# Cocycle propre à valeurs dans une représentation uniformément bornée de $Sp(n,1)$, d'après S. Nishikawa

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16 avril 2020
J 31 du confinement

Séminaire Algèbres d'Opérateurs, Sophie Germain

# Affine representations and 1-cocycles

**Definition**

Let $\pi$ be a representation of a group $G$ in a vector space $H$. A 1-cocycle is a map $b : G \to H$ such that $b(gg') = b(g) + \pi(g)b(g')$.
In other words, $\pi$ and $b$ define an affine representation of $G$ in $H$ by the formula $g.x = \pi(g)x + b(g)$.

**Definition**

A 1-cocycle as above is a coboundary if there exists $v \in H$ such that $b(g) = \pi(g)v - v$, or equivalently if the associated affine representation has a fixed point.

Let $X$ be a set equipped with an action of $G$ and $c : X \times X \to H$ such that $c(x, y) + c(y, z) = c(x, z)$ for any $x, y, z \in X$ and $c(gx, gy) = \pi(g)c(x, y)$ for $g \in G$,$x \in X$. Then for any fixed $x \in X$, $b(g) = c(gx, x)$ is a 1-cocycle.

**Example**

$X$ is the set of vertices of a tree, $H$ is the space of odd $\ell^2$ functions on the set of arrows, and $c(x, y)$ is the caracteristic function of the set of edges on the oriented path from $x$ to $y$. If $G$ has a fixed vertex then $c$ is a coboundary.

# Haagerup vs. Kazhdan

**Definition**

A locally compact second countable group $G$ has the **Haagerup property** if the following equivalent conditions are satisfied:

(i) There exists an action of $G$ by affine isometries on a Hilbert space which is metrically proper.

(ii) There exists a unitary representation $\pi$ of $G$ on a Hilbert space $H$, and a 1-cocycle which is proper.

(iii) There exists a function of conditional negative type on $G$ which is proper.

**Definition**

A locally compact second countable group $G$ has **Kazhdan's property (T)** if the following equivalent conditions are satisfied:

(i) Any action of $G$ by affine isometries on a Hilbert space admits a fixed point (or, equivalently, has a bounded orbit)

(ii) For any unitary representation $\pi$ of $G$ on a Hilbert space $H$, any 1-cocycle is a coboundary (or, equivalently, is bounded).

(iii) Any function of conditional negative type on $G$ is bounded.

# The case of simple Lie groups

Let $G$ be a simple Lie group of rank $r$, with Iwasawa decomposition $G = KAN$ ($K$ compact, $AN$ semidirect product of $N$ nilpotent by an action of $A \simeq \mathbf{R}^r$).
Example: $G = SL_n(\mathbf{R})$ or $SL_n(\mathbf{C})$, then $r = n - 1$

**Theorem**

(i) *If $r \geq 2$ then $G$ is Kazhdan.*

(ii) *If $r = 1$ and $G$ isomorphic to $SO_0(n, 1)$ or $SU(n, 1)$ then $G$ is Haagerup.*

(iii) *If $r = 1$ and $G$ is isomorphic to $Sp(n, 1)$ or $F_{4(-20)}$ then $G$ is Kazhdan.*

The associated riemannian symmetric space is $X = G/K$. Let $\partial X$ be the boundary of $X$. The action of $G$ extends to an action by diffeomorphisms on $\partial X$, and the $G$-orbit decomposition has a stratification described by the set of parabolic subgroups.
We restrict to the rank 1 case: $\partial X = G/P$ is homogeneous, where $P = MAN$ is the Borel subgroup or minimal parabolic subgroup, with $M = K \cap P$.

# A cocycle with values in $\Omega(\partial X)^{\mathrm{top}}_{\int=0}$

Let $\pi$ be the representation of $G$ in the vector space $\Omega(\partial X)^{\mathrm{top}}_{\int=0}$ of differential forms on $\partial X$ with maximal degree and vanishing integral.
Consider the following $G$- invariant 1-cocycle with value in $\Omega(\partial X)^{\mathrm{top}}_{\int=0}$.

$$c : X \times X \to \Omega(\partial X)^{\mathrm{top}}_{\int=0}$$

defined by $c(x,y) = \mu_y - \mu_x$ where for $x \in X$ we denote $\mu_x$ the integral 1 volume form which is $K_x$ invariant ($K_x$ is the stabilizer of $x$, it is $gKg^{-1}$ if $x = gx_0$).

**Theorem**

(i) *There exists a quadratic form $q$ on $\Omega(\partial X)^{\mathrm{top}}_{\int=0}$ which is $G$- invariant such that the kernel $q(c(x,y))$ is proper. For $G = SO_0(n,1)$ ou $SU(n,1)$, the quadratic form $q$ is positive definite. For $G = Sp(n,1)$ or $F_{4(-20)}$, it is not positive definite.*

(ii) *There exists a $K$-invariant euclidian norm on $\Omega(\partial X)^{\mathrm{top}}_{\int=0}$ satisfying: the kernel $\|c(x,y)\|^2$ is proper and the representation $\pi$ of $G$ in $\Omega(\partial X)^{\mathrm{top}}_{\int=0}$ is uniformly bounded for the associated norm.*

# Construction of a quadratic form on $\Omega^{\mathrm{top}}(\partial X)_{\int=0}$

In hyperbolic geometry, it is convenient for 3 points $z$, $w$ and $x \in X$ to define the **Gromov scalar product** of $z$ et $w$ with respect to $x$ as

$$(z, w)_x = d(z, x) + d(w, x) - d(z, w)$$

*Basic example:* for a tree, $(z, w)_x$ is (twice) the distance from $x$ to the segment $[z, w]$.
**Limit case:** fix $x \in X$ and let $z$ et $w$ tend to *distinct* points of the boundary $\partial X$, then the Gromov product has a limit. defining $(z, w)_x$ for $x \in X$ et $(z, w) \in \partial X \times \partial X \setminus \Delta$.
In the case of a tree, $(z, w)_x$ is the distance from $x$ to the path $(z, w)$.
**Change of basis $x$:** introduce the **Buseman functions** $\beta_z(x, y) = \lim(d(z', x) - d(z', y))$ when $z' \to z \in \partial X$. Then

$$(z, w)_y = (y, w)_x + \beta_z(x, y) + \beta_w(x, y)$$

**Proposition**

*For $\alpha \in \Omega^{\mathrm{top}}(\partial X)$, the integral*

$$q_x(\alpha) = \int_{\partial X \times \partial X} (z, w)_x \alpha_z \otimes \alpha_w$$

*is convergent. If moreover $\alpha \in \Omega(\partial X)^{\mathrm{top}}_{\int=0}$ then $q_x(\alpha) = q_y(\alpha$ for any $x, y \in X$, i.e. $q_x$ on $\Omega(\partial X)^{\mathrm{top}}_{\int=0}$ is independent of the choice of the origin $x$..*

**Corollary**

*The formula $q(\alpha) = \int_{\partial X \times \partial X} (z, w)_x \alpha_z \otimes \alpha_w$ defines a $G$-invariant quadratic form on $\Omega(\partial X)^{\mathrm{top}}_{\int=0}$.*

# Complementary series on $\Omega^{\mathrm{top}}(\partial X)$

The Buseman functions $\beta(x,y) : z \mapsto \beta_z(x,y)$ satisfy an obvious cocycle relation $\beta(x,y) + \beta(y,z) = \beta(x,z)$ .
Define a family of representations of $G$ in $\Omega(\partial X)^{\mathrm{top}}$ (fixing an origin $x \in X$)

$$\pi_s(g)\alpha = e^{s\beta(x,gx)} g^{-1*}\alpha$$

for any $\alpha \in \Omega^{\mathrm{top}}(\partial X)$.
Note that the representation $\pi_0$ leaves $\Omega^{\mathrm{top}}(\partial X)_{\int=0}$ invariant and restricts to $\pi$.

**Proposition**

*The integral*

$$q_{s,x}(\alpha) = \int_{\partial X \times \partial X} e^{s(z,w)_x} \alpha_z \otimes \alpha_w$$

*is convergent for $0 < \mathrm{Re} s < \nu$ where $\nu$ is a certain critical exponent to be calculated expliciteley below. Moreover changing the origin yields $q_{s,y}(\alpha) = q_{s,x}(e^{s\beta(x,y)}\alpha)$.*

**Corollary**

*The formula $q_s(\alpha) = \int_{\partial X \times \partial X} e^{s(z,w)_x} \alpha_z \otimes \alpha_w$ defines a quadratic form on $\Omega(\partial X)^{\mathrm{top}}$ invariant by $\pi_s$*

## Limits at $s = 0$ and $\nu$

**Proposition**

(i) *Limit for $s \to 0$: for any $\alpha \in \Omega^{\mathrm{top}}(\partial X)_{\int=0}$,*

$$\lim_{s\to 0} \frac{q_s(\alpha)}{s} = q(\alpha)$$

.

(ii) *Limit for $s \to \nu$: consider the normalisation constant $c_s = q_s(\mu_x)$, then for any $\alpha \in \Omega^{\mathrm{top}}(\partial X)$,*

$$\lim_{s\to\nu} \frac{1}{c_s} q_s(\alpha) = \int_{\partial X} \|\alpha\|_x^2 d\mu_x$$

*the $L^2$-norm of the form $\alpha$ for the metric associated to $x$.*

The limit at 0 simply follows from the fact that $e^{s(z,w)_x} - 1 \sim s(z,w)_x$, leading to consider $\alpha$'s with $\int_{\partial X} \alpha = 0$:
The limit at $\nu$ says that after normalization, the distribution kernel $(z,w)_x$ on $\partial X \times \partial X$ tends to a delta distribution on the diagonal.

# Projective model

The hyperbolic space of dimension $n$ over the division ring (corps non nécessairement commutatif) $K$ with $K = \mathbf{R}$,$\mathbf{C}$ or $\mathbf{H}$ is defined as follows. Consider $K^{n+1}$ as a right $K$-vector space, equipped with a $K$-hermitian scalar product (one has $< xa, yb >= \bar{a} < x, y > b$ and $< y, x >= \overline{< x, y >}$, cf Hibert modules over $C^*$-algebras: consider here $K$ as a real $C^*$-algebra!) We take the signature $(1, n)$ hermitian product:

$$< x, y >= \bar{x}_0 y_0 - \sum_{i=1}^{n} \bar{x}_i y_i$$

and $G$ is the group of right $K$-linear bijections of $K^{n+1}$ preserving the hermitian product (or a subgroup thereof). $X$ and $\partial X$ are subsets of the projective space $P(K^{n+1})$ (set of dimension one subspaces of the right $K$-vector space $K^{n+1}$):
$X = \{\mathbf{x} = xK \in P(K^{n+1}), < x, x >> 0\}$ and $\partial X = \{\mathbf{z} = zK \in P(K^{n+1}), < z, z >= 0\}$

**Proposition**

*The Riemannian distance between* $\mathbf{x}$ *and* $\mathbf{y} \in X$ *is given by the formula*

$$\cosh d(\mathbf{x}, \mathbf{y}) = \frac{| < x, y > |}{< x, x >^{1/2} < y, y >^{1/2}}$$

**Corollary**

*The Gromov scalar product and the Buseman functions are given for* $\mathbf{x}, \mathbf{y} \in X$ *and* $\mathbf{z}, \mathbf{w} \in \partial X$ *by the formulae:*

$$(\mathbf{z}, \mathbf{w})_{\mathbf{x}} = -\ln \frac{< x, x > | < z, w > |}{| < z, x > || < w, x > |} \ , \ \beta_{\mathbf{z}}(\mathbf{x}, \mathbf{y}) = \ln \frac{| < y, z > | < x, x >}{| < x, z > | < y, y >}$$

# Checking positivity for $K$ commutative

Let us use the ball model, i.e. normalize the elements of $X$ and $\partial X \subset P(K^{n+1})$ to be represented by vectors in the affine hyperplane $< x, . >= 1$ where $x = (1, 0)$ is chosen as origin. Denote the hermitian product on $K^n$ by $(\zeta, \omega) = \sum \bar{\zeta}_i \omega_i$.

**Corollary**

*Let and $z = (1, \zeta)$,$w = (1, \omega)$ and $y = (1, \eta)$ with $(\zeta, \zeta) = (\omega, \omega) = 1$ and $(\eta, \eta) < 1$. The above formulae become:*

$$(z, w)_x = -\ln |1 - (\zeta, \omega)|$$

$$\beta_z(x, y) = \ln \frac{1 - (\eta, \eta)}{|1 - (\eta, \zeta)|}$$

Let us expand

$$(\zeta, \omega)_x = -\ln |1 - (\zeta, \omega)| = \sum_{r=1}^{+\infty} \frac{\mathrm{Re}(\zeta, \omega)^r}{r}$$

$$e^{s(z,w)_x} = |1 - (\zeta, \omega)|^{-s} = \sum_{r=0}^{+\infty} (-1)^r \binom{-s/2}{r} \mathrm{Re}(\zeta, \omega)^r$$

SInce the coefficients are positive, it is enough to show that the kernels $\mathrm{Re}(\zeta, \omega)^r$ are positive definite? it is equal to $(\mathrm{Re}(\zeta^{\otimes r}, \omega^{\otimes r})$ where $\zeta^{\otimes r} = \zeta \otimes_K \ldots \otimes_K \zeta$ belongs to the tensor product $K^n \otimes_K \ldots \otimes_K K^n$, which is defined only for commutative $K$ because $K^n$ is both right and left $K$-vector space!

# Checking convergence of the kernel $e^{s(z,w)_x}$ for $0 < \mathrm{Re} s < \nu$, and computing $\nu$.

The convergence of $q_x(\alpha) = \int_{\partial X \times \partial X} (z,w)_x \alpha_z \otimes \alpha_w$ follows from the logarithmic behaviour of $(z,w)_x = -\ln|1-(\zeta,\omega)|$. More subtle is the case of $e^{s(z,w)_x} = |1-(\zeta,\omega)|^{-s}$.

**Proposition**

*The critical value is $\nu = p + 2q$ where $p = \dim K^{n-1} = k(n-1)$ and $q = \dim \mathrm{Im} K = k-1$. One has $\nu = n-1$ for $K = \mathbf{R}$, $2n$ for $K = \mathbf{C}$ and $4n+2$ for $K = \mathbf{H}$.*

For the proof, note that $\|\zeta-\omega\|^2 = 2(1-\mathrm{Re}(\zeta,\omega))$ so that

$$|1-(\zeta,\omega)|^2 = (1-\mathrm{Re}(\zeta,\omega))^2 + (\mathrm{Im}(\zeta,\omega))^2 = \frac{1}{4}\|\zeta-\omega\|^4 + (\mathrm{Im}(\zeta,\omega))^2$$

To check convergence, we are reduced to consider the function $(x,t) \mapsto N(x,t) = (\|x\|^4 + \|t\|^2)^{-1/4}$ on $\mathbf{R}^p \times \mathbf{R}^q$. It is an exercice to show that $N^{-s}$ is locally $L^1$ iff $s < p + 2q$. Here $p = \dim K^{n-1} = k(n-1)$ and $q = \dim \mathrm{Im} K = k-1$.

# Commutative vs. non commutative case

**Theorem**

(i) *The quadratic form* $q(\alpha) = \int_{\partial X \times \partial X} (z,w)_x \alpha_z \otimes \alpha_w$ *on* $\Omega(\partial X)^{\mathrm{top}}_{\int = 0}$ *is positive if* $K = \mathbf{R}$ *or* $\mathbf{C}$. *The representation* $\pi$ *defined by* $\pi(g)\alpha = g^{*-1}$ *is unitary for the associated Hilbert space structure.*

(ii) *The quadratic form* $q_s(\alpha) = \int_{\partial X \times \partial X} e^{s(z,w)_x} \alpha_z \otimes \alpha_w$ *on* $\Omega(\partial X)^{\mathrm{top}}$ *is positive for any real* $s$ *with* $0 < s < \nu$ *if* $K = \mathbf{R}$ *and* $\nu = n-1$, *or or* $K = \mathbf{C}$ *and* $\nu = 2n$ . *The representation* $\pi_s$ *defined by* $\pi_s(g)\alpha = e^{s\beta(x,gx)} g^{-1*}\alpha$ *is unitary for the Hilbert space structure associated to* $q_s$.

**Theorem**

*If* $K = \mathbf{H}$ *the quadratic form* $q$ *on* $\Omega(\partial X)^{\mathrm{top}}_{\int = 0}$ *is not positive. The form* $q_s$ *on* $\Omega(\partial X)^{\mathrm{top}}$ *is defined for* $0 < \mathrm{Re} s < \nu = 4n+2$, *but it is positive definite iff* $s$ *is real and* $1 < s < 4n-1$.

# Riemannian and sub-riemannian structures

**Proposition**

(i) *There is a $G$-invariant riemannian metric on $X$.*

(ii) *The cotangent space $T_z^*\partial X$ is canonically identified to the Lie algebra $\mathrm{Lie}(N_z)$ which is isomorphic to the generalized Heisenberg Lie algebra $K^{n-1} \oplus \mathrm{Im}K$ with $[q, q'] = \mathrm{Im} < q, q' >$ for $q, q' \in K^{n-1}$ and $\mathrm{Im}K$ is central.*

(iii) *The tangent space $T_z\partial Z$ (of dimension $kn - 1$) has a canonically defined subspace $E_z$ of dimension $k(n-1)$ which is the orthogonal (for duality) of the centre of $T_z^*\partial X = \mathrm{Lie}(N_z)$.*

(iv) *To any $x \in X$ one can associate a metric on $T_z\partial X = (\mathrm{Lie}(N_z))^*$ such that different values of $x$ give metrics transported by each other via an automorphism of the Heisenberg group $N_z$. Concretely*

$$T_z\partial X = E_z \oplus F_z^x$$

*where $F_z^x$ is the orthogonal of $E_z$ in $T_z\partial X$. Then the metric on $\partial X$ associated with a point $y \in X$ is the image of the metric associated to $x$ by a map of the form*

$$\begin{pmatrix} e^{-\beta_z(x,y)} & * \\ 0 & e^{-2\beta_z(x,y)} \end{pmatrix}$$

(v) *The subbundle $E$ of $T\partial X$ satisfies the Hörmander condition, i.e. the sections of $E$ and their brackets generate all vector fields. To each $x \in X$ is associated a sublaplacian $\Delta_{x,E} = -\sum X_i^2$ on $\partial X$ which is hypoelliptic.*

# Volume forms on $\partial X$ and Buseman functions

Let us recall that to any $x \in X$ is associated a $K_x$-invariant volume form $\mu_x$ of mass 1 on $\partial X$. Il is related to the $G$-invariant volume form $\mathrm{vol}$ on $X$ as follows: in polar coordinates centered at $x$, the volume form on $X \setminus \{x\}$ is related to $\mu_x$ and to the distance function $\rho_x$ to $x$ by the formula:

$$d\mathrm{vol} = b_{kn}(\sinh \rho_x)^p(\sinh 2\rho_x)^q d\rho_x d\mu_x$$

where $p = k(n-1)$ is the real dimension of $K^{n-1}$, $q = k-1$ the real dimension of $\mathrm{Im}\ K$, and $b_{kn}$ is the volume of the unit ball in $K^n = \mathbf{R}^{kn}$ Asymptotically

$$d\mathrm{vol} \sim b_{kn}e^{-\nu\rho_x} d\rho_x d\mu_x$$

where the number $\nu = p + 2q$ is the critical exponent in the sense that $e^{-\lambda\rho_x}$ is $L^1$ iff $\lambda > \nu$.
One has $\nu = n-1$ for $K = \mathbf{R}$, $2n$ for $K = \mathbf{C}$ and $4n+2$ for $K = \mathbf{H}$.

**Proposition**

*For $x$ and $y \in X$,*

$$d\mu_y = e^{-\nu\beta(x,y)} d\mu_x$$

*where $\beta(x,y)$ is the function $z \mapsto \beta_z(x,y)$ on $\partial x$.*

# Duality functions/top forms, $L^p$-norms...

**Proposition**

(i) *There is a $G$-invariant non degenerate duality between $\Omega^0(\partial X)$ and $\Omega^{\mathrm{top}}(\partial X)$ given by $(f, \alpha) \mapsto \int_{\partial X} f\alpha$.*

(ii) *Under the above duality, the family of representations on $\Omega^{\mathrm{top}}(\partial X)$ defined $\pi_s(g)\alpha = e^{s\beta(x,gx)} g^{-1*}\alpha$ admits as contragredient the representations defined by $\pi_s(g)f = e^{-s\beta(x,gx)} g^{-1*} f$ on $\Omega^0(\partial X)$.*

(iii) *The map $f \mapsto f\mu_x$ intertwines $\pi_s$ on $\Omega^0(\partial X)$ with $\pi_{\nu-s}$ on $\Omega^{\mathrm{top}}(\partial X)$.*

(iv) *Consider the $L^p$ norm on $\Omega^0(\partial X)$ defined by $\|f\|_p^p = \int_{\partial X} f^p d\mu_x$. Then for any $s$ with $0 < \mathrm{Re} s < \nu$, the representation $\pi_s$ acts by isometries on $L^p$ of functions for $\mathrm{Re} s = \frac{\nu}{p}$ $(1 < p < +\infty)$ In particular, $\pi_{\nu/2}$ acts unitarily on $L^2(\partial X, \mu_x)$.*

(v) *Consider the $L^q$ norm on $\Omega^{\mathrm{top}}(\partial X)$ defined by $\|\alpha\|_q^q = \int_{\partial X} \|\alpha\|_x^q d\mu_x$. The above duality between $\Omega^0(\partial X)$ and $\Omega^{\mathrm{top}}(\partial X)$ extends to a duality between $L^p$ and $L^q$ norms if $1/p + 1/q = 1$. Moreover, the representation $\pi_{\nu-s}$ acts by isometries on $L^q$ of top forms for $\mathrm{Re} s = \frac{\nu}{p}$, i.e. $\nu - s = \frac{\nu}{q}$.*

# Cowling's theorem, case of functions

**Theorem**

*Let us equip $\Omega^0(\partial X)$ with the representation $\pi_s(g) = e^{-s\beta(x,gx)} g^{-1*}$ and with the Sobolev norm*

$$\|f\|_{W^\sigma} = \|(1+\Delta_x)^{\sigma/2} f\|_{L^2(\mu_x)}$$

*defined by the sublaplacian associated to the subbundle $E$ and the metric associated to $x$.*
*Then for $\sigma = \nu/2 - \mathrm{Re} s$ and $0 < \mathrm{Re} s < \nu$ (i.e. $-\nu/2 < \sigma < \nu/2$) there is a constant $C_s$ such $\pi_s$ is uniformly bounded by $C_s$:*

$$\|\pi_s(g) f\|_{W^\sigma} \leq C_s \|f\|_{W^\sigma}$$

*for any $g \in G$.*

Similarly the representation $\pi_s$ defined by $\pi_s(g)\alpha = e^{s\beta(x,gx)} g^{-1*}\alpha$ on $\Omega^{\mathrm{top}}(\partial X)$ is uniformly bounded for the norm $\|(1+\Delta_x)^{-\sigma/2}\alpha\|_{L^2(\mu_x)}$. One would naively expect an analogue for the degree $k$ forms for $0 \leq k \leq \mathrm{top}$.
But this is not the case since the represention of $G$ on $\Omega^k(\partial X) = \Omega^k(G/P)$ is not in the principal series, except in the $SO(n,1)$ case. Indeed let $G = KAN$ the Iwasawa decomposition associated to a point $x \in X$ and a point at infinity $z \in \partial X$: $K$ is the stabilizer of $x$ and $P = MAN$ the stabilizer of $z$ ($M = K \cap P$). then the action of $P_x$ on the cotangent space $T^*\partial X$ does not factorize through $MA$, i.e. is not trivial on $N$. This comes from the non commutativity of the nilpotent group $N$: indeed one can canonically identify $T^*\partial X$ with the Lie algebra of $N$ which acts on it by the coadjoint representation!

# Cowling's theorem, case of principal series bundles

More generaly, consider a principal series representation, i.e. a space of smooth sections $\Gamma(\partial X, \mathcal{V})$ of a bundle $\mathcal{V} = G \times_P V$ where $V$ is a representation of $P = MAN$ trivial on $N$, equal to $a \mapsto a^w$ on $A = \mathbf{R}_+^*$ for some $w \in \mathbf{Z}$ and to some unitary representation on the compact group $M$.
For $f \in \Gamma(\partial X, \mathcal{V})$ let $\pi_s(g)f = e^{-s\beta(x,gx)} g^{-1*} f$. This define a family of representations of $G$ on the space $\Gamma(\partial X, \mathcal{V})$. The $L^2$-norm on $\Gamma(\partial X, \mathcal{V})$ is defined by $\|f\|_{L^2}^2 = \int_{\partial X} \|f\|^2 d\mu_x$. Then the representation $\pi_s$ is unitary for $s = \nu/2 - w$.

**Theorem**

*Let us equip $\Gamma(\partial X, \mathcal{V})$ with the Sobolev norm*

$$\|f\|_{W^\sigma} = \|(1+\Delta_x)^{\sigma/2} f\|_{L^2(\mu_x)}$$

*defined by the sublaplacian associated to the subbundle $E$ and the metric defined by Then for $\sigma = \nu/2 - w - \mathrm{Re} s$ and $0 < \mathrm{Re} s + w < \nu$ (i.e. $-\nu/2 < \sigma < \nu/2$) there is a constant $C_s$ such $\pi_s$ is uniformly bounded by $C_s$:*

$$\|\pi_s(g) f\|_{W^\sigma} \leq C_s \|f\|_{W^\sigma}$$

*for any $g \in G$.*

# From Cowling to Nashikawa

Consider the bundle $E$ defined above. It is associated to a representation of $MAN$ trivial on $N$ and to a character of $A$ with $w = 1$.
The representation $\pi_0$ on $\Gamma(\partial X, E^*)$ (i.e. the natural representation of $G$ on the space of sections of $E^*$) is uniformly bounded for the norm $\|(1+\Delta_x)^{\sigma/2} f\|_{L^2}$ for $\sigma = \nu/2 - 1$. The theorem of Cowling applies since $\sigma < \nu/2$. On the contrary, the Cowling theorem does not apply to $\pi_0$ since $\sigma = \nu/2$ is critical. But Nashikawa considers the quotient $\Omega^0(\partial X)/\mathbf{C}$ by constant functions.

**Corollary**

*Let us equip the quotient $\Omega^0(\partial X)/\mathbf{C}$ with the norm*

$$\|(1+\Delta_x)^{\sigma/2} d_E f\|_{L^2}$$

*with $\sigma = \nu/2 - 1$. then the representation $\pi$ on $\Omega^0(\partial X)/\mathbf{C}$ is uniformly bounded.*

Consider the representation $\pi_0$ on $\Omega_0$ and the $G$-equivariant map

$$d_E : \Omega^0(\partial X) \to \Gamma(E^*)$$

obtained by composing the de Rham map $d : \Omega^0(\partial X) \to \Omega^1(\partial X)$ with the restriction map $\Omega^1(\partial X) \to \Gamma(E^*)$.
Then the kernel of $d_E$ is equal to the subspace $\mathbf{C}$ of constant functions. The quotient $\Omega^0(\partial X)/\mathbf{C}$ is equipped with the above norm.

# Nashikawa's theorem

Note that the space $\Omega^0(\partial X)/\mathbf{C}$ considered above is dual (via the pairing $(f,\alpha) \mapsto \int_{\partial X} f\alpha$) to the space $\Omega(\partial X)^{\mathrm{top}}_{\int=0}$ in which the cocycle $c(x,y)$ lives. We shall note $\|.\|$ the norm on $\Omega(\partial X)^{\mathrm{top}}_{\int=0}$ obtained by duality from the norm in the above corollary.

**Theorem**

*The cocycle $c : (x,y) \mapsto \mu_y - \mu_x$ is proper for the above defined norm. In other words, $\|c(x,y\|^2$ tends to infinity when the distance $d(x,y)$ does so.*

**Corollary**

*Let $H$ be the Hilbert space obtained by completing $\Omega(\partial X)^{\mathrm{top}}_{\int=0}$ for the Nashikawa norm $\|.\|$. Then the representation $\pi$ of $G$ is uniformly bounded and admits a proper 1-cocycle.*

Thank you very much
Merci beaucoup
Vielen Dank
Grazie mille
Muchas gracias