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On the Cohomology of Algebraic Varieties

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Abstract

An algebraic variety is an object which can be defined in a purely algebraic way, starting from polynomials or more generally from finitely generated algebras over fields. When the base field is the field of complex numbers, it can also be seen as a complex manifold, and more precisely a Kähler manifold. We will review a number of notions and results related to these two aspects of complex algebraic geometry. A crucial notion is that of Hodge structure, which already appears in the Kähler context, but seems to be meaningful and interpretable only in the context of algebraic geometry.

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1. Introduction

1.1. From topology to geometry and vice-versa. An algebraic variety V is defined by polynomial equations which are polynomials with coefficients in some field K. For any field K' containing K, one considers the set $V(K')$ of solutions with coefficients in K'. In particular, if $K \subset \mathbb{C}$, one can consider $V(\mathbb{C})$ which will be a subvariety of an affine or projective space. When the equations defining $V(\mathbb{C})$ locally satisfy the Jacobian criterion, $V(\mathbb{C})$ can also be seen as a complex manifold, and in particular a topological space, which is compact if the original V is projective. In fact, it is endowed with a so-called Kähler metric, which happens to be extremely restrictive topologically. So we get a first set of forgetting maps:

$$\{\text{Algebraic varieties over } K \subset \mathbb{C}\} \rightsquigarrow \{\text{Algebraic varieties over } \mathbb{C}\}$$
$$\rightsquigarrow \{\text{Compact Kähler manifolds}\}.$$

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Each step above can be seen as an enlargement of the category of functions used on the space $V(\mathbb{C})$: in the first case, rational functions with K-coefficients, in the second case, rational functions with complex coefficients, and in the third case, holomorphic functions rather than polynomials or rational functions.

That some structure is lost at each step is obvious, but it is not so clear whether these changes of category of pairs consisting of a space plus a class of functions, also correspond to relaxing topological restrictions. However, it was shown by Serre [39] in the 60's, for the first inclusion, and by the author [49] in 2004 for the second one, that at each step we get a strict inclusion at the level of topological spaces, even modulo homotopy equivalence. This will be the subject of section 3.3.

Starting with a compact Kähler manifold, we can forget some of its geometric structure. Indeed, a Kähler manifold is at the same time a complex manifold, a symplectic manifold and a Riemannian manifold, the three structures being compatible in a very nice way. It has been known for a very long time that compact Kähler manifolds are more restricted topologically than complex or symplectic manifolds. We will show in section 3.2 that there are in fact many more topological restrictions than the classical ones, obtained by introducing and exploiting the notion of *Hodge structure on a cohomology algebra* introduced in [54].

Continuing further, we can also forget about the complex structure or the symplectic structure, and then keep the differentiable manifold. All these operations again enlarge the class of topological spaces considered. Finally we can even forget about the differentiable structure and consider only the underlying topological space, which is a topological manifold. Its homotopy type or cohomology can be computed by combinatorial data: it is determined by the combinatorics of a good covering by open balls. A major result due to Donaldson [23] says that some topological manifolds do not admit any differentiable structure, so that in this last step, we still enlarge the category of topological spaces involved. Doing so, we also loose a tool which makes the essential bridge between geometry and topology, namely the use of differential forms to compute cohomology (and even homotopy, according to Sullivan [42]), which can be summarized under the name of de Rham theory and will be a guiding theme of this paper.

While we made a long walk from algebraic varieties to topological spaces, de Rham theorems appear to be crucial to understand partially the cohomology of a smooth complex algebraic variety V defined over a field $K \subset \mathbb{C}$, using only its structure as an algebraic variety (eg the ideal of polynomials vanishing on it), and not the topology of $V(\mathbb{C})$. The key point here is the fact that differentiating polynomial or rational functions is a formal operation. This way we can speak of algebraic differential forms and use them to "compute" the cohomology of our algebraic variety (cf. [29]).

A very mysterious and crucial fact is the following: according to whether we consider our complex algebraic variety over $K \subset \mathbb{C}$ as a topological space with its complex of singular cochains, or as a differentiable manifold with its complex of differential forms, or as a K-variety with its complex of algebraic differential forms with K-coefficients, we compute the "same" cohomology groups, but with *different coefficients* ($\mathbb{Q}$-coefficients for Betti cohomology, $\mathbb{R}$ or $\mathbb{C}$-coefficients for differentiable de Rham cohomology, K-coefficients for algebraic de Rham cohomology). Comparing these various groups is crucial in the theory of motives, or of periods (cf. [1]).

We will put the emphasis in this text on the following fact: Hodge theory on a compact Kähler manifold X provides beautiful objects attached to X, namely a Hodge structure of weight k on its rational cohomology of degree k, for any $k \geq 0$. We will show how to extract from the existence of such Hodge structures topological restrictions on X. When X is projective, it is furthermore expected that these Hodge structures reflect faithfully certain algebro-geometric properties of X, related to the structure of its algebraic subvarieties. The simplest example of such expectation is the Hodge conjecture, which predicts from the shape of the Hodge structure on $H^{2k}(X, \mathbb{Q})$ which degree $2k$ rational cohomology classes are generated over $\mathbb{Q}$ by classes of algebraic subvarieties of codimension k of X. This conjecture cannot be extended in the Kähler context (cf. [47] and section 2.1), which suggests that this is not a conjecture in complex differential topology, and that some extra structure existing on the cohomology of algebraic varieties, compatible with Hodge theory, has to be exploited. We will try to give an idea of what can be done in this direction in section 4. The rest of this introduction makes more precise the various tools and notions alluded to above.

1.2. De Rham theorems and Hodge theory. The degree i Betti cohomology $H^i(X, A)$ of a reasonable topological space (say a topological manifold) with value in any abelian group A can be computed in several ways, which correspond to various choices of acyclic resolutions of the constant sheaf A on X. Concretely, one can choose a triangulation of X and consider the simplicial cohomology of the associated simplicial complex. A more general approach uses singular cohomology, built from continuous cochains and their boundaries. The last one involves a good covering by open balls and the associated Čech complex.

The last approach, which is also the most natural from the viewpoint of sheaf cohomology, led Weil [56] to a new proof of the fundamental de Rham theorem [22], which says that in the differentiable case, cohomology with real coefficients can be computed using the complex of differential forms:

Theorem 1.1. *(de Rham) If X is a differentiable manifold, one has*

$$H^i(X, \mathbb{R}) = \frac{\{\text{closed real } i\text{−forms on } X\}}{\{\text{exact real } i\text{−forms on } X\}}. \tag{1.1}$$

An important point however is the fact that de Rham cohomology does not detect cohomology with rational coefficients.

The next step in relating topology and geometry is the major advance in differential topology due to Hodge (cf. [27], [48, Chapter 5]), which provides canonical representatives for the cohomology of a compact differentiable manifold endowed with a Riemannian metric.

For a general oriented Riemannian manifold (X, g), with corresponding volume form Vol_g, one has the Laplacian Δ_d acting on differential forms, preserving the degree, and given by the formula $\Delta_d = dd^* + d^*d$, where d^* is the formal adjoint of d with respect to the L^2-metric $(\alpha, \beta)_{L^2} := \int_X \langle\alpha, \beta\rangle Vol_g$ on compactly supported forms. A differential form α is said to be harmonic if $\Delta_g \alpha = 0$, or equivalently in the compact case, $d\alpha = d^*\alpha = 0$. When X is compact, a harmonic form on X has thus a de Rham cohomology class.

Theorem 1.2. *(Hodge) Let X be a compact orientable differentiable manifold. Then the map $\mathcal{H}^i(X) \to H^i(X, \mathbb{R})$ from the space of harmonic i-forms on X to real cohomology of degree i, which to a harmonic form associates its de Rham class, is an isomorphism.*

1.3. Kähler geometry and algebraic geometry.

A complex manifold (of complex dimension n) is a differentiable manifold of real dimension $2n$ with a set of charts with values in open sets of $\mathbb{C}^n$ such that the transition diffeomorphisms are holomorphic. Its tangent space has then a natural structure of complex vector bundle, given by its local identifications to the tangent space of $\mathbb{C}^n$.

A Kähler metric is a Hermitian metric on the tangent bundle of a complex manifold X which fits very nicely with the complex structure on X: The Hermitian metric h being locally written in holomorphic coordinates as $\sum_{i,j} h_{ij} dz_i \otimes d\overline{z}_j$, there is the corresponding real $(1,1)$-form

$$\omega = \frac{\iota}{2} \sum_{i,j} h_{ij} dz_i \wedge d\overline{z}_j,$$

(the Kähler form), and the Kähler condition is simply $d\omega = 0$. The closed 2-form ω has a de Rham class $[\omega] \in H^2(X, \mathbb{R})$, called the Kähler class of the metric.

A projective complex variety X (defined over a field $K \subset \mathbb{C}$) is the set of solutions of a finite number of equations $P_i(x) = 0$, $x = (x_0, \ldots, x_N) \in \mathbb{P}^N(\mathbb{C})$, where the P_i are homogeneous polynomials (with coefficients in K) in the coordinates x_i.

The P_i's give local rational, hence holomorphic, equations for X, which is thus a closed analytic subset of $\mathbb{P}^N(\mathbb{C})$ as well. A remarkable result due to Chow and generalized later on by Serre [38] says that any closed analytic subset of $\mathbb{P}^N(\mathbb{C})$ is in fact algebraic. When the local defining equations of X can be chosen to have independent differentials, X is a complex submanifold of $\mathbb{P}^N(\mathbb{C})$. We will say that X is a complex projective manifold (defined over K).

The Kodaira criterion [32] characterizes projective complex manifolds inside the class of compact Kähler manifolds.

Theorem 1.3. *A compact complex manifold X is projective if and only if X admits a Kähler class $[\omega]$ which is rational, that is belongs to*

$$H^2(X,\mathbb{Q}) \subset H^2(X,\mathbb{R}).$$

The "only if" is easy. It comes from the fact that if X is projective, one gets a Kähler form on X by restricting the Fubini-Study Kähler form on some projective space $\mathbb{P}^N$ in which X is imbedded as a complex submanifold. But the Fubini-Study Kähler form has integral cohomology class, as its class is the first Chern class of the holomorphic line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ on $\mathbb{P}^N$.

The converse is a beautiful application of the Kodaira vanishing theorem for line bundles endowed with a metric whose Chern form is positive.

1.4. Topology and algebraic geometry. As we mentioned already, one way to put a topology on a complex algebraic variety is to use the topology on the ambient space $\mathbb{C}^n$ or $\mathbb{CP}^n$. This is what we will call the classical topology. There is however another topology, the Zariski topology, which has the property that the closed subsets are the closed algebraic subsets of X, that is, subsets defined by the vanishing of polynomial equations restricted to X. These sets are closed for the classical topology, so this topology is weaker than the classical topology.

This topology is in fact very weak. Indeed, if the variety is "irreducible" (for example smooth and connected), any two Zariski open sets intersect non trivially by analytic continuation. It easily follows that the cohomology of X, endowed with the Zariski topology, with constant coefficients, (that is, with value in a constant sheaf) is trivial. However, the Zariski topology is excellent to compute the cohomology of X with values in other softer sheaves, namely the "coherent sheaves": There is the notion of algebraic vector bundle on X, and even algebraic vector bundle defined over K if X is. Namely, in some Zariski open cover (defined over K), it is trivialized, and the transition matrices are matrices of algebraic functions with K-coefficients. The simplest coherent sheaves are sheaves of algebraic sections of such vector bundles. The general ones allow singularities.

Let us assume that $K = \mathbb{C}$ and let E be such an algebraic vector bundle. There are two things we can do to compute the "cohomology of X with value in E".

1) X is endowed with the Zariski topology and one considers the sheaf $\mathcal{E}$ of algebraic sections of E. Then we compute cohomology of the sheaf $\mathcal{E}$ by general methods of sheaf cohomology, using acyclic resolutions. Concretely, it suffices to compute Čech cohomology with respect to an affine covering. Let us denote these groups $H^l(X_{Zar},\mathcal{E})$.

2) We put on X the classical topology and consider the sheaf of *holomorphic* sections $\mathcal{E}^{an}$ of E in the classical topology. Let us denote these groups $H^l(X_{cl},\mathcal{E}^{an})$.

It is a remarquable fact (the "GAGA principle", [38]), proved by Serre, that the resulting cohomology groups are the same.

Theorem 1.4. *(Serre) For any algebraic coherent sheaf $\mathcal{E}$ on X, one has a canonical (inverse image) isomorphism $H^l(X_{Zar},\mathcal{E}) \to H^l(X_{cl},\mathcal{E}^{an})$.*

Why then to care about the Zariski topology and the algebraic vector bundles? One reason is the fact that staying in the algebraic geometry setting allows to take care of the fields of definition of a variety X and a vector bundle E on it; such a field of definition contains the coefficients of defining equations of X, or the coefficients of rational functions involved in the transition matrices of E. If X, E are defined over a subfield $K \subset \mathbb{C}$, then we can compute the cohomology of X_K, endowed with the "K-Zariski topology" (for which closed subsets are closed algebraic subsets defined by polynomial equations with K-coefficients), with value in $\mathcal{E}_K$ (the sheaf of sections defined over K), and there is an isomorphism (which is called a K-structure on $H^i(X,\mathcal{E})$):

$$H^i(X,\mathcal{E}) = H^i(X_K,\mathcal{E}_K) \otimes_K \mathbb{C}.$$

We already mentioned that the Zariski topology is not good at all to compute Betti cohomology of X endowed with its classical topology. However, holomorphic de Rham theory combined with GAGA allows in fact to compute Betti cohomology of X, at least with complex coefficients, using algebraic differentials and the Zariski topology. This result due to Grothendieck [29] is crucial to understand the notion of absolute Hodge class [19] that will be discussed in section 4.2.

Étale cohomology invented by Grothendieck is another way of constructing an intrinsic cohomology theory, not depending on the topology of the field $\mathbb{C}$. It depends on introducing étale topology which is a refinement of Zariski topology, and is not actually a topology: Roughly speaking, one adds to the Zariski open sets their étale covers. Furthermore, Artin's comparison theorems allow to compare it to Betti cohomology. However, this theory does not allow to recover Betti cohomology with *rational* coefficients (see for example [13]) of our classical topological space X, but only its Betti cohomology with finite or l-adic coefficients.

The presence of various cohomology theories with comparison theorems between them is at the heart of Grothendieck's theory of *Motives* (cf. [1]).

Our last topic in section 4.4 will be another way to go around the fact that the Zariski topology is too weak to compute Betti cohomology of the corresponding complex manifold. This is by looking at the spectral sequence associated to the obviously continuous map

$$X_{cl} \to X_{Zar}$$

which is the identity on points. This study is done by Bloch-Ogus [8] and leads to beautiful results when combined with algebraic K-theory.

2. Hodge Theory in Kähler or Projective Geometry

2.1. Hodge structures. Let us start with the notion of cohomology class of type (p,q) on a complex manifold X. On such an X, we have the notion of differential form of type (p,q): these are the complex differential forms α (say of class $\mathcal{C}^\infty$), which can be written in local holomorphic coordinates $z_1,\ldots,z_n$, $n = dim_{\mathbb{C}}X$, and in the multiindex notation:

$$\alpha = \sum_{I,J} \alpha_{I,J} dz_I \wedge d\overline{z}_J,\ |I| = p,\ |J| = q,$$

where $\alpha_{I,J}$ are $\mathcal{C}^\infty$ functions. Let us denote $A^{p,q}(X)$ the space of (p,q)-forms on X. Thus $A^{p,q}(X) \subset A^k(X)$, $p+q=k$, where $A^k(X)$ is the space of C^∞ complex differential k-forms on X. The cohomology $H^k(X,\mathbb{C})$ with complex coefficients can be computed by de Rham theorem as

$$H^k(X,\mathbb{C}) = \frac{\{\text{closed complex } k\text{−forms on } X\}}{\{\text{exact complex } k\text{−forms on } X\}},$$

and it is natural to define for a complex manifold X and for each (p,q) the space of cohomology classes of type (p,q) by the formula

$$H^{p,q}(X) := \frac{\{\text{closed forms of type } (p,q) \text{ on } X\}}{\{\text{exact forms of type } (p,q) \text{ on } X\}}.$$

The following result is a famous result due to Hodge.

Theorem 2.1. *(The Hodge decomposition theorem) Let X be a compact Kähler manifold. Then for any integer k, one has $H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$.*

The proof of this theorem uses the representation of cohomology classes by harmonic forms (Theorem 1.2 above), together with the fact that the (p,q)-components of harmonic forms are harmonic, a fact which is specific to the Kähler case.

The decomposition above satisfies *Hodge symmetry*, which says that

$$\overline{H^{p,q}(X)} = H^{q,p}(X), \tag{2.2}$$

where complex conjugation acts naturally on $H^k(X,\mathbb{C}) = H^k(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. This is because the complex conjugate of a closed form of type (p,q) is a closed form of type (q,p).

We have the change of coefficients theorems $H^k(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^k(X,\mathbb{C})$. On the other hand, Theorem 2.1 gives the decomposition $H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$, satisfying Hodge symmetry (2.2). These data are summarized in the following definition.

Definition 2.2. *A rational (resp. integral) Hodge structure of weight k is a finite dimensional $\mathbb{Q}$-vector space (resp. a lattice, that is a free $\mathbb{Z}$-module of finite rank) V, together with a decomposition:*

$$V_{\mathbb{C}} := V \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_{p+q=k} V^{p,q}$$

satisfying Hodge symmetry.

The cohomology $H^k(X, \mathbb{Q})$, X compact Kähler, carries such a structure. Note that the Hodge decomposition depends on the Kähler complex structure. Furthermore, the dimensions $h^{p,q}(X) := rk\, H^{p,q}(X)$ are not topological invariants of X, although they are constant under deformations of the Kähler complex structure. However, the following classical remark shows that the Hodge decomposition provides topological restrictions on compact Kähler manifolds.

Remark 2.3. A Hodge structure of odd weight has its underlying $\mathbb{Q}$-vector space of even dimension. Hence a compact Kähler manifold X has

$$b_{2i+1}(X) := \dim_{\mathbb{Q}} H^{2i+1}(X, \mathbb{Q})$$

even for any i.

Example 2.4. There is an equivalence of categories between the set of (integral) Hodge structures of weight 1 and the set of complex tori. To $L, L_{\mathbb{C}} = L^{1,0} \oplus \overline{L^{1,0}}$ corresponds $T := \frac{L_{\mathbb{C}}}{L^{1,0} \oplus L}$. In the reverse direction, associate to T the Hodge structure on $H^{2g-1}(T, \mathbb{Z})$, $g = \dim T$.

Example 2.5. *(Trivial Hodge structure)* A Hodge structure $(V, V^{p,q})$ of weight $2k$ is trivial if $V_{\mathbb{C}} = V^{k,k}$.

The following definition is crucial:

Definition 2.6. *If $(V, V^{p,q})$ is a rational Hodge structure of weight $2k$, Hodge classes of V are elements of $V \cap V^{k,k}$.*

The simplest examples of Hodge classes on a compact Kähler manifold are given by the cohomology classes of closed analytic subspaces $Z \subset X$ of codimension k. The singular locus Z_{sing} of such a Z is then a closed analytic subset of X which has codimension $\geq k+1$ and thus real codimension $\geq 2k+2$. Thus one can define

$$[Z] \in H^{2k}(X, \mathbb{Z})$$

by taking the cohomology class $[Z \setminus Z_{sing}] \in H^{2k}(X \setminus Z_{sing}, \mathbb{Z})$ of the closed complex submanifold

$$Z \setminus Z_{sing} \subset X \setminus Z_{sing}$$

and by observing that $H^{2k}(X \setminus Z_{sing}, \mathbb{Z}) \cong H^{2k}(X, \mathbb{Z})$.

The class $[Z]$ is an integral Hodge class. This can be seen using Lelong's theorem, showing that the current of integration over $Z \setminus Z_{sing}$ is well defined and closed, with cohomology class equal to the image of $[Z]$ in $H^{2k}(X, \mathbb{C})$. On the other hand, this current anihilates all forms of type (p, q), $p \neq q$, $p + q = 2n - 2k$, $n = dim\, X$, and it follows dually that its class is of type (k, k).

The Hodge conjecture is the following statement:

Conjecture 2.7. *Let X be a complex projective manifold. Then the space $Hdg^{2k}(X)$ of degree $2k$ rational Hodge classes on X is generated over $\mathbb{Q}$ by classes $[Z]$ constructed above.*

It would be natural to try to formulate the Hodge conjecture in the Kähler context. However, it seems that there is no way to do this, and this is the reason why we will focus on the interplay between Hodge theory and algebraic geometry in section 4. First of all, it has been known for a long time that Hodge classes on compact Kähler manifolds are not in general generated over $\mathbb{Q}$ by classes of closed analytic subsets. The simplest such example is provided by a complex torus T admitting a holomorphic line bundle $\mathcal{L}$ of indefinite curvature. This means in this case that the harmonic de Rham representative of the Chern class $c_1(\mathcal{L})$ is given by a real $(1, 1)$-form with constant coefficients on T having the property that the corresponding Hermitian form on the tangent space of T is indefinite. If the torus T satisfying this condition is chosen general enough, its space $Hdg^2(T)$ will be generated by $c_1(\mathcal{L})$, as one shows by a deformation argument. It follows that T will not contain any analytic hypersurface, hence no non zero degree 2 Hodge class can be constructed as the Hodge class of a codimension 1 closed analytic subset, while $c_1(\mathcal{L})$ provides a non zero Hodge class on T.

However, there are two other constructions of Hodge classes starting from analytic objects:

1) Chern classes of holomorphic vector bundles: one uses the Chern connection and Chern-Weil theory to show that they are indeed Hodge classes.

2) Chern classes of analytic coherent sheaves (that is, roughly speaking, sheaves of sections of singular holomorphic vector bundles): the construction is much more delicate. We refer to [28] for a recent elegant construction.

In the projective case, it is known that the three constructions generate over $\mathbb{Q}$ the same space of Hodge classes (cf. [38] and [9]). In the general Kähler case, the torus example above shows that Chern classes of holomorphic vector bundles or coherent sheaves may provide more Hodge classes than cycle classes. The fact that Chern classes of coherent sheaves allow in some cases to construct strictly more Hodge classes than Chern classes of holomorphic vector bundles was proved in [47]. This is something which cannot be detected in degree 2, as in degree 2, Chern classes of holomorphic line bundles generate all integral Hodge classes, a fact which is known as the Lefschetz theorem on $(1, 1)$-classes. To conclude, one can prove as in [9] that cycle classes are generated by Chern classes of analytic coherent sheaves.

If we want to extend the Hodge conjecture to the Kähler case, we therefore are led to consider the following question:

Question 2.8. *Are Hodge classes on compact Kähler manifolds generated over* $\mathbb{Q}$ *by Chern classes of coherent sheaves?*

We provided in [47] a negative answer to this question. Weil tori [57] are complex tori T of even dimension $2n$ admitting an endomorphism ϕ satisfying $\phi^2 = -d\,\mathrm{Id}_T$, $d > 0$ and a certain sign assumption concerning the action of ϕ on holomorphic forms on T, which implies that T carries a 2-dimensional $\mathbb{Q}$-vector space of Hodge classes of degree $2n$. The following result shows that they provide a counterexample to question 2.8, thus showing that the projectivity assumption is crucial in the statement of the Hodge conjecture.

Theorem 2.9. *[47] Let* T *be a general Weil torus of dimension* 4*. Then any analytic coherent sheaf* $\mathcal{F}$ *on* T *satisfies* $c_2(\mathcal{F}) = 0$*. Thus the Weil Hodge classes on them are not in the space generated by Chern classes of coherent sheaves.*

2.2. Hard Lefschetz theorem and Hodge-Riemann relations.

Another very deep application of Hodge theory is the hard Lefschetz theorem, which says the following: let X be a compact Kähler manifold of dimension n and $[\omega] \in H^2(X, \mathbb{R})$ be the class of a Kähler form ω on X. Cup-product with $[\omega]$ gives an operator usually denoted by $L : H^*(X, \mathbb{R}) \to H^{*+2}(X, \mathbb{R})$.

Theorem 2.10. *For any* $k \leq n$*,*

$$L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$$

is an isomorphism.

A first formal consequence of the hard Lefschetz theorem 2.10 is the so-called Lefschetz decomposition. With the same notations as before, define for $k \leq n$ the primitive degree k cohomology of X by

$$H^k(X, \mathbb{R})_{prim} := Ker\,(L^{n-k+1} : H^k(X, \mathbb{R}) \to H^{2n+2-k}(X, \mathbb{R})).$$

For example, if $k = 1$, the whole cohomology is primitive, and if $k = 2$, primitive cohomology is the same as the orthogonal subspace, with respect to Poincaré duality, of $[\omega]^{n-1} \in H^{2n-2}(X, \mathbb{R})$.

The Lefschetz decomposition is given in the following theorem (it can also be extended to $k > n$ using the hard Lefschetz isomorphism).

Theorem 2.11. *The cohomology groups* $H^k(X, \mathbb{R})$ *for* $k \leq n$ *decompose as*

$$H^k(X, \mathbb{R}) = \oplus_{2r \leq k} L^r H^{k-2r}(X, \mathbb{R})_{prim}.$$

2.2.1. Hodge-Riemann bilinear relations. We consider a Kähler compact manifold X with Kähler class $[\omega]$. We can define an intersection form q_ω on each $H^k(X,\mathbb{R})$ by the formula

$$q_\omega(\alpha,\beta)=\int_X [\omega]^{n-k}\cup\alpha\cup\beta.$$

By hard Lefschetz theorem and Poincaré duality, q_ω is a non-degenerate bilinear form. It is skew-symmetric if k is odd and symmetric if k is even. Furthermore, the extension of q_ω to $H^k(X,\mathbb{C})$ satisfies the property that

$$q_\omega(\alpha,\beta)=0,\ \alpha\in H^{p,q},\ \beta\in H^{p',q'},\ (p',q')\neq(q,p).$$

This property is indeed an immediate consequence of Lemma 3.1 and the fact that $H^{2n}(X,\mathbb{C})=H^{n,n}(X)$, $n=dim_{\mathbb{C}}X$.

Equivalently, the Hermitian pairing h_ω on $H^k(X,\mathbb{C})$ defined by

$$h_\omega(\alpha,\beta)=\iota^k q_\omega(\alpha,\overline{\beta})$$

has the property that the Hodge decomposition is orthogonal with respect to h_ω. This property is summarized under the name of first Hodge-Riemann bilinear relations.

It is also an easy fact that the Lefschetz decomposition is orthogonal with respect to q_ω. To state the second Hodge-Riemann bilinear relations, note that, because the operator L shifts the Hodge decomposition by $(1,1)$, the primitive cohomology has an induced Hodge decomposition:

$$H^k(X,\mathbb{C})_{prim}=\oplus_{p+q=k}H^{p,q}(X)_{prim},$$

with $H^{p,q}(X)_{prim}:=H^{p,q}(X)\cap H^{p+q}(X,\mathbb{C})_{prim}$. We have now

Theorem 2.12. *The Hermitian form h_ω is definite of sign $(-1)^{\frac{k(k-1)}{2}}\iota^{p-q-k}=:\epsilon(p,q,r)$ on the component $L^rH^{p,q}(X)_{prim}$, $2r+p+q=k$, of $H^k(X,\mathbb{C})$.*

The Hodge-Lefschetz decomposition is particularly interesting when $[\omega]$ can be chosen to be rational, so that X is projective by Kodaira's embedding Theorem 1.3. Indeed, in this case, this is a decomposition into a direct sum of rational Hodge substructures. Furthermore the intersection form q_ω is rational. On each piece of the Lefschetz decomposition, it induces up to sign a *polarization* of the considered Hodge substructure. This notion will come back later on. Let us just say that we mentioned in Example 2.4 the equivalence of categories

$$\{\text{Weight 1 Hodge structures}\}\leftrightarrow\{\text{Complex tori}\}.$$

This can be completed by saying that there is an equivalence of categories

$$\{\text{Weight 1 polarized Hodge Structures}\}\leftrightarrow\{\text{Abelian varieties}\}.$$

Indeed, an intersection form q on L, $L_{\mathbb{C}}=L^{1,0}\oplus\overline{L^{1,0}}$ satisfying the Hodge-Riemann bilinear relations as in Theorem 2.12 provides an integral Kähler class on the torus $T=L_{\mathbb{C}}/(L^{1,0}\oplus L)$ which is thus projective by Theorem 1.3.

3. Hodge Structures on Cohomology Algebras

The following simple Lemma 3.1 is a direct consequence of the following two facts:

1) Under the de Rham isomorphism (1.1), the cup-product is induced by wedge product of differential forms.

2) The wedge product of a closed (p,q)-form and of a closed (p',q')-form is a closed $(p+p',q+q')$-form.

Lemma 3.1. *If X is a compact Kähler manifold, the Hodge decomposition on $H^*(X,\mathbb{C})$ is compatible with cup-product:*

$$H^{p,q}(X) \cup H^{p',q'}(X) \subset H^{p+p',q+q'}(X).$$

Below, *a cohomology algebra* A^* is the rational cohomology algebra of a connected orientable compact manifold, or more generally any graded commutative finite dimensional $\mathbb{Q}$-algebra satisfying Poincaré duality: for some integer $d \geq 0$, $A^0 = A^d = \mathbb{Q}$ and $A^k \otimes A^{d-k} \to A^d$ is perfect for any k. By analogy, d will be called the dimension of A^*.

Definition 3.2. *(Voisin 2008) A Hodge structure on A^* is the data of a Hodge structure of weight k on each A^k (i.e. a Hodge decomposition on $A^k_{\mathbb{C}}$, satisfying Hodge symmetry), such that:*

$$A^{p,q}_{\mathbb{C}} \cup A^{p',q'}_{\mathbb{C}} \subset A^{p+p',q+q'}_{\mathbb{C}}.$$

Let us state a number of obvious properties:

1. By Remark 2.3, if A^* admits a Hodge structure, $\dim A^{2k+1}$ is even, $\forall k$.

2. If A^* admits a Hodge structure, then d is even. This follows from 1 because $\dim A^d = 1$ and A^d carries a Hodge structure of weight d.

3. Any cohomology algebra with trivial odd part admits a Hodge structure, namely the trivial one:
$$A^{2k}_{\mathbb{C}} = A^{k,k},\ \forall k.$$

Definition 3.3. *(Hodge class on A^*) A Hodge class in A^* is an element of $A^{2k} \cap A^{k,k}$ for some k.*

Hodge classes $a \in A^{2k}$ act by multiplication on A^*, sending A^l to A^{l+2k}. These morphisms are morphisms of Hodge structures, hence "special". For example, the simplest restriction on them is the following: they are of even rank if l is odd. Indeed, the image of a morphism of Hodge structures is a Hodge substructure (see [48, I,7.3.1]) hence, in particular, of even dimension if it has odd weight (cf. Remark 2.3).

3.0.2. Deligne's lemma. This lemma implies that, given a cohomology algebra A^*, some classes $a \in A^*$ must be Hodge classes for any Hodge structure on A^*.

Let $Z \subset A^k_{\mathbb{C}}$ be a closed algebraic subset defined by homogeneous equations depending only on the structure of multiplication on A^*. Concretely, the following examples will be interesting for applications: fixing l and s, let

$$Z := \{z \in A^k_{\mathbb{C}},\, \mathrm{rank}\, z : A^l \to A^{l+k} \leq s\}.$$

A second kind of examples is as follows: Fixing l, let

$$Z := \{z \in A^k_{\mathbb{C}},\, z^l = 0 \text{ in } A^{kl}\}.$$

Lemma 3.4. *(Deligne) Let Z' be a an irreducible component of Z, and $V := \langle Z' \rangle \subset A^k_{\mathbb{C}}$ be the complex vector subspace generated by Z'. Then V is stable under Hodge decomposition, i.e. $V = \oplus V^{p,q}$, where $V^{p,q} = V \cap A^{p,q}$.*

Corollary 3.5. *Under the same assumptions, if V is defined over $\mathbb{Q}$, this is a Hodge substructure of A^k.*

Corollary 3.6. *Under the same assumptions, if $\dim V = 1$ and V is defined over $\mathbb{Q}$, it is generated by a Hodge class.*

3.1. Polarized Hodge structures on cohomology algebras.

Let A^* be a cohomology algebra with Hodge structure.

Definition 3.7. *A Hodge structure on A^* admits a real polarization if some $\alpha \in A^{1,1}_{\mathbb{R}}$ satisfies the hard Lefschetz property and the Hodge-Riemann bilinear relations.*

We will say that the Hodge structure on A^* admits a rational polarization if furthermore α can be chosen in $A^2_{\mathbb{Q}} \cap A^{1,1}$.

Here the hard Lefschetz property and the Hodge-Riemann bilinear relations are the analogs of their geometric counterparts described in section 2.2. The hard Lefschetz property implies formally the Lefschetz decomposition (cf. [48, I,6.2.2]): $A^k_{\mathbb{R}} = \oplus_{k-2r \geq 0} \alpha^r A^{k-2r}_{\mathbb{R},prim}$, $k \leq n$, $2n = dim\, A^*$. When α is real of type $(1,1)$ with respect to a Hodge structure on A^*, this is a decomposition into real Hodge sub-structures thus giving a *Hodge-Lefschetz decomposition* of $A^k_{\mathbb{C}}$ into terms of type $\alpha^r A^{p,q}_{prim}$, $2r + p + q = k$.

The Hodge-Riemann relations (cf. Theorem 2.12) say in this context that

$$h_\alpha(a,b) := \iota^k \alpha^{n-k} \cdot a \cdot \overline{b} \in A^{2n}_{\mathbb{C}} \cong \mathbb{C}\alpha^n$$

has a definite sign $\epsilon(p,q,r)$ on each piece of this Hodge-Lefschetz decomposition.

In the rest of this section, we are going to apply these notions to prove the following results:

1. There are very simple examples of compact symplectic manifolds satisfying all the "classical" restrictions (i.e they are formal, satisfy the hard Lefschetz property, have abelian fundamental group), but which are topologically non Kähler (that is, are not homotopically equivalent to a compact Kähler manifold). Such manifolds can be constructed as complex projective bundles over complex tori.

2. (The Kodaira problem) There exist compact Kähler manifolds which are not homeomorphic (and, in fact, not homotopically equivalent) to complex projective manifolds.

The criterion that we will use to prove that the constructed examples as in 1 are topologically non Kähler is the following:

Criterion 3.8. *The cohomology algebra of a compact Kähler manifold carries a Hodge structure. (We can also use as a strengthened criterion the existence of a Hodge structure with real polarization to get more sophisticated examples, eg simply connected examples).*

The criterion that we will use to prove that the constructed examples as in 2 are topologically non projective is the following version of criterion 3.8, where the rational polarization plays now a crucial role, as in Kodaira's embedding Theorem 1.3:

Criterion 3.9. *The cohomology algebra of a complex projective manifold carries a Hodge structure with rational polarization.*

3.2. Symplectic versus Kähler manifolds.

There is a close geometric relation between symplectic geometry and Kähler geometry. If X is compact Kähler, forgetting the complex structure on X and keeping a Kähler form provides a pair (X, ω) which is a symplectic manifold.

On the other hand, numerous topological restrictions are satisfied by compact Kähler manifolds, and not by general symplectic manifolds (cf. [44]). For example, very strong restrictions on fundamental groups of compact Kähler manifolds have been found (see [2]) while Gompf proves in [24] that fundamental groups of compact symplectic manifolds are unrestricted in the class of finitely generated groups.

Hodge theory provides two classical restrictions which come directly from what we discussed in section 2.

1. The odd degree Betti numbers $b_{2i+1}(X)$ are even for X compact Kähler (see Remark 2.3).

2. The hard Lefschetz property (cf. Theorem 2.10), saying that the cup-product maps $[\omega]^{n-k}\cup : H^k(X, \mathbb{R}) \cong H^{2n-k}(X, \mathbb{R})$, $2n = dim_{\mathbb{R}}X$ are isomorphisms, is satisfied.

Another topological restriction on compact Kähler manifolds is the so-called formality property [17]. A number of methods to produce examples of symplectic topologically non Kähler manifolds were found by Thurston [43], McDuff [34], Gompf [24]. On these examples, one of the properties above was not satified.

We want to exhibit here further topological restrictions on compact Kähler manifolds, coming from Criterion 3.8. One of the difficulties to exploit this criterion is the fact that the $h^{p,q}$-numbers of the Hodge decomposition are not determined topologically. Thus we have to analyse abstractly the constraints imposed by the existence of a polarized Hodge structure on the cohomology algebra without knowing the $h^{p,q}$-numbers or the set of polarization classes. Let us give a sample of results in this direction. The proofs, which are purely algebraic, are all based on Deligne's lemma 3.4.

We start with an orientable compact manifold X and consider a complex vector bundle E on X. We denote by $p : \mathbb{P}(E) \to X$ the corresponding projective bundle.

We make the following assumptions on (X, E):

$$H^*(X) \textit{ generated in degree} \leq 2 \textit{ and } c_1(E) = 0.$$

As a consequence of Leray-Hirsch theorem, one has an injection (of algebras) $p^* : H^k(X, \mathbb{Q}) \hookrightarrow H^*(\mathbb{P}(E), \mathbb{Q})$ (cf. [48, 7.3.3]).

Theorem 3.10. *[54] If $H^*(\mathbb{P}(E), \mathbb{Q})$ admits a Hodge structure, then each subspace $H^k(X, \mathbb{C}) \subset H^k(\mathbb{P}(E), \mathbb{C})$ has an induced Hodge decomposition (and thus the cohomology algebra $H^*(X, \mathbb{Q})$ also admits a Hodge structure).*

Furthermore each $c_i(E) \in H^{2i}(X, \mathbb{Q})$, $i \geq 2$, is of type (i, i) for this Hodge structure on $H^{2i}(X, \mathbb{Q})$.

This allows the construction of symplectic manifolds with abelian fundamental group satisfying formality (cf [17]) and the hard Lefschetz property, but not having the cohomology algebra of a compact Kähler manifold. These manifolds are produced as complex projective bundles over compact Kähler manifolds (eg complex tori), which easily implies that all the properties above are satisfied. We start with a compact Kähler manifold X having a given class $\alpha \in H^4(X, \mathbb{Q})$ such that for any Hodge structure on $H^*(X, \mathbb{Q})$, α is not of type $(2, 2)$. Then if E is any complex vector bundle on X satisfying $c_1(E) = 0$, $c_2(E) = N\alpha$, for some integer $N \neq 0$, $\mathbb{P}(E)$ is topologically non Kähler by Theorem 3.10, using Criterion 3.8.

The simplest example of such a pair (X, α) is obtained by choosing for X a complex torus of dimension at least 4 and for α a class satisfying the property that the cup-product map $\alpha\cup : H^1(X, \mathbb{Q}) \to H^5(X, \mathbb{Q})$ has odd rank. Indeed, if α was Hodge for some Hodge structure on the cohomology algebra of X, this morphism would be a morphism of Hodge structures, and its kernel would be a Hodge substructure of $H^1(X, \mathbb{Q})$, hence of even rank by Remark 2.3.

3.3. Kähler versus projective manifolds.

Kodaira's characterization Theorem 1.3 can be used to show that certain compact Kähler manifolds X become projective after a small deformations of their complex structure. The point is that the Kähler classes belong to $H^{1,1}(X)_{\mathbb{R}}$, the set of degree 2 cohomology classes which can be represented by a real closed $(1,1)$-form. They even form an open cone, the Kähler cone, in this real vector subspace of $H^2(X, \mathbb{R})$. This subspace deforms differentiably with the complex structure, and by Kodaira's criterion we are reduced to see whether one can arrange that, after a small deformation, the Kähler cone contains a rational cohomology class.

Example 3.11. Complex tori admit arbitrarily small deformations which are projective.

The following beautiful theorem of Kodaira is at the origin of the work [49].

Theorem 3.12. *[33] Let S be a compact Kähler surface. Then there is an arbitrarily small deformation of the complex structure on S which is projective.*

Kodaira proved this theorem using his classification of complex surfaces. Buchdahl ([10], [11]) gives a proof of Kodaira theorem which does not use the classification. His proof is infinitesimal and shows for example that a rigid compact Kähler surface is projective.

3.3.1. Various forms of the Kodaira problem.

Kodaira's theorem 3.12 immediately leads to ask a number of questions in higher dimensions:

Question 3.13. *(The Kodaira problem) Does any compact Kähler manifold admit an arbitrarily small deformation which is projective?*

In order to disprove this, it suffices to find rigid Kähler manifolds which are not projective. However, the paper [21] shows that it is not so easy: if a complex torus T carries three holomorphic line bundles L_1, L_2, L_3 such that the deformations of T preserving the L_i are trivial, then T is projective. The relation with the previous problem is the fact that from (T, L_1, L_2, L_3), one can construct a compact Kähler manifold whose deformations identify to the deformations of the quadruple (T, L_1, L_2, L_3).

A weaker question concerns global deformations.

Question 3.14. *(The global Kodaira problem) Does any compact Kähler manifold X admit a deformation which is projective?*

Here we consider any deformation parameterized by a connected analytic space B, that is any smooth proper map $\pi : \mathcal{X} \to B$ between connected analytic spaces, with $X_0 = X$ for some $0 \in B$. Then any fiber X_t will be said to be a deformation of X_0. In that case, even the existence of rigid Kähler manifolds which are not projective would not suffice to provide a negative answer, as there exist families of compact Kähler manifolds $\pi : \mathcal{X} \to B$ all of whose fibers X_t for $t \neq 0$ are isomorphic but are not isomorphic to the central fiber X_0.

Note that if X is a deformation of Y, then X and Y are diffeomorphic, because the base B is path connected, and by the Ehresmann theorem (cf. [48, 9.1.1]), the family of deformations $\mathcal{X} \to B$ can be trivialized in the $\mathcal{C}^\infty$-category over any path in B.

In particular, X and Y should be homeomorphic, hence have the same homotopy type, hence also the same cohomology ring. Thus Question 2 can be weakened as follows:

Question 3.15. *(The topological Kodaira problem) Is any compact Kähler manifold X diffeomorphic or homeomorphic to a projective complex manifold?*

Does any compact Kähler manifold X have the homotopy type of a projective complex manifold?

The following theorem answers negatively the questions above.

Theorem 3.16. *There exist, in any complex dimension ≥ 4, compact Kähler manifolds which do not have the rational cohomology algebra of a projective complex manifold.*

Our first proof used the integral cohomology ring. Deligne provided then us with lemma 3.4, which allowed him to extend the result to cohomology with rational coefficients, and even, after modification of our original example, complex coefficients, (see [49]). We in turn used this lemma to construct simply connected examples.

The examples in [49] were built by blowing-up in an adequate way compact Kähler manifolds which had themselves the property of deforming to projective ones, namely self-products of complex tori, or self-products of Kummer varieties. This left open the possibility suggested by Buchdahl, Campana and Yau, that under bimeromorphic transformations, the topological obstructions we obtained above for a Kähler manifold to admit a projective complex structure would disappear. However we proved in [50] the following result.

Theorem 3.17. *In dimensions ≥ 10, there exist compact Kähler manifolds, no smooth bimeromorphic model of which has the rational cohomology algebra of a projective complex manifold.*

The following questions remain open (cf. [30]):

1. What happens in dimension 3?

2. Do there exist compact Kähler manifolds whose π_1 is not isomorphic to the π_1 of a complex projective manifold ? (See [55] for one step in this direction.)

3. Is it true that a compact Kähler manifold with nonnegative Kodaira dimension has a bimeromorphic model which deforms to a complex projective manifold?

3.3.2. Construction of examples. The simplest example of a topologically non projective compact Kähler manifold is based on the existence of endomorphisms of complex tori which prevent the complex tori in question to be algebraic. Let Γ be a rank $2n$ lattice, and let ϕ be an endomorphism of Γ. Assume that the eigenvalues of ϕ are all distinct and none is real. Choosing n of these eigenvalues $\lambda_1, \dots, \lambda_n$, so that no two of them are complex conjugate to each other, one can then define $\Gamma^{1,0} \subset \Gamma_{\mathbb{C}}$ as the eigenspace associated to the λ_i's, and $T = \Gamma_{\mathbb{C}}/(\Gamma^{1,0} \oplus \Gamma)$. Clearly, the extended endomorphism $\phi_{\mathbb{C}}$ of $\Gamma_{\mathbb{C}}$ preserves both $\Gamma^{1,0}$ and Γ, and thus descends to an endomorphism ϕ_T of T.

Our first example was the following. Let (T, ϕ_T) be as above a complex torus with endomorphism. Inside $T \times T$, we have the four subtori

$$T_1 = T \times 0,\ T_2 = 0 \times T,\ T_3 = \text{Diagonal},\ T_4 = \text{Graph}(\phi_T),$$

which are all isomorphic to T. These tori meet pairwise transversally in finitely many points $x_1, \dots, x_N$. Blowing-up these points, the proper transforms $\widetilde{T}_i$ are smooth and do not meet anymore. We can thus blow-up all the $\widetilde{T}_i$'s to get a compact Kähler manifold X. This is our example.

Theorem 3.18. *[49] This compact Kähler manifold X does not have the cohomology algebra of a projective complex manifold. More precisely, the cohomology algebra $H^*(X, \mathbb{Q})$ does not admit a Hodge structure with rational polarization.*

Let us give an idea of the proof. The degree 2 cohomology of the manifold X contains the classes e_i of the exceptional divisors over the $\widetilde{T}_i$. The first step is to use Deligne's Lemma 3.4, or rather its corollary 3.6 to show that these classes have to be Hodge classes for any Hodge structure on $H^*(X, \mathbb{Q})$. The second step consists then in examining the morphisms of Hodge structures

$$\cup e_i : H^1(X, \mathbb{Q}) \to H^3(X, \mathbb{Q})$$

given by cup-product with the e_i's. The conclusion is the following: For any Hodge structure on $H^*(X, \mathbb{Q})$, the weight 1 Hodge structure on $H^1(X, \mathbb{Q})$ is the direct sum of two copies of a weight 1 Hodge structure L, which admits an endomorphism conjugate to ${}^t\phi$. One concludes then with the following easy result:

Proposition 3.19. *[49] If $n \geq 2$ and the Galois group of the splitting field of $\mathbb{Q}(\phi)$ acts as the full symmetric group $\mathfrak{S}_{2n}$ on the eigenvalues of ϕ, then a weight 1 Hodge structure admitting an endomorphism conjugate to ${}^t\phi$ does not admit a rational polarization.*

4. Cohomology of Algebraic Varieties; Algebraic Data

4.1. Algebraic de Rham cohomology.

Let X be a smooth projective variety defined over a field K of characteristic 0. One has the sheaf of

Kähler (or algebraic) differentials $\Omega_{X/K}$ which is a locally free algebraic coherent sheaf on X, locally generated by differentials df_i, where the f_i are algebraic functions on X defined near x, the relations being given by $da = 0$, $a \in K$ and Leibniz rule $d(fg) = fdg + gdf$.

We can form the locally free sheaves $\Omega^l_{X/K} := \bigwedge^l \Omega_{X/K}$ and, by the definition of $\Omega_{X/K}$ and using Leibniz rule, we get the differentials $d : \mathcal{O}_X \to \Omega_{X/K},\ d : \Omega^l_{X/K} \to \Omega^{l+1}_{X/K}$ satisfying $d \circ d = 0$.

Definition 4.1. *The algebraic de Rham cohomology of X is defined as the hypercohomology of the algebraic de Rham complex: $H^k_{dR}(X/K) := \mathbb{H}^k(X, \Omega^*_{X/K})$.*

Note that this finite dimensional K-vector space depends on K. However, when $K \subset L$ (field extension), one has $H^k_{dR}(X_L/L) = H^k_{dR}(X/K) \otimes_K L$. This construction led Grothendieck to the following remarkable conclusion: The cohomology with *complex* coefficients of a smooth complex projective variety (endowed with its classical topology) X_{cl} can be computed as an algebraic invariant of the algebraic variety X.

Note that this is not at all true if we change the field of coefficients or the definition field. Even with $\mathbb{R}$ instead of $\mathbb{C}$, and even if the variety X is defined over $\mathbb{R}$, the cohomology $H^*(X_{cl}, \mathbb{R})$ cannot be computed by algebraic means. It is furthermore known by work of Serre (see also [13], [40] for further refined versions of this phenomenon) that the homotopy types (and even the real cohomology algebra) of X_{cl} indeed is not determined by the abstract algebraic variety X. In fact, a field automorphism of $\mathbb{C}$ will provide another complex algebraic variety, thus another complex manifold, which is usually not homeomorphic or even homotopically equivalent to the original one.

The precise statement of Grothendieck's Theorem is the following:

Theorem 4.2. *[29] Let X be a smooth algebraic variety defined over $\mathbb{C}$. Then there is a canonical isomorphism*

$$H^k_{dR}(X/\mathbb{C}) = H^k(X_{cl}, \mathbb{C}). \tag{4.3}$$

When X is projective, this is a direct consequence of Serre's theorem 1.4 and of the fact that the *holomorphic* de Rham complex Ω^*_X, which is the analytic counterpart of the algebraic de Rham complex, is a resolution of the constant sheaf $\mathbb{C}$ on X_{cl}. The quasi-projective case involves a projective completion $\overline{X}$ of X with a boundary $D = \overline{X} \setminus X$ which is a *normal crossing divisor*, and the introduction of the logarithmic (algebraic and holomorphic) de Rham complexes $\Omega^*_X(\log D)$.

Remark 4.3. What makes Theorem 4.2 striking is the fact that the algebraic de Rham complex, unlike the holomorphic de Rham complex in the classical topology, is not at all acyclic in positive degree in the Zariski topology, so that the proof above is completely indirect. In fact, by the affine version of Theorem 4.2, its degree i cohomology sheaf is the complexified version of the sheaf $\mathcal{H}^i$ studied by Bloch and Ogus [8], (cf. Section 4.4).

4.1.1. Cycle classes. Let X be a smooth projective variety defined over K and $Z \subset X$ be a local complete intersection closed algebraic subset of X, also defined over K. Following Bloch [7], one can construct an algebraic cycle class

$$[Z]_{alg} \in H^{2k}_{dR}(X/K).$$

Assume now that X is defined over $\mathbb{C}$. We denote by $Hdg^{2k}(X)$ the set of Hodge classes of the corresponding complex manifold. This is naturally a subspace of $H^{2k}(X_{cl}, \mathbb{Q})$, hence of $H^{2k}(X_{cl}, \mathbb{C})$. We mentioned in section 2.1 that one can define for any closed algebraic or analytic subset $Z \subset X$ of codimension k a topological cycle class $[Z] \in Hdg^{2k}(X)$. The following result compares the algebraic and topological constructions.

Theorem 4.4. *Via the isomorphism (4.3) in degree $2k$, one has*

$$[Z]_{alg} = (2\iota\pi)^k [Z].$$

Remark 4.5. The coefficient $(2\iota\pi)^k$ is not formal there, or just a matter of definition. It is forced on us, due to the fact that the algebraic cycle class is compatible with definition fields (eg, if Z, X are defined over K, so is $[Z]_{alg}$), while the topological cycle class is rational for the Betti cohomology theory.

4.2. Absolute Hodge classes.

Here we enter one of the most fascinating aspects of the Hodge conjecture, which seriously involves the fact that the complex manifolds we are considering are algebraic.

Let us first introduce the notion of (de Rham) absolute Hodge class (cf. [19]). First of all, let us make a change of definition: a Hodge class of degree $2k$ on X will be in this section a class $\alpha \in (2\iota\pi)^k H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X)$. The reason for this shift is the fact that we want to use the algebraic cycle class $[Z]_{alg}$ introduced in section 4.1.1, which takes value in algebraic de Rham cohomology, and which, by Theorem 4.4, equals $(2\iota\pi)^k[Z]$ via the isomorphism (4.3).

Let X_{cl} be a complex projective manifold endowed with its classical topology and $\alpha \in Hdg^{2k}(X)$ be a Hodge class. Thus $\alpha \in (2\iota\pi)^k H^{2k}(X_{cl}, \mathbb{Q}) \subset H^{2k}(X_{cl}, \mathbb{C})$ and we can use Theorem 4.2 to compute the right hand side as the hypercohomology of the algebraic variety X with value in the complex of algebraic differentials:

$$H^{2k}(X_{cl}, \mathbb{C}) \cong \mathbb{H}^{2k}(X, \Omega^*_{X/\mathbb{C}}). \tag{4.4}$$

For each field automorphism σ of $\mathbb{C}$, we get a new algebraic variety X_σ defined over $\mathbb{C}$, obtained from X by applying σ to the coefficients of the defining equations of X. The corresponding complex manifold $X_{\sigma,cl}$ is called a "conjugate variety" of X_{cl}(cf. [39]). It is in general not homotopically equivalent to X_{cl}. However, as an algebraic variety, X_σ is deduced from X by applying σ, and it follows that there is a natural (only $\sigma(\mathbb{C})$-linear) isomorphism between algebraic de Rham cohomology groups:

$$\mathbb{H}^{2k}(X, \Omega^*_{X/\mathbb{C}}) \cong \mathbb{H}^{2k}(X_\sigma, \Omega^*_{X_\sigma/\mathbb{C}}).$$

Applying the comparison isomorphism (4.4) in the reverse way, the class α thus provides for each σ a (de Rham or Betti) complex cohomology class

$$\alpha_\sigma \in \mathbb{H}^{2k}(X_\sigma, \Omega^*_{X_\sigma}) = H^{2k}(X_{\sigma,cl}, \mathbb{C}).$$

Definition 4.6. *(cf [19]) The class α is said to be (de Rham) absolute Hodge if α_σ is a Hodge class for each σ. Concretely, as α_σ has the right Hodge type, it suffices to check that $\alpha_\sigma = (2\iota\pi)^k\beta_\sigma$, for some rational cohomology class $\beta_\sigma \in H^{2k}(X_{\sigma,cl}, \mathbb{Q})$.*

The main reason for introducing this definition is the following, which is an immediate consequence of the comparison theorem 4.4 and of the naturality of the algebraic cycle class:

Proposition 4.7. *If $Z \subset X$ is an algebraic subvariety of codimension k, then $(2\iota\pi)^k[Z] \in (2\iota\pi)^k H^{2k}(X, \mathbb{Q})$ is an absolute Hodge class.*

Proposition 4.7 shows that the Hodge conjecture contains naturally the following subconjectures:

Conjecture 4.8. *Hodge classes on smooth complex projective varieties are absolute Hodge.*

Conjecture 4.9. *Let X be smooth complex projective. Absolute Hodge classes on X are generated over $\mathbb{Q}$ by algebraic cycles classes.*

Conjecture 4.8 is solved affirmatively by Deligne for Hodge classes on abelian varieties (cf. [19]). An important but easy point in this proof is the fact that Weil classes (cf. section 2.1) on Weil abelian varieties are absolute Hodge.

To conclude this section, let us mention a crucial example of absolute Hodge class. It plays an important role in the theory of algebraic cycles (cf. [31]) and is not known in general to be algebraic (that is to satisfy the Hodge conjecture).

Example 4.10. Let X be smooth projective of dimension n. Recall from Theorem 2.10 that if $h = c_1(H)$, where H is an ample line bundle on X, there is for each $k \leq n$ an isomorphism of Hodge structures $h^{n-k}\cup : H^k(X, \mathbb{Q}) \cong H^{2n-k}(X, \mathbb{Q})$. Consider now the inverse of the Lefschetz isomorphism above: $(h^{n-k}\cup)^{-1} : H^{2n-k}(X, \mathbb{Q}) \cong H^k(X, \mathbb{Q})$. By Poincaré duality and Künneth decomposition, the space $\mathrm{Hom}\,(H^{2n-k}(X, \mathbb{Q}), H^k(X, \mathbb{Q}))$ is contained in $H^{2k}(X \times X, \mathbb{Q})$. The corresponding Hodge of degree $2k$ on $X \times X$ is absolute Hodge, and is not known in general to be algebraic.

4.3. Hodge loci and absolute Hodge classes.

The key point in which algebraic geometry differs from Kähler geometry is the fact that a smooth complex projective variety X does not come alone, but accompanied by a full family of deformations $\pi : \mathcal{X} \to T$, where π is smooth and projective (that is $\mathcal{X} \subset T \times \mathbb{P}^N$ over T, for some integer N), and where the base T is quasi-projective smooth and defined over $\mathbb{Q}$ (T is not supposed to be geometrically irreducible). Indeed, one can take for T a desingularization of a Zariski open set of the reduced Hilbert scheme parameterizing subschemes of $\mathbb{P}^N$ with same Hilbert polynomial as X. The Hilbert scheme and its universal family are known to be defined over $\mathbb{Q}$. The existence of this family of deformations is reflected in the transformations $X \mapsto X_\sigma$ considered above. Namely, the variety T being defined over $\mathbb{Q}$, σ acts on its complex points, and if X is the fiber over some complex point $0 \in T(\mathbb{C})$, then X_σ is the fiber over the complex point $\sigma(0)$ of $T(\mathbb{C})$.

The total space $\mathcal{X}$ is thus an algebraic variety defined over $\mathbb{Q}$ (and in fact we may even complete it to a smooth projective variety defined over $\mathbb{Q}$), but for the moment, let us consider it as a family of smooth complex varieties, that is, let us work with $\pi : \mathcal{X}_{cl} \to T_{cl}$.

Associated to this family are the Hodge bundles H^l on T, which are described set theoretically as follows: $H^l = \{(t, \alpha_t),\, t \in T,\, \alpha_t \in H^l(X_t, \mathbb{C})\}$. Using a relative version of Grothendieck's theorem 4.2, one can show that H^l is an algebraic vector bundle on T, defined over $\mathbb{Q}$.

Definition 4.11. *(cf [12]) The locus of Hodge classes for the family $\mathcal{X} \to T$ and in degree $2k$ is the subset $Z \subset H^{2k}$ consisting of pairs (t, α) where $t \in T(\mathbb{C})$ and α_t is a Hodge class on X_t.*

This locus is thus the set of all Hodge classes in fibers of π. For $\alpha \in Z$ we shall denote by Z_α the connected component of Z passing through α and by T_α the projection of Z_α to T. T_α is the Hodge locus of α, that is the locus of deformations of X where α deforms as a Hodge class.

Observing that the transport map $H^l(X_t, \mathbb{C}) \ni \alpha_t \mapsto \alpha_{t,\sigma} \in H^l(X_{t,\sigma}, \mathbb{C})$ associated to a field automorphism σ of $\mathbb{C}$ in the previous section is nothing but the action of $Aut\,\mathbb{C}$ on the complex points of the total space of the vector bundle H^l, seen as a variety defined over $\mathbb{Q}$, we get the following "geometric" interpretation of the notion of absolute Hodge class.

Lemma 4.12. *(cf. [52]) i) To saying that Hodge classes of degree $2k$ on fibers of the family $\mathcal{X} \to T$ are absolute Hodge is equivalent to say that the locus Z is a countable union of closed algebraic subsets of H^{2k} defined over $\mathbb{Q}$.*

ii) To saying that $\alpha \in Hdg^{2k}(X)$ is an absolute Hodge class is equivalent to say that Z_α is a closed algebraic subset of H^{2k} defined over $\overline{\mathbb{Q}}$ and that its images under $Gal\,(\overline{\mathbb{Q}} : \mathbb{Q})$ are again components of the locus of Hodge classes.

This lemma rephrases Conjecture 4.8 as a structure statement for the locus of Hodge classes. The following result, due to Cattani, Deligne and Kaplan, establishes part of the predicted structure of the locus of Hodge classes. It is a strong evidence for Conjecture 4.8, hence for the Hodge conjecture itself.

Theorem 4.13. *[12] The connected components Z_α of Z are closed algebraic subsets of H^{2k}. Hence the Hodge loci T_α are closed algebraic subsets of T.*

Let us now investigate the arithmetic aspect of the notion of absolute Hodge class, exploiting its relation with the definition field of the component of the

Hodge loci. The following result is obtained in [52] as a consequence of Deligne's global invariant cycle theorem (cf. [18]). This result says that for absolute Hodge classes, or under the much weaker assumption ii), the Hodge conjecture can be reduced to the case of Hodge classes on varieties defined over a number field.

Theorem 4.14. *[52] i) Let $\alpha \in Hdg^{2k}(X)$ be an absolute Hodge class. Then the Hodge conjecture is true for α if it is true for absolute Hodge classes on varieties defined over $\overline{\mathbb{Q}}$.*

ii) Let $\alpha \in Hdg^{2k}(X)$ be a Hodge class, such that the Hodge locus T_α is defined over $\overline{\mathbb{Q}}$. Then the Hodge conjecture is true for α if it is true for Hodge classes on varieties defined over $\overline{\mathbb{Q}}$.

The second statement of Theorem 4.14 is one motivation to investigate the question whether the Hodge loci T_α are defined over $\overline{\mathbb{Q}}$, which by Lemma 4.12 is weaker than the question whether Hodge class are absolute.

We have the following criterion, proved in [52]:

Theorem 4.15. *Let $\alpha \in H^{2k}(X, \mathbb{C})$ be a Hodge class. Suppose that any locally constant Hodge substructure defined along T_α, say $L \subset H^{2k}(X_t, \mathbb{Q})$, $t \in T_\alpha$, is purely of type (k, k). Then T_α is defined over $\overline{\mathbb{Q}}$, and its translates under $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ are again of the form T_β.*

The assumptions in the theorem are reasonably easy to check in practice, for example by infinitesimal methods. On the other hand, they are clearly not satisfied in the case where the component T_α of the Hodge locus consists of one isolated point, if the Hodge structure on $H^{2k}(X)$ is not trivial. In this case, what predicts the Hodge conjecture is that this point should be defined over $\overline{\mathbb{Q}}$. But our criterion does not give this: in fact our criterion applies only when we actually have a non trivial variation of Hodge structure along T_α.

4.4. Bloch-Ogus theory and K-theory. Let X be a smooth complex algebraic variety. As before X_{cl} denotes $X(\mathbb{C})$ endowed with the classical topology, while X_{Zar} denotes $X(\mathbb{C})$ endowed with the Zariski topology. We denote $\pi : X_{cl} \to X_{Zar}$ the identity map, which is continuous. Bloch-Ogus theory is the study of the spectral sequence associated to π. It appears to be one of the best ways to relate the cohomology of X_{cl} to the structure of its spaces of subvarieties or rather algebraic cycles.

Let us start with a notation: Let A be an abelian group; the sheaves $\mathcal{H}^i_X(A)$ are the sheaves on X_{Zar} defined by $\mathcal{H}^i_X(A) := R^i\pi_*A$. More concretely, $\mathcal{H}^i_X(A)$ is the sheaf on X_{Zar} associated to the presheaf $U \mapsto H^i(U_{cl}, A)$. The Leray spectral sequence for π starts at E_2

$$E_2^{p,q} = H^p(X_{Zar}, \mathcal{H}^q_X(A)) \Rightarrow H^{p+q}(X_{cl}, A).$$

There is one simple thing that can be said about the sheaves $\mathcal{H}^i_X(A)$: namely they vanish for $i > n = \dim_{\mathbb{C}} X$. Indeed, this is a consequence of the fact that the homotopy type of a smooth complex affine algebraic variety of dimension n is a CW complex of real dimension $\leq n$ (cf. [48, II,1.2.1]).

However, much more can be said about the shape of the above spectral sequence, as a consequence of Bloch-Ogus theorem providing a Gersten-Quillen's type resolution for the sheaves $\mathcal{H}^i_X$.

Theorem 4.16. *(Bloch-Ogus, [8]) One has $H^p(X_{Zar}, \mathcal{H}^q_X(A)) = 0$ for $p > q$.*

Another spectacular consequence of this resolution is the following formula due to Bloch-Ogus for groups of cycles modulo algebraic equivalence:

Theorem 4.17. *[8] One has, for any $p \geq 0$, the formula $H^p(X_{Zar}, \mathcal{H}^p(\mathbb{Z})) = \mathcal{Z}^p(X)/\mathrm{alg}$.*

Here $\mathcal{Z}^p(X)$ is the free abelian group with basis the irreducible closed algebraic subsets of X of codimension p. The algebraic equivalence relation is generated by the deformation relation: two closed algebraic subsets of X are deformation equivalent if they are the fibers over two points of a codimension p closed algebraic subset $\mathcal{Z} \subset C \times X$, parameterized by a smooth connected curve C.

Finally, the most impressive applications of Bloch-Ogus theory are obtained via the Bloch-Kato conjecture which had been partially established by Merkur'ev and Suslin in [35], [36], by Voevodsky in [46], and is now fully announced by Voevodsky [45]. This conjecture relates Milnor K-theory of a field modulo n to Galois cohomology of this field with twisted $\mathbb{Z}/n\mathbb{Z}$-coefficients. Combined with Bloch-Ogus resolution for finite coefficients and in the étale setting on one hand, and with the Gersten-Quillen resolution for K-theory on the other hand, it leads to beautiful results concerning groups of algebraic cycles modulo certain equivalence relations, and more precisely to their torsion part or their version with finite coefficients (we refer to [15], [37] for reviews of them).

The following beautiful consequence of Bloch-Kato conjecture was obtained by Bloch and Srinivas [6].

Theorem 4.18. *The Bloch-Kato conjecture implies that the sheaves $\mathcal{H}^i_X(\mathbb{Z})$ have no torsion, which is also equivalent to the fact that for any i and n, there are exact sequences:*

$$0 \to \mathcal{H}^i_X(\mathbb{Z}) \stackrel{n}{\to} \mathcal{H}^i_X(\mathbb{Z}) \to \mathcal{H}^i_X(\mathbb{Z}/n\mathbb{Z}) \to 0.$$

Let us state a simple application, which is related to the defect of the Hodge conjecture for integral Hodge classes (already observed by Atiyah and Hirzebruch [3] in 1962). We introduce first the following invariant, which is shown in [14] to be a birational invariant, allowing to detect non rationality of certain unirational varieties. Here we use the following notions: A *rational* variety is birationally equivalent to a projective space, while a *unirational* variety X admits a rational dominating map $\mathbb{P}^N \dashrightarrow X$. Deciding whether a unirational variety is rational or not is a version of the Lüroth problem, which has a long history [5].

Definition 4.19. *The i-th unramified cohomology group of X with coefficients in A is defined by the formula $H^i_{nr}(X,A) = H^0(X_{Zar}, \mathcal{H}^i_X(A))$.*

On the other hand, the defect of the integral Hodge conjecture for X is measured by the groups $Z^{2i}(X) := Hdg^{2i}(X,\mathbb{Z})/\langle [Z], codim\, Z = i\rangle$. The group $Z^4(X)$ was shown by Kollár to be non trivial for very general hypersurfaces of high degree in $\mathbb{P}^4$. However it was shown in [51] that $Z^4(X)$ is trivial if X is a threefold swept-out by rational curves, i.e curves isomorphic to $\mathbb{P}^1$. In higher dimensions, the question whether $Z^4(X) = 0$ for rationally connected varieties (i.e. varieties for which any two points can be joined by a rational curve) was asked in [53]. We disprove this using the main result of [14] and comparing $H^3_{nr}(X, \mathbb{Z}/n\mathbb{Z})$ and the n-torsion of $Z^4(X)$ (see also [4]).

Theorem 4.20. *[16] There exist rationally connected (and even unirational) varieties of dimension 6 for which $Z^4(X) \neq 0$.*

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