Sectional algebras of semigroupoid bundles

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May 9, 2019

Motivation

Theorem

Let R and S be unital, discrete, indecomposable rings and X and Y be zero-dimensional, locally compact, Hausdorff spaces. Any ring isomorphism $\varphi\colon C_c(X,R) \to C_c(Y,S)$ is of the form

$$\varphi(f)(y) = \chi_{\phi(y)}(f(\phi(y)))$$

for some homeomorphism $\phi\colon Y \to X$ and a field $\chi = \{\chi_x\}_{x\in X}$ of ring isomorphisms $\xi_x\colon R \to S$.

- $X \times R$ "is graded" over X; Formalize connection with graded algebra?
- Godement, Kaplansky, Fell, Kumjian,. . . : C^*-algebraic bundles.
- Is it possible to have a setting which also emcompasses possibly *Steinberg* algebras?

Definition

A *semigroupoid* is a graph $\mathfrak{s}, \mathfrak{r} \colon \Lambda \to \Lambda^{(0)}$ with a product $\Lambda^{(2)} = \{(x,y) \in \Lambda \times \Lambda : \mathfrak{s}(x) = \mathfrak{r}(y)\}$, $(x,y) \mapsto xy$, satisfying

- $\mathfrak{s}(xy) = \mathfrak{s}(y)$, $\mathfrak{r}(xy) = \mathfrak{r}(x)$;
- $x(yz) = (xy)z$ whenever sensible.

Example

Categories, groupoids, semigroups, are semigroupoids.

- A semigroupoid Λ is *inverse* if for every $a \in \Lambda$ there exists a unique $a^* \in \Lambda$ such that $aa^*a = a$ and $a^*aa^* = a^*$.
- Homomorphisms=functors: $\phi \colon \Lambda \to \Gamma$ such that $\Lambda^{(2)} \subseteq (\phi \times \phi)^{-1}(\Gamma^{(2)})$ and $\phi(xy) = \phi(x)\phi(y)$ whenever sensible.

Definition

Homomorphism $\phi \colon \Lambda \to \Gamma$ is *rigid* if $(\phi \times \phi)^{-1}(\Gamma^{(2)}) = \Lambda^{(2)}$.

R-bundles

Let R be a unital ring.

Definition

An *R-bundle* is a surjective rigid semigroupoid homomorphism $\pi\colon \Lambda \to \Gamma$ such that

- $\pi^{-1}(\gamma)$ is an R-bimodule for all $\gamma \in \Gamma$;
- The product map of Λ

$$\pi^{-1}(\gamma_1) \times \pi^{-1}(\gamma_2) \to \pi^{-1}(\gamma_1\gamma_2)$$

  is R-balanced for all $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$ (equivalently it is an R-bimodule homomorphism $\pi^{-1}(\gamma_1) \otimes \pi^{-1}(\gamma_2) \to \pi^{-1}(\gamma_1\gamma_2)$).

Sectional algebras

Conventions

All semigroupoids are *étale* (all relevant maps are local homeomorphisms, including bundles), Hausdorff, and *ample* (zero-dimensional). All rings are discrete.

The *sectional algebra* $\mathcal{A}(\pi) \equiv \mathcal{A}(\Lambda \to \Gamma)$ of $\pi \colon \Lambda \to \Gamma$ is the set of continuous functions $\alpha \colon \Gamma \to \Lambda$ such that

- α is continuous;
- $\operatorname{supp}(\alpha) := \{\gamma \in \Gamma : \alpha(\gamma) \neq 0\}$ is compact.

The bimodule structure of $\mathcal{A}(\pi)$ is pointwise. The product is

$$\alpha\beta(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} \alpha(\gamma_1)\beta(\gamma_2).$$

Two examples

Example

If $A = \oplus_{s\in S} A_s$ is a semigroup graded algebra, then $A = \mathcal{A}(\sqcup_{s\in S} A_s \to S)$. This also works for groupoid (and semigroupoid) graded algebras.

Example

If A is any algebra, the semigroupoid algebra of Γ is $A\Gamma = \mathcal{A}(\Gamma \times A \to \Gamma)$.

"Naïve" crossed products

An *action* $\theta\colon \mathcal{S} \curvearrowright \Lambda$ of an inverse semigroupoid $\mathcal{S}$ on a semigroupoid Λ is a collection $\{\theta_s : s \in \mathcal{S}\}$ of maps such that

- for all $s \in \mathcal{S}$, $\theta_s\colon \operatorname{dom}(\theta_s) \to \operatorname{ran}(\theta_s)$ is an isomorphism of ideals of Λ;
- for all $(s,t) \in \mathcal{S}^{(2)}$, $\theta_{st} = \theta_s \circ \theta_t\colon \theta_t^{-1}(\operatorname{dom}(\theta_s)) \to \theta_s(\operatorname{ran}(\theta_t))$.

The semidirect product $\mathcal{S} \ltimes \Lambda = \{(s,\lambda) : \lambda \in \operatorname{dom}(\theta_s)\}$ has product

$$(s,\lambda)(t,\mu) = \left(st, \theta_t^{-1}(\lambda\theta_t(\mu))\right)$$

whenever sensible. We only consider the case when it is associative.

If A is an algebra and $\theta\colon \mathcal{S} \curvearrowright A$ by algebra isomorphisms of algebra ideals, the "naïve" crossed product is

$$\mathcal{S} \star A = \mathcal{A}(\mathcal{S} \ltimes A \to \mathcal{S}).$$

- All of this works for *partial actions* and $\wedge$-*preactions* (Exel; McAllister-Reilly).

Naïve crossed products and semidirect bundles

An action θ of $\mathcal{S}$ on a bundle $\pi\colon \Lambda \to \Gamma$ consists of two actions $\theta^{\Lambda}\colon \mathcal{S} \curvearrowright \Lambda$ and $\theta^{\Gamma}\colon \mathcal{S} \curvearrowright \Gamma$ which are intertwined by π:

$$\theta^{\Gamma}_s \circ \pi = \pi \circ \theta^{\Lambda}_s \qquad \text{for all } s \in \mathcal{S}.$$

(and θ^{Λ} respects the R-bimodule structures of Λ).

- Suppose that $\mathcal{S}$ is discrete and $\mathcal{S} \ltimes \Gamma$ is open in $\mathcal{S} \times \Gamma$. We define an action $\Theta\colon \mathcal{S} \curvearrowright \mathcal{A}(\pi)$ by

$$\operatorname{dom}(\Theta_s) = \left\{\alpha \in \mathcal{A}(\pi) : \operatorname{supp}(\alpha) \subseteq \operatorname{dom}(\theta^{\Gamma}_s)\right\}$$

$$\Theta_s(\alpha) = (\theta^{\Lambda}_s)^{-1} \circ \alpha \circ \theta^{\Gamma}_{s^*} \text{ on } \operatorname{dom}(\theta_{s^*}), \qquad 0 \text{ everywhere else.}$$

- At the bundle level, we create a new bundle $\mathcal{S} \ltimes \pi\colon \mathcal{S} \ltimes \Lambda \to \mathcal{S} \ltimes \Gamma$, $(s, \lambda) \mapsto (s, \pi(\lambda))$.

Theorem

$\mathcal{S} \star \mathcal{A}(\pi) \cong \mathcal{A}(\mathcal{S} \ltimes \pi)$.

Quotients of semigroupoids

A *rigid congruence* $\sim$ on a semigroupoid Λ is an equivalence relation $\sim$ such that

- $x \sim y$ implies $\mathfrak{s}(x) = \mathfrak{s}(y)$ and $\mathfrak{r}(x) = \mathfrak{r}(y)$;
- $x \sim y$ and $z \sim w$ implies $xz \sim yw$ (whenever sensible).

The quotient $\Lambda/\sim$ has a natural semigroupoid structure over $\Lambda^{(0)}$. If Λ is topological/étale/ample and $\sim$ is *open* then $\Lambda/\sim$ is topological/étale/ample.

Quotients of R-bundles

A congruence $\sim$ on an R-bundle $\pi\colon \Lambda \to \Gamma$ consists of rigid open congruences $\sim^\Lambda$ and $\sim^\Gamma$ such that

- π is a morphism from $\sim^\Lambda$ to $\sim^\Gamma$;
- $\sim^\Lambda$ is a congruence for the R-bimodule structures of $\pi^{-1}(\gamma)$ ($\gamma \in \Gamma$);
- If $x, y \in \Lambda$ and $\pi(x) \sim \pi(y)$ in Γ, then there exists $y' \in \Lambda$ such that $y \sim y'$ in Λ and $\pi(x) = \pi(y')$

The first property gives us a quotient map $\pi/\sim\colon \Lambda/\sim \to \Gamma/\sim$. The second and third properties give the R-bimodule structure of the fibers of $\pi/\sim$, which determines an R-bundle.

Quotients vs. quotients

The appropriate class of and element x of Γ or Λ is denoted $\widetilde{x}$.

Theorem

The map

$$T : \mathcal{A}(\pi) \to \mathcal{A}(\pi/\sim)$$

$$T(\alpha)(\widetilde{\gamma}) = \sum_{\delta\sim\gamma} \widetilde{\alpha(\delta)}$$

is a surjective algebra homomorphism.

Question

Is there a better description of the kernel of T, other than the obvious one?

The kernel of T

The *full semigroup* (or pseudogroup) $[[\sim]]$ of $\sim$ (in either Γ or Λ) is the set of all homeomorphisms $\varphi\colon V \to W$, where V and W are open, such that $\varphi(v) \sim v$ for all $v \in V$.

If $\alpha \in \mathcal{A}(\pi)$, we may "conjugate" α by appropriate elements of $[[\sim^\Gamma]]$ and $[[\sim^\Lambda]]$:

Suppose $\psi\colon A \to B$ in $[[\sim^\Lambda]]$ and $\varphi\colon \pi(B) \to \pi(A)$ in $[[\sim^\Gamma]]$ such that $\varphi \circ \pi = \pi \circ \psi$ on A. If $\alpha \in \mathcal{A}(\pi)$ is such that $\operatorname{supp}(\alpha) \subseteq \pi(B)$, then we may define

$$\psi\alpha\varphi = \psi \circ \alpha \circ \varphi \text{ on } \pi(A) \text{ and } 0 \text{ everywhere else.}$$

Then $\psi\alpha\varphi$ is *conjugate* to α, and $\alpha - \psi\alpha\varphi \in \ker T$.

Theorem

If $0(\Gamma)$ is $\sim^\Lambda$-saturated, then the kernel of T is generated as an additive group by all $\alpha - \psi\alpha\varphi$ as above.

An application

The natural order of an inverse semigroupoid is $s \leq t$ iff $s = ts^* s$.

Let θ be an open action of a discrete inverse semigroupoid $\mathcal{S}$ on a groupoid $\mathcal{G}$. The *groupoid of germs* of $\mathrm{Germ}(\theta)$ is the quotient of the semidirect product $\mathcal{S} \ltimes \mathcal{G}$ by the relation

$$(s, g) \sim (t, h) \iff \exists h = g \text{ and } u \leq s, t \text{ such that} u \in \mathrm{dom}(\theta_u).$$

(it is universal for morphisms from $\mathcal{S} \ltimes \mathcal{G}$ to groupoids).

Let $\Theta \colon \mathcal{S} \curvearrowright R\mathcal{G}$ by

$$\mathrm{dom}(\Theta_s) = \{f \in R\mathcal{G} : \mathrm{supp}(f) \subseteq \mathrm{dom}(\theta_s)\}$$

$$\Theta_s(f) = f \circ \theta_s^{-1} \text{ on } \mathrm{dom}(\theta_{s^*}), \quad 0 \text{ everywhere else.}$$

An application

Theorem

$$R\mathrm{Germ}(\theta) \cong \frac{\mathcal{S} \star R\mathcal{G}}{\langle f\delta_s - f\delta_t : s \leq t, f \in \mathrm{dom}(\Theta_s)\rangle}$$

(the right side is the "non-naïve" crossed product), where $f\delta_s$ *is the function taking* $s \mapsto f$ *in* $\mathcal{S} \star R\mathcal{G}$*, and all* $t \neq s$ *to* 0.

Remark

This is valid more generally for ample, non-Hausdorff semigroupoids.

This is a far-reaching generalization of known results of the theory of Steinberg algebras (Beuter-C., Hazrat-Li).

Tensor products and product bundles

Let $\pi\colon \Lambda \to \Gamma$ be an R-bundle, and $\mathcal{E}$ another (ample, Hausdorff) semigroupoid. Construct the new bundle $\pi \times \mathcal{E} = \pi \times \mathrm{id}_{\mathcal{E}}\colon \Lambda \times \mathcal{E} \to \Gamma \times \mathcal{E}$.

Theorem

The map

$$T\colon \mathcal{A}(\pi)\ {}_R{\otimes}_R\ R\mathcal{E} \to \mathcal{A}(\pi \times \mathcal{E})$$

$$T(\alpha \otimes f)(\gamma, e) = (\alpha(\gamma) f(e), e)$$

is an isomorphism.

Tensor products

Corollary

$R(\Gamma_1 \times \Gamma_2) \cong R\Gamma_1 \otimes R\Gamma_2$, for any ring R and ample Hausdorff groupoid Γ_1, Γ_2.

Corollary

If A is an R-algebra, then $A\Gamma \cong A\ {}_R{\otimes}_R\ R\Gamma$.

(Generalizes results of Rigby '18)

Smash products of algebras

Let G be a group and $A = \oplus_{g \in G} A_g$ a G-graded algebra. Let $p_g : A \to A_g$ be the projection. The *smash product* $A \# G$ is the algebra generated by symbols $a \# g$, $a \in A$, $g \in G$, with entrywise module structure and product

$$(a \# g)(b \# h) = a p_{gh^{-1}}(b) \# h$$

Then G acts on $A \# G$ by $g(a \# h) = a \# (hg^{-1})$.

Theorem (Duality theorems, Cohen-Montgomery '84)

$M_{G,0}(A) \cong G \ltimes (A \# G)$ and $A \cong (A \# G)^G$ (fixed algebra), where $M_{G,0}(A)$ is the algebra of finitely supported matrices indexed by $G \times G$, and coefficients in A.

Remark (C. 'Tuesday)

There is a version for groupoid gradings as well.

“Skew products”

Let G be a group, Γ a semigroupoid and $c\colon \Gamma \to G$ a homomorphism. Then Γ “acts on G by left multiplication”: $\gamma \cdot g = c(\gamma)g$; and we can create the skew product

$$\Gamma \# G = \Gamma \times G$$

with product

$$(\gamma_1, g)(\gamma_2, h) = (\gamma_1\gamma_2, h) \text{ whenever } g = c(\gamma_2)h$$

If we have an R-bundle $\pi\colon \Lambda \to \Gamma$ then $c \circ \pi\colon \Lambda \to G$.

Consider the “skew bundle” $\pi \# G = \pi \times \mathrm{id}_G\colon \Lambda \# G \to \Gamma \# G$.

Theorem

$\mathcal{A}(\pi \# G) \cong \mathcal{A}(\pi) \# G$.

Generalizes a result of Hazrat-Li ’18 for Steinberg algebras.

Natural questions

1. Isomorphism theorem for $\mathcal{A}(\pi)$ (analogous to the one in the motivation)?
2. If $\mathcal{S}$ is inverse, is $R\mathcal{S} \cong R\mathcal{G}$ for some groupoid $\mathcal{G}$ (true in the "trivial" case of a product of an inverse semigroup and a groupoid).