Représentations uniformément bornées d'après M. Cowling

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Symmetric spaces of rank one

Let G be a semi-simple group of real rank one, X the associated symmetric space and ∂X its boundary.
Choose a base point point $x_0 \in X$, a point $z_0 \in \partial X$ and z_0' be the opposite of z_0 with respect to x_0.

Notation

1) Let K be the stabiliser of x_0. K is a maximal compact subgroup of G, and X is the homogeneous G-space G/K.
2) Let P be the stabilizer of z_0. P is a (minimal) parabolic subgroup or Borel subgroup of G, and ∂X is the homogeneous G-space G/P.
3) Let $M = K \cap P$ the stabilizer of x_0 and z_0, i.e. the stabilizer of the half-line $[x_0, z_0)$. and $L \subset P$ be the stabilizer of z_0 and z_0', i.e. of the oriented geodesic line (z_0, z_0') (Levi subgroup). There is a one parameter subgroup A of G acting by translations on (z_0, z_0'), such that L is the direct product $L = MA$.

Theorem

The parabolic group P has a maximal nilpotent subgroup N. Then N is normal and P is the semi-direct product of N by the action of $L = MA$ by conjugation.
The group G admits the decomposition property $G = PK$, i.e. any $g \in G$ is of the form $g = pk$ with $p \in P$, $k \in K$. The decomposition is not unique since $P \cap K = M$. However G has the unique (Iwasawa) decompositon $G = ANK$.

Generelized Heisenberg groups

For $G = SO_0(n,1), SU(n,1)$ or $Sp(n,1)$, let $K = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$ respectively, the nilpotent group N is given as follows:
At the Lie algebra level,

$$\mathrm{Lie}(N) = K^{n-1} \oplus \mathrm{Im}K$$

where the right K-vector space K^{n-1} is equipped with the K-hermitian product $< x, y >= \sum \bar{x}_i y_i$. The bracket is given by

$$[(x,t),(y,s)] = (0, \mathrm{Im} < x, y >).$$

At the group level $N = K^{n-1} \oplus \mathrm{Im}K$ where the product is given by the formula

$$(x,t).(y,s) = (x+y, t+s+\frac{1}{2}\mathrm{Im} < x, y >).$$

The action of $L = MA$ by automorphisms on N is as follows:
M is the group of K-linear isometries of the K-hermitian product $< x, y >= \sum \bar{x}_i y_i$ on the right K-vector space K^{n-1}. It acts naturally on K^{n-1} and trivially on $\mathrm{Im}K$.
$A \simeq \mathbf{R}_+^*$ acts by the dilation automorphisms

$$\lambda(x,t) = (\lambda x, \lambda^2 t)$$

The open model

Fix $x_0 \in X$ and $z_0 \in \partial X$ as above, or equivalently fix K maximal compact and P a parabolic, as usual $P = MAN$.
The open model for $X = G/K$ is simply the bijection $N \times A \simeq X$ defined by $(n, a) \mapsto nax_0$. In other words consider $N \times A$ as the subgroup NA of G and map it to the quotient space G/K. The Iwasawa decomposition says that it is bijective. It is equivariant for the action of the subgroup $NA = AN$. The action of $P = MAN$ on $X = G/K$ is easily described on the open model: for $\nu \in N$, $t \in A$ and $m \in M$

$$\nu.(n, a) = (\nu n, a)$$

$$\lambda.(n, a) = (\delta_\lambda(n), \lambda a)$$

where δ is the one parameter dilation group:

$$\delta_\lambda(x, z) = (\lambda x, \lambda^2 z)$$

for $\lambda \in A = \mathbf{R}_+^*$.

$$m.(n, a) = (mnm^{-1}, a)$$

Bruhat decomposition and the boundary as $N \cup \{\infty\}$

In the open model, the boundary ∂X is $N \cup \{\infty\}$ where the bijection maps ∞ to z_0 and n to nz_0'.
Explanation: The Bruhat decomposition (in rank one)

$$G = P \cup PwP$$

where w is in the centre of K, and $w^2 = 1$. Geometrically w acts on X or on ∂X by central symmetry with respect to x_0.
Since $P = MAN$, the Bruhat decomposition is conveniently written

$$G = P \cup NwP$$

and any element $g \in G \setminus P$ has a unique form $g = nwp$ with $n \in N$ and $p \in P$.

Proposition

(i) *The action of Pon $\partial X = G/P$ has two orbits, z_0 and its complement, i.e. the orbit of $z_0' = wz_0$. The action of N on the orbit of wz_0 is simply transitive.*

(ii) *The map $N \cup \{\infty\} \mapsto G/P$ defined by $n \mapsto nwz_0$ and $\infty \mapsto z_0$ is a bijection, whose inverse is the map $\mathbf{n} : G/P \to N \cup \{\infty\}$ defined as follows: if $g \in P$, $\mathbf{n}(g) = \infty$; if not, $\mathbf{n}(g) \in N$ is uniquely determined by $g = \mathbf{n}(g)wp$ with $p \in P$.*

Description of the action of G on $N \cup \{\infty\}$

Proposition

The action of G on G/P is transported on $N \cup \{\infty\}$ as follows: let $g \in G$, then $g.n = \mathbf{n}(gnw)$ for any $n \in G$ and $g.\infty = \mathbf{n}(g)$

In particular the action of w on $N \cup \{\infty\}$ is given by the map $n \mapsto \mathbf{n}(wnw)$.

Example: $G = PSL(2, \mathbf{R})$ the action of $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on $\mathbf{R} \cup \{\infty\}$ is $x \mapsto -1/x$.

Proof: for $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ we have

$$wnw = \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1/x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & -1 \\ 0 & 1/x \end{pmatrix}$$

Definition

The inversion on $N \setminus \{0\}$ is the map $\sigma : n \mapsto \mathbf{n}(wnw)$. It is involutive and extends to $N \cup \{\infty\}$ by $\sigma(0) = \infty$ and $\sigma(\infty) = 0$.

The involution σ commutes with the action of M and satisfies $\sigma \circ \delta_t = \delta_{t^{-1}} \circ \sigma$.

Inversion on generalized Heisenberg groups

Recall that $N = K^{n-1} \oplus \mathrm{Im}K$.
Define for any $Z \in \mathrm{Im}K$ the endomorphism J_Z of K^{n-1} defined by right multiplication by Z. One has $J_Z^2 = -|Z|^2$.
For any $(X, Z) \in N$ let

$$A(X,Z) = \frac{\|X\|^2}{4} + J_Z, \ \overline{A}(X,Z) = \frac{\|X\|^2}{4} - J_Z$$

so that $A(X,Z)\overline{A}(X,Z) = \overline{A}(X,Z)A(X,Z) = B(X,Z)$ where

$$B(X,Z) = \frac{\|X\|^4}{16} + |Z|^2.$$

Note the homogeneity properties $A(\delta_t(X,Z)) = t^2 A(X,Z)$ and $B(\delta_t(X,Z)) = t^4 B(X,Z)$ for $t \in \mathbf{R}_+^*$.

Theorem

The action of w on $N \cup \{\infty\}$ exchanges 0 and ∞, and is given on non zero elements by the map $\sigma : (X,Z) \mapsto (X', Z')$ where

$$X' = -A(X,Z)^{-1}X = -B(X,Z)^{-1}\overline{A}(X,Z)X$$

$$Z' = -B(X,Z)^{-1}Z$$

The Jacobian of the map σ is $B(X,Z)^{-\nu/2}$

Link between Iwasawa and Buseman in rank one

Assume G has real rank one. Recall that for $x, y \in X$ and $z \in \partial X$,
$\beta_z(x, y) = \lim(d(z', x) - d(z', y))$ when $z' \to z \in \partial X$.
Recall the obvious properties of β:
(i) G-invariance

$$\beta_{gz}(gx, gy) = \beta_z(x, y)$$

(ii) cocycle relation

$$\beta_z(x_1, x_2) + \beta_z(x_2, x_3) = \beta_z(x_1, x_3)$$

Moreover :
if P is the stabilizer of z and $P = MAN$ then for any $n \in N$, $\beta_z(x, nx) = 0$.

Lemma

For $g = ank$, one has

$$a = e^{\beta_{z_0}(x_0, gx_0)}$$

In other words if we note $H(g) = \ln a$ for $g = ank$, we have $\beta_{z_0}(x_0, gx_0) = H(g)$

Proof: $\beta_{z_0}(x_0, gx_0) = \beta_{z_0}(x_0, ankx_0) = \beta_{z_0}(x_0, anx_0) = \beta_{z_0}(x_0, n'ax_0) = \beta_{z_0}(x_0, ax_0) = d(x_0, ax_0) = \ln a$.
In the open model, the Buseman function reads as follows: if $x \simeq (n, a)$ then $\beta_{z_0}(x_0, x) = \ln a$. The level function for the Buseman functions (horocycles) are the horizontal surfaces $a = \text{constant}$.

Principal series representations

Definition

A principal series representation is a representation of G on a space $\Gamma(G/P, \mathcal{V})$ of smooth sections of a G-equivariant vector bundle $\mathcal{V}$ such that for any $z \in G/P$, the representation of the parabolic subgroup P_z on the fiber $\mathcal{V}_z$ is trivial on the nilpotent subgroup N_z. In other words, $\mathcal{V} = G \times_P V$ where the vector space V carries a representation of $P = MAN$ which is trivial on N, i.e. factorizes through $P/N \simeq MA$.

Note that a representation of MA decomposes of representations $\tau \otimes \chi_s$ where τ is an irreductible representation of the compact group M and χ_s ($s \in \mathbf{C}$) the character $a \mapsto a^s$ of $A \simeq \mathbf{R}_+^*$.
When τ is the one dimensional trivial representation (i.e. $\mathcal{V}$ is a line bundle), the principal series is said to be spherical. Denote π_s the representation of G on the space of sections of the line bundle $\mathcal{L}_s = G \times_P \mathbf{R}$ where $P = MAN$ acts by the character $a \mapsto a^{-s}$ of $A \simeq \mathbf{R}_+^*$.
The representation π_s can be represented on the space $C^\infty(\partial X)$ as follows:

$$\pi_s(g)f = e^{-s\beta(x_0, gx_0)} g^{-1*} f$$

twisting the natural representation π_0 by the cocycles given by the Buseman functions.
The natural represention of G on the space $\Omega^{\mathrm{top}}(\partial X)$ is equivalent to the representation π_ν where $\nu = \mathrm{dim} K^{n-1} + 2\mathrm{dim}\mathrm{Im} K = k(n-1) + 2(k-1) = kn + k - 2$ is the Hausdorff dimension of ∂X, bigger than the naive dimension $\mathrm{top} = \mathrm{dim}\partial X = kn - 1$.

Spherical principal series in the open model

We can transport the representations π_s on the space $C^\infty(N \cup \{\infty\})$ via the bijection between ∂X and $N \cup \{\infty\}$.

Proposition

For $f \in C^\infty(N \cup \{\infty\})$ and $g \in G$, one has

$$\pi_s(g)f(n) = e^{-s\beta_g(n)} f(g^{-1}.n)$$

where $g^{-1}.n = \mathbf{n}(g^{-1}nw)$ and $\beta_g(n) = H(wn^{-1}) - H(wn^{-1}g)$

Remark

Considering the representation π_ν gives the following formula $g_* d\mu = e^{-\nu\beta_g} d\mu$ if μ is the Haar measure on N.

Proposition

π_s is isometric on $L^p(N)$ iff $\mathrm{Re}\, s = \nu/p$.
In particular π_s is unitary on $L^2(N)$ iff $\mathrm{Re}\, s = \nu/2$.

Kohn's sublaplacian on N

Let $\mathrm{Lie}N = n = \mathbf{n}^{(1)} \oplus \mathbf{n}^{(2)}$ be a generalized Heisenberg Lie algebra: $\mathbf{n}^{(1)} = K^{n-1}$ and $\mathbf{n}^{(2)} = \mathrm{Im}K$ with $[\mathbf{n}^{(1)}, \mathbf{n}^{(1)}] = \mathbf{n}^{(2)} = \mathcal{Z}(n)$.
Let X_j ($j = 1$ to $d_1 = \dim \mathbf{n}^{(1)} = k(n-1)$) be an orthonormal basis of $\mathbf{n}^{(1)} = K^{n-1}$,
let Z_k ($k = 1$ to $d_2 = \dim \mathbf{n}^{(2)} = k - 1$) be an orthonormal basis of $\mathbf{n}^{(2)} = \mathrm{Im}K$.
As left invariant vector fields:

$$\tilde{X}_j f(X,Z) = \frac{\partial f}{\partial x_j}(X,Z) + \frac{1}{2}\sum_k < J_{Z_k}X, X_j > \frac{\partial f}{\partial z_k}(X,Z)$$

where J_Z is as above right multiplication by $Z \in \mathrm{Im}K$ in K^{n-1}, and

$$\tilde{Z}_k f(X,Z) = \frac{\partial f}{\partial z_k}(X,Z).$$

The element $-\sum_i X_i^2$ of the envelopping algebra gives rise to the left invariant differential operator

$$\Delta = -\sum_i \tilde{X}_i^2$$

We consider Δ as a densely defined operator on $L^2(N)$. It is essentially self-adjoint and positive.

Cowling's theorem

Let W^α be the Hilbert space obtained by completing the space $C^\infty(N \cup \{\infty\})$ by the Sobolev norm

$$\|f\|_{W^\alpha} = \|\Delta^{\alpha/2} f\|_{L^2(N)}$$

As a consequence of the hypoellipticity property of Δ, we have the crucial result (Folland):

Lemma

$f \in W^{\alpha+1}$ iff $\tilde{X}_j f \in W^\alpha$ for any $j = 1$ to d_1.

The result of M. Cowling with F. Astengo and B. Di Blasio is the following. Recall that $\nu = d_1 + 2d_2 = k(n-1) + 2(k-1) = kn + k - 2$ is $n-1$ for $SO(n,1)$, $2n$ for $SU(n,1)$ and $4n+2$ for $Sp(n,1)$.

Theorem

For any complex number s in the strip

$$0 < \mathrm{Re} s < \nu$$

there is a constant C_s such π_s is uniformly bounded by C_s on the Sobolev space W^α for $\alpha = \nu/2 - \mathrm{Re} s$:

$$\|\pi_s(g) f\|_{W^\alpha} \leq C_s \|f\|_{W^\alpha}$$

for any $g \in G$.

Steps of Cowling's proof

1) **Reduction to the boundedness of a single operator**
The group G is generated by the subgroups M, N, A and the single element w.
We shall prove that
For $n \in N$ and $m \in M$, $\pi_s(n)$ and $\pi_s(m)$ are unitary on any W^α
For $a \in A$, $\pi_s(a)$ is unitary on W^α provided $\alpha = \nu/2 - \mathrm{Re} s$.
The question is now whether the operator $\pi_s(w)$ is bounded on on W^α for $\alpha = \nu/2 - \mathrm{Re} s$? The answer is yes provided $-\nu/2 < \alpha < \nu/2$, i.e. $0 < \mathrm{Re} s < \nu$.

2) **Homogeneous functions acting as multiplicators on the Sobolev spaces W^α**

Theorem

Let m be a function on $N \setminus \{0\}$ homogeneous of degree $d \in \mathbf{C}$ with $\mathrm{Re} d \leq 0$. Then for any α and β such that $-\nu/2 < \alpha \leq \beta < \nu/2$ and $\alpha - \beta = \mathrm{Re} d$, the multiplication $f \mapsto mf$ extends to a bounded operator W^β to W^α

3) **Boundedness of the operator $\pi_s(w)$ on W^α for $\alpha = \nu/2 - \mathrm{Re} s$ and $-\nu/2 < \alpha < \nu/2$.**

Reduction to the boundedness of a single operator

Proposition

The representation π_s restricted to N is unitary on L^2 any W^σ

Proof: $\pi_s(n) = \pi_0(n)$ acts by isometries on $L^2(N)$ and commutes with the left invariant $\Delta = -\sum_i \tilde{X}_i^2$.

Proposition

The representation π_s restricted to M is unitary on L^2 any W^σ

Proof: $\pi_s(m) = \pi_0(m)$ acts by isometries on $L^2(N)$ and commutes with $\Delta = -\sum_i \tilde{X}_i^2$.

Proposition

The representation π_s restricted to A is unitary on W^σ provided $\alpha = \nu/2 - \mathrm{Re}s$.

Proof: Let $t \in A = \mathbf{R}_+^*$, note that $\Delta(f \circ \delta_t) = t^2 \Delta f$ and by functional calculus $\Delta^{\alpha/2}(f \circ \delta_t) = t^\alpha \Delta^{\alpha/2} f \circ \delta_t$. On the other hand $\pi_s(t)f = t^s(f \circ \delta_t)$. It follows immediately that $\Delta^{\alpha/2} \circ \pi_s(t) = \pi_{s+\alpha}(t) \circ \Delta^{\alpha/2}$, with $\pi_{s+\alpha}$ unitary for $\mathrm{Re}(s+\alpha) = \nu/2$.

The Lorentz spaces $L^{p,2}(N)$

Define the spaces $L^{p,2}(N)$ and $L^{p,\infty}(N)$ as the set of measurable functions on N for which the following norms are respectively finite:

$$\|f\|_{p,2} = (\frac{2}{p}\int_0^{+\infty}(s^{1/p}f^*(s))^2\frac{ds}{s})^{1/2}$$

$$\|f\|_{p,\infty} = \sup_{s\in(0,+\infty)} s^{1/p}f^*(s)$$

where the function f^* on $(0,+\infty)$ is the non increasing rearrangement of f:

$$f^*(s) = \inf\{t\in(0,+\infty), \mu(\{n\in N, |f(n)|>t\}\leq s\}$$

Proposition

(i) *Let $f\in L^{p,2}(N)$ and $m\in L^{r,\infty}(N)$, then $\|mf\|_{L^{q,2}(N)}\leq C\|m\|_{L^{r,\infty}(N)}\|f\|_{L^{p,2}(N)}$ for $\frac{1}{p}+\frac{1}{r}=\frac{1}{q}$*

(ii) *Let $f\in L^{p,2}(N)$ and $g\in L^{r,\infty}(N)$ then $\|f*g\|_{L^{q,2}(N)}\leq C\|g\|_{L^{r,\infty}(N)}\|f\|_{L^{p,2}(N)}$ for $\frac{1}{p}+\frac{1}{r}=\frac{1}{q}+1$.*

Technical lemmas

Lemma

Let f be a homogenerous function on N of degree $d \in \mathbf{C}$ with $\mathrm{Re} d \leq 0$. Then $f \in L^{p,\infty}(N)$ with $\frac{1}{p} = -\frac{\mathrm{Re} d}{\nu}$.

Proof: By hypothesis $f(\delta_t n) = t^d f(n)$ hence $|f(\delta_t n)| == t^{\mathrm{Re} d} f(n)$. Compute

$$\mu(\{n \in N, |f(n)| > t\} \leq s\} == t^{-\frac{\nu}{\mathrm{Re} d}} \mu(\{n \in N, |f(n)| > 1\})$$

which shows that $f^*(s) \leq Cs^{-\frac{\mathrm{Re} d}{\nu}}$ and $f \in L^{p,\infty}(N)$ for $\frac{1}{p} = -\frac{\mathrm{Re} d}{\nu}$.

Lemma

(i) *If $-\nu/2 < \alpha \leq 0$ and $\frac{1}{p} = \frac{1}{2} - \alpha/\nu$, then $\|f\|_{W^\alpha} \leq C(\alpha)\|f\|_{p,2}$.*

(ii) *IIf $0 \leq \alpha < \nu/2$ and $\frac{1}{p} = \frac{1}{2} - \alpha/\nu$, then $\|f\|_{p,2} \leq C(\alpha)\|f\|_{W^\alpha}$.*

Proof: the left invariant operator $\Delta^{\alpha/2}$ for negative α is given by right convolution by a kernel $R_{-\alpha}$ homogeneous of degree $-\alpha - \nu$.

$$\Delta^{\alpha/2} f = f * R_{-\alpha}$$

By the first lemma, $R_{-\alpha} \in L^{r,\infty}(N)$ with $\frac{1}{r} = \frac{\alpha+\nu}{\nu} = \frac{\alpha}{\nu} + 1$.

By the proposition above, item (ii), $f \mapsto f * g$ maps $L^2 = L^{2,2}$ to $L^{p,2}$ where $\frac{1}{p} + \frac{1}{r} = \frac{1}{2} + 1$, i.e $\frac{1}{p} = \frac{1}{2} - \alpha/\nu$. One has then $\|\Delta^{\alpha/2} f\|_{L^2} \leq C\|f\|_{p,2}$, which proves item (i).

Item (ii) follows from (i) by duality since $L^{p,2}(N)$ is dual of $L^{q,2}(N)$ with $\frac{1}{p} + \frac{1}{q} = 1$, and W^α dual of $W^{-\alpha}$.

Homogeneous functions multiplying the W^α's

Theorem

Let m be a function on $N \setminus \{0\}$ homogeneous of degree $d \in \mathbf{C}$ with $\mathrm{Re} d \leq 0$. Then for any α and β such that $-\nu/2 < \alpha \leq \beta < \nu/2$ and $\alpha - \beta = \mathrm{Re} d$, the multiplication $f \mapsto mf$ extends to a bounded operator W^β to W^α

Sketch of proof:

(i) Case $-\nu/2 < \alpha \leq 0 \leq \beta < \nu/2$
By the first lemma $m \in L^{r,\infty}(N)$ with $\frac{1}{r} = -\frac{\mathrm{Re} d}{\nu}$. Then apply items (i) and (ii) of the second lemma:

$$\|mf\|_{W^\alpha} \leq C\|mf\|_{p,2} \leq C\|f\|_{q,2} \leq \|f\|_{W^\beta}$$

with $\frac{1}{p} = \frac{1}{2} - \frac{\alpha}{\nu}$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{r} = \frac{1}{2} - \frac{\beta}{\nu}$.

(ii) Case $-1 \leq \alpha = \beta \leq 1$
We want to show that $f \mapsto mf$ (with $\mathrm{Re} d = 0$) is bounded on W^α. Begin with $\alpha = 1$: if $f \in W^1$ then $X_j(f) \in L^2$. Consider

$$X_j(mf) = mX_j(f) + X_j(m)f.$$

The first term is obviously L^2 as a product of a L^2 function by a bounded function. As to the second, note that the functions $X_j(m)$ are homodegneous of degree $d - 1$ with $\mathrm{Re}(d-1) = -1$,so that the second term is also L^2 by applying case (i) to $\alpha = 0$, $\beta = 1$. This implies that mf is in W^1. The general case follows by duality and interpolation.

Proof continued

(iii) Case $-\nu/2 < \alpha = \beta < \nu/2$
Assume $0 \leq \alpha = \beta < \nu/2$, the symmetric case will follow by duality. Let k the integer such that $-1 \leq \alpha - k < 0$. For any homogeneous invariant differential operator D of degree k, one has $Df \in W^{\alpha - k}$. Let us show that $D(mf)$ also belongs to $W^{\alpha - k}$.
First note that $mDf \in W^{\alpha - k}$ by case (ii) applied to $\alpha - k$. On the other hand $D(mf) - mDf$ is a sum of terms of the form $m_j D_j f$ with m_j homogeneous of degree $d - j$ and D_j homogeneous invariant differential operator of degree $k - j$, with $j = 1, 2, \ldots, k$. One has $D_j f \in W^{\alpha - k + j}$ (with $0 < \alpha - k + j \leq \alpha < \nu/2$) so that $m_j D_j f \in W^{\alpha - k}$ (with $-\nu/2 \leq -1 \leq \alpha - k < 0$) by case (i).
We conclude that $D(mf) \in W^{\alpha - k}$ for all D of order k, which implies that $mf \in W^{\alpha}$. The case $-\nu/2 < \alpha = \beta \leq 0$ follows by dualilty.

(iv) Case $-\nu/2 < \alpha < \beta < \nu/2$
Assume as usual by $0 < \alpha < \beta < \nu/2$, the symmetric case will follow by duality. Consider the function $m^{-\frac{z\beta}{d}}$ which has homogeneity degree $-z\beta$.
For $\mathrm{Re} z = 0$ it multiplies W^{β} to itself by case (iii).
For $\mathrm{Re} z = 1$ it multiplies W^{β} to $W^0 = L^2$ by case (i).
By complex interpolation $m^{-\frac{z\beta}{d}}$ multiplies W^{β} to $W^{\beta(1 - \mathrm{Re} z)}$. In particular for $z = -\frac{d}{\beta}$, multiplication by m maps W^{β} to $W^{\beta + \mathrm{Re} d} = W^{\alpha}$.

Cowling's main result

Theorem

Let σ be the inversion on N defined above, S its Jacobian and T_α be the operator defined on functions on N by the formula:

$$T_\alpha f = S^{1/2-\alpha/\nu}(f \circ \sigma)$$

Then T_α is bounded on W^α for any α satisfying $-\nu/2 < \alpha < \nu/2$.

Let us summarize the facts we need about σ and S. First σ is involutive, and for m homogeneous of degree d, $m \circ \sigma$ is homogeneous of degree $-d$.
Recall the fact that $S = \mathcal{B}^{-\nu/2}$ is homogeneous of degree -2ν, so that $S^{1/2-\alpha/\nu}$ has degree $-\nu + 2\alpha$.
The change of variable formula says that T_0 is unitary on L^2:

$$\|S^{1/2}(f \circ \sigma)\|_{L^2} = \|f\|_{L^2}$$

There is a magical fomula which is special to nilpotent groups satisfying a certain J^2 property, which include the case of the groups $K^{n-1} \oplus \mathrm{Im} K$ we consider here:

Lemma

For any $j = 1$ to $k(n-1)$,

$$X_j(f \circ \sigma) = \sum_i (X_j \sigma_{X_i})(X_i f) \circ \sigma)$$

where for any $X \in \mathrm{Lie}(N)$, the function σ_X on N is defined by $n \mapsto < \sigma(n), X >$

Reducing T_α to $T_{\alpha-1}$

Start from $f \in W^\alpha$ and try to prove that $T_\alpha f = S^{1/2-\alpha/\nu}(f \circ \sigma)$ is also in W^α. What we want is to show that for all j's, the function $X_j T\alpha$ are in $W^{\alpha-1}$. Let us compute $X_j T_\alpha f = X_j(S^{1/2-\alpha/\nu})(f \circ \sigma) + S^{1/2-\alpha/\nu} X_j(f \circ \sigma)$.
Using the lemma above, we get

$$X_j T_\alpha f = S^{-1/2-\alpha/\nu}(X_j S)(f \circ \sigma) + \sum_i S^{1/2-\alpha/\nu}(X_j \sigma_{X_i})(X_i f \circ \sigma)$$

This can also be written $X_j T_\alpha f = T_{\alpha-1}\Theta_\alpha f$
where

$$\Theta_\alpha f = S^{1+\alpha/\nu}(X_j S \circ \sigma) f + \sum_i S^{1/\nu}(X_j \sigma_{X_i} \circ \sigma) X_i f$$

Lemma

The operator Θ_α maps W^α to $W^{\alpha-1}$ if $0 < \alpha < \nu/2$

Proof: if $f \in W^\alpha$, then all the $X_i f$ are in $W^{\alpha-1}$. Use the following observations:

(i) The function $S^{1+\alpha/\nu}(X_j S \circ \sigma) f$ has homogeneity degree -1 and therefore multiplies W^α to $W^{\alpha-1}$

(ii) The functions $S^{1/\nu}(X_j \sigma_{X_i} \circ \sigma)$ have homogeneity degree 0 and therefore mutiply $W^{\alpha-1}$ to itself.

It follows that $\Theta_\alpha f$ is in $W^{\alpha-1}$.

Proof of Cowling's theorem

(i) Case $0 \leq \alpha \leq 1$
The case $\alpha = 0$ is already known: indeed $T_0 : f \mapsto S^{1/2}(f \circ \sigma)$ is an isometry. Let us treat the case $\alpha = 1$. By the lemma above $X_j T_1 = T_0 \Theta_1$ where Θ_1 maps W^1 to L^2. It follows that for $f \in W^1$, all the $X_j T_1 f$'s are in L^2, i.e. $T_1 f \in W^1$. By interpollation, $T\alpha$ is bounded on W_α for $0 \leq \alpha \leq 1$.

(ii) Case $0 \leq \alpha < \nu/2$
More generally consider the interval $[0, \nu/2[$ as the union of $[k-1, k[$ for $k = 1$ to $\nu/2$. We proceed by (finite) induction on k. For any $f \in W_\alpha$ we have as above $X_j T_\alpha f = T_{\alpha-1} \Theta_\alpha f$ with $\Theta_\alpha f \in W^{\alpha-1}$ and $T_{\alpha-1}$ continuous in $W^{\alpha-1}$ by induction hypothesis. Hence $X_j T_\alpha f \in W^{\alpha-1}$ and $T_\alpha f \in W^\alpha$.

(iii) General case: $-\nu/2 < \alpha < \nu/2$
it follows by duality between $W^{-\alpha}$ and W^α with $T_{-\alpha} = T_\alpha^*$.

This ends the proof.

Thank you very much
Merci beaucoup
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Grazie mille
Muchas gracias