Discrete Series and Characters of the Component Group

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Suppose $\phi : W_{\mathbb{R}} \to {}^LG$ is an L-homomorphism. There is a close relationship between the L-packet associated to ϕ and characters the component group of the centralizer of ϕ. This is reinterpreted in [2], see also [1], in part to make it more canonical and a bijection. This involves a number of changes, including using the notion of strong real form, several strong real forms at once, and taking a cover of the component group. This cover is not necessarily a two-group, so the values of the character may not be just signs.

Over the years a number of people have asked me how to relate an L-packet of discrete series to characters of S_ϕ in this language. While this is a special case of [2] and [1], it isn't so easy to extract it. In these notes I work out this case, and give some details in the case of $SU(p,q)$ and $U(p,q)$. There are no complete proofs; see the references for more details.

I also address a question raised by Michael Harris about endoscopy.

Besides the basic references cited above, I make some use of [3].

1 Basic Setup

We fix a pair (G,γ) consisting of a reductive, algebraic (or complex) group and an outer automorphism of G. This corresponds to an inner class of real forms of G. We are interested in discrete series representations of real forms of G. A real form in this inner class has discrete series representations if and only if $\gamma = 1$, so we assume this.

The extended group is a semidirect product $G^\Gamma = G \rtimes \Gamma$ where $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Since $\gamma = 1$ it is a direct product, and we may safely drop Γ from the notation. The constructions of [2],[1],[3] involving the "twist" simplify in this setting.

We fix Cartan and Borel subgroups H and B of G. Let $G^\vee$ be the dual group, and fix Cartan and Borel subgroups $H^\vee$ and $B^\vee$ of $G^\vee$. By construction we have canonical identifications

$$X^*(H) = X_*(H^\vee), \quad X_*(H) = X^*(H^\vee) \tag{1.1}$$

where X^* and X_* denote the character and co-character lattices, respectively. Let $R(G,H), P(G,H), R^\vee(G,H)$ and $P^\vee(G,H)$ be the root, weight, coroot, and coweight lattices of G with respect to H. Recall

$$\begin{aligned} P &= \{\lambda \in X^*(H) \otimes \mathbb{R} \,|\, \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \quad \text{for all } \alpha^\vee \in R^\vee\} \\ P^\vee &= \{\gamma^\vee \in X_*(H) \otimes \mathbb{R} \,|\, \langle \alpha, \gamma^\vee \rangle \in \mathbb{Z} \quad \text{for all } \alpha \in R\} \end{aligned} \tag{1.2}$$

and these are lattices if and only if G is semisimple. We abbreviate these $R, P, R^\vee$ and $P^\vee$. Define $R(G^\vee, H^\vee)$ etc. similarly. Then, for example $R(G,H) = R^\vee(G^\vee, H^\vee)$. Let $W = W(G,H)$ be the Weyl group.

Let $Z = Z(G)$ be the center of G. Then

$$\begin{aligned} P^\vee &= \{\gamma^\vee \in X_*(H) \otimes \mathbb{R} \,|\, \exp(2\pi i \gamma^\vee) \in Z\} \\ X_*(H) &= \{\gamma^\vee \in X_*(H) \otimes \mathbb{R} \,|\, \exp(2\pi i \gamma^\vee) = 1\} \end{aligned} \tag{1.3}$$

Write $G^{\vee\Gamma}$, instead of ${}^LG$, for the L-group of G. This is a semidirect product $G^\vee \rtimes \Gamma$.

2 Strong real forms

We are working entirely in the inner class of real forms given by $\gamma = 1$.

A *strong involution* of G (in this inner class) is an element $x \in G$ satisfying $x^2 \in Z$. A *strong real form* of G (in this inner class) is a conjugacy class of strong involutions.

An (inner) *involution* of G is an element $\theta \in \mathrm{Int}(G)$ (algebraic, or holomorphic, involutions of G) satisfying $\theta^2 = 1$. A real form of G (in this inner class) is a conjugacy class of involutions. The map $x \to \mathrm{int}(x)$ defines a surjection

$$\{\text{strong involutions}\} \to \{\text{involutions}\} \tag{2.1}$$

and factors to a surjection

(2.2) $$\{\text{strong real forms of } G\} \to \{\text{real forms of } G\}.$$

(since we have fixed $\gamma = 1$ we drop the qualifier *inner* everywhere). If G is adjoint this is a bijection.

If x is a strong involution we let $\theta_x = \text{int}(x)$ and $K_x = G^{\theta_x} = \text{Cent}_G(x)$.

Every strong involution x is conjugate to an element of H. Let

(2.3) $$\mathcal{X}_1 = \{x \in H \mid x^2 \in Z\}.$$

The Weyl group $W = W(G, H)$ acts on $\mathcal{X}_1$. From (1.3) it is easy to see:

Lemma 2.4 *For $\gamma^\vee \in P^\vee$ let*

(2.5) $$x(\gamma^\vee) = \exp(\pi i \gamma^\vee) \in \mathcal{X}_1.$$

Then $P^\vee \ni \gamma^\vee \to x(\gamma^\vee) \in \mathcal{X}_1$ factors to an isomorphism

(2.6) $$P^\vee/2X_* \simeq \mathcal{X}_1.$$

If G is semisimple this is a finite set.

Furthermore there is a canonical bijection

(2.7) $$\{\textit{strong real forms of } G\}/\sim \overset{1-1}{\longleftrightarrow} \mathcal{X}_1/W.$$

We will sometimes choose a set of representatives $\mathcal{X}_1' \subset H$ of $\mathcal{X}_1/W$. It is often convenient consider only those x with x^2 fixed; this is always a finite set. So fix $z \in Z$ and let

(2.8) $$\mathcal{X}_1[z] = \{x \in H \mid x^2 = z\}, \quad \mathcal{X}_1'[z] = \{x \in \mathcal{X}_1' \mid x^2 = z\}.$$

An important special class of strong real forms are the following. Let $z(\rho^\vee) = \exp(2\pi i \rho^\vee) \in Z$. This is independent of the choice of positive roots. We say a strong real form is *pure* if $x \in \mathcal{X}_1[z(\rho^\vee)]$.

3 Representations of strong real forms

A *representation of a strong involution* is a pair (x, π) where x is a strong involution and π is a $(\mathfrak{g}, K_x)$-module. We say (x, π) is equivalent to (x', π') if there exists $g \in G$ such that$gxg^{-1} = x'$ and $\pi' = \pi^g$. A *representation of a strong real form* of G is an equivalence class of representations of strong involutions.

Example 3.1 Here is a key example. Let $G = SL(2,\mathbb{C})$ and $x = \mathrm{diag}(i, -i)$. Then $K_x = \mathbb{C}^\times$ and the corresponding real group is $SU(1,1) = SL(2,\mathbb{R})$. Let π be the $(\mathfrak{g}, K_x)$-module corresponding to a discrete series representation of $SL(2,\mathbb{R})$; that is π is a $(\mathfrak{g}, \mathbb{C}^\times)$-module.

Thus (x, π) is a representation of the strong involution x. Let $\overline{\pi}$ be the contragredient of π. Then

$$\Pi = \{(x, \pi), (x, \overline{\pi})\} \tag{3.2}$$

can be thought of as an L-packet of $SL(2,\mathbb{R})$.

Choose $g \in G$ satisfying $gxg^{-1} = -x$ (i.e. the Weyl group element). Note that $\overline{\pi}^g \simeq \pi$, and therefore $(x, \overline{\pi}) \equiv (gxg^{-1}, \overline{\pi}^g) \equiv (-x, \pi)$. Thus we can think of our L-packet for $SL(2,\mathbb{R})$ instead as

$$\Pi = \{(x, \pi), (-x, \pi)\} \tag{3.3}$$

where π is fixed and we vary x.

4 L-parameters

Suppose ϕ is an admissible homomorphism of the Weil group of $\mathbb{R}$ into $G^{\vee\Gamma}$, and fix a strong involution x. Associated to (x, ϕ) is the L-packet (possibly empty) $\Pi(x, \phi)$ of $(\mathfrak{g}, K_x)$-modules. Define

$$\Pi(\phi) = \{(x, \pi)\}/\sim \tag{4.1}$$

consisting of equivalence classes of pairs (x, π), where x is a strong involution and $\pi \in \Pi(x, \phi)$. Note that $\Pi(x, \phi) = \Pi(x', \phi)$ if and only if x is G-conjugate to x'.

For fixed x the representations (x, π), up to this notion of equivalence, are in bijection with equivalence classes of admissible representations of the corresponding real group $G(\mathbb{R})$.

This is a little bit of a slippery definition, and it is helpful to make some choices to pin it down. Recall (2.7) if $\mathcal{X}_1'$ is a set of representatives of $\mathcal{X}_1/W$ then every strong involution x is conjugate to precisely one element of $\mathcal{X}_1'$. Therefore there is a canonical bijection

$$\Pi(\phi) \overset{1-1}{\longleftrightarrow} \coprod_{x \in \mathcal{X}_1'} \Pi(x, \phi), \tag{4.2}$$

For fixed x, $\Pi(x,\phi)$ is in bijection with an L-packet of admissible representations of a real group $G(\mathbb{R})$. In these terms we have

$$\Pi(x,\phi)=\{(x', \tag{4.3}$$

However (see Example 3.1) it is often helpful to allow x to vary with a given conjugacy class. See the examples in Section 10.

Unless G is adjoint these sets are often larger than necessary. For one thing if G is not semisimple they are infinite. Secondly, multiple strong real forms corresponding to the same real form can occur. See the examples.

For these reasons it is sometimes helpful to fix an element $z\in Z$, and define

$$\Pi_z(\phi)=\{(x,\pi)\,|\,\pi\in\Pi(x,\phi),x^2=z\}/\sim \tag{4.4}$$

These sets are finite, and the number of strong real forms mapping to single real form is small, and often 1. On the other hand for a fixed z some real forms may fail to occur. See the examples.

5 Discrete Series L-packets

Now suppose ϕ is a discrete series L-parameter. Let λ be the infinitesimal character for G determined by ϕ. For the purposes of this discussion this infinitesimal character is not important, and we can take it to be ρ, the infinitesimal character of the trivial representation.

Proposition 5.1 *Suppose ϕ is a discrete series L-parameter for G. There are canonical bijections*

$$\mathcal{X}_1\overset{1-1}{\longleftrightarrow}\Pi(\phi) \tag{5.2}$$

and

$$\mathcal{X}_1[z]\overset{1-1}{\longleftrightarrow}\Pi_z(\phi) \tag{5.3}$$

Sketch of proof. The parameter ϕ specifies central and infinitesimal characters. This determines an L-packet of discrete series of any real form of G: those with the given central and infinitesimal characters.

Write the infinitesimal charcter as $\lambda \in \mathfrak{h}^*$, dominant with respect to the fixed choice of Borel subgroup B. Let $\pi_x(\lambda)$ be the discrete series $(\mathfrak{g}, K_x)$-module with the given central character, and Harish-Chandra parameter λ.

The key point is that if $x' = wx$ with $w \in W$, there is a natural way to identify the representation $\pi_{wx}(\lambda)$ with $\pi_x(w^{-1}\lambda)$, and the rest is formal. See [3] for details. ■

Fix $x \in \mathcal{X}_1$, and consider the set $\Pi(x, \phi)$. This is an L-packet of $(\mathfrak{g}, K_x)$-modules, or equivalently an L-packet of admissible representations of the corresponding real form of G. Under the bijection (5.2),

$$\begin{aligned} \Pi(x,\phi) &\overset{1-1}{\longleftrightarrow} \{x' \in \mathcal{X}_1, \mid x' = wx \text{ for some } w \in W\} \\ &= \{\pi_{wx}(\lambda) \mid w \in W\} \\ &= \{\pi_x(w^{-1}\lambda) \mid w \in W\} \end{aligned} \tag{5.4}$$

It is important to know which elements of $\mathcal{X}_1$ correspond via (5.2) to generic discrete series representation of quasipslit strong real forms of G.

Lemma 5.5 *Suppose (x, π) is a discrete series representation of a strong involution $x \in \mathcal{X}_1$. Then π is generic if and only if*

$$\alpha(x) = -1 \quad \textit{for all simple roots } \alpha. \tag{5.6}$$

It might be helpful to explain what this means. Recall π is a $(\mathfrak{g}, K_x)$-module. This corresponds to an admissible representation $\pi_{\mathbb{R}}$ of a real form $G(\mathbb{R})$ of G. We say π is *generic* if $\pi_{\mathbb{R}}$ is generic, i.e. admits a Whittaker model. This is equivalent to: $\pi_{\mathbb{R}}$ is *large*. If $\lambda_{\mathbb{R}}$ is the Harish-Chandra parameter of $\pi_{\mathbb{R}}$ this is equivalent to: every real simple root in the chamber defined by $\lambda_{\mathbb{R}}$ is non-compact. In the $(\mathfrak{g}, K_x)$ picture an imaginary root α is compact if $\alpha(x) = 1$, and non-compact otherwise.

6 The groups S_ϕ and $\widetilde{S_\phi}$

Fix a discrete series L-homomorphism ϕ. Let

$$S_\phi = \mathrm{Cent}_{G^\vee}(\phi). \tag{6.1}$$

Without loss of generality we may assume $\phi(\mathbb{C}^\times) \in H^\vee$ and $\phi(j)h\phi(j)^{-1} = h^{-1}$ for all $h \in H^\vee$. Then

$$S_\phi = \{h \in H^\vee \mid h^2 = 1\}. \tag{6.2}$$

Note that S_ϕ is in fact independent of ϕ.

Let $G^\vee_{sc}$ be the *topologically* simply connected cover of $G^\vee$. Thus $G^\vee_{sc} = \mathbb{C}^n \times G^\vee_{d,sc}$ where $n = \dim(Z(G^\vee))$ and $G^\vee_{d,sc}$ is the simply connected cover of the derived group $G^\vee_d$. Let $p : H^\vee_{sc} \to H^\vee$ be the inverse image of $H^\vee$ in $G^\vee_{sc}$, and define

$$\widetilde{S}_\phi = p^{-1}(S_\phi) \subset G^\vee_{sc} \tag{6.3}$$

Lemma 6.4 *Let $R = R(G,H) \subset X^*(H) = X_*(H^\vee)$. There is a natural isomorphism*

$$X^*(H)/2R \simeq \widetilde{S}_\phi. \tag{6.5}$$

For $\lambda \in X^*(H)$ write $s(\lambda)$ for the corresponding element of $\widetilde{S_\phi}$.

Lemma 6.6 *There is a perfect pairing*

$$\langle\, ,\, \rangle : \mathcal{X}_1 \times \widetilde{S}_\phi \to S^1. \tag{6.7}$$

In particular we obtain a group isomorphism

$$\mathcal{X}_1 \simeq \widetilde{S_\phi}\,\widehat{}\,, \tag{6.8}$$

where $\widetilde{S_\phi}\,\widehat{}$ is the Pontriagin dual of $\widetilde{S_\phi}$. For $x \in \mathcal{X}_1$ write χ_x for the corresponding character of $\widetilde{S_\phi}$ via the isomorphism (6.8)*. We also obtain a canonical bijection of sets*

$$\Pi(\phi) \xleftrightarrow{1-1} \widetilde{S_\phi}\,\widehat{}\,. \tag{6.9}$$

Proof. By Lemmas 2.4 and 6.4 the pairing $\mathcal{X}_1 \times \widetilde{S_\phi}$ amounts to a pairing

$$P^\vee/2X_*(H) \times X^*(H)/2R \to S^1. \tag{6.10}$$

This is the obvious one. The pairing $X^*(H) \times X_*(H) \to \mathbb{Z}$ extends to a pairing $\langle\, ,\, \rangle$ between $X_*(H) \otimes \mathbb{R}$ and $X^*(H) \otimes \mathbb{R}$. For $\gamma^\vee \in P^\vee \subset X_*(H) \otimes \mathbb{R}, \lambda \in X^*(H)$ define

$$(\gamma^\vee, \lambda) = e^{\pi i \langle \gamma^\vee, \lambda\rangle} \in S^1. \tag{6.11}$$

This factors to the desired pairing (6.10). ■

Note that in the notation of Lemmas 2.4 and 6.4 we can write the pairing as

$$(x(\gamma^\vee), s(\lambda)) = e^{\pi i \langle \gamma^\vee, \lambda \rangle} \quad (\gamma^\vee \in P^\vee, \lambda \in X^*(H)) \tag{6.12}$$

or equivalently

$$\chi_{x(\gamma^\vee)}(s(\lambda)) = e^{\pi i \langle \gamma^\vee, \lambda \rangle}. \tag{6.13}$$

Note that the bijection (6.9) takes the trivial character of $\widetilde{S_\phi}$ to the trivial representation of a compact strong real form of G. This is not the right normalization: we prefer the basepoint to be a generic discrete series of a quasisplit strong real form. By Lemma 5.5 we need to choose x so that $\alpha(x) = -1$ for all simple roots. This is provided (canonically) by the element $x(\rho^\vee)$.

Definition 6.14 *For $x \in \mathcal{X}_1$ define $\tau_x \in \widetilde{S_\phi}\widehat{\ }$ by*

$$\tau_x = \chi_x \chi_{x(\rho^\vee)}^{-1}. \tag{6.15}$$

In other words (cf. (6.13))

$$\tau_{x(\gamma^\vee)}(s(\lambda)) = e^{\pi i \langle \gamma^\vee - \rho^\vee, \lambda \rangle} \tag{6.16}$$

Proposition 6.17 *The map $x \to \tau_x$ induces canonical bijections of sets*

$$\mathcal{X}_1 \overset{1-1}{\longleftrightarrow} \widetilde{S_\phi}\widehat{\ }. \tag{6.18)(a}$$

and

$$\Pi(\phi) \overset{1-1}{\longleftrightarrow} \widetilde{S_\phi}\widehat{\ }. \tag{6.18)(b}$$

The element $x(\rho^\vee)$, and the trivial character of $\widetilde{S_\phi}$, correspond to a generic discrete series representation of a quasisplit strong real form of G.

Recall $\mathcal{X}_1[z] = \{x \in H \mid x^2 = z\} \subset \mathcal{X}_1$. We want to identify the image of the various subsets $\mathcal{X}_1[z]$ under this bijection. Since S_ϕ is a quotient of $\widetilde{S_\phi}$, $\widehat{S_\phi}$ is a subset of $\widetilde{S_\phi}\widehat{\ }$.

Lemma 6.19 *Fix $z \in Z$. Choose $x_0 \in \mathcal{X}_1[z]$. Then the bijection $x \to \tau_x$ of Proposition 6.17 restricts to bijections*

$$\mathcal{X}_1[z] \overset{1-1}{\longleftrightarrow} \tau_{x_0}\widehat{S_\phi}. \tag{6.20}$$

and

$$\Pi_z(\phi) \overset{1-1}{\longleftrightarrow} \tau_{x_0}\widehat{S_\phi}. \tag{6.21}$$

The right hand sides are independent of the choice of x_0, but the bijections depend on this choice.

An alternative version is:

Lemma 6.22 *Fix $z \in Z$. Choose $x_0 \in \mathcal{X}_1[z]$. Then the map $x \to \tau_x/\tau_{x_0}$ induces bijections*

$$\mathcal{X}_1[z] \overset{1-1}{\longleftrightarrow} \widehat{S_\phi}. \tag{6.23}$$

and

$$\Pi_z(\phi) \overset{1-1}{\longleftrightarrow} \widehat{S_\phi}. \tag{6.24}$$

depending on the choice of x_0.

Definition 6.25 *For $\chi \in \widehat{S_\phi}$ we write $(x(\chi), \pi(\chi)) \in \Pi_z(\phi)$ via the bijection* (6.24). *This depends on the choice of x_0.*

To make this map explicit, suppose $x_0 = x(\gamma_0^\vee)$. Then for $\gamma^\vee \in P^\vee$ and $\lambda \in X^*(H)$ we have

$$(\tau_{x(\gamma^\vee)}/\tau_{x_0})(s(\lambda)) = e^{\pi i \langle \gamma^\vee - \gamma_0^\vee, \lambda\rangle} \tag{6.26}$$

The one case in which these bijections are canonical is if $z = z(\rho^\vee)$, in which case we can take $x_0 = x(\rho^\vee)$:

$$\Pi_{z(\rho^\vee)}(\phi) \overset{1-1}{\longleftrightarrow} \widehat{S_\phi}. \tag{6.27}$$

The problem is that (in some cases) neither (6.18)(b) nor (6.27) are perfectly suited to the usual theory. In (6.18)(b) $\widetilde{S_\phi}$ is not necessarily a two group, and each weak real form may occur as the image of many strong real forms. On the other hand in (6.27) it may happen that not every (weak) real form is the image of a strong real form in $\mathcal{X}_1[z(\rho^\vee)]$. We address this in the next section.

7 Endoscopic Lifting

Fix a discrete series parameter ϕ. Let $\widetilde{S_\phi}$ be as in Section 6. Recall (6.8) and (6.9)

$$\Pi(\phi) \overset{1-1}{\longleftrightarrow} \mathcal{X}_1 \overset{1-1}{\longleftrightarrow} \widetilde{S_\phi}\widehat{\ }. \tag{7.1}$$

For $\chi \in \widetilde{S_\phi}\widehat{\ }$ write x_χ for the corresponding element of $\mathcal{X}_1$, and $\pi(\chi)$ for the corresponding discrete series representation of the strong real form x_χ.

If $G(\mathbb{R})$ is a real group let $\kappa(G(\mathbb{R})) = \pm 1$ be the Kottwitz invariant of $G(\mathbb{R})$ [4]. For $x \in \mathcal{X}_1$ let $\kappa(x)$ be the Kottwitz invariant of the corresponding (weak) real form of G.

Definition 7.2 *Fix $\tilde{s} \in \widetilde{S_\phi}$. Let*

$$\widetilde{\eta}(\tilde{s}) = \sum_{\chi \in \widetilde{S_\phi}\widehat{\ }} \kappa(x_\chi)\chi(\tilde{s})\pi(\chi). \tag{7.3}$$

Fix a strong real form x. The part of the sum (7.9) *on this strong real form is*

$$\widetilde{\eta}_x(\tilde{s}) = \kappa(x) \sum_{\{\chi \,|\, x_\chi \equiv x\}} \chi(\tilde{s})\pi(\chi). \tag{7.4}$$

This is a special case of the discussion on pages 20-21 of [2]; see [2, Theorem 26.8] for details. This is a collection of virtual representations of strong real forms of G, parametrized by $\widetilde{S_\phi}$. Now [2, Theorem 1.39] relates these virtual characters to characters of an endoscopic group, and is the version of endoscopic lifting of [2]. The precise relationship of this lifting to that of Langlands, Kottwitz and Shelstad is complicated. For the purposes of these notes we use only Definition 7.2.

Note that if $x_\chi \equiv x_{\chi'}$ then

$$\chi(\tilde{s}) = \pm\chi'(\tilde{s}). \tag{7.5}$$

Therefore (7.4) could be written

$$\widetilde{\eta}_x(\tilde{s}) = \kappa(x)\mu(\tilde{s}) \sum_{\{\chi \,|\, x_\chi \equiv x\}} \epsilon(\chi)\pi(\chi). \tag{7.6}$$

for some $\mu(\tilde{s}) \in \mathbb{C}^\times$, and $\epsilon(\chi) \pm 1$.

Definition 7.2 is entirely canonical, but involves the group $\widetilde{S_\phi}$. As usual to pass to S_ϕ a choice is needed, so fix $z \in Z$ and $x_0 \in \mathcal{X}_1[z]$. Then there are bijections

$$\Pi_z(\phi) \overset{1-1}{\longleftrightarrow} \mathcal{X}_1[z] \overset{1-1}{\longleftrightarrow} \widehat{S_\phi}, \tag{7.7}$$

depending on the choice of x_0.

Definition 7.8 *Fix $s \in S_\phi$, and define*

$$\eta(s) = \sum_{\chi \in \widehat{S_\phi}} \kappa(x_\chi)\chi(s)\pi(\chi). \tag{7.9}$$

This depends on the choice of x_0.

Fix a strong real form x. The part of the sum (7.9) *on this strong real form is*

$$\eta_x(s) = \kappa(x) \sum_{\{\chi \,|\, x_\chi \equiv x\}} \chi(s)\pi(\chi). \tag{7.10}$$

Recall that in the case of pure strong real forms $x_0 = x(\rho^\vee)$ is a canonical basepoint.

A question arising in applications to automorphic forms is to describe the sets

$$\widehat{S_\phi}(x, s, \pm 1) = \{\chi \in \widehat{S_\phi} \,|\, x_\chi \equiv x, \chi(s) = \pm 1\}. \tag{7.11}$$

Note that

$$\eta_x(s) = \kappa(x) \left(\sum_{\widehat{S_\phi}(x,s,+1)} \pi(\chi) - \sum_{\widehat{S_\phi}(x,s,-1)} \pi(\chi) \right). \tag{7.12}$$

This was brought to my attention by Michael Harris; see [5].

8 Example: Strong Real forms of $SU(p,q)$

Let $n = p + q$.

First assume n is odd. Then $z(\rho^\vee) = I$. In this case the map from pure strong real forms to weak real forms is a bijection. Thus $x \in \mathcal{X}_1[I] = \{\mathrm{diag}(\pm 1, \dots, \pm 1)\}$ with an even number of minus signs. There is a bijection between real forms and strong real forms. For example if $n = 5$ we have $SU(5,0), SU(3,2)$ and $SU(1,4)$.

Suppose $n = 2m$ is even. Then $z(\rho^\vee) = -I$, and

$$\mathcal{X}_1[z(\rho^\vee)] = \{i\mathrm{diag}(\epsilon_1, \dots, \epsilon_n)\} \tag{8.1}$$

with $\epsilon_i = \pm 1$ and $\prod \epsilon_i = (-1)^m$. In particular $x(\rho^\vee) = \mathrm{diag}(\overbrace{i, \dots, i}^{m}, \overbrace{-i, \dots, -i}^{m})$ and this corresponds to the quasisplit form $SU(m,m)$. The map from pure strong real forms to weak real forms is not surjective, and two-to-one except for the quasisplit form. If m is even we can think of these as $SU(2m,0), SU(2m-2,2), \dots SU(0,2m)$. If m is odd we have $SU(2m-1,1)$, $SU(2m-3,3)$, $\dots$, $SU(1,2m-1)$.

For example if $n = 2m = 4$ we can think of the strong real forms as $SU(4,0), SU(2,2)$ and $SU(0,4)$. If $n = 2m = 6$ we get $SU(5,1), SU(3,3)$ and $SU(1,5)$.

To get all weak real forms we must choose another element z. The most convenient such choice is $z = I$ if m is odd, or $z = iI$ if m is even.

Suppose m is odd. Let $z = I$. Then the elements $x = \mathrm{diag}(\pm 1, \dots, \pm 1)$ with and even number of minus signs give the groups $SU(2m,0), SU(2m-2)$,..., $SU(0,2m)$; each real form occurs twice. For example for $n = 2m = 6$ we have $SU(6,0), SU(4,2), SU(2,4)$ and $SU(0,6)$.

Now assume m is even. Then take $z = iI$ and let $\alpha = e^{\pi i/4}$. Then $x = \alpha(\pm 1, \dots, \pm 1)$ with an odd number of minus signs. We get each $SU(p,q)$ with p,q odd counted twice. For example if $n = 2m = 8$ we have $SU(7,1), SU(5,3), SU(3,5)$ and $SU(1,7)$.

The following table summarizes the situation. The number of groups in the row labelled by z is the number of inequivalent strong real forms corresponding to z. The occurence of two groups $SU(p,q)$ and $SU(q,p)$ means that the map from strong real forms to this real forms is two to one. If only one of these groups occurs it is one to one.

1. $n = 2$
   (a) $z = z(\rho^\vee) = -I$: $SU(1,1)$
   (b) $z = I$: $SU(2,0)$, $SU(0,2)$

2. $n = 3$

   (a) $z = z(\rho^\vee) = I$: $SU(3,0), SU(1,2)$

3. $n = 4$:

   (a) $z = z(\rho^\vee) = -I$: $SU(4,0), SU(2,2), SU(0,4)$
   (b) $z = iI$: $SU(3,1), SU(1,3)$

4. $n = 5$:

   (a) $z = z(\rho^\vee) = I$: $SU(5,0), SU(3,2), SU(1,4)$

5. $n = 6$:

   (a) $z = z(\rho^\vee) = -I$: $SU(5,1), SU(3,3), SU(1,5)$,
   (b) $z = I$: $SU(6,0), SU(4,2), SU(2,4), SU(0,6)$

9 Example: Strong Real Forms of $U(p,q)$

The equal rank real forms of $GL(n,\mathbb{C})$ are the unitary groups $U(p,q)$ with $p+q=n$. We consider strong real forms in this case.

Recall the map from $\mathcal{X}_1$ to (weak) real forms is surjective. In this case there there are infinitely many strong real forms mapping to each real form. For example if $p+q=n$ then

$$x = \mathrm{diag}(\overbrace{\alpha,\dots,\alpha}^{p},\overbrace{-\alpha,\dots,-\alpha}^{q}) \tag{9.1}$$

maps to $U(p,q)$ for any $\alpha \in \mathbb{C}^\times$.

A more reasonable theory is obtained by fixing $z = \beta I \in Z$. Recall $\mathcal{X}_1 = \coprod_{z\in Z}\mathcal{X}_1[z]$. In this case the map from $\mathcal{X}_1[z]$ to weak real forms is surjective, for any $z = \beta I$. In fact, choose α with $\alpha^2 = \beta$. Then

$$\mathcal{X}_1[z] = \alpha\mathrm{diag}(\epsilon_1,\dots,\epsilon_n) \tag{9.2}$$

where $\epsilon_i = \pm 1$.

Suppose n is odd. The elements x conjugate to

$$x = \mathrm{diag}(\overbrace{\alpha,\dots,\alpha}^{p},\overbrace{-\alpha,\dots,-\alpha}^{q}) \tag{9.3}$$

constitute a single strong real form. Those conjugate to

$$x = \mathrm{diag}(\overbrace{-\alpha,\dots,-\alpha}^{p},\overbrace{\alpha,\dots,\alpha}^{q}) \tag{9.4}$$

are a distinct strong real form. It is convenient to think of these strong real forms as $U(p,q)$ and $U(q,p)$, although this notation has no intrinsic meaning: which to call $U(p,q)$ and which to call $U(q,p)$, amounts to a choice of $\sqrt{\beta}$.

If n is even a similar statement holds, except that there is only one strong real form mapping to $U(n/2,n/2)$.

There is one important special case in which the distinction between $U(p,q)$ and $U(q,p)$ is canonical. Note that $x(\rho^\vee) = i^{n-1}(1,-1,\dots,(-1)^{n-1})$ and $z(\rho^\vee) = (-1)^{n-1}I$. So if $\beta = (-1)^{n-1}$ then it is natural to take $\alpha = i^{n-1}$. We then say x is the strong real form $U(p,q)$ where p is the multiplicity of the i^{n-1} eigenvalue. Note that if n is odd then

$$x = \mathrm{diag}(\overbrace{1,\dots,1}^{p},\overbrace{-1,\dots,-1}^{q}) \tag{9.5}$$

corresponds to $U(p,q)$ if $n = 1 \pmod 4$, and $U(q,p)$ if $n = 3 \pmod 4$.

10 Discrete series of $U(p,q)$

Let $G = GL(n,C)$ and suppose ϕ is a discrete series L-parameter. We make the bijection $\Pi_z(\phi) \overset{1-1}{\longleftrightarrow} \widehat{S_\phi}$ of (6.24) explicit in this case.

Fix $z = \beta I$ for some $\beta \in \mathbb{C}^\times$. We have to choose $x_0 \in \mathcal{X}_1[z]$. We would like to choose x_0 so that the trivial character of S_ϕ corresponds to a generic discrete series representation. In this case by (5.6) this is possible: choose α satisfying $\alpha^2 = (-1)^{n-1}\beta$ and let

$$x_0 = \alpha(1,-1,\dots,(-1)^n). \tag{10.1}$$

Note that there are two choices of α; see the end of this section.

We have $S_\phi \simeq \mathbb{Z}/2\mathbb{Z}^n$. For $\epsilon_i = \pm 1$ write $\chi = \chi(\epsilon_1,\dots,\epsilon_n)$ for the corresponding character of S_ϕ.

Proposition 10.2 *Fix $z = \beta I$. Choose α satisfying $\alpha^2 = (-1)^{n-1}\beta$ and let $x_0 = \alpha x(\rho^\vee)$. Then the bijection*

$$\widehat{S_\phi} \overset{1-1}{\longleftrightarrow} \mathcal{X}_1[z] \tag{10.3}$$

is given by

$$\chi(\epsilon_1,\dots,\epsilon_n)\to x = diag(\epsilon_1,\dots,\epsilon_n)\alpha x(\rho^\vee) \qquad (10.4)$$
$$= \alpha i^{n-1}(\epsilon_1,-\epsilon_2,\dots,(-1)^{n-1}\epsilon_n)$$

Thus in the bijection $\chi\to(x(\chi),\pi(\chi))$ *of Definition 6.25,* $\chi(\epsilon_1,\dots,\epsilon_n)$ *goes to a representation of the strong real form* $\alpha i^{n-1}(\epsilon_1,-\epsilon_2,\dots,(-1)^{n-1}\epsilon_n)$.

Suppose $\chi=\chi(\epsilon_1,\dots,\epsilon_n)\in\widehat{S_\phi}$. Let $p(\chi)$ (resp. $q(\chi)$) be the number of times $+1$ (resp. -1) occurs in $(\epsilon_1,-\epsilon_2,\epsilon_3,\dots,(-1)^{n-1}\epsilon_n)$.

Corollary 10.5 *In the setting of the Proposition suppose* $\chi,\chi'\in\widehat{S_\phi}$ *correspond to* $x,x'\in\mathcal{X}_1[z]$*, respectively. Then* x,x' *are equivalent strong real forms if and only if* $p(\chi)=p(\chi')$*. They map to equivalent weak real forms if and only if* $p(\chi)=p(\chi')$ *or* $p(\chi)=q(\chi')$.

As discussed in Section 9 the choice of $\alpha=\sqrt{\beta}$ amounts to a labelling of the strong real forms as $U(p,q)$ or $U(q,p)$. That is, we may *define* the strong real form $U(p,q)$ to be those $x\in\mathcal{X}_1[z]$ for which the dimension of the αi^{n-1}-eigenspace is p.

Corollary 10.6 *In the setting of the Proposition,* $\chi\in\widehat{S_\phi}$ *goes to a discrete series representation of the strong real form* $U(p(\chi),q(\chi))$.

The most natural case is $z=z(\rho^\vee)$, i.e. $\beta=(-1)^{n-1}$, in which case we can take $\alpha=1$ and $x_0=x(\rho^\vee)$. Then $U(p,q)$ is the strong real form corresponding to those x with i^{n-1} occuring as an eigenvalue of multiplicity p. See (9.5).

Note that each weak real form $U(p,q)$ occurs twice (as the strong real forms $U(p,q)$ and $U(q,p)$) unless $p=q$. The total number of discrete series representations is therefore $\sum_{p=0}^{n}\binom{n}{p}=2^n=|\widehat{S_\phi}|$.

Note that $x\to -x$ is an automorphism of $\mathcal{X}_1[z]$. It is helpful to understand the corresponding involution of $\Pi_z(\phi)$. A related issue is the number of basepoints in $\Pi_z(\phi)$, i.e. generic discrete series representations.

Suppose $\chi\in\widehat{S_\phi}$. The representation $\pi(\chi)$ is generic if and only if $x(\chi)=\pm\alpha x(\rho^\vee)$, in which case $x(\chi)$ is quasisplit.

Note that x represent $-x$ are distinct strong real forms ($U(p,q)$ and $U(q,p)$, so to speak), unless $n=2m$ and this is the quasisplit form $U(m,m)$.

If n is even there is one quasisplit strong real form $U(m,m)$. This has two generic discrete series representations $\pi(\pm\alpha x(\rho^\vee))$, which are interchanged by an outer automorphism. If n is odd there are two quasisplit strong real forms $U(m+1,m)$ and $U(m,m+1)$. Each has a unique generic discrete series representations; these are $\pi(\pm\alpha x(\rho^\vee))$.

The connection between $\Pi_z(\phi)$ and admissible representations of real groups is only well defined up to this automorphism. For example suppose $n=2m$, $\alpha=1$, and $G(\mathbb{R})=U(m,m)$. Write π,π' for the two generic discrete series representations of $G(\mathbb{R})$. Then we can say $x(\rho^\vee)$ corresponds to π, and $-x(\rho^\vee)$ to π', or vice versa. Since π,π' are interchanged by an automorphism of $G(\mathbb{R})$, there is no intrinsic way to specify this choice.

It is also helpful to make the connection with more familiar parameters more explicit. This is primarily an exercise in the definitions; see (5.4).

10.1 Example: $U(4)$

Consider pure strong real forms, i.e. $x_0 = x(\rho^\vee) = -i\text{diag}(1,-1,1,-1)$. We work at infinitesimal character $(4,3,2,1)$, which is a central shift of $\rho = (3/2,1/2,-1/2,-3/2)$.

The quasisplit group is $U(2,2)$. We need to fix a generic discrete series representation of $U(2,2)$. Let's choose the Harish-Chandra parameter to be $(4,2;3,1)$ (the other choice is $(3,1;4,2)$). Then we decree that $x(\rho^\vee)$ correspond to $\pi(4,2;3,1)$.

By (5.4) we derive the following table of x which are conjugate to $x(\rho^\vee)$, and the corresponding Harish-Chandra paramter of a discrete series representation of $U(2,2)$

x	G	λ
-i(1,-1,1,-1)	$U(2,2)$	(4,2;3,1)
-i(1,1,-1,-1)	$U(2,2)$	(4,3;2,1)
-i(1,-1,-1,1)	$U(2,2)$	(4,1;3,2)
-i(-1,1,1,-1)	$U(2,2)$	(3,1;4,2)
-i(-1,1,-1,1)	$U(2,2)$	(3,2;4,1)
-i(-1,-1,1,1)	$U(2,2)$	(2,1;4,3)

Now consider the weak real form $U(3,1)$. We obtain the following table:

x	G	λ
-i(1,1,1,-1)	$U(3,1)$	(4,3,2;1)
-i(1,1,-1,1)	$U(3,1)$	(4,3,1;2)
-i(1,-1,1,1)	$U(3,1)$	(4,2,1;3)

Now the negatives of these three parameters also give representations of the weak real form $U(3,1)$:

x	G	λ
-i(-1,-1,-1,1)	$U(3,1)$	(4,3,2;1)
-i(-1,-1,1,-1)	$U(3,1)$	(4,3,1;2)
-i(-1,1,-1,-1)	$U(3,1)$	(4,2,1;3)

If we're careful we can think of these are representations of $U(1,3)$:

x	G	λ
-i(-1,-1,-1,1)	$U(1,3)$	(1;4,3,2)
-i(-1,-1,1,-1)	$U(1,3)$	(2;4,3,1)
-i(-1,1,-1,-1)	$U(1,3)$	(3;4,3,1)

although his only really makes sense at the strong real form level.

Finally $x = \pm i(1,1,1,1)$ each gives the one dimensional representation of $U(4,0)$, alternatively thought of as $U(4,0)$ and $U(0,4)$.

10.2 Example: $U(3)$

This is a little different from the preceding example since 3 is odd. In this case $x(\rho^\vee) = -1(1,-1,1)$. Fix infinitesimal character $(3,2,1)$. In this case there is only one generic discrete series representation of $U(2,1)$, so there is no choice to make.

Note that x and $-x$ are distinct strong real forms, mapping to the same weak real form. Thus $\pm x$ both map to the same representation of the weak real form. In this sense we have the following table:

x	G	λ
±(1,-1,1)	$U(2,1)$	(3,1;2)
±(1,1,-1)	$U(2,1)$	(3,2,;1)
±(-11,1,1)	$U(2,1)$	(2,1;3)
±(1,1,1)	$U(3,0)$	(3,2,1;)

Again if we're careful, and work with strong real forms, we can think of this as:

x	G	λ
-(1,-1,1)	$U(2,1)$	(3,1;2)
-(1,1,-1)	$U(2,1)$	(3,2,;1)
-(-11,1,1)	$U(2,1)$	(2,1;3)
-(1,1,1)	$U(3,0)$	(3,2,1;)
-(-1,1,-1)	$U(1,2)$	(2;3,1)
-(-1,-1,1)	$U(1,2)$	(1;3,2)
-(1,-1,-1)	$U(1,2)$	(3;2,1)
-(-1,-1,-1)	$U(0,3)$	(;3,2,1)

11 Discrete series of $SU(p,q)$

This case is a bit more complicated than $U(p,q)$.

As in the preceding section write $\chi = \chi(\epsilon_1, \dots, \epsilon_n)$, where now the number of -1's is even. Again let $p(\chi)$ (resp. $q(\chi)$) be the number of $+1$s (resp. -1s) in $(\epsilon_1, -\epsilon_2, \dots, (-1)^n\epsilon_n)$.

If $n = 2m+1$ things are straightforward. Recall (Section 8) the map from pure strong real forms to real forms is a bijection. We consider only pure strong real forms.

Proposition 11.1 *Let $G = SL(n,\mathbb{C})$ with $n = 2m+1$. There is a canonical bijection between $\widehat{S_\phi}$ and the discrete series representations of the real forms, equivalently strong real forms,*

$$\{SU(p,q) \,|\, p+q = n, q \equiv m \pmod 2\}. \qquad (11.2)$$

The character $\chi(\epsilon_1, \dots, \epsilon_n)$ corresponds to a discrete series representation of $SU(p(\chi), q(\chi))$

Now suppose $n = 2m$. Recall (Section 8) not every real form is pure in this case, and the map from strong real forms to real forms is not bijective. We have to distinguish the strong real forms $SU(p,q)$ and $SU(q,p)$ in this case.

We first consider the pure case. Note that $z(\rho^\vee) = -I$ and $x(\rho^\vee) = i^{n-1}(1,-1,\dots,1)$.

Proposition 11.3 *Let $G = SL(n,\mathbb{C})$ with $n = 2m$. The case of pure strong real forms givs a a canonical bijection between $\widehat{S_\phi}$ and the discrete series* *representations of the strong real forms*

$$\{SU(p,q)\,|\,p+q=n, p\equiv q\equiv m \pmod 2\}. \tag{11.4}$$

Again χ *corresponds to a representation of the strong real form* $SU(p(\chi),q(\chi))$.

To get discrete series representation of the missing groups we need to consider non-pure real forms.

Proposition 11.5 *Suppose* $n=2m$. *Let*

$$\alpha=\begin{cases} e^{\pi i/4} & m \text{ even} \\ I & m \text{ odd}\end{cases} \tag{11.6}$$

and

$$z=\alpha^2 I=\begin{cases} iI & m \text{ even} \\ I & m \text{ odd}\end{cases} \tag{11.7}$$

Let

(11.8)

Then there is a bijection between $\widehat{S_\phi}$ *and the discrete series representations of the strong real forms*

$$\{SU(p,q)\,|\,p+q=n, p\equiv q\equiv m+1 \pmod 2\}. \tag{11.9}$$

In this case $\chi=\chi(\epsilon_1,\dots,\chi_n)$ *corresponds to a representation of* $SU(p,q)$ *where ... (I'm not sure about the right choice here...)*

12 Endoscopy for $U(p,q)$

We consider the question at the end of Section 7 in the case of $U(p,q)$. Recall this concerns the set

$$\widehat{S_\phi}(x,s,\pm 1)=\{\chi\in\widehat{S_\phi}\,|\,x_\chi\equiv x,\ \chi(s)=\pm 1\}. \tag{12.1}$$

Fix $z = \beta I$, α satisfying $\alpha^2 = (-1)^{n-1}\beta$, and let $x_0 = \alpha x(\rho^\vee)$. Then $\Pi_z(\phi) \overset{1-1}{\longleftrightarrow} \widehat{S_\phi}$, with the trivial character corresponding to a generic discrete series representation of a quasisplit strong real form.

Fix $x \in \mathcal{X}_1[z]$, so $x = \alpha i^{n-1}(\epsilon_1, \dots, \epsilon_n)$. Let r be the dimension of the αi^{n-1}-eigenspace of x.

Let $\chi_0 = \chi(1, -1, \dots, (-1)^n) \in \widehat{S_\phi}$. Let V be the standard module of $GL(n, \mathbb{C})$.

Proposition 12.2 *There is a natural bijection between $\widehat{S_\phi}(x, s, \chi_0(s))$ and a basis of the* 1*-eigenspace of s acting on $\Lambda^r(V)$.*

We specialize to the case of pure strong real forms. Let $z = (-1)^n I$ and $x_0 = i^{n-1}(1, -1, \dots, (-1)^n)$. In the preceding notation we have $\beta = (-1)^{n-1}$ and $\alpha = 1$. Also r is the dimension of the i^{n-1}-eigenspace of x.

As in Section 10 let $p(\chi)$ be the number of j for which $\epsilon_j = 1$ (resp. $\epsilon_j = -1$).

Lemma 12.3 *Fix $0 \le r \le n$. Let $\chi = \chi(\epsilon_1, \dots, \epsilon_n)$ be a character of $(\mathbb{Z}/2\mathbb{Z})^n$. Suppose $s \in (\mathbb{Z}/2\mathbb{Z})^n$.*

There is a natural bijection between a basis of

(12.4) $$\{\gamma \in \Lambda^r(V) \mid \Lambda^r(s)\gamma = \gamma\}$$

and

(12.5) $$\{\chi \mid p(\chi) = r,\ \chi(s) = 1\}$$

Proof. The proof is elementary. Suppose $\chi = \chi(\epsilon_1, \dots, \epsilon_n)$ is in the right hand side. Suppose $1 \le i_1 < \cdots < i_r \le n$ are the indices for which $\epsilon_i = -1$. Map this to the vector $w = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}$. The condition that $\chi(s) = 1$ becomes the condition $\Lambda^r(s)(w) = w$, and it is easy to see this is a bijection. ∎

As in Section 7 let $\chi_0 = \chi(1, -1, \dots, (-1)^n)$.

By definition we have

(12.6)(a) $$\widehat{S_\phi}(x, s, \chi_0(s)) = \{\chi \mid x_\chi = x, \chi(s) = \chi_0(s)\}.$$

Suppose the dimension of the αi^{n-1}-eigenspace of x is r. Then the right hand side is

(12.6)(b) $$\{\chi \mid \dim(\alpha i^{n-1}) - \text{eigenspace of } x_\chi \text{ is } r, \chi(s) = \chi_0(s)\}.$$

By Proposition 10.2 this is

$$\{\chi \mid p(\chi\chi_0) = r, \chi(s) = \chi_0(s)\}. \tag{12.6)(c}$$

Multiplication by χ_0 gives a bijection with

$$\{\chi \mid p(\chi) = r, \chi(s) = 1\}. \tag{12.6)(d}$$

The result follows from Lemma (12.3).

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