\bar{\partial}$ -method.

If $E = \mathbb{C}$, the existence of the expansion (2.16) was first established by Bordemann, Meinrenken and Schlichenmaier [9], Schlichenmaier [60], [31]. They used the theory of Toeplitz structures of Boutet de Monvel and Guillemin [12].

Lu's computation for \mathbf{b}_1 plays an important role in Donaldson's work [23] on Kähler metrics with constant scalar curvature. We refer to [5], [41] for further information. In [42], we computed the coefficients $\mathbf{b}_{1,f}$, $\mathbf{b}_{2,f}$, $C_1(f, g)$, $C_2(f, g)$ from (2.15), (2.16). These computations are also relevant in Kähler geometry (cf. [26], [27], [35]).

Theorem 2.8 (Ma-Marinescu [42]). *If $E = \mathbb{C}$, for any $f \in \mathcal{C}^\infty(X)$, we have:*

$$\begin{aligned} \mathbf{b}_{0,f} &= f, & \mathbf{b}_{1,f} &= \frac{r^X}{8\pi} f - \frac{1}{4\pi} \Delta f, \\ \mathbf{b}_{2,f} &= \mathbf{b}_2 f + \frac{1}{32\pi^2} \Delta^2 f - \frac{1}{32\pi^2} r^X \Delta f - \frac{\sqrt{-1}}{8\pi^2} \langle \text{Ric}, \partial\bar{\partial} f \rangle. \end{aligned} \tag{2.23}$$

3. Quantization and Symplectic Reduction

We explain briefly the Guillemin-Sternberg conjecture in Section 3.1, then we review the asymptotic expansion of the G -invariant part of the Bergman kernel in Section 3.2, and we specialize the results in the Kähler case in Section 3.3. In particular, we show how to obtain the scalar curvature on the reduction from the G -invariant Bergman kernel on the total space, and we compare the metrics on the two sides of the “quantization commutes with reduction”.

We use the same notation and assumptions as in Section 1.1.

3.1. Quantization commutes with reduction. Recall that (X, ω, J) is a compact symplectic manifold of real dimension $2n$ with compatible almost complex structure J , and (L, h^L, ∇^L) is a prequantum line bundle on X (cf. (1.5)).

Let G be a compact connected Lie group of dimension n_0 with Lie algebra \mathfrak{g} . We assume that G acts on the left on X and that this action lifts to L . Moreover, we assume that G preserves g^{TX} , J , h^L and ∇^L .

The G -action commutes with the Dirac operator D^L , and $\text{Ker}(D^L_\pm)$ are finite dimensional G -representations. The quantization space $Q(L)$ of L (cf. (1.3)) is an element in the representation ring $R(G)$ of G .

For $K \in \mathfrak{g}$, let K^X be the vector field on X generated by K , and let L_K be the corresponding Lie derivative. Let $\Lambda^*_+ \subset \mathfrak{g}^*$ be the set of dominant weights, and let V_γ^G be the irreducible representation of G with highest weight $\gamma \in \Lambda^*_+$. Let $Q(L)^\gamma \in \mathbb{Z}$ be the multiplicity of V_γ^G in $Q(L)$. Then we have

$$Q(L) = \bigoplus_{\gamma \in \Lambda^*_+} Q(L)^\gamma \cdot V_\gamma^G \in R(G), \tag{3.1}$$

and there are only finitely many $\gamma \in \Lambda^*_+$ such that $Q(L)^\gamma \neq 0$.

It is not easy to read off $Q(L)^\gamma$ directly from the Atiyah-Bott-Segal-Singer equivariant index theorem for its character. Guillemin and Sternberg [29] suggested a geometric way to compute $Q(L)^\gamma$, by using the associated moment map.

Definition 3.1. The moment map $\mu : X \rightarrow \mathfrak{g}^*$ is defined by the Kostant formula [33],

$$2\sqrt{-1}\pi\mu(K) = \nabla^L_{K^X} - L_K, \quad \text{for } K \in \mathfrak{g}. \tag{3.2}$$

Then μ is G -equivariant and one has $i_{K^X}\omega = d\mu(K)$.

For a regular value $\nu \in \mathfrak{g}^*$ of μ , the Marsden-Weinstein symplectic reduction $X_\nu := \mu^{-1}(G \cdot \nu)/G$ is a compact symplectic orbifold with the symplectic form ω_ν induced by ω . Moreover, L (resp. J) induces a prequantum line bundle L_ν (resp. an almost complex structure J_ν) over (X_ν, ω_ν) . One can then construct the associated spin^c Dirac operator (twisted by L_ν), $D_+^{L_\nu}$ on X_ν , of which the index $Q(L_\nu) \in \mathbb{Z}$ (identified as the virtual dimension of $Q(L_\nu)$ in (1.3)).

If $\gamma \in \Lambda_+^*$ is not a regular value of μ , then by [49] (cf. [54, §7.4], [43, §3.5] for a standard perturbation definition), $Q(L_\gamma)$ is still well defined. Now we can state:

Guillemin-Sternberg conjecture: For any $\gamma \in \Lambda_+^*$,

$$Q(L)^\gamma = Q(L_\gamma). \quad (3.3)$$

By the classical shifting trick (i.e., by working on $X \times \mathcal{O}_\gamma$, where $\mathcal{O}_\gamma = G \cdot \gamma$ is the orbit of the co-adjoint action of G on \mathfrak{g}^*), we only need to prove (3.3) for $\gamma = 0$.

This conjecture was proved by Meinrenken [47] and Vergne [67] when G is abelian; by Meinrenken [48], Meinrenken-Sjamaar [49] for non-abelian groups G , by using the technique of symplectic cut of Lerman [34].

Tian and Zhang [65] gave an analytic proof of the Guillemin-Sternberg conjecture, using a deformation of the Dirac operator, which is associated with the function $|\mu|^2$. Their approach works for a general vector bundle E satisfying certain positivity conditions [65, (4.2)] (used afterwards by Paradan [54, p. 445] and Teleman [63, p. 6]), and also for manifolds with boundary [66]. Paradan [54] developed later a K -theoretic approach by making use of the theory of transversally elliptic operators. See [68] for a survey and complete references on this subject.

3.2. Berezin-Toeplitz quantization and reduction. We use the same notation and assumptions as in Sections 1.1 and 3.1. We assume also that the G -action lifts on E and preserves h^E and ∇^E .

Then G -action commutes with the Dirac operator D_p in (1.6). Let $\text{Ker}(D_p)^G$ be the G -trivial component of $\text{Ker}(D_p)$. Let P_p^G be the orthogonal projection from $\mathcal{C}^\infty(X, E_p)$ onto $\text{Ker}(D_p)^G$. The G -invariant Bergman kernel is the \mathcal{C}^∞ kernel $P_p^G(x, x')$, ($x, x' \in X$) of P_p^G associated to $dv_X(x')$.

Assume for simplicity that G acts freely on $\mu^{-1}(0)$, and $g^{TX}(\cdot, \cdot) = \omega(\cdot, J\cdot)$. We will denote by $X_G = \mu^{-1}(0)/G$, and we add a subscript G to denote the objects on X_G induced by the corresponding objects on X .

By a result of Tian and Zhang [65, Th. 0.2], and (1.7b), we have

$$\dim \text{Ker}(D_p)^G = \dim \text{Ker}(D_{G,p}) \quad \text{for } p \gg 1. \quad (3.4)$$

We will describe how $P_p^G(x, x')$ “concentrates” on the Bergman kernel $P_{G,p}(x_0, x'_0)$ on X_G , when $p \rightarrow +\infty$.

Theorem 3.2 (Ma-Zhang [43, Th.0.1]). *For any open G -neighborhood U of $\mu^{-1}(0)$ and any $\varepsilon_0 > 0$, we have*

$$P_p^G(x, x') = \mathcal{O}(p^{-\infty}) \text{ if } (x, x') \notin U \times U \text{ or if } d^X(Gx, x') \geq \varepsilon_0. \tag{3.5}$$

Let U be an open G -neighborhood of $\mu^{-1}(0)$ such that G acts freely on U . For any G -equivariant vector bundle with connection (F, ∇^F) on U , we denote by (F_B, ∇^{F_B}) the bundle on $B := U/G$ induced by G -invariant sections of F on U .

For $x \in U$ denote by $\text{vol}(Gx)$ the volume of the orbit Gx equipped with the metric induced by g^{TX} . Following [65, (3.10)], let $h(x)$ be the function on U defined by

$$h(x) = (\text{vol}(Gx))^{1/2}. \tag{3.6}$$

Then h descends to a function on B .

Let pr_1 and pr_2 be the projections from $X \times X$ onto the first and the second factor X respectively. Then we can view $P_p^G(x, x')$ ($x, x' \in U$) as a smooth section of $\text{pr}_1^*(E_p)_B \otimes \text{pr}_2^*(E_p^*)_B$ on $B \times B$.

We introduce the following coordinates: for any $x_0 \in X_G$, $Z \in T_{x_0}B$, we write $Z = Z^0 + Z^\perp$, with $Z^0 \in T_{x_0}X_G$, $Z^\perp \in N_{G,x_0}$, where N_G is the normal bundle of X_G in B . For $\varepsilon_0 > 0$ small enough, we identify $Z \in T_{x_0}B$, $|Z| < \varepsilon_0$ with $\exp_{\exp_{x_0}^{X_G}(Z^0)}^B(Z^\perp) \in B$, here we still denote by $Z^\perp \in N_{G, \exp_{x_0}^{X_G}(Z^0)}$, the parallel transport of Z^\perp along the curve $u \rightarrow \exp_{x_0}^{X_G}(uZ^0)$ with respect to the connection on N_G induced by projecting the Levi-Civita connection on TB .

We identify $(E_p)_{B,Z}$ with $(E_p)_{B,x_0}$ by using parallel transport with respect to $\nabla^{(E_p)_B}$ (cf. §2.1) along the curve $[0, 1] \ni u \rightarrow uZ$.

Let dv_B, dv_{X_G}, dv_{N_G} be the Riemannian volume forms on TB, TX_G, N_G induced by g^{TX} . Let $\varrho \in \mathcal{C}^\infty(TB|_{X_G}, \mathbb{R})$, with $\varrho = 1$ on X_G , be defined by

$$dv_B(x_0, Z) = \varrho(x_0, Z)dv_{X_G}(x_0)dv_{N_G,x_0} \text{ for } Z \in T_{x_0}B, x_0 \in X_G. \tag{3.7}$$

For $x_0 \in X_G$, $Z = (Z^0, Z^\perp)$, $Z' = (Z'^0, Z'^\perp) \in T_{x_0}X_G \oplus N_{G,x_0} = T_{x_0}B$, set

$$\begin{aligned} \mathcal{P}(Z, Z') = & 2^{\frac{n_0}{2}} \exp\left(-\frac{\pi}{2} \sum_i (|z_i^0|^2 + |z_i'^0|^2 - 2z_i^0 \bar{z}_i'^0)\right) \\ & \times \exp\left(-\pi|Z^\perp|^2 - \pi|Z'^\perp|^2\right), \end{aligned} \tag{3.8}$$

with $n_0 = \dim G$. As in (1.16) and (2.9), \mathcal{P} is the Bergman kernel of a limit operator, which itself is sum of two terms: one is defined on $T_{x_0}X_G$, and is equal \mathcal{L} (cf. (1.13)); the other is defined on N_{G,x_0} , it is equal to a harmonic oscillator. This explains why we expect the G -invariant Bergman kernel $P_p^G(x, x')$ to exhibit the same sort of behavior, see (3.11).

Let $\{\Theta_p^G\}_{p \in \mathbb{N}}$ be a sequence of linear operators $\Theta_p^G : L^2(X, E_p) \rightarrow L^2(X, E_p)$ with smooth kernel $\Theta_p^G(x, y)$ with respect to $dv_X(y)$. We assume

that $\Theta_p^G(x, y)$ is $G \times G$ -invariant. Let $\pi_B : TB \times_{X_G} TB \rightarrow X_G$ be the obvious projection. Relative to our trivialization, $\Theta_p^G(x, y)$ induces a smooth section $\Theta_{p, x_0}^G(Z, Z')$ of $\pi_B^*(\text{End}(\Lambda^{0, \bullet} \otimes E)_B)$ over $TB \times_{X_G} TB$ with $Z, Z' \in T_{x_0}B$. We introduce the following notation in analogy to Notation 2.1.

Notation 3.3. We write

$$p^{-n+\frac{n_0}{2}} \Theta_{p, x_0}^G(Z, Z') \stackrel{h}{\approx} \sum_{r=0}^k (Q_{r, x_0}^G \mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}), \quad (3.9)$$

if there exists a family $\{Q_{r, x_0}^G\}_{0 \leq r \leq k, x_0 \in X_G}$ with $Q_{r, x_0}^G \in \text{End}(\Lambda^{0, \bullet} \otimes E)_{B, x_0}[Z, Z']$ smooth with respect to the parameter $x_0 \in X_G$, and there exist $\varepsilon' \in]0, a^X[$ and $C_0 > 0$ with the following property: for any $l, m \in \mathbb{N}$, there exist $C > 0, M > 0$ such that for any $x_0 \in X_G, Z, Z' \in T_{x_0}B, |Z|, |Z'| < \varepsilon'$ and $p \in \mathbb{N}^*$, the following estimate holds:

$$\begin{aligned} & (1 + \sqrt{p}|Z^\perp| + \sqrt{p}|Z'^\perp|)^m \left| p^{-n+\frac{n_0}{2}} \Theta_{p, x_0}^G(Z, Z')(h\rho^{\frac{1}{2}})(Z)(h\rho^{\frac{1}{2}})(Z') \right. \\ & \quad \left. - \sum_{r=0}^k (Q_{r, x_0}^G \mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} \right|_{\mathcal{C}^l(X_G)} \\ & \leq C p^{-\frac{k+1}{2}} (1 + \sqrt{p}|Z^0| + \sqrt{p}|Z'^0|)^M \exp(-\sqrt{C_0 p}|Z - Z'|) + \mathcal{O}(p^{-\infty}). \end{aligned} \quad (3.10)$$

Theorem 3.4 (Ma-Zhang [43, Th. 0.2]). *There exists a family of polynomials $\{Q_{r, x_0}\}_{r \in \mathbb{N}, x_0 \in X_G} \in \text{End}(\Lambda^{0, \bullet} \otimes E)_{B, x_0}[Z, Z']$ on Z, Z' with the same parity as r , such that $Q_{0, x_0} = I_{C \otimes E, G}$, and for any $k \in \mathbb{N}$ the following expansion holds in the sense of Notation 3.3,*

$$p^{-n+\frac{n_0}{2}} P_{p, x_0}^G(Z, Z') \stackrel{h}{\approx} \sum_{r=0}^k (Q_{r, x_0} \mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}). \quad (3.11)$$

To read off the scalar curvature on the reduction from P_p^G , we define $\mathcal{I}_p(x_0) \in \text{End}(\Lambda^{0, \bullet} \otimes E)_{G, x_0}$ for $x_0 \in X_G$ by :

$$\mathcal{I}_p(x_0) = \int_{\substack{|Z| \leq \varepsilon_0 \\ Z \in N_G}} (\rho h^2)(x_0, Z) P_p^G((x_0, Z), (x_0, Z)) dv_{N_G}(Z). \quad (3.12)$$

By (3.4), (3.5), $\mathcal{I}_p(x_0)$ does not depend on ε_0 modulo $\mathcal{O}(p^{-\infty})$, and

$$\dim \text{Ker}(D_{G, p}) = \int_{X_G} \text{Tr}[\mathcal{I}_p(x_0)] dv_{X_G}(x_0) + \mathcal{O}(p^{-\infty}). \quad (3.13)$$

From Theorem 3.4, we infer the existence of $\Phi_r \in \mathcal{C}^\infty(X_G, \text{End}(\Lambda^{0, \bullet} \otimes E)_G)$, and $\Phi_0 = I_{C \otimes E, G}$, with the property that for all $k, m \in \mathbb{N}$, there exists $C_{k, m} > 0$

such that for all $p \in \mathbb{N}^*$,

$$\left| p^{-n+n_0} \mathcal{J}_p(x_0) - \sum_{r=0}^k \Phi_r(x_0) p^{-r} \right|_{\mathcal{C}^m(X_G)} \leq C_{k,m} p^{-k-1}. \tag{3.14}$$

Using Theorems 3.2, 3.4, and the same argument as in Remark 2.5, we see that the analogue of Theorems 3.2, 3.4 still holds for the kernel $T_{f,p}^G(x, x')$ of the operator $T_{f,p}^G := P_p^G f P_p^G$, for $f \in \mathcal{C}^\infty(X, \text{End}(E))$.

Theorem 3.5 (Ma-Zhang [43, p. 86-88]). *Let $f \in \mathcal{C}^\infty(X, \text{End}(E))$. For any open G -neighborhood U of $\mu^{-1}(0)$, $\varepsilon_0 > 0$, we have*

$$T_{f,p}^G(x, x') = \mathcal{O}(p^{-\infty}) \text{ if } (x, x') \notin U \times U \text{ or if } d^X(Gx, x') \geq \varepsilon_0. \tag{3.15}$$

Moreover, there exists a family $\{\mathcal{Q}_{r,x_0}^G(f)\}_{r \in \mathbb{N}, x_0 \in X_G} \in \text{End}(\Lambda^{0,\bullet} \otimes E)_{B,x_0}[Z, Z']$ of polynomials in Z, Z' with the same parity as r such that for any $k \in \mathbb{N}$, we have in the sense of Notation 3.3,

$$p^{-n+\frac{n_0}{2}} T_{f,p,x_0}^G(Z, Z') \overset{h}{\approx} \sum_{r=0}^k (\mathcal{Q}_{r,x_0}^G(f) \mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}). \tag{3.16}$$

Moreover, $\mathcal{Q}_{0,x_0}^G(f) = f^G(x_0) I_{\mathbb{C} \otimes E, G}$, where f^G is the G -invariant component of f .

Since $\text{Tr}[T_{f,p}^G] = \int_X \text{Tr}[T_{f,p}^G(x, x)] dv_X(x)$, we deduce from Theorem 3.5 that there exists a sequence $B_{r,f}$ with $B_{0,f} = \int_{X_G} \text{Tr}[f^G(x_0)] dv_{X_G}(x_0)$ and for any $k \in \mathbb{N}$,

$$p^{-n+n_0} \text{Tr}[T_{f,p}^G] = \sum_{r=0}^k B_{r,f} p^{-r} + \mathcal{O}(p^{-k-1}). \tag{3.17}$$

Note that in [43, §4.1, §4.5] the case where 0 is a regular value of μ (so that X_G is an orbifold) is treated in detail. In [43, §4.2], it is shown by a shifting trick that Theorems 3.2 and 3.4 imply the expansion of the kernel of the orthogonal projection $P_p^{V_\gamma^G}$ from $\Omega^{0,\bullet}(X, L^p \otimes E)$ onto the V_γ^G -component of $\text{Ker}(D_p)$ for any $\gamma \in \Lambda_+^*$.

3.3. The Kähler case. In this subsection, as in Section 2.3, we assume that (X, ω, J) is a compact Kähler manifold carrying a holomorphic Hermitian line bundle (L, h^L) and a holomorphic Hermitian vector bundle (E, h^E) and moreover $\omega = \frac{\sqrt{-1}}{2\pi} R^L$ is the Kähler form of (X, g^{TX}) . We assume also that the G -action on X, L, E is holomorphic, and preserves the metrics.

By (2.19), we see as in Section 2.3 that the G -invariant Bergman projection P_p^G reduces to a projection from $\mathcal{C}^\infty(X, L^p \otimes E)$ onto $H^0(X, L^p \otimes E)^G$, and

the Toeplitz operator $\{T_{f,p}^G\}$ reduces to a sequence of linear operators acting on $\mathcal{C}^\infty(X, L^p \otimes E)$. In particular, $\mathcal{Q}_{r,x_0}, \mathcal{I}_p(x_0), \Phi_r(x_0), \mathcal{Q}_{r,x_0}^G(f)$ in (3.11), (3.14) and (3.16) take values in $\text{End}(E_G)_{x_0}$.

Let \tilde{h} be the restriction of h on X_G . Let r^{X_G} be the scalar curvature on (X_G, ω_G, J_G) , and Δ_{X_G} be the positive Laplacian on X_G . Let $\{w_j^0\}$ be an orthonormal frame of $T^{(1,0)}X_G$. The following result generalizes formula (2.13) for the coefficient \mathbf{b}_1 of the expansion (2.7).

Theorem 3.6 (Ma-Zhang [43, Th. 0.6]). *The coefficients Φ_0 and Φ_1 from (3.14) are given by,*

$$\Phi_0 = \text{Id}_{E_G}, \quad \Phi_1(x_0) = \frac{1}{8\pi} r_{x_0}^{X_G} + \frac{3}{4\pi} \Delta_{X_G} \log \tilde{h} + \frac{1}{2\pi} \sum_j R_{x_0}^{E_G}(w_j^0, \bar{w}_j^0). \quad (3.18)$$

We discuss now the metric aspect of quantization. Let $i : \mu^{-1}(0) \hookrightarrow X$ be the natural injection. Let $\pi_G : \mathcal{C}^\infty(\mu^{-1}(0), L^p \otimes E)^G \rightarrow \mathcal{C}^\infty(X_G, L_G^p \otimes E_G)$ be the natural identification. By a result of Zhang [73, Th. 1.1 and Prop. 1.2], for $p \gg 1$, the map $\pi_G \circ i^* : \mathcal{C}^\infty(X, L^p \otimes E)^G \rightarrow \mathcal{C}^\infty(X_G, L_G^p \otimes E_G)$ induces a natural isomorphism

$$\sigma_p = \pi_G \circ i^* : H^0(X, L^p \otimes E)^G \rightarrow H^0(X_G, L_G^p \otimes E_G). \quad (3.19)$$

(When $E = \mathbb{C}$, this result was first proved in [29, Th. 3.8] for $p \geq 1$). We denote by $\langle \cdot, \cdot \rangle$ the L^2 -Hermitian products on these spaces. A corollary of Theorem 3.5 is as follows.

Theorem 3.7 (Ma-Zhang [43, Th. 4.8]). *Set $\sigma_p^G = \sigma_p \circ P_p^G$ and let σ_p^{G*} be the adjoint of σ_p^G . Then $\mathcal{T}_{f,p} = p^{-\frac{n_0}{2}} \sigma_p^G f \sigma_p^{G*} \in \text{End}(H^0(X_G, L_G^p \otimes E_G))$ is a Toeplitz operator with principal symbol $2^{\frac{n_0}{2}} f^G / \tilde{h}^2$, for any $f \in \mathcal{C}^\infty(X, \text{End}(E))$.*

The natural Hermitian product $\langle \cdot, \cdot \rangle_{\tilde{h}}$ on $\mathcal{C}^\infty(X_G, L_G^p \otimes E_G)$ is given by

$$\langle s_1, s_2 \rangle_{\tilde{h}} = \int_{X_G} \langle s_1, s_2 \rangle(x_0) \tilde{h}^2(x_0) dv_{X_G}(x_0). \quad (3.20)$$

Theorem 3.8 (Ma-Zhang [43, Th. 0.10]). *The isomorphism $(2p)^{-\frac{n_0}{4}} \sigma_p$ is an asymptotic isometry from $(H^0(X, L^p \otimes E)^G, \langle \cdot, \cdot \rangle)$ onto $(H^0(X_G, L_G^p \otimes E_G), \langle \cdot, \cdot \rangle_{\tilde{h}})$, i.e., if $\{s_i^p\}_{i=1}^{d_p}$ is an orthonormal basis of $(H^0(X, L^p \otimes E)^G, \langle \cdot, \cdot \rangle)$, then*

$$(2p)^{-\frac{n_0}{2}} \langle \sigma_p s_i^p, \sigma_p s_j^p \rangle_{\tilde{h}} = \delta_{ij} + \mathcal{O}(p^{-1}). \quad (3.21)$$

In [43, Remark 0.11], we find a natural symplectic extension of Theorem 3.8.

When $E = \mathbb{C}$ and G is a torus, Charles [19] first showed that $\mathcal{T}_{f,p}$ in Theorem 3.7 is a Toeplitz operator, and obtained (3.21).

Assume that $E = \mathbb{C}$. Then $P_p^G(x_0, x_0)$ becomes a positive function. By setting $Z = Z' = 0$ in (3.11), we get the following expansion on X_G for any k ,

$$p^{-n+\frac{n_0}{2}} h^2(x_0) P_p^G(x_0, x_0) = \sum_{r=0}^k c_r(x_0) p^{-r} + \mathcal{O}(p^{-k-1}), \quad c_0(x_0) = 2^{n_0/2}. \tag{3.22}$$

Paoletti [50, Th. 1], [51, Th. 1] had obtained the expansion (3.22), but he claimed that $c_0(x_0) = 1$. After our preprint [43] was posted, Hall-Kirwin [30], Paoletti [52], [53] and Burns-Guillemin-Wang [16] have established related results.

4. Noncompact Case: Vergne’s Conjecture

In this section, we use the same notation and assumptions as in Sections 1, 3.1, except that we assume now that X is noncompact. One asks naturally the following question: what is the quantization formula in this situation?

When (X, g^{TX}) is a complete Riemannian manifold, it is shown in [38, §3.5], [40, §5], [41, §6.1, §7.5], [43, §4.6] that under natural (positivity) conditions on R^L, R^E , the asymptotic expansion of the Bergman kernel holds. However, in this section, we do not assume (X, g^{TX}) to be complete.

In Section 4.1, the quantization formula is explained for the model example \mathbb{C}^n . In Section 4.2, we review briefly our solution with Zhang of Vergne’s conjecture: “quantization commutes with reduction” in the noncompact setting.

4.1. Quantization formula on \mathbb{C}^n . We continue the discussion of Section 1.2. Let’s assume now that $a_j = 2\pi$ for $j = 1, \dots, n$. Then (L, h^L, ∇^L) is a prequantum line bundle on $(\mathbb{C}^n, \omega = \frac{\sqrt{-1}}{2} \sum_j dz_j \wedge d\bar{z}_j)$.

Let T^n be the n -dimensional torus with Lie algebra \mathfrak{t}^n . We define a holomorphic action of T^n on \mathbb{C}^n by $e^{i\theta} \cdot z = (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n)$, with $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ and $e^{i\theta} = (e^{i\theta_1}, \dots, e^{i\theta_n}) \in T^n$. For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, we define a holomorphic T^n -action on L by $e^{i\theta} \cdot \mathbf{1} = e^{i\theta \cdot \lambda} \mathbf{1}$ with $\theta \cdot \lambda = \sum_j \theta_j \lambda_j$. Then the associated moment map $\mu : \mathbb{C}^n \rightarrow \mathbb{R}^{n*}$ (cf. (3.2)) is given by

$$\mu(z) = \frac{1}{2}(|z_1|^2, \dots, |z_n|^2) + \lambda. \tag{4.1}$$

Given $\{u_i\}_{i=1}^n \subset \mathbb{Z}^m$, the *Delzant polytope* $\Delta \subset \mathbb{R}^{m*}$ [2, §VII. 1.c., 2.a.] is defined by

$$\Delta = \{x \in \mathbb{R}^{m*} : (u_i, x) \geq \lambda_i \text{ for } 1 \leq i \leq n\}, \tag{4.2}$$

if the vertices have integer coordinates and each vertex q has exactly m -edges, and the u_i such that $(u_i, q) = \lambda_i$ form a basis of \mathbb{Z}^m .

Let $j : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be the linear map defined by $j(e_i) = u_i$ with $\{e_i\}$ the canonical basis of \mathbb{R}^n . Let $N = \text{Ker}(j)/(\text{Ker}(j) \cap (2\pi\mathbb{Z})^n) \subset \mathbb{R}^n/(2\pi\mathbb{Z})^n \simeq T^n$, so that N is a $(n - m)$ -dimensional torus with Lie algebra $\mathfrak{n} \xrightarrow{i} \mathbb{R}^n \simeq \mathfrak{t}^n$. Thus we have the exact sequence: $0 \rightarrow \mathbb{R}^{m*} \xrightarrow{j^*} \mathbb{R}^{n*} \xrightarrow{i^*} \mathfrak{n}^* \rightarrow 0$.

Now N acts naturally on \mathbb{C}^n and L , the associated moment map is $\Phi = i^* \circ \mu : \mathbb{C}^n \rightarrow \mathfrak{n}^*$. Its symplectic reduction $X_\Delta = \Phi^{-1}(0)/N$ is a m -dimensional compact Kähler manifold, and L descends naturally to a positive holomorphic line bundle L_Δ on X_Δ . Then X_Δ is the toric variety associated to the Delzant polytope Δ .

Observe that if N acts trivially on a holomorphic section $z^\alpha \mathbf{1}$ of L for some $\alpha \in \mathbb{N}^n$, then $z^\alpha \mathbf{1}$ descends to a holomorphic section of L_Δ on X_Δ .

For $e^{i\theta} \in T^n$, we have $e^{i\theta} \cdot z^\alpha \mathbf{1} = e^{i\theta \cdot (\alpha + \lambda)} z^\alpha \mathbf{1}$. Thus N acts trivially on the holomorphic section $z^\alpha \mathbf{1}$ if and only if $i^*(\alpha + \lambda) = 0$, and this is equivalent to the existence of a $\nu \in \mathbb{R}^{m*}$ such that $\alpha_i + \lambda_i = (\nu, u_i)$, i.e., $\nu \in \Delta \cap \mathbb{Z}^m$ and $\alpha_i + \lambda_i = (\nu, u_i)$.

For $\nu \in \Delta \cap \mathbb{Z}^m$, we denote by s_ν the holomorphic section of L_Δ on X_Δ induced by $z^\alpha \mathbf{1}$, where $\alpha_i = (\nu, u_i) - \lambda_i$.

Theorem 4.1 ([28, §3.5]). *The cohomology of L_Δ on X_Δ is given by*

$$H^0(X_\Delta, L_\Delta) = \bigoplus_{\nu \in \Delta \cap \mathbb{Z}^m} \mathbb{C} s_\nu, \quad H^j(X_\Delta, L_\Delta) = 0 \text{ if } j > 0. \quad (4.3)$$

By Theorem 1.5, we see that the kernel of D^L on the noncompact space \mathbb{C}^n is an infinite dimensional vector space. Moreover, by the discussion after (1.13) we deduce that all higher L^2 cohomology groups of \mathbb{C}^n with values in L vanish. Theorem 4.1 implies that “quantization commutes with reduction” still holds. Note that the moment map $\Phi = i^* \circ \mu$ is proper here.

Example 4.2. Set $m = n - 1$, $u_i = e_i$ for $i \leq m$, $u_n = -(1, \dots, 1) = -\sum_{i=1}^m e_i$, $\lambda = (0, \dots, 0, -1)$. Then $\text{Ker}(j) = \mathbb{R}(1, \dots, 1)$, $\Phi(z) = \frac{1}{2} \sum_{i=1}^n |z_i|^2 - 1$. In this case, $(X_\Delta, L_\Delta) \simeq (\mathbb{C}\mathbb{P}^{n-1}, \mathcal{O}(1))$ with $\mathcal{O}(1)$ the hyperplane line bundle on $\mathbb{C}\mathbb{P}^{n-1}$.

4.2. Vergne’s conjecture. Recall that (X, ω, J) is a noncompact symplectic manifold with the prequantum line bundle (L, h^L, ∇^L) , and g^{TX} is a J -invariant Riemannian metric on X . Let $\tau : TX \rightarrow X$ be the natural projection. Following [1, p. 7] (cf. [54, §3]), set $T_G X = \{(x, v) \in T_x X : \langle v, K^X(x) \rangle = 0 \text{ for all } K \in \mathfrak{g}\}$.

Then the quantization space $Q(L) = \text{Ind}(D^L)$ of L is not well defined, since usually D^L is not a Fredholm operator, and we need to make precise the self-adjoint extension of $D^L|_{\Omega_0^{0,\bullet}(X,L)}$, where $\Omega_0^{0,\bullet}(X, L)$ denotes the space of sections with compact support.

We suppose that the moment map $\mu : X \rightarrow \mathfrak{g}^*$ is proper. Then the right hand side of (3.3) is well defined.

We identify \mathfrak{g} with \mathfrak{g}^* by using an Ad_G -invariant metric on \mathfrak{g} . Let $\mu^X(x) := (\mu(x))^X(x)$ ($x \in X$)¹ be the vector field induced by $\mu : X \rightarrow \mathfrak{g}$.

We suppose for the moment that $\{x \in X : \mu^X(x) = 0\}$ is compact.

Recall that $\mathbf{c}(\cdot)$ is the Clifford action defined in (1.1). For $x \in X$, $\xi \in T_x X$, set²

$$\begin{aligned} \sigma_{L,\mu}^X(x, \xi) &= \tau^* (\sqrt{-1}\mathbf{c}(\xi + \mu^X) \otimes \text{Id}_L)|_{(x,\xi)} \\ &: \tau^*(\Lambda^{\text{even}}(T^{*(0,1)}X) \otimes L) \rightarrow \tau^*(\Lambda^{\text{odd}}(T^{*(0,1)}X) \otimes L). \end{aligned} \tag{4.4}$$

Then $\sigma_{L,\mu}^X$ is a transversally elliptic symbol on $T_G X$ in the sense of Atiyah [1, §1, §3] and Paradan [54, §3], [55, §3], which determines a transversal index $\text{Ind}(\sigma_{L,\mu}^X)$ in the formal representation ring $R[G]$ of G ,

$$\text{Ind}(\sigma_{L,\mu}^X) = \bigoplus_{\gamma \in \Lambda_+^*} \text{Ind}_\gamma(\sigma_{L,\mu}^X) \cdot V_\gamma^G \in R[G]. \tag{4.5}$$

The index $\text{Ind}(\sigma_{L,\mu}^X)$ does not depend on g^{TX}, h^L, ∇^L , and it depends only on the homotopy classes of J, μ^X . The set $\{\gamma \in \Lambda_+^* : \text{Ind}_\gamma(\sigma_{L,\mu}^X) \neq 0\}$ can be infinite. Michèle Vergne suggested to use $\text{Ind}_\gamma(\sigma_{L,\mu}^X)$ to replace the left hand side of (3.3).

Vergne’s conjecture (ICM 2006 plenary lecture [69, §4.3]) : If $\mu : X \rightarrow \mathfrak{g}^*$ is proper and if $\{x \in X : \mu^X(x) = 0\}$ is compact, then for any $\gamma \in \Lambda_+^*$,

$$\text{Ind}_\gamma(\sigma_{L,\mu}^X) = Q(L_\gamma). \tag{4.6}$$

Special cases of this conjecture, related to the discrete series of semi-simple Lie groups, have been proved by Paradan [55], [57].

For $a > 0$, set $X_a = \{x \in X : |\mu|^2(x) \leq a\}$. If a is a regular value of $|\mu|^2$, then X_a is a compact manifold with boundary ∂X_a , and μ^X is nowhere zero on ∂X_a . Thus $\sigma_{L,\mu}^{X_a}$ is a transversally elliptic symbol on X_a .

Theorem 4.3 (Quantization commutes with reduction, Ma-Zhang [45, Th.0.2,0.3]). *Suppose that $\mu : X \rightarrow \mathfrak{g}^*$ is proper. For any $\gamma \in \Lambda_+^*$, there exists $a_\gamma > 0$ such that the function $a \mapsto \text{Ind}_\gamma(\sigma_{L,\mu}^{X_a})$ is constant on $\{a > a_\gamma : a \text{ is regular value of } |\mu|^2\}$. Denote by $Q(L)^\gamma$ this constant. Then for any $\gamma \in \Lambda_+^*$, we have*

$$Q(L)^\gamma = Q(L_\gamma). \tag{4.7}$$

¹The vector field μ^X is also called Kirwan vector field in view of [32].

²The symbol $\sigma_{L,\mu}^X$ is the (semi-classical) symbol of Tian-Zhang’s [65] deformed Dirac operator (4.8) in their approach to the Guillemin-Sternberg geometric quantization conjecture.

If $\{x \in X : \mu^X(x) = 0\}$ is compact, then $Q(L)^\gamma = \text{Ind}_\gamma(\sigma_{L,\mu}^X)$. Therefore Theorem 4.3 implies Vergne's conjecture. Note that Paradan [58] gives a new proof of Theorem 4.3 by using symplectic cuts and the wonderful compactifications of [56].

Idea of the proof. 1) Assume that $\{x \in X : \mu^X(x) = 0\}$ is compact. For $T > 0$, let D_T^L be the deformed Dirac operator introduced by Tian-Zhang [65, (1.20)]:

$$D_T^L = D^L + \sqrt{-1}Tc(\mu^X) : \Omega^{0,\bullet}(X, L) \rightarrow \Omega^{0,\bullet}(X, L). \quad (4.8)$$

A first step is to interpret the transversal index as the Atiyah-Patodi-Singer index of D_T^L for a manifold with boundary defined as in [66]. The proof uses Braverman's L^2 -interpretation of the transversal index [15, §5]. The proof of (4.7) for $\gamma = 0$ is then easy.

2) A second key result is as follows. Let (N, ω^N, J^N) be a compact symplectic manifold with a prequantum line bundle (F, h^F, ∇^F) (see Section 1.1). We suppose that G acts on N and the action lifts to F as above with the associated moment map $\eta : N \rightarrow \mathfrak{g}^*$, etc. For $\gamma \in \Lambda_+^*$, set $Q(F)^{-\gamma} = \dim \text{Hom}_G((V_\gamma^G)^*, Q(F))$, where Hom_G is the linear space of G -homomorphisms. Let $L \otimes F$ be the obvious prequantum line bundle over $X \times N$. Then we have

$$Q(L \otimes F)^{\gamma=0} = \sum_{\gamma \in \Lambda_+^*} Q(L)^\gamma \cdot Q(F)^{-\gamma}. \quad (4.9)$$

The proof of Theorem 4.3 is obtained in [45] by combining these two arguments. \square

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