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CONSTRUCTION OF AUTOMORPHIC GALOIS REPRESENTATIONS, I

Introduction

Our goal in this concluding chapter is to apply the results of [L.IV.A] and [CHL.IV.B], together with Kottwitz' results in [K1,K2], to obtain a rapid construction of the compatible system of ℓ-adic Galois representations attached to a cohomological cuspidal automorphic representation Π of $GL(n)$ over a CM field, satisfying the symmetry condition corresponding to base change from a unitary group. The Galois representations are shown to be *weakly associated* to Π, in the sense that the characteristic polynomials of Frobenius at almost all places at which Π is unramified bear the expected relation to Π. We thus extend the results of [K3] to Shimura varieties attached to untwisted unitary groups, where endoscopy is an issue; the results of [C1] are generalized to Π not assumed to satisfy a local ramification hypothesis.

We have restricted our attention to Π satisfying an excessively strong regularity hypothesis at archimedean places, in order to guarantee the property (*) of [L.IV.A], which implies that the corresponding representations occur in the spectrum of unitary groups with multiplicity one, leading to simplifications in statements as well as proofs. The construction can be extended to cohomological representations not satisfying this hypothesis by a p-adic continuity argument sketched several years ago by one of the authors [H].

Kottwitz' approach has been reconsidered and updated in the book [M] of S. Morel. The emphasis of [M], however, is on the new phenomena arising in noncompact Shimura varieties, and only the case of imaginary quadratic fields is treated there.

The construction of even-dimensional Galois representations follows the strategy used in [BR] when $n = 2$, and necessarily breaks down when Π fails to satisfy a weak regularity condition. This condition arises as well, and for the same reason, in the article [S] of Shin, which gives a somewhat different construction of the Galois representations attached to Π, under precisely this weak version of regularity. Shin's results are more complete than ours: his Galois representations are shown to satisfy a strong version of compatibility with the local Langlands correspondence at finite places. His results thus generalize the combined results of [HT] and [TY]. An expository version of [S] should appear in Book 2.

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1. Cohomology of unitary Shimura varieties, following Kottwitz

Notation is as in [CHL.IV.B]. In particular, F is a totally real field, $\mathcal{K}/F$ is a totally imaginary quadratic extension, $\eta = \eta_{\mathcal{K}/F}$ the corresponding quadratic character of $\mathbb{A}_F^\times$, $d = [F:\mathbb{Q}]$, $c \in Gal(\mathcal{K}/F)$ the non-trivial Galois automorphism. We assume $\mathcal{K} = F \cdot E$ for some imaginary quadratic extension E, chosen compatibly with the simplifying hypotheses of [CHL.IV.B, (1.2)]; we make this more explicit in (1.2) below. The quadratic character for $E/\mathbb{Q}$, a character of $\mathbb{A}^\times/\mathbb{Q}^\times$, is denoted $\eta_{E/\mathbb{Q}}$. Except where otherwise indicated, n denotes a positive even integer.

1.1. Automorphic representations of similitude and unitary groups. The results of [L.IV.A] and [CHL.IV.B] relate automorphic representations of unitary groups and θ-stable representations of linear groups, under the simplifying hypotheses. We will be constructing Galois representations in the cohomology of Shimura varieties attached to unitary similitude groups, and the formalism developed by Kottwitz in [K1] and [K2] is adapted to similitude groups rather than unitary groups. Automorphic representations of similitude and unitary groups can be related without redoing the comparisons of the two previous chapters, as in [C1] and [HT].

In this section we let G be the unitary group denoted $U(V_1)$ in [CHL.IV.B, 1.4.1], the stabilizer of the hermitian form $< \bullet, \bullet >$ on the $n+1$-dimensional space V_1. Let $GU = GU(V_1)$ be the algebraic group over $\mathbb{Q}$ defined by

$$GU(R) = \{g \in GL(V_1 \otimes_{\mathbb{Q}} R) \mid\ < gv, gw >= \nu(g) < v, w >,\ v, w \in V_1 \otimes_{\mathbb{Q}} R\}$$

for any $\mathbb{Q}$-algebra R, where $\nu(g) \in R^\times$. There is a short exact sequence

$$1 \to G \to GU \xrightarrow{\nu} \mathbb{G}_m \to 1 \tag{1.1.1}$$

of algebraic groups over $\mathbb{Q}$, but G is also the restriction of scalars of a group over F that we also denote G.

Let $T = R_{E/\mathbb{Q}}\mathbb{G}_{m,E}$ and let $T^1 = \ker N : T \to \mathbb{G}_{m,\mathbb{Q}}$ where $N = N_{E/\mathbb{Q}}$ is the norm homomorphism. Thus T^1 is the group $U(1)$ for the quadratic extension $E/\mathbb{Q}$ and in particular can be identified with a subgroup of the center of G. There is an injective homomorphism

$$T^1 \to T \times G;\ t \mapsto (t, t^{-1})$$

that gives rise to a short exact sequence of algebraic groups

$$1 \to T^1 \to T \times G \xrightarrow{\phi} GU \to 1 \tag{1.1.2}$$

where $T \to GU$ is the obvious inclusion in the center. Let Z denote the center of GU. Then

$$Z(\mathbb{Q}) = \{x \in \mathcal{K}^\times \mid N_{\mathcal{K}/F}(x) \in \mathbb{Q}^\times\}$$

and $Z(R)$ is determined similarly for any $\mathbb{Q}$-algebra R. Let $A = \mathbb{R}_+^\times$ be the obvious diagonal subgroup of $Z(\mathbb{A})$, associated to $\mathbb{Q}^\times \subset \mathcal{K}^\times$. This is also the identity component of the split center of GU. The group $Z(\mathbb{Q})A$ is closed in $Z(\mathbb{A})$ and $Z(\mathbb{A})/Z(\mathbb{Q})A$ is compact. Write $\Gamma = GU(\mathbb{Q}) \cdot Z(\mathbb{A})$. Then $\Gamma = GU(\mathbb{Q})A\omega$, for some compact subset $\omega \subset Z(\mathbb{A})$, and it follows that Γ is closed. Let

$$\mathbf{S} = G(\mathbb{A})GU(\mathbb{Q})Z(\mathbb{A}) = \Gamma.G(\mathbb{A});$$

this is an open (so closed) subgroup of $GU(\mathbb{A})$, because the map $Z \times G \to GU$ is open locally, and surjective on the groups of units at unramified primes by Lang's theorem.

Lemma 1.1.3. *When $n+1$ is odd*

$$GU(\mathbb{A}) = \mathbf{S} \ .$$

Proof. By the above discussion the quotient

$$C = GU(\mathbb{A})/\mathbf{S}$$

is the quotient of a subgroup of $\mathbb{G}_m(\mathbb{A})$, the image of ν, by a subgroup containing the image of rational elements and norms of idèles of E. Since $\dim V_1$ is odd and at least 3 the map ϕ is surjective locally at all but a finite set S of finite primes, namely the primes of $\mathbb{Q}$ that are not split in E and ramify in $\mathcal{K}$ (cf. [HT], p. 201; a proof is given in the appendix.) Hence C is a finite group of exponent 2. A non trivial element in C can be represented by an element of $GU(\mathbb{Q}_S)$. But by weak approximation it is in the image of $GU(\mathbb{Q})$. In other words C is trivial, and $GU(\mathbb{A}) = \mathbf{S}$.

In particular, if we temporarily let $\Sigma = G(\mathbb{A}) \cap \Gamma$ then, for $n+1$ odd

$$\Gamma\backslash GU(\mathbb{A}) \simeq \Sigma\backslash G(\mathbb{A})$$

Proposition 1.1.4. *Assume $n+1$ is odd. Let π be an irreducible automorphic representation of GU whose restriction to G contains an irreducible representation σ. Suppose σ occurs with multiplicity one in the discrete spectrum for G. Then π occurs with multiplicity one in the discrete spectrum for GU.*

Moreover, if χ is the central character of π, then π is the only automorphic representation of GU containing σ with central character χ.

Proof. For any Hecke character χ of $Z(\mathbb{A})$, let

$$\mathcal{A}(GU,\chi) = L^2(\Gamma\backslash GU(\mathbb{A}),\chi)$$

be the space of automorphic forms on $GU(\mathbb{A})$, square-integrable modulo $Z(\mathbb{A})$, on which $Z(\mathbb{A})$ acts by χ, or equivalently Γ acts by χ extended trivially to $GU(\mathbb{Q})$. We have

$$\mathcal{A}(G,\chi) = I_\chi := Ind_\Gamma^{\mathbf{S}}\chi,$$

since $\mathbf{S} = GU(\mathbb{A})$. We may assume that π has central character χ, thus the multiplicity we need to determine is

$$m_{GU}(\pi) = \dim Hom_{GU(\mathbb{A})}(\pi,\mathcal{A}(GU,\chi)) = \dim Hom_{\mathbf{S}}(\pi, Ind_\Gamma^{\mathbf{S}}\chi).$$

Similarly, letting $\Sigma = G(\mathbb{A}) \cap \Gamma$, we have, $\mathcal{A}(G)$ being $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$:

$$\mathcal{A}(G) = \oplus_\xi \mathcal{A}(G,\xi)$$

where for any character ξ of Σ trivial on $G(F)$, $\mathcal{A}(G,\xi) = L^2(\Sigma\backslash G(\mathbb{A}),\xi)$ is defined by analogy with $\mathcal{A}(GU,\chi)$. As above, we can identify

$$\mathcal{A}(G,\xi) = Ind_\Sigma^{G(\mathbb{A})}\xi.$$

Now suppose ξ is the restriction to Σ of χ and that σ occurs in $\mathcal{A}(G,\xi)$. Observe that since

$$\Gamma\backslash GU(\mathbb{A}) \simeq \Sigma\backslash G(\mathbb{A})$$

then

$$\mathcal{A}(G,\xi) = Res_{G(\mathbb{A})} I_{\chi}.$$

and

$$m_G(\sigma) := \dim Hom_{G(\mathbb{A})}(\sigma, \mathcal{A}(G,\xi)) = \dim Hom_{G(\mathbb{A})}(\sigma, Res_{G(\mathbb{A})} I_{\chi}).$$

But

$$\dim Hom_{G(\mathbb{A})}(\sigma, Res_{G(\mathbb{A})} I_{\chi}) = \sum_{\pi} \dim Hom_{G(\mathbb{A})}(\sigma, Res_{G(\mathbb{A})}\pi)$$

where π runs over representations of $GU(\mathbb{A})$ in I_{χ}, counted with multiplicity. Since, by hypothesis

$$\dim Hom_{G(\mathbb{A})}(\sigma, \mathcal{A}(G,\xi)) = 1$$

there is exactly one such representation π and

$$\dim Hom_{GU(\mathbb{A})}(\pi, I_{\chi}) = 1$$

and we are done.

1.1.5. Remark. If n is even, then the group C defined as above is still a group of exponent 2 but is not in general trivial, cf. the discussion at the end of the appendix. The argument above breaks down if $Res_{\mathbf{S}}\pi$ is reducible. If this happens, then it is necessarily the case that $\pi \xrightarrow{\sim} \pi \otimes \varepsilon$ for some non-trivial character of C. One expects then that π arises from some endoscopic group of GU, as in [LL].

1.2. Base change and descent for similitude groups.

By the results of the previous section, any pair consisting of an automorphic representation σ of G and a Hecke character ξ of T (called χ in §1) with the property that

$$\xi(x^c) = \xi(x)^{-1}, \forall x \in T^1(\mathbb{A}), \tag{1.2.1}$$

defines a representation $\sigma * \xi$ of the open subgroup $\mathbf{V} = im(\phi) \subset GU(\mathbb{A})$, occurring in $L^2(\phi(G(\mathbb{Q}) \times T(\mathbb{Q})\backslash\mathbf{V})$. Condition (1.2.1) corresponds to the condition that the representation $\sigma \otimes \xi$ of $G(\mathbb{A}) \times T(\mathbb{A})$ factors via the map ϕ.

In what follows, n is even and π will be any automorphic representation of $GU(\mathbb{A})$ whose restriction to $im(\phi)$ contains $\sigma * \xi$ (and hence its restriction to $G(\mathbb{A})$ contains σ). We recall the special hypotheses in force in the two previous chapters.

Special Hypotheses 1.2.2.

(1.2.2.1) *$\mathcal{K}/F$ is unramified at all finite places (in particular $d > 1$).*
(1.2.2.2) *Π_v is spherical (unramified) at all non-split non-archimedean places v of $\mathcal{K}$.*
(1.2.2.3) *The degree $d = [F : \mathbb{Q}]$ is even.*

Special Hypothesis 1.2.3. *The highest weight $\mu(v)$ is sufficiently far from the walls. This comes in three strengths:*

(i) *(Regularity) For at least one $v \in \Sigma_F$, $\mu(v)$ is regular, i.e. $\mu_i(\tilde{v}) \neq \mu_j(\tilde{v})$ if $i \neq j$.*
(ii) *(Strong regularity) $\mu(v)$ is regular for all $v \in \Sigma_F$;*
(iii) *(Weak regularity) For at least one $v \in \Sigma_F$, there is at least one* **odd** *i such that $\mu_i(\tilde{v}) > \mu_{i+1}(\tilde{v})$.*

Proposition 1.2.4. *Let Π be a cuspidal automorphic representation of $GL(n+1)_{\mathcal{K}}$ that is θ_{n+1}-stable, cohomological at archimedean places, and satisfies the property (*) of [L.IV.A, §5]. Moreover, assume Π is spherical at all places inert in $\mathcal{K}/F$, and let σ be an automorphic representation of G, whose existence is guaranteed by [L.IV.A, 5.3], whose weak base change is isomorphic to Π. Assume the infinitesimal character of Π_∞ is strongly regular. Then*

(1) *σ_∞ is in the discrete series.*
(2) *For any σ'_∞ in the same L-packet, $\sigma'_\infty \otimes \sigma_f$ occurs in the automorphic spectrum of G with multiplicity one. Moreover, any automorphic representation σ' of G with weak base change isomorphic to Π is of this kind.*
(3) *There is a Hecke character ξ of T such that the central character ξ_Π of Π satisfies*

$$\xi_\Pi(t) = \xi(t^c/t), \ \forall t \in T(\mathbb{A}) \subset \mathcal{K}^\times(\mathbb{A}) = Z_{GL(n+1)}(\mathbb{A});$$

(4) *Let σ' be any automorphic representation of G with weak base change isomorphic to Π. There exists an automorphic representation π of $GU(\mathbb{A})$ whose pullback to $G(\mathbb{A}) \times T(\mathbb{A})$ via the map ϕ of (1.1.2) contains $\sigma' \otimes \xi$. Any such π occurs in the discrete automorphic spectrum of GU with multiplicity 1.*

Proof. We know that σ_∞ is cohomological, so the first assertion follows from the strong regularity hypothesis. Assertion (2) is a consequence of [L.IV.A,5.3]. The claim (3) is proved in [HT, VI.2.10]. Finally, (4) follows from (2) and Proposition 1.1.4.

Corollary 1.2.5. *Notation is as in Proposition 1.2.4. The set of automorphic representations π' of $GU(\mathbb{A})$ with $\pi'_f \xrightarrow{\sim} \pi_f$ has $n+1$ members. The restrictions of these π'_∞ to $G(\mathbb{R})$ are precisely the $n+1$ elements of the discrete series L-packet of σ_∞.*

The analogue for the endoscopic representations considered in [CHL.IV.B] is the following. Notation is as in [CHL.IV.B].

Proposition 1.2.6. *Let τ be a cuspidal automorphic representation of $GL(n)_{\mathcal{K}}$ (resp. $GL(1)_{\mathcal{K}}$) that is θ_n-stable, cohomological at archimedean places, and satisfies the property (*) of [L.IV.A, §5]. Moreover, assume Π is spherical at all places inert in $\mathcal{K}/F$. Assume the infinitesimal character of Π_∞ is strongly regular. Let χ be a θ_1-stable cohomological Hecke character of $GL(1)_{\mathcal{K}}$ chosen as in Theorem 4.6 of [CHL.IV.B]. Let σ be an automorphic representation of G, whose existence is guaranteed by [CHL.IV.B, 4.6], whose weak base change to $\mathcal{G}_{n+1}$ is isomorphic to $\tau \times \chi$.*

Then

(1) σ_∞ *is in the discrete series. Let* $\Pi_G(\phi_G)$ *denote the corresponding discrete series L-packet.*
(2) *For any* σ'_∞ *in* $\Pi^+ \subset \Pi_G(\phi_G)$, $\sigma'_\infty \otimes \sigma_f$ *occurs in the automorphic spectrum of G with multiplicity one. Moreover, any automorphic representation* σ' *of G with weak base change isomorphic to* Π *is of this kind. In particular, if* $\sigma'_\infty \in \Pi^-$, *then* $\sigma'_\infty \otimes \sigma_f$ *does not occur in the automorphic spectrum of G.*
(3) *There is a Hecke character* ξ *of T such that the central character* ξ_Π *of* Π *satisfies*

$$\xi_\Pi(t) = \xi(t^c/t),\ \forall t \in T(\mathbb{A}) \subset \mathcal{K}^\times(\mathbb{A}) = Z_{GL(n+1)}(\mathbb{A});$$

(4) *Let* σ' *be any automorphic representation of G with weak base change isomorphic to* $\tau \times \chi$. *There exists an automorphic representation* π *of* $GU(\mathbb{A})$ *whose pullback to* $G(\mathbb{A}) \times T(\mathbb{A})$ *via the map* ϕ *of (1.1.2) contains* $\sigma' \otimes \xi$. *Any such* π *occurs in the discrete automorphic spectrum of GU with multiplicity 1.*

Corollary 1.2.7. *Notation is as in Proposition 1.2.6. The set of automorphic representations* π' *of* $GU(\mathbb{A})$ *with* $\pi'_f \xrightarrow{\sim} \pi_f$ *has n members. The restrictions of these* π'_∞ *to* $G(\mathbb{R})$ *are precisely the n elements of the subset* Π^+ *of the discrete series L-packet of* σ_∞.

1.3. Endoscopic groups of similitude groups.

The L-group of GU in the Galois version is a semidirect product of $GL(n+1,\mathbb{C})^{[F:\mathbb{Q}]} \times GL(1,\mathbb{C})$ with $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. The factors of $GL(n+1,\mathbb{C})^{[F:\mathbb{Q}]}$ are indexed by embeddings of $\alpha : \mathcal{K} \to \overline{\mathbb{Q}}$ above a fixed embedding of E, and the elements of $Gal(\overline{\mathbb{Q}}/E)$ act transitively on the factors of $GL(n+1,\mathbb{C})$ by permuting the indices in the tautological way, while leaving the factor $GL(1,\mathbb{C})$ untouched. Any complex conjugation $c \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts by

$$(g_\alpha, \lambda) \mapsto (\lambda\, {}^t g_\alpha^{-1}, \lambda). \tag{1.3.1}$$

The homomorphism $T \times G \to GU$ identifies by duality ${}^LGU$ with a subgroup of ${}^LT \times {}^LG$ that maps surjectively to ${}^LG$; let ϕ^* denote this map. Let (H', s', ξ') be an elliptic endoscopic datum for ${}^LGU$. Then $\xi = \phi^* \circ \xi' : H' \to {}^LG$ defines an elliptic endoscopic datum (H, s, ξ) with $s = \phi^*(s')$ and $H = \xi(H')$. This establishes a bijection between equivalence classes of elliptic endoscopic triples for GU and for G.[4]

Specifically, for any partition $n+1 = a+b$, there is an elliptic endoscopic group $GH = GH_{a,b}$ of GU which can be identified with the subgroup

$$\{\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \in GL(a) \times GL(b) \cap GU^*\} \tag{1.3.2}$$

where GU^* is the quasi-split inner form of GU and the identification with (1.3.2) depends on the choice of Hecke characters μ_a and μ_b, as in [H.I.A, (5.5)]. More

[4]This fails for $GU(n)$ for n even.

precisely, the identification of the group (1.3.2) with the centralizer in ${}^LGU$ of the element

$$s'_{a,b} = ((diag(I_a, -I_b)_\alpha, 1),$$

in the above coordinates of $\widehat{GU}$, is given by a map ξ_{μ_a,μ_b} defined as in [H.I.A, (5.5)].

Letting H be the corresponding elliptic endoscopic group for G, there is a short exact sequence

$$1 \to T^1 \to T \times H \overset{\phi_H}{\to} GH \to 1 \qquad (1.3.3)$$

just as in (1.1.2).

Lemma 1.3.4. *The image of $\phi_H : T(\mathbb{A}) \times H(\mathbb{A}) \to GH(\mathbb{A})$ is of finite index in $GH(\mathbb{A})$. In particular, $GH(\mathbb{Q}) \cdot im(\phi_H) = GH(\mathbb{A})$.*

Proof. Let $\nu : GH \to \mathbb{G}_m$ be the similitude group. Since $n+1$ is odd, one of (a,b), say b, is also odd. Thus

$$\nu(GH(\mathbb{A})) \supset \nu_b(GU(b)^*(\mathbb{A}))$$

where $GU(b)^*$ is the quasi-split group of rational unitary similitudes of size b and ν_b is the corresponding similitude factor. But this group contains $N(T(\mathbb{A}))$ as a subgroup of finite index, as we have already seen in §1.1 (cf. appendix for details). This implies the first statement, and the second statement follows by weak approximation, as in §1.1.

The proof of Proposition 1.1.4 then applies without change to GH, and we obtain

Proposition 1.3.5. *Let π be an automorphic representation of GH whose restriction to H contains a representation σ. Suppose σ occurs with multiplicity one in the discrete spectrum for H. Then π occurs with multiplicity one in the discrete spectrum for GH. Moreover, π is uniquely determined by its central character among automorphic representations containing σ.*

In their study of Galois representations on the cohomology of Shimura varieties, Morel and Shin [M,S] do not verify the multiplicity one property. The multiplicity is left undetermined in Morel's formulas, whereas Shin makes use of Taylor's argument, published as [HT, VII.1.8], to obtain a Galois representation of the right dimension.

2. Statement of the main theorem

2.1. Global parameters.

The following discussion has been strongly influenced by Arthur's ongoing work on functoriality for classical groups and Moeglin's analysis of local L-packets in the same setting. In this section, n is not assumed even, $G = U(V)$ and $GU = GU(V)$ (rational similitudes) for some n-dimensional hermitian space (*not* $n+1$-dimensional).

As in §4 below, Kottwitz' conjectural formula for the intersection cohomology of Shimura varieties is expressed in terms of Arthur's conjectural parametrization of automorphic representations of the group GU by homomorphisms

$$\psi : \mathcal{L}_\mathbb{Q} \times SL(2,\mathbb{C}) \to {}^LGU,$$

where $\mathcal{L}_{\mathbb{Q}}$ is the hypothetical Langlands group. (Such a map is generally called an *Arthur parameter.*) As Arthur and Moeglin have emphasized in their respective contexts, for a classical group H whose L-group naturally embeds in ${}^LGL(n)$, representations of $\mathcal{L}$ can be replaced by collections of cuspidal automorphic representations Π_i of $GL(n_i)$ for some partition $n = \sum_i n_i$, where the collection satisfies a group-theoretic symmetry condition replacing the condition that the (hypothetical) corresponding homomorphism $\psi = \oplus_i \psi_{\Pi_i}$ take values in the image of the L-group of H in ${}^LGL(n)$. This description, valid for (conjecturally) tempered parameters, can be extended to parameters that are non-trivial on the factor $SL(2,\mathbb{C})$, using the Moeglin-Waldspurger determination of the discrete automorphic spectrum of $GL(n)$.

The work of Arthur and Moeglin (for example [A1, Mo]) shows that conjectures of Arthur that seemed to depend on the existence and properties of $\mathcal{L}_{\mathbb{Q}}$ can be phrased unconditionally (in terms of packets of cuspidal representations Π_i) and then *proved*, at least for special orthogonal or symplectic groups (Arthur [A1], see also the final chapter of [A2]). Our job, in §4, will therefore consist in replacing, in Kottwitz' formulation, statements apparently depending on $\mathcal{L}_{\mathbb{Q}}$ by propositions that, in our simple cases, can be stated unconditionally and proved.

For example, let U temporarily be any inner form of $U(n)$ over F – for example $U = G$ in the notation of §1 – and let σ be an automorphic representation of U occurring in the discrete spectrum, with σ_∞ in the discrete series. Corollaire 5.3 of [L.IV.A] states that σ then admits a base change to $GL(n)$ of the form $\Pi_1 \boxplus \Pi_2 \boxplus \cdots \boxplus \Pi_r$ for some partition $n = \sum_i n_i$ as above, with each Π_i in the automomorphic discrete spectrum of $GL(n_i)$. The symmetry condition is that each Π_i is $\theta^*_{n_i}$-stable.

Moreover, since $\boxplus_i \Pi_i$ is cohomological (cf. [L.IV.A, §5]), its infinitesimal character is regular, which implies in particular that

$$\Pi_i \not\simeq \Pi_j \text{ for } i \neq j.$$

Since σ_∞ is tempered, it is presumably the case that each Π_i is in fact cuspidal, but Labesse does not verify this; however, his proof actually does provide enough information to draw this conclusion. The unordered collection

$$\{[\Pi_1], [\Pi_2], \ldots, [\Pi_r]\} \tag{2.1.1}$$

can be taken to be the parameter of σ; the brackets have been added to make the notation more parameter-like. The $\theta^*_{n_i}$-stability of Π_i corresponds – formally, since one has no $\mathcal{L}$ – to the possibility of extending the parameter to a parameter with values in ${}^LU$. This begs the question of the uniqueness of such an extension, but the requirement that σ_∞ be in the discrete series determines the extension of the archimedean parameter in the Langlands classification (cf. the discussion in (3.2) of [C.III.A]), and this in turn (formally) determines the extension of the parameter of Π uniquely.

The parameter for an automomorphic representation of GU has in addition a Hecke character ξ (not dignified with brackets) of the imaginary quadratic field E – of the torus T of §1 – subject to the compatibility condition (1.2.1). It follows from the last part of Proposition 1.1.4 that the pair (σ, ξ) determines a unique automorphic extension π of the representation $\sigma * \xi$ to $GU(\mathbb{A})$. Thus if (2.1.1) is the parameter of σ, we can take

$$\psi = \psi_\pi = \{[\Pi_1], [\Pi_2], \ldots, [\Pi_r], \xi\} \tag{2.1.2}$$

to be the parameter of the automorphic representation π of GU. The discussion in §1 shows that (2.1.2) is sufficient to determine ψ uniquely when n is odd. When n is even we leave the possible ambiguity open.

In order to apply Kottwitz' formulas one needs to be able to calculate with the group $\mathfrak{S}_\psi$, where ψ is a global Arthur parameter with values in the L-group of GU, where

$$S_\psi = Cent(\psi, {}^L GU) Z(\widehat{GU}) \subset {}^L(GU)$$

where $Cent(\psi, {}^L GU)$ is the centralizer of ψ in the L-group, and

$$\mathfrak{S}_\psi = S_\psi / S_\psi^0 Z(\widehat{GU}),$$

the quotient of the component group of S_ψ by the image of $Z(\widehat{GU})$.

For a parameter $\psi_{\mathcal{K}} = \{[\Pi_1], [\Pi_2], \ldots, [\Pi_r]\}$ for $GL(n)$ as in (2.1.1), it is natural to define $S_{\psi_{\mathcal{K}}} = \prod_i Z({}^L GL(n_i, \mathbb{C}))$; this is appropriate because, as we have seen, the different Π_i are pairwise non-isomorphic. The centralizer of ψ viewed as parameter for the representation of U has to take into account the θ-stability. This translates into the condition that $z \in S_\psi$ commute with the action of complex conjugation as defined in a number of earlier chapters, for example in (3.1) of [C.III.A]. One verifies that complex conjugation in the L-group acts on each $Z({}^L GL(n_i, \mathbb{C})) \equiv \mathbb{C}^\times$ by $t \mapsto t^{-1}$. Thus the identity component of S_ψ is just the center of $GL(n, \mathbb{C})$, and for the parameter ψ extending $\psi_{\mathcal{K}} = \{[\Pi_1], [\Pi_2], \ldots, [\Pi_r]\}$ as above, we have

$$\mathfrak{S}_\psi = (\pm 1)^r / (\pm I_n) \simeq \mathbb{Z}/2\mathbb{Z}^{r-1}, \tag{2.1.3}$$

with $(\pm I_n) \subset Z(GL(n, \mathbb{C}))$ embedded diagonally. One views the elements of $\mathfrak{S}_\psi$ as diagonal matrices in $GL(n, \mathbb{C})$, up to multiplication by $\pm I_n$. When $\psi = \{[\Pi_1], [\Pi_2], \ldots, [\Pi_r], \xi\}$ is a parameter for GU, $\mathfrak{S}_\psi$ is the same as in (2.1.3); in other words, the presence of ξ makes no difference.

2.1.4. In the applications, we only consider $r = 1$ (the *stable case*), in which case $\mathfrak{S}_\psi$ is trivial, or $r = 2$ (the *endoscopic case*), with $\mathfrak{S}_\psi = \mathbb{Z}/2\mathbb{Z}$. In keeping with the conventions of previous chapters, we write $[\Pi]$ instead of $[\Pi_1]$ in the stable case and $([\tau], [\chi])$ instead of $\{[\Pi_1], [\Pi_2]\}$ in the endoscopic case, with τ cuspidal and θ_n^*-stable on $GL(n)$ and χ a θ_1^*-stable Hecke character of $GL(1)$. We always assume χ chosen to satisfy the hypotheses of [CHL.IV.B, Theorem 4.6].

Our special hypotheses are in force here and prescribe that $[\Pi]$, $[\tau]$, and $[\chi]$ all be spherical at places inert in $\mathcal{K}/F$. Then we can apply the results of §1.2. If $\psi = \{[\Pi_1], [\Pi_2], \ldots, [\Pi_r], \xi\}$, $r \leq 2$, then we fix π as in Propositions 1.2.4 and 1.2.6 and let $\Pi(\psi)$ denote the set of representations π' of $GU(\mathbb{A})$ with $\pi'_f \xrightarrow{\sim} \pi_f$. We have

$$|\Pi(\psi)| = \begin{cases} n+1 & r = 1 \\ n & r = 2 \end{cases}. \tag{2.1.4.1}$$

Remark 2.1.4.2. The choice of π fixes an element of the *near equivalence class* of π_f, that is to say the class of representations of $GU(\mathbf{A}^f)$ that are locally isomorphic to π_f almost everywhere. We believe this near equivalence class is a singleton under our hypotheses. For even dimensional unitary groups this need not be the case, but the choice of a hyperspecial maximal compact subgroup at all finite places should still single out an element of the near equivalence class. This question may have been treated explicitly in the literature, but we have been unable to find a complete reference.

2.2. Conjectures on Galois representations.

For the purposes of this section we let n be any positive integer and let $\mathcal{G} = \mathcal{G}_n$ be the algebraic group $R_{\mathcal{K}/\mathbb{Q}}GL(n)_{\mathcal{K}}$. Let $\mathfrak{g} = Lie(\mathcal{G}(\mathbb{R}))$, $K_\infty \subset \mathcal{G}(\mathbb{R})$ the product of a maximal compact subgroup with the center $Z_{\mathcal{G}}(\mathbb{R})$. We consider cuspidal automorphic representations Π of $\mathcal{G}$ satisfying the following two hypotheses:

General Hypotheses 2.2.1. *Writing* $\Pi = \Pi_\infty \otimes \Pi_f$, *where* Π_∞ *is an admissible* $(\mathfrak{g}, K_\infty)$*-module, we have*

(i) *(Regularity) There is a finite-dimensional irreducible representation* $W(\Pi) = W_\infty$ *of* $\mathcal{G}(\mathbb{R})$ *such that*

$$H^*(\mathfrak{g}, K_\infty; \Pi_\infty \otimes W_\infty) \neq 0.$$

(ii) *(Polarization) The contragredient* $\Pi^\vee$ *of* Π *satisfies*

$$\Pi^\vee \xrightarrow{\sim} \Pi \circ c.$$

A special case of the conjectural global Langlands correspondence between automorphic representations of algebraic type [C2] and compatible systems of Galois representations is given by the following conjecture:

Conjecture 2.2.2. *Let* Π *be a cuspidal automorphic representation of* $\mathcal{G}$ *satisfying Hypotheses 2.2.1. There is a compatible family of continuous representations*

$$\{\rho_{\Pi,\lambda}\} : Gal(\overline{\mathbb{Q}}/\mathcal{K}) \to GL(n, E(\Pi)_\lambda),$$

as λ *runs through non-archimedean completions of a certain number field* $E(\Pi)$, *satisfying*

(a) $\rho_{\Pi,\lambda}$ *is* **geometric** *in the sense of Fontaine and Mazur [FM]. More precisely,* $\rho_{\Pi,\lambda}$ *is unramified outside a finite set of places of* $\mathcal{K}$, *namely the set of places of ramification of* Π *and the set of primes of* $\mathcal{K}$ *dividing the residue characteristic* ℓ *of* λ. *Moreover, at places dividing* ℓ, $\rho_{\Pi,\lambda}$ *is de Rham, in Fontaine's sense, and in particular is Hodge-Tate.*

(b) $\rho_{\Pi,\lambda}$ *is* **HT regular***: the Hodge-Tate weights of* $\rho_{\Pi,\lambda}$ *have multiplicity one.*

(c) *There is a non-degenerate bilinear pairing*

$$\rho \otimes \rho \circ c \;\to E(\Pi)_\lambda \otimes \mathbb{Q}_\ell(1-n).$$

This correspondence has the following properties:

(i) *For any finite place* v *prime to the residue characteristic* ℓ *of* λ,

$$(\rho_{\Pi,\lambda}\mid_{\Gamma_v})^{Frob-ss} = \mathcal{L}(\Pi_v).$$

Here Γ_v *is a decomposition group at* v *and* $\mathcal{L}$ *is the* **normalized** *local Langlands correspondence (denoted* $r_\ell(\Pi_v)$ *in [HT]). The superscript on the left denotes Frobenius semisimplification.*

(ii) *The representation* $\rho_{\Pi,\lambda}\mid_{\Gamma_v}$ *is de Rham for any* v *dividing* ℓ *and the Hodge-Tate numbers at* v *are explicitly determined by the archimedean weight of* $L(\Pi_\infty)$, *by the formula given in* [HT,Theorem VII.1.9], *cf. §4.5, below.*

(iii) *If* Π_v *is unramified then* $\rho_{\Pi,\lambda}\mid_{\Gamma_v}$ *is crystalline.*

Note that the correspondence $\mathcal{L}$ in (i) must include a "half-integral twist" for reasons of purity. We refer to the introduction of [HT] for a complete discussion.

Through the efforts of Shin, Bellaïche, Chenevier, and Sorensen, as well as the authors of the present article, this conjecture is now nearly completely established. The statement and an account of the proof of a slighly weaker version – Frobenius semisimplification in (i) is replaced by semisimplification when Π is not weakly regular, in the sense of 1.2.3 – should appear in Book 2. The main theorem of the present article is

Theorem 2.2.3. *Let* Π *be as in Conjecture 2.2.2. Assume* Π *is strongly regular, in the sense of 1.2.3. Then there is a number field* $E(\Pi)$ *and a compatible family of continuous representations*

$$\{\rho_{\Pi,\lambda}\} : Gal(\overline{\mathbb{Q}}/\mathcal{K}) \rightarrow GL(n, E(\Pi)_\lambda)$$

satisfying (a), (b), (c), and (ii) of Conjecture 2.2.2. Moreover, for almost all finite places v *of* $\mathcal{K}$ *prime to the residue characteristic* ℓ *of* λ *at which* Π *is unramified,*

$$(\rho_{\Pi,\lambda}\,|_{\Gamma_v})^{Frob-ss} = \mathcal{L}(\Pi_v). \tag{2.2.4}$$

Finally, if Π_v *and* $\mathcal{K}$ *are unramified at all primes* v *dividing the residue characterstic* ℓ*, then* $\rho_{\Pi,\lambda}\,|_{\Gamma_v}$ *is crystalline at* v*.*

This is a direct consequence of the approach developed by Kottwitz in [K1] and [K2], together with the results of [L.IV.A] and [CHL.IV.B]. The bulk of §4 is devoted to reformulating the latter results as a substitute for the appeal to the (still conjectural) Arthur parameters in [K1].

The following section constructs Galois representations of the appropriate dimension.

2.3. Local systems and Galois representations.

In this section $\mathcal{G} = R_{\mathcal{K}/\mathbb{Q}}GL(n+1)_{\mathcal{K}}$ with n even, as in §1. We fix a complex embedding ι of the imaginary quadratic field E and let Σ be the CM type of $\mathcal{K}$ consisting of all extensions of ι to $\mathcal{K}$ (so the notation of (1.1) is no longer in force). The irreducible representation $W(\Pi)_{\mathbb{R}}$ of $R_{\mathcal{K}/\mathbb{Q}}GL(n+1)_{\mathcal{K}}(\mathbb{R})$ factors over the set Σ_F of real embeddings of F:

$$W(\Pi) = \otimes_{v\in\Sigma_F} W_v,$$

where W_v is an irreducible representation of $\mathcal{G}(\mathcal{K}\otimes_{F,v}\mathbb{R}) \xrightarrow{\sim} GL(n+1,\mathbb{C})\times GL(n+1,\mathbb{C})$. The highest weight of W_v is denoted $\mu(v)$; it can be identified in the usual way with a pair of non-increasing n-tuples of non-negative integers $(\mu(\tilde{v}), \mu(\tilde{v}^c))$, where $\tilde{v}$ is the extension of $v \in \Sigma_F$ to an element of Σ and we write

$$\mu(\tilde{v}) = (\mu_1(\tilde{v}) \geq \mu_2(\tilde{v}) \geq \cdots \geq \mu_n(\tilde{v})). \tag{2.3.1}$$

We assume $W(\Pi)$ is strongly regular, as in (1.2.3.ii), so that all the inequalities in (2.3.1) are in fact strict.

Let W_+ be the corresponding irreducible representation of $G(\mathbb{R})$, as in the discussion following Theorem 2.3 of [CHL.IV.B]. We use the same notation to denote an extension to $GU(\mathbb{R})$. This representation has a rational model over a finite extension $E(W)$ of E. For any rational prime ℓ and any ℓ-adic place $\lambda*$ of $E(W)$, we obtain a representation $GU(E(W)_{\lambda*}) \rightarrow GL(W_{+,\lambda*})$ that defines a local system

$$\tilde{W}_{\lambda*} = \tilde{W}(\Pi)_{\lambda*}$$

in $E(W)_{\lambda*}$-vector spaces over the locally symmetric variety

$$Sh(GU) = \varprojlim_{K_f} Sh(GU)_{K_f} = \varprojlim_{K_f} GU(\mathbb{Q})\backslash GU(\mathbb{A})/K_{\infty,GU} \cdot K_f. \tag{2.3.2}$$

This is a complex projective pro-variety of dimension n which can be endowed with a canonical model over the field $\mathcal{K}$, as we recall briefly in §3, below; this will be discussed at greater length in Book 2.

Now let $\psi = \{[\Pi_1], [\Pi_2], \ldots, [\Pi_r], \xi\}$ be a parameter for GU, as in §2.1. We assume $r \leq 2$; if $r = 2$ we assume $\Pi_1 = \tau$ is a θ_n-stable representation of $GL(n)$ and $\Pi_2 = \chi$ a θ_1-stable Hecke character of $GL(1)$. We assume the special hypotheses (1.2.2); in particular each Π_i satisfies (1.2.2.2). Define $\Pi(\psi)$ as in (2.1.4); by construction all the π_∞ are cohomological for the representation W_+. Moreover, all the automorphic representations π of GU in $\Pi(\psi)$ share a common finite part π_f. This representation has a model over a finite extension $E(\pi)$ of $E(W)$.[5]

Proposition 2.3.3. *Assume the coefficient representation $W(\Pi)$ is strongly regular, as in (1.2.3.ii).*

Suppose $r = 1$, so that $\psi = \{[\Pi], \xi\}$. Then the space

$$H(Sh(GU), \tilde{W}_{\lambda*})[\pi_f] = Hom_{GU(\mathbf{A}^f)}(\pi_f, H^n(Sh(GU), \tilde{W}_{\lambda*}) \otimes_{E(W)} E(\pi_f))$$

is a free $E(\Pi_\infty)_{\lambda*} \otimes_{E(W)} E(\pi_f)$-module of rank $n+1$. For each finite place λ of $E(\pi_f)$ dividing $\lambda*$, we thus have an $n+1$-dimensional representation

$$\rho_{\Pi,\xi,\lambda} : Gal(\overline{\mathbb{Q}}/\mathcal{K}) \rightarrow GL(n+1, E(\pi_f)_\lambda)$$

realized on the λ-component of $H(Sh(GU), \tilde{W}_{\lambda*})[\pi_f]$.

If $r = 2$, so that $\psi = \{[\tau], [\chi], \xi\}$, the space $H(Sh(GU), \tilde{W}_{\lambda*})[\pi_f]$ is a free $E(\Pi_\infty)_{\lambda*} \otimes_{E(W)} E(\pi_f)$-module of rank n. For each finite place λ of $E(\pi_f)$ dividing $\lambda*$, we thus have an n-dimensional representation

$$\rho_{\tau,\xi,\lambda} : Gal(\overline{\mathbb{Q}}/\mathcal{K}) \rightarrow GL(n, E(\pi_f)_\lambda)$$

realized on the λ-component of $H(Sh(GU), \tilde{W}_{\lambda*})[\pi_f]$.

Proof. The representation $W(\Pi, \Sigma)$ defines a local system $\tilde{W}$ in $E(W)$ vector spaces over $Sh(GU)$, and for any complex embedding $\beta : E(W) \hookrightarrow \mathbb{C}$, we can consider the complex vector space

$$H(Sh(GU), \tilde{W}(\mathbb{C}))[\pi_f] = Hom_{GU(\mathbf{A}^f)}(\pi_f, H^n(Sh(GU)\tilde{W} \otimes_{E(W)} \mathbb{C})$$

[5] In fact, because all the ramification is concentrated at places split in $\mathcal{K}/F$, it follows from the existence of the conductor for representations of $GL(n+1)$ that we can take $E(\pi)$ to be the field of definition of the isomorphism class of π_f. The ramified representations come, in fact, from $GL(n, F_v)$-factors, by hypothesis (1.2.2.2), and we apply Proposition 3.2 of [C2]. At inert primes the unitary group is unramified and the local unramified representation is defined over its field of rationality because the space of invariants of a hyperspecial subgroup has multiplicity one by an argument of Waldspurger, cf. [C2, p. 103].

This is isomorphic by Matsushima's formula to

$$Hom_{GU(\mathbf{A}^f)}(\pi_f, H^n(\mathfrak{g}, K_\infty; \mathcal{A}(GU) \otimes W_+) = \bigoplus_{\pi'; \pi'_f \simeq \pi_f} H^n(\mathfrak{g}, K_\infty; \pi'_\infty \otimes W_+).$$

Here $\mathcal{A}(GU)$ is the space of automorphic forms on GU, which we break up as a sum over π' with fixed finite part. Recall here that GU is anisotropic modulo center, so all automorphic representations of $GU(\mathbb{A})$ are essentially unitary. Since $W(\Pi)$ is assumed strongly regular, so is W_+, hence all cohomology is concentrated in degree n and the only automorphic representations that contribute non-trivially are those π' for which π'_∞ is in the discrete series L-packet of GU associated to W_+. The result now follows from Corollaries 1.2.5 and 1.2.7 and the comparison theorem between Betti and étale cohomology.

3. Review of Shimura varieties attached to anisotropic unitary groups

We follow Kottwitz' conventions [K2, §5] in attaching a Shimura variety $Sh(V_1)$ to the PEL type defined by the group $GU(V_1)$ and an appropriate Shimura datum. For the sake of completeness, we briefly review the definition of $Sh(V_1)$ as a moduli space of abelian varieties with PEL structures. A more leisurely account of these Shimura varieties is contained in Book 2.

The definition of V_1 includes the choice of a unique real prime v_0 of F such that $U(V_{1,v_0})$ is not compact. We choose an extension of v_0 to a complex prime w_0 of $\mathcal{K}$ or, what is the same thing, a complex embedding of the quadratic imaginary field E. With our choices, the reflex field $E(GU(V_1), X(V_1))$ of $Sh(V_1)$ can be identified with $\mathcal{K}$, or rather, since it is by definition a subfield of $\mathbb{C}$, with the image of $\mathcal{K}$ under the complex embedding w_0. We assume the CM type Σ is chosen to contain w_0 and so that

1. $V_1 \otimes_{\mathcal{K}, w_0} \mathbb{C}$ is of signature $(1, n)$, whereas
2. $V_1 \otimes_{\mathcal{K}, w} \mathbb{C}$ is of signature $(0, 1+n)$ for $w \in \Sigma$, $w \neq w_0$.

We write GU instead of $GU(V_1)$. Fix an open compact subgroup $K = \prod_p K_p \subset GU(\mathbf{A}^f)$, and let $Sh_K = Sh(V_1)_K$ denote the corresponding Shimura variety at finite level. By this we mean the collection of complex points

$$GU(\mathbb{Q})\backslash X(V_1) \times GU(\mathbf{A}^f)/K.$$

Here K_∞ is the product of a maximal compact subgroup of $GU(\mathbb{R})$ with its center and $X(V_1)$ can be identified with the hermitian symmetric space $GU(\mathbb{R})/K_\infty$, and Sh_K is endowed with its canonical model. The constructions in [K2] allow us to idenfity Sh_K with (a part of) the set of complex points of a moduli space of abelian varieties of PEL type. This is determined as follows. Let C_1 be the matrix algebra $End_{\mathcal{K}}(V_1)$, viewed as a simple $\mathbb{Q}$-algebra, and let $*$ be the anti-involution on C_1 defined by the hermitian form on V_1. Thus GU is the algebraic group over $\mathbb{Q}$ whose points in any $\mathbb{Q}$-algebra R are given by

$$\{x \in C_1 \otimes_{\mathbb{Q}} R \mid xx^* \in R^\times\}.$$

Let $h_1 : \mathbb{C} \to GU(\mathbb{R})$ be a homomorphism such that $h_1(\bar{z}) = *(h_1(z))$ such that the symmetric real-valued bilinear form $(v, h_1(i)w)$ on $V_1(\mathbb{R})$ is positive definite. Writing

$$V_1(\mathbb{R}) \xrightarrow{\sim} \otimes_{w \in \Sigma} V_{1,\sigma} = \otimes_{w \in \Sigma} V_1 \otimes_{\mathcal{K}, w} \mathbb{C}, \tag{3.1}$$

it follows from our choice of Σ above that we can take

$$h_1(z) = (diag(z, \bar{z}I_n), \bar{z}I_{n+1}, \dots, \bar{z}I_{n+1}). \tag{3.2}$$

We follow [K2] in defining the Shimura variety $Sh(V_1)$ to be that attached to the Shimura datum $(GU, X(V_1))$ where $X(V_1)$ is the conjugacy class of h_1^{-1}. In particular, $X(V_1)$ is not the conjugacy class of h_1, which was used mistakenly in [K1]. However, it is important to point out that the formula (19.6) of the number of fixed points in [K2] does yield the formula (mistakenly applied to $Sh(G, h_1)$) that is used as the basis of the stabilization in [K1], which is the basis of the calculations in the following section. Thus (3.2) is still the basis for the calculations in our §4.3.

As hermitian symmetric space $X(V_1)$ can be identified with the unit ball in $\mathbb{C}^{n-1}$. The data C_1 and h are used in [K2] to define a moduli space of abelian varieties of PEL type, denoted simply S_K, which is smooth and projective over $Spec(\mathcal{O}_{\mathcal{K},p})$ provided p is unramified in $\mathcal{K}$ and K_p is a hyperspecial maximal compact subgroup. Then $S_K(\mathbb{C})$ can be identified with a disjoint union of $|ker^1(\mathbb{Q}, GU(V_1)|$ copies of $Sh_K(\mathbb{C})$.

We also introduce the similitude group $GU(V_2)$, defined as in (1.1) with V_1 replaced by the hermitian space V_2 of [CHL.IV.B, Lemma 1.4.3]. We can define a Shimura datum $(GU(V_2), X_2)$ and obtain a Shimura variety $Sh(V_2)$ whose reflex field is again identified with the image of $\mathcal{K}$ under w_0. Now $\dim V_2 = n$, the signatures are determined by

(1) $V_2 \otimes_{\mathcal{K},w_0} \mathbb{C}$ is of signature $(2, n-2)$, whereas
(2) $V_2 \otimes_{\mathcal{K},w} \mathbb{C}$ is of signature $(0, n)$ for $w \in \Sigma$, $w \neq w_0$.

and the homomorphism h_1 is replaced by a homomorphism h_2 which, in appropriate coordinates and with respect to the decomposition analogous to (3.1), is given by

$$h_2(z) = (diag(zI_2, \bar{z}I_{n-2}), \bar{z}I_n, \dots, \bar{z}I_n). \tag{3.3}$$

In what follows we follow Kottwitz in determining the unramified factors of the zeta function of $Sh(V_1)$ associated to certain parameters for $GU(V_1)$, with the goal of determining the corresponding ℓ-adic Galois representations at almost all unramified places. The corresponding analysis for $Sh(V_2)$ will be carried out in the second book of this series, where it will be used to study p-adic Hodge-theoretic properties of the automorphic Galois representations constructed by p-adic analytic continuation of the representations constructed here.

4. Kottwitz' formula for Shimura varieties attached to anisotropic unitary groups

4.1 Preliminaries. Let $W_\infty = W(\Pi)$ and W_+ be as in §2.3. We use the same notation to denote an extension to $GU(\mathbb{R})$. This representation has a rational model over some number field $E(W)$. Let λ denote an ℓ-adic place of $E(W)$ and let W_λ denote the corresponding local system over $Sh(V_1)$, as in [K1] (where it is called $\mathcal{F}_\lambda$).

Let p be a rational prime such that K_p is a hyperspecial maximal compact subgroup of $GU(\mathbb{Q}_p)$. Choose a prime v of $\mathcal{K}$ – viewed as the reflex field of $Sh(V_1)$ – dividing p. In §10 of [K1] Kottwitz obtains what he calls a "conjectural description" of the cohomology

$$H^\bullet(Sh_K, W_\lambda) = \bigoplus_i (-1)^i H^i(Sh_K, W_\lambda)$$

where the cohomology in question is étale (ℓ-adic) cohomology and the description is as a representation of the group $Gal(\bar{\mathcal{K}}_v/\mathcal{K}_v) \times \mathcal{H}^p_K$, where $\mathcal{H}^p_K$ is the Hecke algebra of K-biinvariant functions on $GU(\mathbb{A}^p_f)$ (adeles trivial at p). Several remarks are in order.

4.2. Remarks.

(4.2.1) Kottwitz' conjectural description applies to the intersection cohomology $IH^\bullet$ of a compactification of Sh_K rather than the cohomology, but since Sh_K is (smooth[6] and) proper there is no difference. One of his conjectures concerns the validity of an extension of his calculations to the boundary of the Baily-Borel compactification of the Shimura variety under consideration and the relation to the non-elliptic terms in the trace formula for GU. This conjecture, which has been verified by S. Morel in a number of cases, is vacuous in the present situation. In particular, where Kottwitz refers to "the contribution of $P = G$" to the cohomology, the reader should bear in mind that in our situation there are no proper parabolic subgroups and $P = G$ represents the only contribution.

(4.2.2) The article [K1] is written for general Shimura varieties. In §3 of [K1] Kottwitz proposes a conjectural expression for the trace of powers of Frobenius, twisted by Hecke operators, based on an axiomatic description of the points on the Shimura variety over finite fields. This expression is proved for Shimura varieties of PEL types A and C in [K2]. In the case at hand, this should be sketched in Book 2.

(4.2.3) In sections 4-7 of [K1] Kottwitz stabilizes this expression, assuming the validity of the fundamental lemma. Actually, Kottwitz assumes two fundamental lemmas: the fundamental lemma for cyclic base change (invoked in the discussion preceding [K1,(7.2)]; cf. [Morel,Lemme 10.4.7] for the transfer factors) and the fundamental lemma for endoscopy (invoked in the discussion preceding [K1, (7.1)]). These are now proved ([C,L, CL] for the first; [LN, N] for the second). This portion of Kottwitz' discussion should be described in more detail, taking these more recent developments into account, in Book 2.

(4.2.4) For our purposes, the final conjectural ingredient in Kottwitz' formula is Arthur's formula for the multiplicities of representations of $GU(\mathbb{A})$ in the discrete spectrum of automorphic forms on GU. This is still conjectural at the time of writing, though it may not be by the time this book appears in print. However, for the test functions we have in mind, a substitute for Arthur's multiplicity conjectures is provided by the explicit computation of the multiplicities for the parameters introduced in §2.1, specifically (2.1.1) and (2.1.2).

(4.2.5) In [K1, (9.6)], Kottwitz invokes a conjectural expression in terms of local (spectral) transfer factors for the (conjectured) pairing between elements of a global L-packet with elements in the corresponding component group $\mathfrak{S}_\psi$ (defined below). In our setting the global L-packets are defined in terms of local L-packets at the real prime, and our pairing is given by Shelstad's spectral transfer factors, so this conjecture is superfluous for us.

[6] This is true for K sufficiently small, which we can assume by taking K_p torsion free for some p split in in E; this is compatible with the standing hypotheses of [CHL.IV.B, 2.2].

4.2.6 Notation for global parameters. As indicated in (4.2.4) above, we work with the parameters $\psi = \{[\Pi_1], \dots, [\Pi_r], \xi\}$ of §2.1. In the applications, we only consider $r = 1$ (the *stable case*), in which case $\mathfrak{S}_\psi$ is trivial, or $r = 2$ (the *endoscopic case*), with $\mathfrak{S}_\psi = \mathbb{Z}/2\mathbb{Z}$. In order to be consistent with the previous chapters, we write $[\Pi]$ instead of $[\Pi_1]$ in the stable case and $([\tau], [\chi])$ instead of $\{[\Pi_1], [\Pi_2]\}$ in the endoscopic case, with τ cuspidal and θ_n^*-stable on $GL(n)$ and χ a θ_1^*-stable Hecke character of $GL(1)$. Our special hypotheses are in force here and prescribe that $[\Pi]$, $[\tau]$, and $[\chi]$ all be spherical at places inert in $\mathcal{K}/F$.

4.3. Translation of Kottwitz' formulas.

We return to the notation of the earlier sections; thus $G = U(V_1)$ with $\dim V_1 = n + 1$. The results of the two previous chapters, together with the discussion of §1, provide unconditional spectral expressions for the traces of our test functions. The remainder of this section reviews the calculations in §10 of [K1] using the spectral parameters introduced in (4.2) in place of Arthur's conjectural multiplicity formulas. The reader should not be surprised to learn that, with this interpretation of Arthur's ψ, our calculations are consistent with his multiplicity conjectures.

Let f be a test function on GU. Then [K1, (8.3)] writes Arthur's conjectural spectral expression for the trace of f in the discrete spectrum as follows

$$T_{disc}(f) = \sum_{[\psi]} |\mathfrak{S}_\psi|^{-1} \sum_{\pi \in [\psi]} \sum_{x \in \mathfrak{S}_\psi} \epsilon_\psi(xs_\psi) < xs_\psi, \pi > tr\ \pi(f) \tag{4.3.1}$$

We write $[\psi]$ instead of $\Pi(\psi)$ for the packet defined by ψ, the letter Π being otherwise employed. Instead of ψ our parameters are pairs $([\Pi], \xi)$ or $([\tau], [\chi], \xi))$ as described in (4.2.3). For uniformity we write Π in both cases when this is convenient; in the endoscopic case we have $\Pi = \Pi(\tau, \chi)$. We fix a sufficiently large set S of rational primes, outside which all our data are unramified, and we choose a function f_S that has trace zero on any representation ρ_S such that $\rho_\infty \otimes \rho_S \otimes \rho^S$ is automorphic for $GU(\mathbb{A})$ for any ρ^S and $\rho_S \neq \Pi_S$; this is possible by the arguments of [CHL.IV.B] and the multiplicity one property. We also assume *trace* $\Pi_S(f_S) = 1$. We then consider functions on $GU(\mathbf{A}^f)$ of the form $f_f = f_S \otimes f^S$ where f^S is a tensor product of functions unramified for the hyperspecial compact subgroup (for all $v \notin S$). For all but at most finitely many p, it is possible to choose f_f in such a way with the local component f_p equal to the characteristic function of the chosen hyperspecial maximal compact at p. This last condition is required for the validity of Kottwitz' formulas, and so we assume p is not in the (finite) set of primes for which it may be impossible to realize this condition.

Moreover, we choose f_∞ to eliminate cohomology outside the middle dimension, or more precisely we only consider automorphic representations with cohomology in regular coefficient systems; this is regularity in the sense of Special Hypotheses 2.3 (i). In particular, our ψ's are all parameters that should be tempered, and in particular are trivial on the $SL(2)$-factor, the terms s_ψ and ϵ_ψ of Arthur's formula are irrelevant, and (4.3.1) becomes

$$T_{disc}(f) = \sum_{(\Pi,\xi)} |\mathfrak{S}_{[\Pi],\xi}|^{-1} \sum_{\pi \in ([\Pi],\xi)} \sum_{x \in \mathfrak{S}_{[\Pi],\xi}} < x, \pi > tr\pi(f) \tag{4.3.2}$$

As explained in (4.2.3), we set $\mathfrak{S}_{[\Pi],\xi} = 1$ if Π in the stable case and $\mathfrak{S}_{[\tau],[\chi],\xi} = \{\pm 1\}$ in the endoscopic case. In our setting, if ψ is a parameter factoring through

$\hat{H}$ for an endoscopic group H then S_ψ is the group of θ-invariants in $Z(\hat{H})$ modulo the center of $\widehat{GU}$, as already seen. Then the pairing $< x, \pi >$ for $BC(\pi) = (\tau \times \chi, \xi)$ should in fact be $< x, \pi_\infty >$, again by our choice of f_f. We can define $< x, \pi_\infty >$ in terms of the transfer factors.

There is a similar expression to (4.3.2) for the discrete trace of test functions on an endoscopic group H, cf. [K1]. Bear in mind that $\Pi = \Pi(\tau, \chi)$ occurs both in the term for GU and in the term for H where H is isogenous to $U(n)^* \times U(1)$. In the expression for the stable trace of H, $([\tau], [\chi], \xi)$ is a stable parameter ψ_H, say, which we denote $([\tau], [\chi], \xi)_H$ so

$$\mathfrak{S}_{\psi_H} = \mathfrak{S}_{([\tau],[\chi],\xi)_H} = 1. \tag{4.3.3}$$

Finally, Kottwitz' (conjectural) expression for the stable trace is a simplified version of (4.3.1) [K1, (8.5)]. Applying the additional simplifications of (4.3.2) and (4.3.3) we find simply

$$ST_{disc}(f) = \sum_{([\Pi],\xi)} \sum_{[\pi]\in([\Pi],\xi)} tr\pi(f). \tag{4.3.4}$$

Kottwitz' expression for the Lefschetz trace [K1, (10.1)] is the following. Let v be a place of $\mathcal{K}$ dividing a rational prime p where data are unramified. Then if $K_f = \prod K_v$ is hyperspecial at v, $Sh(GU)_{K_f} := S_{K_f}$ has good reduction at v [K2], and

$$trace(Frob_v^\alpha \times f_S \mid H^\bullet_{et}(S_{K_f}, \bar{\mathbb{Q}}_\ell))$$

is given by

$$\sum_{\mathcal{E}} \iota(GU, H) ST^H_{disc}(h) \tag{4.3.5}$$

where $h = h^p h_p^\alpha h_\infty$ is a certain test function on H with $h^p = f_f^H$ in our previous notation (where $f_f = f_S \times f_0^{S,p}$, f_S as above and f_0^S the unit of the Hecke algebra), h_∞ determined by Shelstad's character formulas, and h_p is the key to Kottwitz' approach to the zeta function. We have added a superscript ^{H} in (4.3.5) for the sake of clarity. Kottwitz then proceeds to rewrite this sum in terms of Arthur's ψ-packets, using Arthur's conjectures, finally deriving (4.3.1). In our case this is easy and has in essence been done in [CHL.IV.B], so we will simply bypass formula (4.3.1) and exhibit its analogue directly. What is important is that the expression (4.3.5) for the trace for Frobenius is *unconditional*.

First, the *geometric* expression for the trace [K1, 3.1] was proved by Kottwitz in [K2]. The fact that it implies (4.3.5) is proved in [K1,§7, Theorem 7.2]. One only has to check the assumptions of local harmonic analysis used in this section, which have now been verified (cf. (4.2.3)). What we now have to do is to check (4.3.4) for the H relevant to us, without relying on the arguments in [K1].

Thus, we first verify Kottwitz' formula (4.3.4) for the stable contribution.

Lemma 4.3.6. *Let Π be a cuspidal automorphic representation of $GL(n+1)_{\mathcal{K}}$ satisfying Special Hypotheses 2.2 and the strong regularity hypothesis 1.2.3(ii).*

Then the conjectural formula (4.3.4) is valid provided f_∞ is a very cuspidal Euler-Poincaré function or pseudocoefficient for the discrete series L-packet $([\Pi],\xi)_\infty$ of GU attached to $([\Pi_\infty],\xi_\infty)$ and f_f is biinvariant under the level subgroup K.

Proof. This is just a reformulation of Labesse's results in §5 of [L.IV.A] and their extension in §1.2 above. Indeed, under our hypotheses, the set of $[\pi]\in([\Pi],\xi)$ with non-zero trace on f is of the form $\pi_\infty\otimes\pi_f$ where π_∞ belongs to the discrete series L-packet $([\Pi],\xi)_\infty$; the strong regularity hypothesis guarantees that there are no non-tempered cohomological representations with non-zero trace under f_∞. Then (4.3.4) is a restatement of Labesse's multiplicity one theorem, or more precisely our extension 1.2.4 of this multiplicity one theorem, in the present setting.

The final step in Kottwitz' remarkable paper consists in rewriting (4.3.5) for the Frobenius trace as an expression (which, obviously, will not be stable) in terms of automorphic representations of $GU(\mathbb{A})$. This is the formula (10.3) in [K1]. It is deduced there from Arthur's conjectural expression (8.1)[7] for multiplicities of representations π of $GU(\mathbb{A})$ in $L^2(A_{GU}GU(\mathbb{Q})\backslash GU(\mathbb{A}))$, as well as the similar expression (8.5) for endoscopic groups. We can use Kottwitz' arguments as soon as (8.1) is verified in our situation, and in our terms which do not depend on conjectures.

Consider first the odd-dimensional case, so ψ is a pair $([\Pi],\xi)$ as in (4.2.6). In this case Kottwitz' (8.1) simply says the contribution of π_f, in the notation of Corollary 1.2.5, is the sum of the $\pi'_\infty\otimes\pi_f$ with multiplicity 1. This is true by [L.IV.A], cf. Proposition 1.2.4 and Corollary 1.2.5.

We choose $f=f_S\otimes f^S$ on $GU(\mathbb{A})$ as explained after (4.3.1). By the arguments in [L.IV.A] and [CHL.IV.B], the proper endoscopic contributions to (4.3.5) are zero.

Since p is not a prime of ramification, we have an unramified parameter $\phi_p : W_{\mathbb{Q}_p}\to {}^LGU$ for the representation π_p.[8] With the preceding remark, and letting $v\mid p$ as above, Kottwitz' expression of the Frobenius trace is simply

$$trace(Frob_v^\alpha\times f_S \mid H^\bullet_{et}(S_{K_f},\bar{\mathbb{Q}}_\ell)) = C(Sh)tr(r_\mu(\phi_p)(Frob_v^\alpha))). \tag{4.3.7}$$

Here

$$C(Sh)=(-1)^{\dim Sh}Nv^{\frac{\dim Sh}{2}},$$

and the representation r_μ of ${}^LGU$ is associated to the datum μ obtained from Shimura's $h:\mathbb{C}^\times\to GU(\mathbb{R})$ defining the Shimura variety (cf. §3). We recall this construction in our case. The group $GU(\mathbb{R})$ is isomorphic to the subgroup of $GU(1,n)\times GU(n+1)^{d-1}$ of d-tuples with equal similitude factor, where $d=[F:\mathbb{Q}]$. We have taken $GU(1,n)$ consistently with the computations in [CHL.IV.B,§3], relying on [C.III.A]. As in §3, we take

$$h(z)=(diag(z,\bar zI_n),\bar zI_{n+1},\dots,\bar zI_{n+1};z\cdot\bar z),$$

where we have added an extra $\mathbb{C}^\times$ term at the end for consistency with the computation in the L-group. (Of course, the last term is determined, for $h:\mathbb{C}^\times\to G(\mathbb{R})$, by the similitude factor.) Now, as in §1.3,

$$GU(\mathbb{C})\simeq GL(n+1,\mathbb{C})^d\times\mathbb{C}^\times$$

[7] Otherwise unexplained formula numbers and notation refer to [K1].

[8] If we restrict our attention to p split in the imaginary quadratic field E, we can separate the contributions of the various places v of $\mathcal{K}$ dividing p, as in [HT]. In particular, we can treat v for which Π_v is unramified even if $\Pi_{v'}$ is ramified for some other v' dividing p.

(with the similitude factor included) and the parameter μ deduced from h is the cocharacter $\mathbb{G}_m \to GU$, defined over $\mathbb{C}$ or $\overline{\mathbb{Q}}$, given by

$$x \mapsto (diag(x, I_n), I_{n+1}, \ldots, I_{n+1}; x).$$

If we recall that the reflex field of Sh is $\mathcal{K}$, we see that this defines a representation of ${}^L GU$ over $\mathcal{K}$, simply given, in the description of §1.3, by

$$(g_\alpha, \lambda) \mapsto g_{\alpha_0} \otimes \lambda \in GL(n+1, \mathbb{C}).$$

(Recall that $\mathcal{K}$ is embedded in $\mathbb{C}$ as a reflex field. We take $\overline{\mathbb{Q}}$ equal to the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and then g_{α_0} is defined in terms of this identification.)

We are now able to prove Theorem 2.2.3 in the odd-dimensional case.

Proposition 4.3.8. *There is a compatible system $r_\lambda(\xi)$ of λ-adic characters of $\mathcal{K}$ such that the representations*

$$\rho_{\Pi,\lambda} = (\rho_{\Pi,\xi,\lambda})^\vee \otimes r_\lambda(\xi),$$

with $\rho_{\Pi,\xi,\lambda}$ as in Proposition 2.3.3, satisfy the conclusions of Theorem 2.2.3.

Proof. Lemma 4.3.6 is the verification that formula (4.3.7) is correct in the stable case. In view of formula (4.3.7), the proof is identical to that of [C1], with the additional simplification that the multiplicity $a = 1$ in [C1, (5.12)]. The character $r_\lambda(\xi)$ is computed explicitly in [HT] in the proofs of Theorems VII.1.3 and VII.1.9, where it is shown that $r_\lambda(\xi) = rec(\xi^c)$ (the character ψ of [HT] is here denoted ξ) with rec the notation of [HT] for the global reciprocity map associating ℓ-adic characters to Hecke characters.

4.4 Proof of Theorem 2.2.3.

We now describe Kottwitz' formula applied to a parameter $\psi = ([\tau], [\chi], \xi)$ in the endoscopic case (cf. (4.2.6)). Here τ is our given θ_n-stable representation of $GL(n, \mathbb{A}_\mathcal{K})$, and n is even. The choice of f cuts out the contribution in the stable discrete trace formula. This has been described in [CHL.IV.B,§4]. Arthur's multiplicity formula, stated by Kottwitz as [K1,(8.1)], has been proved as Theorem 4.6 of [CHL.IV.B], and extended to the similitude group in Proposition 1.2.6, for our standing choice of f_S. The formula is simply

$$\begin{aligned} m(\pi_\infty \otimes \pi_f) &= \frac{1}{\mathfrak{S}_\psi}[< \pi_\infty, 1 > + < \pi_\infty, x >] \\ &= 0 \text{ or } 1. \end{aligned} \tag{4.4.1}$$

The integer $m(\pi_\infty \otimes \pi_f)$ is the multiplicity in $L^2(GU(\mathbb{Q})\backslash GU(\mathbb{A}))$. As described above, $\mathfrak{S}_\psi \simeq \{\pm 1\}$, and it can be seen as the group generated by

$$s'_{n,1} = ((diag(I_n, -1)), 1),$$

cf. §1.3. In (4.4.1),π_∞ runs over the L-packet $\Pi_\infty^+ \cup \pi_\infty^-$ of Proposition 1.2.6. Formula (4.4.1) means that

$$\begin{aligned} &< \pi_\infty, 1 > = 1 \\ &< \pi_\infty, x > = \pm 1 \ (\pi_\infty \in \Pi_\infty^\pm). \end{aligned}$$

For the compatibility with Kottwitz' computations, we note that this coincides with his formula (9.6):

$$< \pi, x >= \Delta_f(\psi_H, \pi_f) \cdot \Delta_\infty(\psi_H, \pi_\infty). \tag{4.4.2}$$

Here the finite factor $\Delta_f(\psi_H, \pi_f)$ is equal to 1, because all data are unramified or $GL(n)$, whereas $\Delta_\infty(\psi_H, \pi_\infty)$ is computed in [C.III.A] and [CHL.IV.B,§3.3].

Using only Kottwitz' geometric formula (4.3.5), Arthur's formula (4.4.1), combined with (4.4.2), and the fact that the only contributions of (4.3.5) for our function f are given by GU and H, we obtain

(4.4.3) $trace(Frob_v^\alpha \mid H^\bullet)$
$$= \frac{1}{2}C(Sh)[trace\ r_\mu(\phi_p(Frob_v^\alpha) + < x, \pi >< \lambda_{\pi_\infty}, x > trace\ r_\mu(\phi_p(x \cdot Frob_v^\alpha))],$$

cf. [K1,(10.3)]. The representation r_μ has already been defined. The coefficient

$$c_x =< x, \pi >< \lambda_{\pi_\infty}, x > \tag{4.4.4}$$

is shown by Kottwitz to be independent of the representation $\pi = \pi_\infty \otimes \pi_f$ where π_f is fixed. We now have to compute c_x.

Lemma 4.4.5. *The coefficient* $c_x = 1$.

4.4.6. Proof of Lemma 4.4.5. Note that in (4.4.4), $< x, \pi >$ depends only on (real) representation theory, whereas $< \lambda_{\pi_\infty}, x >$ depends on the choice of data for the Shimura variety. We have $< x, \pi >=< \pi_\infty, x >$, and its value has been described above.

The pairing $< \lambda_{\pi_\infty}, x >$ is defined in §7 of [K1]. We have to return to the calculations of [C.III.A] and the description of the L-packet $\{\pi_\infty\}$. We work at the place $v_0 \mid \infty$ where G is of signature $(1, n)$. In [C.III.A,§3] and [CHL.IV.B], we have taken $G_{v_0} = U(1, n)$. For the purposes of this paragraph, we let GU denote the full similitude group, so that

$$GU(\mathbb{R}) = GU_{v_0} \times \prod_{v \neq v_0} GU(n+1);\ GU_{v_0} = GU(1, n).$$

The additional split torus factors do not affect the calculation. We have chosen an elliptic torus $T \subset G_{v_0}$, the diagonal torus, which defined $\tilde{T} \subset GU$. The representations $\pi_\infty \in \Pi_\infty$ were given by

$$\pi = \pi(\phi, \sigma^{-1}B_0),$$

where B_0 is the standard Borel subgroup in $G(\mathbb{C}) = GL(n, \mathbb{C})$ and $\sigma \in \Omega/\Omega_{\mathbb{R}} = \mathfrak{S}_{n+1}/\mathfrak{S}_1 \times \mathfrak{S}_n$. In [CHL.IV.B,§3], we had used the natural bijection

$$\Omega/\Omega_{\mathbb{R}} \leftrightarrow \{1, \ldots, n+1\};\ \sigma \mapsto \sigma \cdot 1.$$

On the other hand, we have (relative to the place v_0) $\hat{G} = GL(n+1, \mathbb{C})$; we denote by $(\hat{B}, \hat{S})$ the standard Borel subgroup and maximal torus. Given $B = \sigma^{-1}B_0$, there is a unique isomorphism $j_B : \hat{T} \xrightarrow{\sim} \hat{S}$ sending the positive coroots for (B, T) to the positive roots for $(\hat{B}, \hat{S})$. Identifying both $\hat{T}$ and $\hat{S}$ with $(\mathbb{C}^\times)^{n+1}$, we see that for $B = \sigma^{-1}B_0$,

$$j_B : t \mapsto \sigma(t) \ (t \in (\mathbb{C}^\times)^{n+1}).$$

Now Shimura's parameter for GU – hence for GU_{v_0} – is given by

$$z \mapsto diag(z, \bar{z}I_n) \in \tilde{T}, z \in \mathbb{C}^\times.$$

This defines in the usual manner a 1-parameter group

$$\mu : \mathbb{G}_m \to \tilde{T}/\mathbb{C}, \ \tilde{T}(\mathbb{C}) = (\mathbb{C}^\times)^{n+1} \times \mathbb{C}^\times$$
$$x \mapsto (x, I_n, x).$$

Kottwitz defines $< \lambda_{\pi_\infty}, x >$ as equal to $< j_B \circ \mu, x >$ for the natural identification between $X_*(\tilde{T}/\mathbb{C})$ and $X^*(\widehat{\tilde{T}})$. Here $\mu = (1, 0, \dots, 0, 1) \in X_*(\tilde{T}) \simeq \mathbb{Z}^{n+2}$ and $j_B \circ \mu = \sigma\mu = (\varepsilon_i, 1) \in \mathbb{Z}^{n+2}$, where

$$\varepsilon_i = 1, \ i = \sigma \cdot 1$$
$$= 0, \ i \neq \sigma \cdot 1.$$

Finally, since $x = (1, \dots, -1, 1)$, we obtain

$$< \lambda_{\pi_\infty}, x > = 1, \ i \leq n$$
$$= -1, \ i = n + 1.$$

Comparing this to our previous expression for $< x, \pi > = < \pi_\infty, x >$, this completes the proof of Lemma 4.4.5. We also see that the similitude factor plays no role in the calculation, justifying our decision to work with the full similitude group.

We finally obtain

Lemma 4.4.7. *There is a compatible system $r_\lambda(\xi, \mu_1)$ of λ-adic characters of $\mathcal{K}$ such that, for all p outside a finite set, the representations*

$$\rho_{\tau,\lambda} = (\rho_{\tau,\xi,\lambda} \otimes r_\lambda(\xi, \mu_1)^{-1})^\vee,$$

with $\rho_{\tau,\xi,\lambda}$ as in Proposition 2.3.3, satisfy the conclusions of Theorem 2.2.3.

Proof. Let $r_\mu^\pm$ be the representation of the commutator ${}^L H$ of $x = x_{n,1}$ in ${}^L GU$ on the $\pm$-eigenspace of x. It follows immediately from Lemma 4.4.5 and the expression (4.4.3) for $trace(Frob_v^\alpha \mid H^\bullet)$ that there is a sign $* \in \pm$ such that, for all p outside a finite set, the expression (4.4.3) is equal to $tr(r_\mu^*(\psi_{\pi_p}(Frob_v^\alpha)))$.

Recall that π_p is an extension to $GU(\mathbb{Q}_p)$ of an irreducible representation of $G(\mathbb{Q}_p) = \prod_{v \mid p} G(F_v)$ whose v component is the representation $\pi_v(\tau_v, \chi_v)$ defined in [CHL.IV.B]. We can identify an unramified local Langlands parameter ψ_p for $R_{F/\mathbb{Q}}G$ at p with the collection of unramified local Langlands parameters $(\psi_v, v \mid p)$ of G. Then the local Langlands parameter ψ_{π_p} defines by the map $\widehat{GU} \to \hat{G}$ the collection $(\psi_v = \psi_{\xi_{\mu_n,\mu_1}} \circ \psi_{\tau_v,\chi_v})$ of parameters of $\pi_v(\tau_v, \chi_v)$ (cf. [H.I.A, (5.5.4)]; this is completed by [H.I.A, (5.5.5)] if p does not split in E). The element $r_\mu(x_{n,1}$

commutes with these parameters, by construction, and its eigenvalue decomposition is the decomposition of the representation space of r_μ as the sum

$$r_\mu^+ \circ \psi_{\tau_v} \otimes \mu_1 \oplus r_\mu^- \circ \psi_{\chi_v} \otimes \mu_n.$$

Now if $* = +$, we have

$$tr(\rho_{\tau,\xi,\lambda}(Frob_v^j) = tr(r_\mu^+ \circ \psi_{\tau_v} \otimes \mu_1)(Frob_v^j),$$

and the remainder of the argument follows as in the proof of Proposition 4.4.1. If $* = -$, on the other hand, we have

$$tr(\rho_{\tau,\xi,\lambda}(Frob_v^j) = tr(r_\mu^- \circ \psi_{\chi_v} \otimes \mu_n)(Frob_v^j),$$

which is the Galois representation attached to a twist, depending on ξ (cf. Proposition 4.3.8), of the Hecke character $(\chi \otimes \mu_n)^{-1}$. But $\rho_{\tau,\xi,\lambda}$ is an n-dimensional representation, hence its trace on a dense subset of $Gal(\overline{\mathbb{Q}}/\mathcal{K})$ cannot equal the value of an abelian twist of a Hecke character. This completes the proof.

4.5. Hodge-Tate numbers of automorphic Galois representations.

In this section we complete the proof of Theorem 2.2.3 by verifying condition (ii) of Conjecture 2.2.2, always assuming strong regularity of Π_∞ as well as Special Hypotheses 1.2.2. We first need to state condition (ii) explicitly.

Fix a prime λ of the coefficient field $E(\pi)$, say of residue characteristic ℓ. Up to twisting by the ℓ-adic realization of an algebraic Hecke character – namely $r_\lambda(\xi)^{-1}$ if $r = 1$, $r_\lambda(\xi, \mu_1)^{-1}$ if $r = 2$ – the automorphic Galois representation $\rho_{\Pi,\lambda}$ constructed above is realized in the cohomology of the geometric λ-adic local system $\tilde{W}(\Pi)_\lambda$ on a Shimura variety, obtained in a standard way from the finite-dimensional representation $W(\Pi)$ introduced above. It is therefore of geometric type, in the sense of Fontaine and Mazur: each $\rho_{\Pi,\lambda}$ is unramified outside a finite set of places of $\mathcal{K}$, and at every place dividing the residue characteristic ℓ of λ, $\rho_{\Pi,\lambda}$ is de Rham. The latter fact is a consequence of the comparison theorems of p-adic Hodge theory. In particular the Hodge-Tate numbers can be read off from the Hodge numbers of the de Rham cohomology of the flat vector bundle $\tilde{W}(\Pi)$ associated to $W(\Pi)$. The comparison of $\tilde{W}(\Pi)_\lambda$ and $\tilde{W}(\Pi)$, and therefore the determination of the Hodge-Tate numbers from the highest weights $\mu(\sigma)$ of W_σ, presupposes a dictionary relating complex and ℓ-adic places of $\mathcal{K}$. In [HT] this is given by an isomorphism $\alpha : \bar{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ (in [HT] the isomorphism is denoted ι, but that letter is already used for a complex embedding of E). For what follows it suffices to identify the algebraic closure of $\mathbb{Q}$ in $\bar{\mathbb{Q}}_\ell$ with the field of algebraic numbers in $\mathbb{C}$. Then the ℓ-adic embeddings of $\overline{\mathbb{Q}}$, and in particular of $\mathcal{K}$, are identified with the complex embeddings; if s is an embedding of $\mathcal{K}$ in $\bar{\mathbb{Q}}_\ell$, we write αs for the corresponding complex embedding.

Let s be an embedding of $\mathcal{K}$ in $\bar{\mathbb{Q}}_\ell$, and let $D_{dR,s}$ denote Fontaine's functor from representations of $\Gamma_s = Gal(\bar{\mathbb{Q}}_\ell/\overline{s(\mathcal{K})})$ to filtered $\bar{\mathbb{Q}}_\ell \otimes_{\mathbb{Q}_\ell} E(\Pi)_\lambda$-modules:

$$D_{dR,s}(R) = (R \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_s}.$$

The Hodge-Tate numbers of R (with respect to v) are the j such that $gr^j D_{dR,s}(R) \neq (0)$. Then in the situation of Theorem 2.2.3, the Hodge-Tate numbers of $\rho_{\Pi,\lambda}$ with respect to s are the j of the form

$$j = i - \mu_{n-i}(\alpha(s)),\ i = 0, \ldots, n-1. \tag{4.5.1}$$

This is to be compared to part 4 of Theorem VII.1.9 of [HT]; we have replaced the index p there (an unfortunate misprint gives p two incompatible meanings) by i.

Scholium 4.5.2. *The set $HT(\Pi,\lambda)$ of Hodge-Tate numbers of $\rho_{\Pi,\lambda}$ satisfies the following properties:*

1. *The set $HT(\Pi,\lambda)$ depends only on the identification α of complex and ℓ-adic places and on the archimedean components Π_∞, ξ_∞, $\mu_{1,\infty}$.*
2. *The choice of $\mu_{1,\infty}$ can be made independently of Π. The choice of ξ_∞ depends only on the central character of the representation $W(\Pi)$.*
3. *Let ρ_1 and ρ_2 be two continuous λ-adic representations of $Gal(\overline{\mathbb{Q}}/\mathcal{K})$ such that, for almost all primes v prime to ℓ,*

$$Tr(\rho_1(Frob_v)) = Tr(\rho_2(Frob_v)).$$

Then the Hodge-Tate numbers of ρ_1 and ρ_2 coincide at all primes dividing ℓ

Proof. Point (2) is easy to check, and point (3) is a simple consequence of Chebotarev density, since the Hodge-Tate numbers of a continuous representation R depend only on its semisimplification.

Point (1) is a bit more subtle. First consider the stable case, and let $\{\pi\}$ be the L-packet for GU indexed by $([\Pi],\xi)$. The calculation of the Hodge-Tate numbers for $Sh(V_1)$ is identical to that for the simple Shimura varieties considered in [HT]. The formula in Proposition III.2.1.6 of [HT]:

$$\dim gr^j D_{DR,\sigma}(R^i_\xi(\pi)) = \#ker^1(\mathbb{Q}, G_{\iota\sigma}) \sum_{\pi_\infty} m_{\iota(\pi)\otimes\pi_\infty} d(\pi_\infty), \tag{4.5.3}$$

where

$$d(\pi_\infty) = \dim H^{i-p}(LieQ_{\iota\sigma}, U_{\iota\sigma}, \pi_\infty \otimes \mu^p(\iota\xi))$$

and the other notation (different from the notation used here) is explained in [HT], shows that $HT(\Pi,\lambda)$ depends only on α and on the set of representations π_∞, which in turn depends only on Π_∞.

In the endoscopic case, we let $\tau = \Pi$, choose χ as above, and let $\{\pi\}$ be the corresponding L-packet for GU. The formula (4.5.3) remains valid and shows that $HT(\Pi,\lambda)$ depends on α (denoted ι in (4.5.3) and on the set of representations π_∞ with non-trivial multiplicity $m_{\iota(\pi)\otimes\pi_\infty}$ in the L-packet $([\tau],[\chi],\xi)$. Our construction above shows that this depends in turn exclusively on $\tau_\infty = \Pi_\infty$ and χ_∞. But our choice of χ_∞ also depends only on Π_∞. So (1) is true in the endoscopic case as well.

It is now easy to verify condition (ii) of Conjecture 2.2.2. It follows from (1) and (2) of (4.5.2) that, in order to determine $HT(\Pi,\lambda)$, it suffices to determine $HT(\Pi',\lambda)$ for any Π' satisfying they hypotheses of Theorem 2.2.3 with $\Pi'_\infty = \Pi_\infty$. In particular, we may assume Π' to satisfy the condition that, at some finite prime v_0, Π'_{v_0} is in the discrete series. Then [HT], following [C1] and [K3], constructs a Galois representation $\rho'_{\Pi',\lambda}$ satisfying the local-global compatibility (2.2.4) at almost all primes. By (3) of (4.5.2), this implies $\rho'_{\Pi',\lambda}$ and $\rho_{\Pi',\lambda}$ have the same Hodge-Tate weights. Condition (ii) thus follows from Theorem VII.1.9 of [HT].

4.5.4. The crystalline property. Finally, if Π_v is unramified at all primes dividing ℓ, and if ℓ is unramified in $\mathcal{K}$, it is claimed in Theorem 2.2.3 that $\rho_{\Pi,\lambda}\mid_{\Gamma_v}$ is crystalline. Indeed, in this case we can assume the compact open subgroup $K_\ell \subset$

$GU(\mathbb{Q}_\ell)$ is hyperspecial maximal, which implies that the Shimura variety $Sh(V_1)$ has good reduction at all primes dividing ℓ (cf. [K2], §5, where Kottwitz relies on previous results of Langlands-Rapoport and Zink). It follows by the crystalline conjecture in p-adic Hodge theory (with $p = \ell$, see [Fa], [Ts]) that any representation in the cohomology $H^*(Sh(V_1), \tilde{W}(\Pi))$ is crystalline; here we also use the fact that $\tilde{W}(\Pi)$ is obtained by tensor operations from the cohomology of an abelian scheme over $Sh(V_1)$, cf. [HT, Lemma III.4.2]. Moreover, we are free to choose our Hecke characters ξ and μ_1 unramified at primes dividing ℓ. Thus the λ-adic characters $r_\lambda(\xi)^{-1}$ and $r_\lambda(\xi, \mu_1)^{-1}$ are also crystalline. This completes the verification of the crystalline property as stated in Theorem 2.2.3.

The stronger property conjectured in Conjecture 2.2.2 (iii) requires a more careful analysis of the integral models of $Sh(V_1)$, and is deduced as in [HT] by Shin for the representations constructed above [S].

5. Removal of Special Hypotheses 1.2.2

Let now $\mathcal{K}/F$ be any CM quadratic extension of a totally real field, and let Π be an automorphic representation of $GL(n, \mathcal{K})$ satisfying Hypothesis 2.2.1. Let $\mathcal{I}$ be a set of cyclic Galois extensions K/F of prime degree q_K. We say $\mathcal{I}$ is *S-general* if, for any $v \notin S$ and any finite extension M/F, there is $K \in \mathcal{I}$ in which v splits completely which is linearly disjoint from M. See [So] for more details.

Proposition 5.1. *There is a finite set S of places of F and an S-general collection $\mathcal{I}$ of totally real quadratic extensions F_i/F such that, for each $F_i \in \mathcal{I}$, letting $\mathcal{K}_i = F_i \cdot \mathcal{K}$, Π_i the base change of Π to $\mathcal{K}_i$, the triple $(F_i, \mathcal{K}_i, \Pi_i)$ satisfies Special Hypotheses 2.2. Moreover, we can assume that, for every $v \in S$ and every $F_i \in \mathcal{I}$, either v splits in $\mathcal{K}/F$ or the unique extension of v to $\mathcal{K}$, denoted $v_\mathcal{K}$, splits in $\mathcal{K}_i$.*

Proof. Let S be the set of primes of v at which (2.2.1), (2.2.2), or (2.2.4) fails: either v ramifies in $\mathcal{K}/F$, or v stays prime in $\mathcal{K}$ and the corresponding component $\Pi_{v_\mathcal{K}}$ is ramified, or the residue characteristic of v is small. We take $\mathcal{I}$ to be the set of totally real quadratic extensions F_i/F with the property that, for all $v \in S$, $F_{i,v} \xrightarrow{\sim} \mathcal{K}_v$. It is obvious that this set has the properties claimed.

Theorem 5.2. *Let Π be an automorphic representation of $GL(n, \mathcal{K})$ satisfying the hypotheses of Conjecture 2.2.2 and the strong regularity hypothesis of 1.2.3 (ii). Then Conjecture 2.2.2 is valid for Π.*

Proof. The theorem is deduced from Theorem 2.2.3 and proposition 5.1. exactly as in [HT], pp. 229-232. We omit the details, since the more complicated case of a general solvable extension is treated in Sorensen's notes [So].

Remark 5.3. We can also extend Theorem 5.2 to CM fields $\mathcal{K}$ that do not contain imaginary quadratic fields. As in [HT], it suffices to replace $\mathcal{K}$ by $\mathcal{K} \cdot E_i$ for a sufficiently general family of imaginary quadratic fields E_i, then to use to patching argument of [HT], pp. 229-232. We omit the details, since the argument is identical to the one developed at length in [HT].

Appendix

We verify the claims about local surjectivity of the map $\phi : T \times G \to GU$ made in the course of the proof of Lemma 1.1.3, and the analogous claims for endoscopic groups in §1.3. Let p be a rational prime, $\Gamma_p = Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. Consider the long exact cohomology sequence attached to the short exact sequence (1.1.1):

$$GU(\mathbb{Q}_p) \xrightarrow{\nu} \mathbb{Q}_p^\times \to H^1(\Gamma_p, G) \to H^1(\Gamma_p, GU) \to \dots. \tag{A.1}$$

If p splits in E then $G \xrightarrow{\sim} GL(n, \mathbb{Q}_p)$ and $H^1(\Gamma_p, G) = 1$ by Hilbert's Theorem 90, so ν is surjective; but one can also show easily that $\phi_p : T(\mathbb{Q}_p) \times G(\mathbb{Q}_p) \to GU(\mathbb{Q}_p)$ is surjective in that case. We therefore assume p is inert in E. Denote by $\mathcal{K}_p$, F_p, E_p the tensor products of the corresponding fields with $\mathbb{Q}_p$. We have in all degrees n the exact sequences of *groups*

$$GU(\mathbb{Q}_p) \xrightarrow{\nu} \mathbb{Q}_p^\times \xrightarrow{g} H^1(\mathbb{Q}_p, G) \to H^1(\mathbb{Q}_p, GU) \to \dots. \tag{A.2}$$

Now $H^1(\mathbb{Q}_p, G)$ classifies hermitian forms on free rank n $\mathcal{K}_p$-algebras (hermitian relative to F_p), so is isomorphic to $F_p^\times/N_{\mathcal{K}/F}\mathcal{K}_p^\times$. An easy check shows that the map $\mathbb{Q}_p^\times \to F_p^\times/N_{\mathcal{K}/F}\mathcal{K}_p^\times$ is the obvious one. Also, $H^1(\mathbb{Q}_p, GU)$ classifies hermitian forms up to $\mathbb{Q}_p$-similitude. If $\alpha \in F_p^\times/N_{\mathcal{K}/F}\mathcal{K}_p^\times$ is the invariant of a form h, the invariant of the similar form $\lambda\alpha$, for $\lambda \in \mathbb{Q}_p^\times$, is $\lambda^n\alpha$. If n is odd and p is unramified in $\mathcal{K}$, we see that the map $g : \mathbb{Q}_p^\times \to H^1(\mathbb{Q}_p, G)$ factors through an injection $\mathbb{Q}_p^\times/N_{E_p/\mathbb{Q}_p}E_p^\times \to H^1(\mathbb{Q}_p, G)$. Thus

$$\nu(GU(\mathbb{Q}_p)) \subset N_{E_p/\mathbb{Q}_p}E_p^\times = \nu \circ \phi(T(\mathbb{Q}_p) \times G(\mathbb{Q}_p))$$

which implies that

$$\phi(T(\mathbb{Q}_p) \times G(\mathbb{Q}_p)) = GU(\mathbb{Q}_p)$$

provided p is unramified in $\mathcal{K}$. In particular, ϕ is locally surjective at all but finitely many places.

It is well-known that the real group $GU(r, s)$ is connected if $r \neq s$, which easily implies that ϕ is locally surjective at the real place.

This completes the verification of the claims made in the proof of 1.1.3. Since $GH_{a,b}$ and $H_{a,b}$ have simply connected derived subgroups and contain the maximal tori GA and A, respectively, the argument above applies just as before.

For completeness, we treat the case of even $n = 2m$. Assume V is a quasi-split hermitian space of dimension n over F_p, so that $GU(\mathbb{Q}_p)$ is the rational similitude group of the form with matrix

$$J_{2m} = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}.$$

The map

$$\mathbb{Q}_p^\times \to GU(\mathbb{Q}_p);\ t \mapsto diag(I_m, t \cdot I_m)$$

splits the similitude map, hence $\nu : GU(\mathbb{Q}_p) \to \mathbb{Q}_p^\times$ is surjective in the even-dimensional quasi-split case. On the other hand, the image of $\nu \circ \phi$ is always contained in $\{a \in \mathbb{Q}_p^\times \mid val_p(a) \in 2\mathbb{Z}\}$ when p is unramified in $\mathcal{K}$. It follows that the quotient $C = GU(\mathbb{A})/\mathbf{S}$, with $\mathbf{S}$ defined as in the proof of Proposition 1.1.4, is in general highly non-trivial in the even case.

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