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CONSTRUCTION OF AUTOMORPHIC GALOIS REPRESENTATIONS, II

Introduction

This is a revised version of the articles "Construction of automorphic Galois representations I,II" by Michael Harris posted at the site

$$http://www.institut.math.jussieu.fr/projets/fa/bp0.html$$

The present version is still provisional, as the second proof of Theorem 3.2.5 depends on a theorem that at the time of this writing has not yet been committed to print; however, the first proof, based on [Ch], is complete.

1. Construction of Galois representations under simplifying hypotheses

Let F be a totally real field, $\mathcal{K}/F$ a totally imaginary quadratic extension, $d = [F:\mathbb{Q}]$, $c \in Gal(\mathcal{K}/F)$ the non-trivial Galois automorphism. Let n be a positive integer and $\mathcal{G} = \mathcal{G}_n$ be the algebraic group $R_{\mathcal{K}/\mathbb{Q}}GL(n)_{\mathcal{K}}$. Let $\mathfrak{g} = Lie(\mathcal{G}(\mathbb{R}))$, $K_\infty \subset \mathcal{G}(\mathbb{R})$ the product of a maximal compact subgroup with the center $Z_{\mathcal{G}}(\mathbb{R})$. We consider cuspidal automorphic representations Π of $\mathcal{G}$ satisfying the following two hypotheses:

General Hypotheses 1.1. *Writing $\Pi = \Pi_\infty \otimes \Pi_f$, where Π_∞ is an admissible $(\mathfrak{g}, K_\infty)$-module, we have*

(i) *(Regularity) There is a finite-dimensional irreducible representation $W(\Pi) = W_\infty$ of $\mathcal{G}(\mathbb{R})$ such that*

$$H^*(\mathfrak{g}, K_\infty; \Pi_\infty \otimes W_\infty) \neq 0.$$

(ii) *(Polarization) The contragredient $\Pi^\vee$ of Π satisfies*

$$\Pi^\vee \xrightarrow{\sim} \Pi \circ c.$$

We next make the following temporary hypotheses:

Special Hypotheses 1.2.
(1.2.1) *$\mathcal{K}/F$ is unramified at all finite places (in particular $d > 1$).*
(1.2.2) *Π_v is spherical (unramified) at all non-split non-archimedean places v of $\mathcal{K}$.*
(1.2.3) *The degree $d = [F:\mathbb{Q}]$ is even.*

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The irreducible representation $W(\Pi)$ factors over the set Σ of real embeddings of F

$$W(\Pi) = \otimes_{\sigma\in\Sigma} W_\sigma,$$

where W_σ is an irreducible representation of $\mathcal{G}(\mathcal{K}\otimes_{F,\sigma}\mathbb{R})\xrightarrow{\sim} GL(n,\mathbb{C})\times GL(n,\mathbb{C})$. The highest weight of W_σ is denoted $\mu(\sigma)$. It can be identified in the usual way with a pair of non-increasing n-tuples of non-negative integers $(\mu(\tilde{\sigma}),\mu(\tilde{\sigma}^c))$, one for each extension $\tilde{\sigma}$ of σ to an embedding of $\mathcal{K}$, where we write

$$\mu(\tilde{\sigma}) = (\mu_1(\tilde{\sigma})\geq\mu_2(\tilde{\sigma})\geq\dots\mu_n(\tilde{\sigma})).$$

Moreover, the polarization condition implies that one of the n-tuples diagrams is dual to the other, in other words that

$$\mu_i(\tilde{\sigma}^c) = -\mu_{n-i-1}(\tilde{\sigma}).$$

Special Hypothesis 1.3. *For at least one $\sigma\in\Sigma$, the highest weight $\mu(\sigma)$ is sufficiently far from the walls; in practice, it suffices to assume $\mu(\sigma)$ is regular, i.e. $\mu_i(\tilde{\sigma})\neq\mu_j(\tilde{\sigma})$ if $i\neq j$.*

Let K be a p-adic field, WD_K its Weil-Deligne group. Let $\mathcal{A}(n,K)$ denote the set of equivalence classes of irreducible admissible representations of $GL(n,K)$, and let $\mathcal{G}(n,K)$ denote the set of equivalence classes of n-dimensional Frobenius semisimple representations of WD_K. We denote by

$$\mathcal{L}:\mathcal{A}(n,K)\to\mathcal{G}(n,K)$$

the local Langlands correspondence, normalized to coincide with local class field theory when $n=1$ in such a way that a uniformizer of $K^\times$ is sent to a geometric Frobenius.

The following result has now been proved in several stages (articles [L.IV.1], [CHL.IV.2], [CHL.IV.3] of Book 1, and especially [S], of which an expository account will appear in Book 2.

Theorem 1.4. *(i) Suppose n is odd and Π satisfies Hypotheses (1.1) and (1.2). Then there is a number field $E(\Pi)$ and a compatible system $\rho_{\lambda,\Pi}:\Gamma_{\mathcal{K}}\to GL(n,E(\Pi)_\lambda)$ of λ-adic representations, where λ runs through the finite places of $E(\Pi)$, such that*

(a) *For all finite primes v of $\mathcal{K}$ of residue characteristic prime to $N_{E(\Pi)/\mathbb{Q}}(\lambda)$,*

$$\rho_{\lambda,\Pi}^{F-ss}\mid_{\Gamma_v}\xrightarrow{\sim}\mathcal{L}(\Pi_v\otimes|\bullet|_v^{\frac{1-n}{2}}).$$

Here the superscript $^{F-ss}$ denotes Frobenius semisimplification.

(b) *For all finite primes v of $\mathcal{K}$ dividing $N_{E(\Pi)/\mathbb{Q}}(\lambda)$, $\rho_{\lambda,\Pi}\mid_{\Gamma_v}$ is de Rham, and its Hodge-Tate numbers have multiplicity at most one (i.e., $\rho_{\lambda,\Pi}$ is* H*odge-Tate regular and are determined by Π_∞, or equivalently by $W(\Pi)$, in accordance with the recipe given in (1.5), below.*

(c) *Let v be a finite prime of $\mathcal{K}$ dividing $N_{E(\Pi)/\mathbb{Q}}(\lambda)$. Suppose Π_v has a vector fixed by a hyperspecial maximal compact subgroup of $GL(n,\mathcal{K}_v)$. Then $\rho_v:=\rho_{\lambda,\Pi}\mid_{\Gamma_v}$ is crystalline, and if φ denotes the smallest linear power of the crystalline Frobenius of $D_{crys}(\rho_v)$ then*

$$\det(T-\varphi)=\det(T-\mathcal{L}(\Pi_v\otimes|\bullet|_v^{\frac{1-n}{2}})(Frob_v)).$$

(d) *Let v be a finite prime of $\mathcal{K}$ dividing $N_{E(\Pi)/\mathbb{Q}}(\lambda)$. Suppose Π_v has a vector fixed by an Iwahori subgroup of $GL(n,\mathcal{K}_v)$. Then $\rho_{\lambda,\Pi}\mid_{\Gamma_v}$ is semistable.*

(ii) If n is even, the same conclusions hold as in (i), provided Π *in addition satisfies Hypothesis 1.3.*

When Π satisfies the additional hypothesis that Π_{v_0} is square-integrable for some finite place v_0, then this theorem is mostly proved in [HT], extending an earlier theorem due to Clozel and Kottwitz [C,K2] obtaining (a) at most places where Π is unramified. The theorem is completed in [TY]. What we here call $\rho_{\lambda,\Pi}$ is the representation denoted $R_\ell(\Pi^\vee)$ in [HT]. The compatibility (a) with the local Langlands correspondence is due in general to Shin [S]. A weaker version of Theorem 1.4 with local compatibility at almost all finite places is deduced in [CHL.IV.3] from the results of [L.IV.1] and [CHL.IV.3], using the methods of [K1].

1.5 Hodge-Tate numbers of automorphic Galois representations.

Fix a prime λ of the coefficient field $E(\pi)$, say of residue characteristic p. The automorphic Galois representation $\rho_{\lambda,\Pi}$ constructed in Book 2 is obtained in the cohomology of a geometric p-adic local system $\tilde{W}_p(\Pi)$ on a Shimura variety, obtained in a standard way from the finite-dimensional representation $W(\Pi)$ introduced above. It is therefore of geometric type, in the sense of Fontaine and Mazur: each $\rho_{\lambda,\Pi}$ is unramified outside a finite set of places of $\mathcal{K}$, and at every place dividing the residue characteristic of λ, $\rho_{\lambda,\Pi}$ is de Rham. The latter fact is a consequence of the comparison theorems of p-adic Hodge theory. For the same reasons, $\rho_{\lambda,\Pi}$ is semistable at a prime v dividing the residue characteristic of λ such that Π_v has Iwahori invariants, and when Π_v is unramified, it is even crystalline and the second part of Theorem 1.4 (c) holds by its part (a) and a theorem of Katz-Messing.

In particular the Hodge-Tate numbers can be read off from the Hodge numbers of the de Rham cohomology of the flat vector bundle $\tilde{W}(\Pi)$ associated to $W(\Pi)$. The comparison of $\tilde{W}_p(\Pi)$ and $\tilde{W}(\Pi)$, and therefore the determination of the Hodge-Tate numbers from the highest weights $\mu(\sigma)$ of W_σ, presupposes a dictionary relating complex and p-adic places of $\mathcal{K}$. In [HT] this is given by an isomorphism $\iota : \bar{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. For what follows it suffices to identify the algebraic closure of $\mathbb{Q}$ in $\bar{\mathbb{Q}}_p$ with the field of algebraic numbers in $\mathbb{C}$. Then the p-adic embeddings of $\overline{\mathbb{Q}}$, and in particular of $\mathcal{K}$, are identified with the complex embeddings; if s is an embedding of $\mathcal{K}$ in $\bar{\mathbb{Q}}_p$, we write ιs for the corresponding complex embedding.

Let s be an embedding of $\mathcal{K}$ in $\bar{\mathbb{Q}}_p$, and let $D_{dR,s}$ denote Fontaine's functor from representations of $\Gamma_s = Gal(\bar{\mathbb{Q}}_p/\overline{s(\mathcal{K})})$ to filtered $\bar{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} E(\pi)_\lambda$-modules:

$$D_{dR,v}(R) = (R \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_s}.$$

The Hodge-Tate numbers of R (with respect to s) are the j such that $gr^j D_{dR,s}(R) \neq (0)$. Then in the situation of Theorem 1.4, the Hodge-Tate numbers of $\rho_{\lambda,\Pi}$ with respect to s are the j of the form

$$j = i - \mu_{n-i}(\iota(s)^c),\ i = 0, \ldots, n-1. \tag{1.6}$$

This is to be compared to part 4 of Theorem VII.1.9 of [HT]; the replacement of $\iota(s)$ by $\iota(s)^c$ corresponds to our replacement of Π by $\Pi^\vee$ in the definition of $\rho_{\lambda,\Pi}$.

Suppose n is even but Π does not satisfy Hypothesis 1.3. Then we expect to prove the following theorem in Book 2.

Expected Theorem 1.7. *Suppose* Π *satisfies Hypotheses (1.1) and (1.2). Then there is a number field* $E(\Pi)$ *and a compatible system* $\rho_{\lambda,\Pi} : \Gamma_{\mathcal{K}} \to GL(\frac{n(n-1)}{2}, E(\Pi)_\lambda)$ *of* λ*-adic representations, where* λ *runs through the finite places of* $E(\Pi)$*, such that*

(a) *For almost all finite primes* v *of* $\mathcal{K}$ *of residue characteristic prime to* $N_{E(\Pi)/\mathbb{Q}}(\lambda)$ *at which* Π_v *is unramified,* $\rho_{\lambda,\Pi}$ *is an unramified representation, and*

$$\rho^{ss}_{\lambda,\Pi} \mid_{\Gamma_v} \xrightarrow{\sim} \wedge^2 \mathcal{L}(\Pi_v)(2-n).$$

(b) *For all finite primes* v *of* $\mathcal{K}$ *dividing* $N_{E(\Pi)/\mathbb{Q}}(\lambda)$*,* $\rho_{\lambda,\Pi} \mid_{\Gamma_v}$ *is de Rham.*

(c) *Let* v *be a finite prime of* $\mathcal{K}$ *dividing* $N_{E(\Pi)/\mathbb{Q}}(\lambda)$*. Suppose* Π_v *has a vector fixed by a hyperspecial maximal compact subgroup of* $GL(n, \mathcal{K}_v)$*. Then* $\rho_{\lambda,\Pi} \mid_{\Gamma_v}$ *is crystalline.*

This theorem is equally valid for odd and even n, but is a consequence of Theorem 1.4 when n is odd.

Expected Theorem 1.7 is the Galois counterpart of one of the theorems about stable base change for unitary groups proved in [L.IV.1]. The consequence for Galois representations, using the techniques of Kottwitz in [K1], will be derived in Book 2.

The remainder of this note explains how to extend the conclusions of Expected Theorem 1.4.

2. An application of eigenvarieties and p-adic families of Galois representations

In this section we recall the main result of [Ch] (using Theorem 1.4 above).

Let $\mathcal{K}/F$ be a CM quadratic extension of a totally real field, satisfying Special Hypotheses (1.2.1) and (1.2.3), and assume n even. These hypotheses imply by a standard Galois cohomological argument (cf. Harris' introduction to Book 1) that

Lemma 2.1. *There exists a hermitian space* $V_0/\mathcal{K}$ *relative to the extension* $\mathcal{K}/F$ *such that the unitary group* $G_0 = U(V_0)$ *satisfies*

(i) *For all finite places* v*,* $G_0(F_v)$ *is quasi-split and splits over an unramified extension of* F_v*; in particular,* $G_0(F_v)$ *contains a hyperspecial maximal compact subgroup.*

(ii) *For all real places* v*,* $G_0(F_v)$ *is compact.*

Moreover, G_0 *is unique up to isomorphism.*

By Labesse [L.IV.1], any Π satisfying Hypotheses (1.1) and (1.2) admits a strong descent to the unitary group G_0.

Fix ℓ a finite prime and consider the following hypothesis on a Π satisfying Hypotheses (1.1).

Special Hypotheses 2.2. *There is a place* v_0 *of* F *dividing the rational prime* ℓ *such that*

(2.2.1) v_0 *splits in* $\mathcal{K}$*,*

(2.2.2) *If* v *is a place of* $\mathcal{K}$ *dividing* v_0*,* Π_v *has nonzero Iwahori-invariants.*

We fix a pair of embeddings $\iota = (\iota_\ell, \iota_\infty)$ of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$. The following theorem is [Ch, Thm. 3.3 & 3.5]. The main ingredient in its proof is the construction of certain ℓ-adic eigenvarieties of the unitary group G_0. The inertial part in (a') below, including the assertion about the monodromy operator, is due to Bellaïche-Chenevier [BC].

Theorem 2.3. *Suppose* Π *satisfies Hypotheses (1.1), (1.2) and Special Hypotheses (2.2). Then there is a semisimple continuous Galois representation* $\rho_{\iota,\Pi} : \Gamma_{\mathcal{K}} \to GL(n, \overline{\mathbb{Q}}_\ell)$ *such that*

(a′) *For all finite primes* v *of* $\mathcal{K}$ *of residue characteristic prime to* ℓ*,*

$$\rho_{\iota,\Pi}^{F-ss} \mid_{\Gamma_v} \prec \mathcal{L}(\Pi_v \otimes \mid \bullet \mid_v^{\frac{1-n}{2}}).$$

Moreover, the assertion about Hodge-Tate numbers in Theorem 1.4. (i) (b) holds, as well as the whole of (b), (c) and (d) for all finite primes v *of* $\mathcal{K}$ *above* ℓ *but not dividing* v_0*.*

If $\rho = (s, N)$ and $\rho' = (s', N')$ are two Weil-Deligne representations, we refer to [Ch] §3.1 for the precise definition of the dominance relation $\rho \prec \rho'$. Let us simply say here that it implies that $s \simeq s'$ and that N is in the Zariski-closure of the conjugacy class of N'.

Remark 2.4. Assertion (a′) of Theorem 2.3 is weaker than compatibility with the local Langlands correspondence. We return to this point at the end of §3.

3. Removal of Special Hypotheses

3.1. Removal of Special Hypotheses 1.2.

Let now $\mathcal{K}/F$ be any CM quadratic extension of a totally real field, and let Π be an automorphic representation of $GL(n, \mathcal{K})$ satisfying Hypothesis 1.1. The term S *general* is defined in [So, Definitions 1 and 3]

Proposition 3.1.1. *There is a finite set* S *of places of* F *and an* S*-general collection.* $\mathcal{I}$ *of totally real quadratic extensions* F_i/F *such that, for each* $F_i \in \mathcal{I}$*, letting* $\mathcal{K}_i = F_i \cdot \mathcal{K}$*,* Π_i *the base change of* Π *to* $\mathcal{K}_i$*, the triple* $(F_i, \mathcal{K}_i, \Pi_i)$ *satisfies Special Hypotheses 1.2. Moreover, we can assume that, for every* $v \in S$ *and every* $F_i \in \mathcal{I}$*, either* v *splits in* $\mathcal{K}/F$ *or the unique extension of* v *to* $\mathcal{K}$*, denoted* $v_{\mathcal{K}}$*, splits in* $\mathcal{K}_i$*.*

Proof. Let S be the set of primes of v at which (1.2.1) or (1.2.2) fails: either v ramifies in $\mathcal{K}/F$, or v stays prime in $\mathcal{K}$ and the corresponding component $\Pi_{v_{\mathcal{K}}}$ is ramified. We take $\mathcal{I}$ to be the set of totally real quadratic extensions F_i/F with the property that, for all $v \in S$, $F_{i,v} \xrightarrow{\sim} \mathcal{K}_v$. It is obvious that this set has the properties claimed.

Theorem 3.1.2. *Let* Π *be an automorphic representation of* $GL(n, \mathcal{K})$ *satisfying Hypothesis 1.1 and Special Hypothesis 1.3 if* n *is even. Then the conclusions of Theorem 1.4 also hold for* Π*.*

(Assuming Expected Theorem 1.7) If n *is even and satisfies Hypothesis 1.1, then the conclusions of Expected Theorem 1.7 also hold for* Π*.*

Proof. The first part is deduced from Theorem 1.4 and Lemma 1 of [So] exactly as in [HT], pp. 229-232. We omit the details, since the more complicated case of a general solvable extension is treated in the next section. The second assertion is deduced from Expected Theorem 1.7 and Proposition A.1 in the same way.

3.2. Removal of Special Hypotheses 1.3.

Since Special Hypotheses 1.3 are only relevant to even n, we assume n to be even. Moreover, the case $n = 2$ is already understood. Thus we assume $n \geq 4$.

We state a variant of Lemma 4.1.2 of [CHT].

Lemma 3.2.1 [CHT]. *Let F be a totally real field, v a place of F, $w \neq v$ a second finite place, M/F any finite extension. Let L be a finite Galois extension of F_v. There exists a totally real solvable Galois extension F'/F in which w splits completely, linearly disjoint from M, such that, for every place v' of F' dividing v, the extension $F'_{v'}/F_v$ is isomorphic to L/F_v. Moreover, there is a constant $\mu(L)$ independent of w such that $[F' : F]$ can be assumed to be less than $\mu(L)$.*

In the statement of [*loc. cit.*], we take $S = \{v, w, \} \cup S_\infty$, where S_∞ is the set of real places of F, $D = M$, $L = E'_v$, and $F_w = E'_w$. For $\mu(L)$ we need to return to the proof of [*loc. cit.*]. Writing the extension L/F_v as a series of cyclic extensions $L \supset L_1 \supset L_2 \cdots \supset L_r \supset F_v$, it suffices by induction to assume L/F_v cyclic of degree d such that $Gal(L/F_v)$ corresponds via class theory to a character $\chi_L : F_v^\times \to \mathbb{C}^\times$. Then we need to extend χ_L to a character χ of the idèle classes of F trivial on $F_w^\times$ such that the class field associated to χ is of degree less than $\mu(L)$. In the proof of Lemma 4.1.1 of [CHT], an open subgroup $U \subset \mathbf{A}_F^{S,\times}$ is chosen to satisfy

$$\prod_{u \in S} \chi_u(x) = \chi_L(x) = 1 \ \forall x \in U \cap F^\times.$$

In [CHT] χ_u is the restriction of the character denoted χ_S to $F_u^\times$; in our case χ_u is trivial for $u \neq v$ and $\chi_v = \chi_L$. In particular, the subgroup U depends only on χ_L. The character χ is an extension of $1 \cdot \chi_L \cdot 1$ from $J = U \cdot F_v^\times \cdot \prod_{u \in S, u \neq v} F_u^\times / (U \cap F^\times)$ to $\mathbf{A}_F^\times / F^\times$. Now the index of J in $\mathbf{A}_F^\times / F^\times$ is bounded by $h_F \cdot i(U)$, where h_F is the class number of F and $i(U)$ is the index of $U \cdot \mathcal{O}_v^\times \cdot \mathcal{O}_w^\times$ in $\prod_x \mathcal{O}_x^\times$ where x runs over all finite primes of F. Thus the class field associated to χ is of degree at most $\mu(L) = d \cdot h_F \cdot i(U)$, which depends only on χ_L since we have already seen this is the case for U.

Corollary 3.2.2. *Let Π be an automorphic representation of $\mathcal{G}$ satisfying Hypotheses 1.1 and let M/F be any finite extension not containing $\mathcal{K}$. Let w be a place of F and let S be the set of places at which Π is ramified. There is a constant $\mu(\Pi)$ and a totally real solvable Galois extension F'/F of degree $\leq \mu(\Pi)$) in which w splits completely, linearly disjoint from M, such that, letting $\mathcal{K}' = \mathcal{K} \cdot F'$ the base change $\Pi_{\mathcal{K}'}$ of Π to $GL(n, \mathcal{K}')$ has the following property: for every prime $v \in S$ not dividing w and every prime v' of $\mathcal{K}'$ dividing v, the local component $\Pi_{\mathcal{K}',v'}$ has an Iwahori-fixed vector.*

Proof. By induction on the number of places in S we may assume S is the set of (one or two) primes above a single place v of F, $v \neq w$. Passing to a quadratic extension if necessary, as in (3.1), we may assume v splits in $\mathcal{K}$ as $u \cdot u^c$. We may thus identify Π_u with an irreducible admissible representation of $GL(n, F_v)$. It follows from the local Langlands correspondence, and indeed from the numerical correspondance proved by Henniart, that there exists a finite Galois extension L_v/F_v, necessarily solvable, such that the base change Π_{u,L_v} of Π_u to $GL(n, L_v)$ has an Iwahori-fixed vector. Equivalently, letting (s, N) be the representation of the Weil-Deligne group of F_v corresponding to Π_u – s is a Frobenius semisimple representation of the Weil group of F_v and N is a nilpotent endomorphism satisfying the usual commutation rules – the restriction of s to the Weil group of L is unramified. We now apply Lemma 3.2.1 to this triple (L_v, w, M). For the constant $\mu(\Pi)$ we can take $\prod_{v \in S} \mu(L_v)$.

Remark. It follows that, letting S be as in the statement of Corollary 3.2.2, the collection $\mathcal{I}$ of solvable extensions F'/F of degree at most $\mu(\Pi)$ for which $\Pi_{\mathcal{K}'}$ has an Iwahori-fixed vector locally above all places in S is S-general in the sense of [So]. Moreover, because of the assumption on the degree, the extensions in the family $\mathcal{I}$ also have *uniformly bounded heights*, in the sense of [So].

In the proof of the next proposition part (b), we admit Expected Theorem 1.7. Recall that $\iota = (\iota_\ell, \iota_\infty)$ is a pair of embeddings.

Proposition 3.2.3. *Suppose Π satisfies Hypotheses (1.1) and Special Hypothesis (2.2), in the sense that, for at least a prime v of $\mathcal{K}$ dividing ℓ and split above F, Π_v has an Iwahori fixed vector. Then there exists a semisimple continuous Galois representation $\rho_{\iota,\Pi} : \Gamma_{\mathcal{K}} \to GL(n, \bar{\mathbb{Q}}_\ell)$ satisfying Theorem 2.4 (a') and Theorem 1.4 (i) (b) for Π.*

Proof. Arguing as in Theorem 3.1.2 we may assume that $\mathcal{K}/F$ satisfies Hypothesis (1.2). Condition (a$'$) follows then from Theorem 2.3. Moreover, the same theorem ensures that $\rho_{\iota,\Pi}$ is at least Hodge-Tate, with the right weights, at primes dividing ℓ. It remains to prove that $\rho_{\iota,\Pi}$ is at least de Rham. But now part (b) of Expected Theorem 1.7, which holds for Π thanks to Theorem 3.1.2, together with condition (a) and Chebotarev density, implies at least that $\wedge^2\rho_{\iota,\Pi}$ is de Rham. Now since $n \geq 4$, the map from $\wedge^2 : GL(n) \to GL(\frac{n(n-1)}{2})$ is an isogeny. A theorem of Wintenberger [Wi] asserts that if L is an ℓ-adic field and $\rho : \Gamma_L \to GL(n, \bar{\mathbb{Q}}_\ell)$ is a Hodge-Tate representation whose image under an isogeny is de Rham, then ρ is itself de Rham. This completes the proof.

Remark 3.2.4. The difference between condition (a$'$) and (a) corresponds precisely to the absence of complete information about local monodromy already mentioned in Remark 2.5.

In the next result we no longer assume Expected Theorem 1.7.

Theorem 3.2.5. *Fix a prime ℓ and embeddings $\iota = (\iota_\ell, \iota_\infty)$ as above. Let Π be an automorphic representation of $GL(n, \mathcal{K})$ satisfying General Hypotheses (1.1). Then there exists a semisimple continuous Galois representation*

$$\rho_{\iota,\Pi} : \Gamma_{\mathcal{K}} \to GL(n, \bar{\mathbb{Q}}_\ell)$$

satisfying the conclusions of Theorem 2.4 (a') and of Theorem 1.4 (i) (b), (c) and (d) for Π.

Proof. We first show the existence of $\rho_{\iota,\Pi}$ satisftying Theorem 2.4 (a'). For that property, it follows from the preceding proposition that, in Theorem 1.4(ii), we can replace the condition "Π in addition satisfies Special Hypothesis (1.3)" by the condition "Π satisfies Special Hypothesis (2.2)" But it follows immediately from the remark following the proof of Corollary 3.2.2 that we can apply Theorem 1 of [So] to reduce to the case of Π satisfying Special Hypothesis (2.2).

Note that by the p-adic monodromy theorem the property of being de Rham is preserved under finite base change, as are Hodge-Tate numbers. Part (b) of the Theorem follows then from Proposition 3.2.3 if we admit Expected Theorem 1.7.

We now give another proof of (b), as well as (c) and (d), without assuming Expected Theorem 1.7. Using a quadratic base change as in Proposition 3.1.1. we may assume that $\mathcal{K}/F$ satisfies Hypothesis (1.2) and that each prime of F dividing

ℓ splits in $\mathcal{K}$. Let v be such a prime. Let F_1 be a quadratic totally real extension of F such that v splits in F_1 and write $v = uu'$ in F_1. By Lemma 3.2.1 we may choose a solvable totally real extension F_2 of F_1 such that :

- u splits completely in F_2,

- there exists a prime w of F_2 above u' such that, if $\Pi_{\mathcal{K}.F_2}$ denotes the (cuspidal) base change of Π to $\mathcal{K}.F_2$, the representation $(\Pi_{\mathcal{K}.F_2})_{w'}$ has Iwahori-invariants for each place w' of $\mathcal{K}.F_2$ dividing w. Thus $\Pi_{\mathcal{K}.F_2}$ satisfies Hypotheses (1.1), (1.2) and (2.2) with respect to the place w. Moreover, if $\tilde{v}$ is a prime of $\mathcal{K}$ above v, we have by construction identifications $\mathcal{K}_v = (\mathcal{K}.F_2)_{\tilde{v}}$ hence

$$\rho_{\iota,\Pi} \mid_{\Gamma_v} = \rho_{\iota,\Pi_{\mathcal{K}.F_2}} \mid_{\Gamma_{\tilde{v}}} .$$

Parts (b), (c) and (d) follow then from Theorem 2.4 as $\tilde{v}$ does not divide w.

Remark 3.2.6. The theorem of Wintenberger invoked in the proof of Proposition 3.2.3, together with (c) of Expected Theorem 1.7, implies in this case that, if Π_v has a vector fixed by a hyperspecial maximal compact subgroup of $GL(n, \mathcal{K}_v)$, for some v dividing ℓ, then the local representation at v is crystalline up to twisting by a quadratic character. This twist could be eliminated if we had a generalization of Kisin's theorem on analytic continuation of crystalline periods when the base field is not $\mathbb{Q}_p$. Indeed, this would imply that the local representation at v has at least one crystalline period. It follows that the possible quadratic character is necessarily unramified, hence the twist preserves the property of being crystalline. The proof we give here is very different.

As far as we know, the idea of the proof given here to get properties (b), (c) and (d) is new, and is the first real application of the theory of eigenvarieties with "some weights fixed". Christopher Skinner informed us that he also had this idea to get part (c) of the result above.

If Π_v has a vector fixed by an Iwahori subgroup, then the local representation at v ought to be semistable. This should follow from the corresponding fact in the setting of Expected Theorem 1.7, but the latter seems to require new information on models of Shimura varieties for $U(2, n-2)$ at Iwahori level. As Haines has pointed out, Faltings has proved that the local models have the required properties, but this is not quite enough to conclude directly.

Remark 3.2.7. The Galois representations constructed above by ℓ-adic continuity form a compatible system in the weak sense that their characteristic polynomials of Frobenius coincide for all ℓ. Since the coefficients of the characteristic polynomials all belong to the coefficient field $E(\Pi)$ of the cohomological representation Π, by (ii) and (v) of Theorem 2.3, it follows that these Galois representations are compatible in the strong sense of having coefficients in the same number field. I omit the formulation of the theorem in the present draft.

4. Totally real fields

With future applications in mind, we state here a version of Theorem 1.4 for cohomological automorphic representations of $GL(n, \mathbf{A}_F)$. General Hypotheses 1.1 are replaced by the following hypothesis.

General Hypotheses 4.1. *Let* Π *be a cuspidal automorphic representation of* $GL(n, \mathbf{A}_F)$ *Writing* $\Pi = \Pi_\infty \otimes \Pi_f$*, we assume*

(i) *(Regularity) There is a finite-dimensional irreducible representation* $W(\Pi) = W_\infty$ *of* $\mathcal{G}(\mathbb{R})$ *such that*

$$H^*(\mathfrak{g}, K_\infty; \Pi_\infty \otimes W_\infty) \neq 0.$$

(ii) *(Polarization) There is a Hecke character*

$$\chi : \mathbf{A}_F^\times/(F)^\times \to \mathbb{C}^\times$$

with $\chi_v(-1)$ *independent of the prime* $v \mid \infty$ *such that he contragredient* $\Pi^\vee$ *of* Π *satisfies*

$$\Pi^\vee \xrightarrow{\sim} \Pi \otimes \chi.$$

Theorem 4.2. *Let* Π *be a cuspidal automorphic representation of* $GL(n, \mathbf{A}_F)$ *satisfying Hypotheses 4.1. Then there is a number field* $E(\Pi)$ *and a compatible system* $\rho_{\lambda,\Pi} : \Gamma_F \to GL(n, E(\Pi)_\lambda)$ *of* λ*-adic representations, where* λ *runs through the finite places of* $E(\Pi)$*, such that*

(a) *For all finite primes* v *of* F *of residue characteristic prime to* $N_{E(\Pi)/\mathbb{Q}}(\lambda)$*,*

$$\rho_{\lambda,\Pi}^{F-ss} \mid_{\Gamma_v} \xrightarrow{\sim} \mathcal{L}(\Pi_v \otimes |\bullet|_v^{\frac{1-n}{2}}).$$

Here the superscript $^{F-ss}$ *denotes Frobenius semisimplification.*

(b) *For all finite primes* v *of* F *dividing* $N_{E(\Pi)/\mathbb{Q}}(\lambda)$*,* $\rho_{\lambda,\Pi} \mid_{\Gamma_v}$ *is de Rham, and its Hodge-Tate numbers have multiplicity at most one (i.e.,* $\rho_{\lambda,\Pi}$ *is* H*odge-Tate regular and are determined by* Π_∞*, or equivalently by* $W(\Pi)$*, in accordance with the recipe given in (1.5).*

(c) *Let* v *be a finite prime of* F *dividing* $N_{E(\Pi)/\mathbb{Q}}(\lambda)$*. Suppose* Π_v *has a vector fixed by a hyperspecial maximal compact subgroup of* $GL(n, \mathcal{K}_v)$*. Then* $\rho_v := \rho_{\lambda,\Pi} \mid_{\Gamma_v}$ *is crystalline, and if* φ *denotes the smallest linear power of the crystalline Frobenius of* $D_{crys}(\rho_v)$ *then*

$$\det(T - \varphi) = \det(T - \mathcal{L}(\Pi_v \otimes |\bullet|_v^{\frac{1-n}{2}})(Frob_v)).$$

(d) *Let* v *be a finite prime of* $\mathcal{K}$ *dividing* $N_{E(\Pi)/\mathbb{Q}}(\lambda)$*. Suppose* Π_v *has a vector fixed by an Iwahori subgroup of* $GL(n, \mathcal{K}_v)$*. Then* $\rho_{\lambda,\Pi} \mid_{\Gamma_v}$ *is semistable.*

Proof. The deduction of this result from Theorem 3.2.5 follows exactly the proof of Proposition 4.3.1 of [CHT]. The claim regarding $E(\Pi)$ is as indicated in Remark 3.2.7.

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