Hodge conjecture and locally residual currents

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Abstract
The conference is divided in three parts. In the first part, we introduce the sheaves of locally residual currents and the subsheaves of principal value currents, with some fundamental exact sequences. In the second part, we formulate the following main results. The proofs can be found in the note [1]. Let be $X$ a projective connected manifold of dimension $n$. Let be $Y_i (1 \leq i \leq n)$ hypersurfaces on $X$, defining ample line bundles, and intersecting properly. Then, the Dolbeault cohomology groups $H^i(\Omega^q)$ of the sheaf of holomorphic $q$–forms on $X$ can be computed as the $i$–th cohomology group of some complex of global sections of locally residual currents on $X$. For $q = n$, we get another theorem by restricting to residual currents obtained from meromorphic forms with simple poles on the $Y_i$. In the conclusion, we discuss open questions related with the Hodge conjecture.

1 Locally residual currents and principal value currents

Let us introduce locally residual currents. For these, the reader can refer to [4], or [5]. We give here a brief account of the construction of residual currents.

Let $X$ be a complex connected manifold of dimension $n$. First, there is, associated to any meromorphic $q$–form $\Psi$ a current, of bidegree $(q, 0)$, denoted $P(\Psi)$ or $[\Psi]$, and called the principal value of $\Psi$, which satisfies the following properties:

1. If $\Psi$ is holomorphic, then $[\Psi]$ coincides with the classical current associated to $\Psi$:
   
   $[\Psi](\phi) = \int_X \Psi \wedge \phi.$

2. Let us denote $\mathcal{C}^{q,0}(\star)$ the sheaf of meromorphic currents, i.e. currents which are locally $[\omega]$, with $\omega$ a meromorphic $q$–form, and...
$C^{q,0}$ the subsheaf of those currents which are $\overline{\partial}$–closed. Then, $P$ induces an isomorphism of $\mathcal{O}$–modules, $P : \mathcal{M}^q \to C^{q,0}(\ast)$, and the kernel of $\overline{\partial}$ is just the image of $\Omega^q$. Moreover, $\partial$ as an operator on meromorphic forms, commutes with $P$: $[\partial \Psi] = \partial [\Psi]$.

Now, let $f_0, \ldots, f_p$ be a sequence of holomorphic functions, defining a complete intersection in an open subset $U \subset X$ (in other words, a regular sequence). There is a current on $U$ denoted $1/f_0 \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p)$ (cf. [5]), such that:

$$\overline{\partial}(1/f_0 \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p)) = \overline{\partial}(1/f_0) \wedge \cdots \wedge \overline{\partial}(1/f_p)$$

We have the following fundamental lemma (cf.[4]):

**Lemma 1** If for an holomorphic $q$–form $\omega$ we have:

$$\omega / f_0 \wedge \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p) = 0,$$

then we can write: $\omega_i = \sum_{i=1}^p f_i \omega_i$, with $\omega_i$ holomorphic $q$–forms.

A locally residual current of bidegree $(q,p)$ is a current which can be written locally as $\omega / f_0 \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p)$, with $(f_0, \ldots, f_p)$ a regular sequence of holomorphic functions and $\omega$ an holomorphic $q$–form. We do not assume here (as for instance in [7]) that a locally residual current is $\overline{\partial}$–closed.

Now let be $Z$ an analytic subset of pure codimension $p$ on $X$. Then, let us denote $C_Z^{q,p}$ the presheaf which associate to any open subset $U$ the set $C_Z^{q,p}(U)$ of $\overline{\partial}$–closed locally residual currents of bidegree $(q,p)$, with support contained in $Z$. If $Y$ is a hypersurface of $Z$, we denote $C_Y^{q,p}(\ast Y)$ the presheaf of locally residual currents with support in $Z$, $\overline{\partial}$–closed outside $Y$. Then, $C_Z^{q,p}(\ast Y)$ is clearly a sheaf. Moreover:

**Lemma 2** $C_Z^{q,p}(\ast Y)$ is an $\mathcal{O}$–module.

**Proof.**

We have to show the stability for the sum. Let us remark that if $T \in C_Z^{k,p}(\ast Y)$ for some $x \in Y$, then if $f$ is an holomorphic function in an open neighborhood $U_x$ of $x$ such that $U_x \cap Y \subset U_x \cap Z \cap \{f = 0\}$, we have that for some integer $k$, $f^k T$ extends to a germ of $C_Z^{k,p}$. Thus, it suffices to show that the subsheaf $C_Z^{k,p}$ is stable for the sum. Since $Z$ is generically smooth, it suffices to show that, for $U$ a Stein open subset and $Z$ an irreducible smooth complete intersection in $U$, defined by a regular sequence of holomorphic coordinates $(z_1, \ldots, z_p)$ the sum of two currents $T = \omega \wedge \overline{\partial}(1/z_1^{k_1}) \wedge \cdots \wedge \overline{\partial}(1/z_p^{k_p})$ and $T' = \omega' \wedge \overline{\partial}(1/z_1^{k_1'}) \wedge \cdots \wedge \overline{\partial}(1/z_p^{k_p'})$ in $C_Z^{k,p}(U)$ is still in $C_Z^{k,p}(U)$. But we have clearly, if $h_i := \max(k_i, k_i')$ and $z_i := z_i^{h_i} \cdot \cdots \cdot z_p^{h_p} \ast I$ for some multindex $I$:

$$T + T' = (z^H \cdot \ast \omega + z^H \ast \omega') \wedge \overline{\partial}(1/z_1^{h_1}) \wedge \cdots \wedge \overline{\partial}(1/z_p^{h_p}),$$

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which concludes the proof.

Let \( Y \) be an hypersurface on \( X \). We say that \( Y \) intersects \( Z \) properly if \( Z \cap Y \) is of pure dimension \( p + 1 \). More generally, hypersurfaces \( Y_1, \ldots, Y_k \) intersect properly on \( Z \) if for any \( i \leq k \), \( Y_i \) intersect \( Z \cap Y_i \cap \ldots \cap Y_{i-1} \) properly. If \( Y \) is an hypersurface intersecting \( Z \) properly, we will identify \( C^p_Z(\star Y) \) with \( C^p_Z(\star (Y \cap Z)) \). From

\[
\overline{\partial}((1/f_p + \overline{\partial}(1/f_1) \wedge \ldots \wedge \overline{\partial}(1/f_p)) = \overline{\partial}(1/f_1) \wedge \ldots \wedge \overline{\partial}(1/f_{p+1}),
\]

we deduce a natural operator:

\[
\overline{\partial} : C^{q+1}_Z(\star (Y \cap Z)) \rightarrow C^q_Z(\star Y)
\]

More generally, if \( Y, Y' \) are two hypersurfaces intersecting properly on \( Z \), we will define an operator:

\[
Res_Y : C^{q+1}_Z(\star (Y \cup Y')) \rightarrow C^q_{Z\cap Y}(\star Y')
\]

by the following way. First, we multiply by a power of the function \( f \) defining \( Y' \), such that \( f^kT \) extends to \( C^p_Z(\star Y) \). Then, we take \( \overline{\partial} \); and finally, we divide by \( f^k \). If \( Y' \) is empty, \( Res_Y \) coincide with \( \overline{\partial} \).

With this notation, we can write, if \( (f_0, \ldots, f_p) \) is some regular sequence on an open set \( U : \omega/(f_0) \wedge \overline{\partial}(1/f_1) \wedge \ldots \wedge \overline{\partial}(1/f_p) \) as:

\[
Res_{Y_1, \ldots, Y_p} : \omega/(f_0f_1 \ldots f_p) \rightarrow \omega/(f_0f_1 \ldots f_p),
\]

where \( Y_i = \{ f_i = 0 \} \) and \( Res_{Y_1, \ldots, Y_p} := Res_{Y_1} \circ \ldots \circ Res_{Y_p} \).

**Lemma 3** If \( Res_{Y_1, \ldots, Y_p} : \omega/(f_0f_1 \ldots f_p) \) is \( \overline{\partial} \)-closed, then it can be written \( Res_{Y_1, \ldots, Y_p} \Psi \), with \( Pol(\Psi) \subset Y_1 \cup \ldots \cup Y_p \).

**Proof.**

Let us assume that \( Res_{Y_1, \ldots, Y_p} : \omega/(f_0 \ldots f_p) \) is \( \overline{\partial} \)-closed. This means that \( \omega \wedge \overline{\partial}(1/f_0) \wedge \ldots \wedge \overline{\partial}(1/f_p) = 0 \). Thus, by the lemma 1, we have:

\[
Res_{Y_1, \ldots, Y_p} : \omega/(f_0 \ldots f_p) = Res_{Y_1, \ldots, Y_p} \omega/(f_1 \ldots f_p)
\]

Now, let be \( Y_1, \ldots, Y_p \) analytic hypersurfaces, intersecting properly. Thus, for \( i < p \), the hypersurfaces \( Y_1, \ldots, Y_i \) also intersect properly. By the preceding, the operator \( \overline{\partial} \) maps \( C^{q+1}_Z(Y_1 \cap \ldots \cap Y_{i+1}) \) into \( C^{q+1}_Z(Y_1 \cap \ldots \cap Y_{i+1}) \).

**Lemma 4** We have the following exact sequence of sheaves:

\[
0 \rightarrow \Omega^q \rightarrow C^{q,0}_Y(\star Y_1) \rightarrow C^{q,1}_Y(\star Y_2) \rightarrow \cdots \rightarrow C^{q,p-1}_Y(Y_1 \cap \ldots \cap Y_{p-1})(\star Y_p) \rightarrow C^{q,p}_Y(Y_1 \cap \ldots \cap Y_p) \rightarrow 0
\]

with as morphisms \( \overline{\partial} : C^{q,i-1}_Y(Y_1 \cap \ldots \cap Y_{i-1})(\star Y_i) \rightarrow C^{q,i}_Y(Y_1 \cap \ldots \cap Y_i)(\star Y_{i+1}) \).
Proof.

By the preceding, we have the composed residue operators $\text{Res}_{Y_1, \ldots, Y_i}(\Psi)$, with $\text{Pol}(\Psi) \subset Y_1 \cup \ldots \cup Y_{i+1}$. The exactitude of the preceding complex comes from the fact that locally, when the locally residual current $\text{Res}_{Y_1, \ldots, Y_i}(\Psi)$, with $\Psi$ meromorphic form with polar locus $\text{Pol}(\Psi) \subset Y_1 \cup \ldots \cup Y_{i+1}$, is $\partial\bar{\partial}$–closed, then it can be written with $\text{Pol}(\Psi) \subset Y_1 \cup \ldots \cup Y_i$ and thus can be also be written as $\partial\text{Res}_{Y_1, \ldots, Y_{i-1}}(\Psi)$. 

We define a subcomplex of the complex of locally residual currents, by the following way. First, let us remark that a special kind of locally residual currents are given by the following lemma (cf. [4]):

**Lemma 5** Let $Y$ be an analytic subvariety of pure codimension $p$ in $X$, and $\omega$ a meromorphic $r$–form on $Y$. There is a natural way to associate to $(\omega, Y)$ a current of bidegree $(r + p, p)$ on $X$, denoted $\omega \wedge [Y]$, which coincides with the Lelong integration current $\omega \wedge [Y](\phi) = \int_Y \omega \wedge \phi$ if $\omega$ is holomorphic.

We will call the currents $\omega \wedge [Y]$, with $\omega$ meromorphic on $Y$, principal value currents.

Then, if $Y' \subset Y$ is the polar hypersurface of $\omega$ (outside which $\omega$ is $\partial\bar{\partial}$–closed), $\partial\bar{\partial}(\omega \wedge [Y])$ is still a locally residual current, of bidegree $(r + p, p + 1)$ with support in $Y'$. We say that $\omega$ has logarithmic pole if this current can still be written $\omega' \wedge [Y']$, with $\omega'$ a meromorphic $(r - 1)$–form on $Y'$. More generally, if $Z$ and $Z'$ are hypersurfaces in $X$ intersecting properly on $Y$, and $\omega$ is a meromorphic form on $Y$, we say that $\omega$ has a logarithmic pole on $Z$ (or on $Z \cap Y$) if $\text{Res}_Z(\omega \wedge [Y])$ is still of the form $\omega' \wedge [Z \cap Y]$.

If $Z$ is of pure codimension $p$, and $Y$ is a hypersurface intersecting $Z$ properly, with define $\mathcal{C}^{r+p,p}_Z(Y)$ the subsheaf of $\mathcal{C}^{r+p,p}(\ast Y)$ of currents locally written as $\omega \wedge [Y]$, with $\omega$ a meromorphic $r$–form with logarithmic pole on $Y$. We denote $\mathcal{C}^{r+p,p}_Z$ the corresponding subsheaf of $\mathcal{C}^{r+p,p}_Z$.

**Lemma 6** For a sequence of hypersurfaces $Y_1, \ldots, Y_p$ intersecting properly, the following subcomplex of the preceding one is also exact:

$$0 \rightarrow \Omega^q_Y \rightarrow \mathcal{C}^{q,0}_Y(Y_1) \rightarrow \mathcal{C}^{q,1}_Y(Y_2) \rightarrow \cdots \rightarrow \mathcal{C}^{q,p-1}_{Y_1 \cap \cdots \cap Y_{p-1}}(Y_p) \rightarrow \mathcal{C}^{q,p}_{Y_1 \cap \cdots \cap Y_p} \rightarrow 0$$

Proof.

It suffices to show that, if $U$ is a Stein open subset and $X_1, \ldots, X_{i+1}$ are analytic hypersurfaces intersecting properly in $U$, a $\partial\bar{\partial}$–closed current $\omega \wedge [Y]$, with $Y = X_1 \cap \ldots \cap X_{i+1}$ and $\omega$ a meromorphic $r$–form, can be written $\partial\omega' \wedge [Y']$, with $Y' = X_1 \cap \ldots \cap X_i$ and $\omega'$ a meromorphic $(r + 1)$–form on $Y'$. But we know that $\omega \wedge [Y]$ can be written $\text{Res}_{X_1, \ldots, X_{i+1}}(\Psi)$, with $\text{Pol}(\Psi) \subset X_1 \cup \ldots \cup X_{i+1}$ and $\Psi$ with
logarithmic poles on each $X_j$. Thus, $\omega \wedge [Y] = \partial \text{Res}_{X_1, \ldots, X_i} \Psi$ and $\text{Res}_{X_1, \ldots, Y_i} \Psi = \omega' \wedge [Y']$ for some $\omega'$ with logarithmic pole.

2 Computing the cohomology by residual currents on projective varieties

Let us now assume that $X$ is compact, and that $Y_1, \ldots, Y_n$ are positive hypersurfaces, in the sense that the corresponding Cartier divisors are ample. By the theorem of Kodaira, it implies that $X$ is projective. Then we have the following:

**Theorem 1**

1. The exact complex:

$$
0 \rightarrow \Omega^q \rightarrow C^q(Y_1) \rightarrow C^q,1(Y_2) \rightarrow \cdots \rightarrow C^q_{Y_1 \cap \cdots \cap Y_n} \rightarrow 0
$$

is acyclic, and thus we have a canonical isomorphism:

$$(\forall i, 0 \leq i \leq n) H^i(\Omega^n) \simeq H^0(C^q_{Y_1 \cap \cdots \cap Y_i}) / \partial H^0(C^q_{Y_1 \cap \cdots \cap Y_{i-1}})$$

2. Moreover, an element $T = H^0(C^q_{Y_1 \cap \cdots \cap Y_i})$ can be written as a global residue: $T = \text{Res}_{Y_1, \ldots, Y_i} (\Psi)$, with $\Psi$ a meromorphic $q$–form with poles in $Y_1 \cup \ldots \cup Y_i$.

3. Moreover, $T$ is $\overline{\partial}$–exact iff we can choose $\Psi$ with poles in $Y_1 \cup \ldots \cup Y_{i-1}$.

The last assertion in the theorem was already contained in [6].

Let us assume $q = n$. Then we have the following variant of the preceding theorem:

**Theorem 2**

1. The exact complex:

$$
0 \rightarrow \Omega^n \rightarrow C^{n,0}(Y_1) \rightarrow C^{n,1}(Y_2) \rightarrow \cdots \rightarrow C^{n,p-1}_{Y_1 \cap \cdots \cap Y_n} \rightarrow 0
$$

is acyclic, and thus we have a canonical isomorphism:

$$(\forall i, 0 \leq i \leq n) H^i(\Omega^n) \simeq H^0(C^{n,i}_{Y_1 \cap \cdots \cap Y_i}) / \overline{\partial} H^0(C^{n,i-1}_{Y_1 \cap \cdots \cap Y_{i-1}})$$

2. Moreover, an element $T = H^0(C^{n,i-1}_{Y_1 \cap \cdots \cap Y_{i-1}}(Y_i))$ can be written as a global residue: $T = \text{Res}_{Y_1, \ldots, Y_{i-1}} (\Psi)$, with $\Psi$ a meromorphic closed $n$–form with poles in $Y_1 \cup \ldots \cup Y_i$.

3. Moreover, $T$ is $d$–exact iff we can choose $\Psi$ with poles in $Y_1 \cup \ldots \cup Y_{i-1}$.
Remark. 1. For \( q < n \), it is not true in general that the complex with logarithmic poles computes the Dolbeault cohomology groups, since it is not in general acyclic. The acyclicity for the logarithmic poles, for \( q = n \), comes from the Kodaira annihilation theorem.

2. In each of the two preceding theorems, the first part is a variant of Dolbeault’s theorem, representing cohomology classes by \( \partial^{-}\)closed currents with fixed supports. The assertions would remain true if we don’t fix for the supports given complete intersections, but consider more general complexes of locally residual currents (resp. principal value currents with logarithmic poles) with arbitrary supports. In fact, let us denote for instance \( C^{n,i} \) (resp. \( C^{n,i} (\ast) \)) the currents locally of the form \( \omega \wedge [Y] \) and moreover \( \bar{\partial}^{-}\)closed (resp. with logarithmic poles). Then, we have a natural morphism:

\[
H^0(C^{n,i})/\partial H^0(C^{n,i-1})(\ast) \to H^i(\Omega^n).
\]

This morphism is clearly surjective, since we know that we even can fix the supports. But it is also injective: in fact, if the image of a current \( T \in H^0(C^{n,i}) \) is zero, we know that \( T \) is \( \bar{\partial}^{-}\)-exact; and we can include the support in a complete intersection. Thus we can apply the preceding theorem, to write \( T \) in the form \( \bar{\partial}\omega' \wedge [Y'] \). Thus we get, expressed in another way, the main theorem of [3].

We get as corollary a theorem of P. Griffiths ([2]):

**Corollary 1** Let the \( n \) positive hypersurfaces \( Y_1, \ldots, Y_n \) intersecting transversally in \( s \) points \( P_1, \ldots, P_s \), and let be \( c_1, \ldots, c_s \) \( s \) complex numbers. A necessary and sufficient condition for the existence of a meromorphic \( n-\)form \( \Psi \), with simple pole contained in \( Y_1 \cup \ldots \cup Y_n \), and residues \( c_i \) at \( P_i \), is that \( \sum_{i=1}^{s} c_i = 0 \).

**Proof.**

By the preceding, we know that \( \sum_{i=1}^{s} c_i [P_i] \) is \( \bar{\partial}^{-}\)-exact iff it is globally residual, and moreover residue with simple poles. Thus, it suffices to show that \( \sum_{i=1}^{s} c_i [P_i] \) is exact iff \( \sum_{i=1}^{s} c_i = 0 \). But we know that an \( (n, n) \)-form \( \omega \) is exact iff \( \int_X \omega = 0 \), since \( H^{n,n}(X) \simeq H^n(\Omega^n) \) is one-dimensional.

3 Conclusion

The Hodge conjecture says that a closed current of bidegree \((p, p)\), with an integral class, that is for which the integral of any smooth cohomologous \(2p\)-form on a smooth real submanifold of dimension \(2p\) is integral, has a representative given by an integration current, that is a current given by a finite sum \( \sum c_i [Y_i] \), with \( c_i \) rational numbers and \( Y_i \) irreducible analytic subvarieties of complex codimension \( p \). Let us define, for any analytic subvariety \( Y \) of pure dimension \( p \), and any
\( \partial \)-closed current \( T \) of bidegree \((p, p)\), the integral \( \int_Y T \) as the integral \( \int_Y \omega \) of any smooth \( \partial \)-cohomologous \( \omega \) of bidegree \((p, p)\). Then, the Hodge conjecture can be reformulated in the following way: for a given \( d \)-closed current \( T \) of bidegree \((p, p)\), with integral cohomology class \((\int_Y \phi \) is integral for any smooth \( d \)-closed representative \( \phi \) of degree \( 2p \) and real \( Y \)), then if \( \int_Y T = 0 \) for any analytic subvariety \( Y \) of complex dimension \( p \), then the current \( T \) is exact. We would like to divide the conjecture into two parts.

**Conjecture 1.** Any current \( T \) of bidegree \((p, p)\), with integral class, has a \( d \)-closed representative as locally residual current.

It is clear that this could not be true without the integral class condition. Without the integral condition, we have shown that \( T \) has a \( \partial \)-closed representative; but not \( d \)-closed.

**Conjecture 2.** If a \( d \)-closed locally residual current of bidegree \((p, p)\) has zero integral on any subvariety \( Y \) of complementary dimension, it is exact.

This second conjecture is very close to the classical conjecture saying that for an integral cohomology class, numerical and cohomological equivalence coincide. We already know, by the preceding, that on each irreducible \( Y \) of dimension \( p \), the "restriction" of the locally residual \( T \) can be written as a global residue. It would be necessary to "glue together" these different global residues.

**References**


