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ENDOSCOPIC TRANSFER

Introduction

The goal of this chapter is to generalize a result of Blasius and Rogawski [BR] on the transfer of cohomological tempered automorphic representations of $H' = U(2)$ to $G = U(3)$, when H' is replaced by $U(n)$ for any even $n = 2m$ and G is a specific inner form of $U(2m+1)$. Endoscopic transfer properly requires an automorphic representation, or rather an L-packet, of an endoscopic group of the form $H = U(a) \times U(b)$, with $a + b = n + 1$; so here H should be $U(n) \times U(1)$. However, the representations of the factors $U(n)$ and $U(1)$ play different roles here. In subsequent books we will want to attach compatible families of ℓ-adic representations to cohomological automorphic representations of $GL(n)$ that descend to H'. The additional character χ of $U(1)$ serves as a factor of adjustment to guarantee that the resulting L-packet of G corresponds to an n-dimensional piece of the ℓ-adic cohomology of the corresponding Shimura variety. This is not always possible, and when $n = 1$ [BR] determines exactly when the character χ can be chosen to obtain an L-packet of G of the desired form. This is the result we generalize. We only treat situations in which the ramification at finite primes is as simple as possible, which allows us to concentrate on the complications at archimedean primes. Our calculation is almost an immediate consequence of the results of [C.III.A] and [L.IV.A].

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1. Conventions for unitary groups

Let F be a totally real field, $\mathcal{K}/F$ a totally imaginary quadratic extension, $\eta = \eta_{\mathcal{K}/F}$ the corresponding quadratic character of $\mathbf{A}_F^\times/F^\times$ (and equivalently of $Gal(\bar{F}/F)$, $d = [F:\mathbb{Q}]$, $c \in Gal(\mathcal{K}/F)$ the non-trivial Galois automorphism. Let n be a positive integer and $\mathcal{G} = \mathcal{G}_n$ be the algebraic group $R_{\mathcal{K}/\mathbb{Q}}GL(n)_{\mathcal{K}}$. Let $\mathfrak{g} = Lie(\mathcal{G}(\mathbb{R}))$, $K_\infty \subset \mathcal{G}(\mathbb{R})$ a maximal compact subgroup. We consider cuspidal automorphic representations Π of $\mathcal{G}$ satisfying the following two hypotheses:

Hypotheses 1.1. *Writing* $\Pi = \Pi_\infty \otimes \Pi_f$, *where* Π_∞ *is an admissible* $(\mathfrak{g}, K_\infty)$-*module, we have*

(i) *(Regularity) There is a finite-dimensional complex algebraic irreducible rep-*

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resentation $W(\Pi) = W_\infty$ *of* $\mathcal{G}(\mathbb{R})$ *such that*

$$H^*(\mathfrak{g}, K_\infty; \Pi_\infty \otimes W_\infty) \neq 0.$$

(ii) *(Polarization) The contragredient* $\Pi^\vee$ *of* Π *satisfies*

$$\Pi^\vee \xrightarrow{\sim} \Pi \circ c.$$

Remark. These are natural generalizations of Hecke characters of type A_0, inasmuch as present methods do not allow us to attach Galois representations to more general *algebraic* representations, in the sense defined by Clozel in [C90]. The polarization condition is unnecessarily restrictive; the more general condition would be

$$\Pi^\vee \xrightarrow{\sim} \Pi \circ c \otimes \xi$$

where ξ is a Hecke character that factors through the norm to the idèles of F. Two questions arise:

1. Can the theory developed below be extended to this more general setting?
2. Can the authors or their friends find a convincing name (rather than acronym such as RP for Regular Polarized) for this class of automorphic representations?

We are hopeful at least with regard to (1).

The following hypotheses will remain in force throughout this chapter.

Simplifying Hypotheses 1.2.

(1.2.0) *The degree* n *is even.*
(1.2.1) $\mathcal{K}/F$ *is unramified at all finite places (in particular* $d > 1$*).*
(1.2.2) Π_v *is spherical (unramified) at all non-split non-archimedean places* v *of* $\mathcal{K}$.
(1.2.3) *The degree* $d = [F : \mathbb{Q}]$ *is even.*

In Book 2, we will associate a compatible system of Galois representations, occurring in the cohomology of Shimura varieties, to representations Π satisfying these hypotheses and (when n is even) those satisfying Hypotheses 1.3 below. If n is odd the Shimura varieties are attached to unitary groups of vector spaces of dimension n. When n is even, and only then, this is not always possible, and we must resort to endoscopy. Thus we assume (1.2.0). However, all the results of this chapter remain true for odd n, when appropriately modified, but the appropriate modifications require the introduction of additional notation. The remaining hypotheses allow us to simplify the analysis of the stable trace formula, and in particular to sidestep the problem of classification of L-packets for p-adic unitary groups, important for the classification of automorphic representations but not essential for the construction of Galois representations.

The irreducible representation $W(\Pi)$ factors over the set Σ_F of real embeddings of F

$$W(\Pi) = \otimes_{v \in \Sigma_F} W_v,$$

where W_v is an irreducible representation of $\mathcal{G}(\mathcal{K} \otimes_{F,v} \mathbb{C}) \xrightarrow{\sim} GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$, each factor associated to a prime (say $\tilde{v}, \tilde{v}^c$) of $\mathcal{K}$ extending v. Each factor is parametrized by its highest weight; thus

$$\mu(\tilde{v}) = (\mu_1(\tilde{v}) \geq \mu_2(\tilde{v}) \geq \cdots \geq \mu_n(\tilde{v}))$$

and $\mu(\tilde{v}^c)$. Moreover, the polarization condition implies that the two factors are dual, or equivalently that

$$\mu_i(\tilde{v}^c) = -\mu_{n+1-i}(\tilde{v}). \tag{1.2.4}$$

Strong Regularity Hypothesis 1.3. *For every $v \in \Sigma_F$, the highest weight $\mu(v)$ is regular, i.e. $\mu_i(\tilde{v}) \neq \mu_j(\tilde{v})$ if $i \neq j$.*

Remark. It should be enough to assume 1.3 for a single $v \in \Sigma_F$, but this probably requires results of Book 2.

1.4. Existence of hermitian spaces with fixed local invariants.

The following lemmas are special cases of the well known classification of hermitian spaces.

Lemma 1.4.1. *There exists a hermitian space $V_1/\mathcal{K}$ of dimension $n+1$ relative to the extension $\mathcal{K}/F$ such that, letting $G = U(V_1)$,*

(1.4.1.1) *For all finite places v of F, $G(F_v)$ is quasi-split and splits over an unramified extension of F_v; in particular, $G(F_v)$ contains a hyperspecial maximal compact subgroup.*

(1.4.1.2) *For all real places v of F , with the exception of one place v_0, $G(F_v)$ is compact; $G(F_{v_0}) \xrightarrow{\sim} U(1,n)$.*

Lemma 1.4.2. *There exists an n-dimensional hermitian space $V_0/\mathcal{K}$ relative to the extension $\mathcal{K}/F$ such that the unitary group $G_0 = U(V_0)$ satisfies*

(1.4.2.1) *For all finite places v, $G_0(F_v)$ is quasi-split and splits over an unramified extension of F_v; in particular, $G_0(F_v)$ contains a hyperspecial maximal compact subgroup.*

(1.4.2.2) *For all real places v, $G_0(F_v)$ is compact.*

Moreover, G_0 is unique up to isomorphism.

Lemma 1.4.3. *Under Hypotheses 1.2 there exists an n-dimensional hermitian space $V_2/\mathcal{K}$ relative to the extension $\mathcal{K}/F$ such that, letting $G_2 = U(V_2)$,*

(1.4.3.1) *For all finite places v of F, $G_2(F_v)$ is quasi-split and splits over an unramified extension of F_v; in particular, $G_2(F_v)$ contains a hyperspecial maximal compact subgroup.*

(1.4.3.2) *For all real places v, with the exception of one place v_0, $G_2(F_v)$ is compact; $G_2(F_{v_0}) \xrightarrow{\sim} U(2, n-2)$.*

Sketch of proofs. In dimension $n+1$, which is odd, it is known that there is no Hasse obstruction to the existence of a unitary group with prescribed local factors.

In Lemmas 1.4.2 and 1.4.3, the hermitian spaces are of even dimension. Kottwitz' explicit description of the Hasse obstruction in the even-dimensional case (cf. [Clozel, IHES §2]) implies the existence of the groups G_0 and G_2.

We let U^* denote the quasi-split inner form of G_0 (or G_2). In Book III G is denoted G_1.

(1.5) Vanishing of the obstruction to stable descent.

The central character ξ_Π of Π is a Hecke character of $\mathcal{K}$ satisfying the analogues of hypotheses 1.1(i) and (ii), and in particular

$$\xi_\Pi^{-1} = \xi_\Pi \circ c.$$

It follows that the restriction of ξ_Π to the idèles of F is a Hecke character trivial on norms from $\mathcal{K}$, in other words

$$\xi_\Pi \;|_{\mathbf{A}_F^\times} = \eta^u, u = 0, 1. \tag{1.5.1}$$

Moreover, ξ_Π is obtained by base change from a Hecke character of $U(1)$ if and only if $u = 0$. This global condition can be checked at any place of F that does not split in $\mathcal{K}$. In particular, it suffices to check the condition at an archimedean place, and it follows from (1.2.4) that $u = 0$. Thus ξ_Π is locally a base change everywhere, in other words:

Lemma 1.5.2. *Let v be an inert place of F, w the place of $\mathcal{K}$ dividing F. Then there is a character β_v of $U(1)_v = \ker N_{\mathcal{K}_w/F_v} : \mathcal{K}_w^\times \to F_v^\times$ such that, for all $z \in \mathcal{K}^\times$*

$$\xi_{\Pi,w}(z) = \beta_v(z/c(z)).$$

2. Review of stable descent

For any finite prime v of F, let U_v^* denote a quasisplit unitary group over v, relative to the extension $\mathcal{K}/F$ (a split étale extension if v is a split prime). With w a prime of $\mathcal{K}$ dividing v, we can construct a formal descent π_v of Π_w as follows:

(2.1) v split. Then

$$GL(n, \mathcal{K}_v) \xrightarrow{\sim} GL(n, F_v) \times GL(n, F_v)$$

and the polarization condition implies for the local components of Π:

$$\Pi_v = \Pi_1 \otimes \Pi_2$$

with $\Pi_2 \xrightarrow{\sim} \Pi_1 \circ {}^t g^{-1}$. The group $U_v^* (\xrightarrow{\sim} GL(n, F_v))$ embeds (up to conjugacy) via

$$g \mapsto (g, {}^t g^{-1})$$

We set $\pi_v = \Pi_1$, which is well-defined. We let $\pi_v = \Pi_w$ with respect to this isomorphism.

(2.2) v inert. Then Π_w is unramified by (1.2.2) and invariant under the outer automorphism corresponding to descent to U_v^*. In the notation of [M.III.C], this implies that Π_w is in the image of either the base change map $\widetilde{BC}$ or the twisted base change $\widetilde{BC}'$. Combining Lemma 1.5.2 and Theorem 4.1 of [M.III.C], and bearing in mind that n is *odd*, one verifies that Π_w is in the image of $\widetilde{BC}$. We let π_v be the spherical representation of U_v^* such that $\widetilde{BC}(\pi_v) = \Pi_w$. The existence and properties of this formal base change are recalled in [M.III.C], where in particular it is proved that *this property characterizes π_v uniquely.*

In each of the above cases we choose an open compact subgroup $K_v \subset U_v^*$ such that $\pi_v^{K_v} \neq 0$. In case (2.2), and for all split v outside a finite set $S = S(\Pi)$ of split primes, π_v is spherical with respect to a hyperspecial maximal compact subgroup K_v, and we choose such a group K_v. For $v \in S(\Pi)$ choose $K_w \subset GL(n, \mathcal{K}_w)$ so that $\Pi_w^{K_w} \neq 0$, and let $K_v \subset U_v^*$ be the corresponding subgroup under the isomorphism $U_v^* \xrightarrow{\sim} GL(n, \mathcal{K}_w)$.

Let $K_f = K_{\Pi,f} = \prod_v K_v$. Letting π_f be the restricted tensor product of the π_v defined above, we then have $\pi_f^{K_f} \neq 0$. Using the theory of the conductor for $GL(n)$, we can even assume $\dim \pi_f^{K_f} = 1$, but this seems to be irrelevant. We recall the results of Labesse on stable descent:

Theorem 2.3 [L.IV.A]. *Let* Π *satisfy Hypotheses 1.1-1.3. Let* $\mathcal{A}(G_?)$ *be the space of (necessarily cuspidal) automorphic forms on (the anisotropic unitary group)* $G_?$*, where* $G_? = G_0, G_2, U^*$.

(a) There is exactly one representation π_∞ *of* $G_{0,\infty}$ *such that* $\pi_\infty \otimes \pi_f$ *occurs in* $\mathcal{A}(G_0)$*, and it occurs with multiplicity one.*

(b) There are $\frac{n(n-1)}{2}$ *distinct representations* π_j *of* $G_{2,\infty}$ *such that* $\pi_j \otimes \pi_f$ *occurs in* $\mathcal{A}(G_2)$*, each with multiplicity one.*

(c) Finally, Π *descends to some L-packet* $\tau = \{\tau_i(\Pi)\}$ *of* U^* *with* $(\tau_i(\Pi))_f = \pi^f$ *for all* i*, each with multiplicity one.*

The representations at the real primes are described as follows. Let v be a real prime of F. The coefficient system W has a factor W_v, a representation of $GL(n, \mathcal{K} \otimes_v \mathbb{C})$, described above. We write $W = W' \otimes W''$. The group $G_?(F \otimes_v \mathbb{C})$ embeds naturally in $GL(n, K \otimes_v \mathbb{C})$ and, by the polarization condition, the restriction W_+ of W' or W'' to $G_?(F_v)$ is the same. Thus we have an irreducible complex representation W_+ of $G_?(F_v)$, or equivalently of its compact inner form.

In case (a), the factor at v of π_∞ is the *dual* of W_+. In cases (b,c), $\tilde{W}_+$ defines, as in [Clozel, C.III.A, beginning of §2], an L-packet of discrete series of $G_2(F_v)$ or $U^*(F_v)$ at the place(s) v where this group is non-compact. The representations π_j of (b) are, with multiplicity one, the constituents of this L-packet. In case (c) we get, with multiplicity one, the tensor products over all real primes of these constituents.

Remark. For this chapter, only (c) of 2.3 is really needed. However, we note here that (a) is verified in [L.IV.A] without hypothesis 1.3.

We write τ_f instead of $(\tau_i(\Pi))_f$; of course τ_f is just another name for the π_f constructed above.

3. Endoscopic parameters

3.1. Conventions. In what follows we let H denote the quasi-split F-group $U^* \times U(1)$, where U^* is the quasi-split form of $U(n)$ introduced at the end of §1. We view H as an elliptic endoscopic group for the group G of Lemma 1.4, cf. [H.I.A], 5.5. The transfer of parameters, and of automorphic representations, depends on the choice of an L-homomorphism

$$\xi : {}^L H \to {}^L G$$

which we normalize as in [H.I.A, (5.5)]. We recall the formulas here. Fix a Hecke character μ of $\mathcal{K}$ whose restriction to the idèles of F is the quadratic character $\eta_{\mathcal{K}/F}$ and which is unramified at all finite places of $\mathcal{K}$ that do not split over F. (This is possible by (1.2.1)). On $\hat{H} = {}^L H^0 = GL(n) \times GL(1)$, we have

$$\xi(h_n, h_1) = \begin{pmatrix} h_n & 0 \\ 0 & h_1 \end{pmatrix} \in \hat{G} \subset {}^L G, \ h_n \in GL(n), h_1 \in GL(1) \tag{3.1.1}$$

The Hecke character μ can be viewed as a character of $W_{\mathcal{K}}$ by class field theory, and for $w \in W_{\mathcal{K}}$ we let

$$\xi(w) = \begin{pmatrix} \mu(w) I_n & 0 \\ 0 & 1 \end{pmatrix} \times w \in \hat{G} \rtimes W_{\mathcal{K}} \tag{3.1.2}$$

Finally, if $w_\sigma \in W_F$ is a representative of the non-trivial coset of $W_{\mathcal{K}}$, we set

$$\xi(w_\sigma) = \begin{pmatrix} \Phi_n & 0 \\ 0 & 1 \end{pmatrix} \cdot \Phi_{n+1}^{-1} \times w_\sigma. \tag{3.1.3}$$

Here for any m, Φ_m is the anti-diagonal $m \times m$-dimensional matrix with entries $(\Phi_m)_{ij} = (-1)^{i-1}\delta_{i,n-j+1}$ [cf. M. III.C., 3.2.5].

Let Π be a representation satisfying Hypotheses 1.1-1.3, and let $\tau = \tau_\Pi$ be the L-packet of U^* of Theorem 2.3 (c). Let χ be a character of $U(1)$, and let

$$\tilde{\chi}(z) = \chi(z/c(z)) : \mathcal{K}^\times \backslash \mathbf{A}_{\mathcal{K}}^\times \to \mathbb{C}^\times$$

be the base change of χ to a Hecke character of $GL(1)_{\mathcal{K}}$. The goal of this chapter is to analyze the endoscopic transfer of the L-packet $\tau \times \chi$ for the endoscopic group H to an L-packet, denoted $\pi(\tau, \chi)$, of G.

3.2. Local endoscopic transfer at non-archimedean places.

Expanding the set $S = S(\Pi)$ if necessary, we may assume that χ (and equivalently $\tilde{\chi}$) is unramified for $v \notin S$. We may also assume that all primes at which χ ramifies are split in $\mathcal{K}/F$; thus S consists only of split primes. We consider factorizable test functions $f = \otimes_v f_v$ on $G(\mathbf{A})$, and write $f = f_\infty \otimes f_f$. Let $G_v = G(F_v)$ and let $K'_v \subset G_v$ be a hyperspecial maximal compact subgroup for each finite $v \notin S$, so that $K'_v = \prod_v K'_v$ is an open compact subgroup of $G((\mathbf{A}^f)^S)$. For $v \in S$ the endoscopic transfer $\pi_v(\tau_v, \chi_v)$ of $\tau_v \times \chi_v$ is defined as an explicit induced representation in [M.III.C, 4.2.3; cf. H.I.A, (9.3.2)]: identifying G_v with $GL(n+1, F_v)$, we have

$$\pi_v(\tau_v, \chi_v) = I_{P_n}^{G_v}(\Pi_v \otimes (\mu_v \circ \det) \times \tilde{\chi}_v), \tag{3.2.1}$$

where P_n is the standard maximal parabolic with Levi factor $GL(n, F_v) \times GL(1, F_v)$, $I_{P_n}^{G_v}$ denotes normalized induction, and the induced representation is necessarily irreducible.[4] Note that the transfer depends on the homomorphism ξ, and particularly on the local component μ_v of μ at v. For $v \notin S$ we assume f_v belongs to the Hecke algebra of K'_v-bi-invariant functions. For $v \in S$ we choose a K'_v so that $\pi_v(\tau_v, \chi_v)^{K'_v} \neq 0$. One can make a more judicious choice using the theory of types, but this is not necessary.

Combining the theorems of Laumon-Ngô, Hales, and Waldspurger recalled in part II (cf. a summary in [H.I.A, §6]), we know that there is a test function f^H on $H(\mathbf{A}^f)$ such that

$$\sum_{\gamma \sim \gamma_H} \Delta_f(\gamma_H, \gamma) O_\gamma(f) = SO_{\gamma_H}(f^H) \tag{3.2.2}$$

whenever $\gamma \in G(\mathbf{A}^f)$ and $\gamma_H \in H(\mathbf{A}^f)$ are associated and $\Delta_f(\bullet, \bullet)$ is the corresponding product of local transfer factors. Moreover, for $v \notin S$, f_v and f_v^H are related by

$$f_v^H = b_v(\xi)(f_v), \; b_v(\xi) : \mathcal{H}(G_v, K'_v) \to \mathcal{H}(H_v, K_v \times K_{1,v}) \tag{3.2.3}$$

[4] Because Π is cuspidal unitary, hence Π_v is generic and unitary, and $\tilde{\chi}_v$ is also unitary.

where, in the notation of [M.III.C, (2.7)], $b_v(\xi)$ is the homomorphism of Hecke algebras corresponding to the L-homomorphism ξ and $K_{1,v} \subset U(1)_v$ is the unique maximal compact subgroup (cf. [H.I.A., (9.3.3.3)] and [M.III.C]). For v split, the relation between f_v and f_v^H is explicitly given by Theorem 9.3.3.4 of [H.1.A] (in terms of distributions), or in terms of functions, by:

$$f_v^H = (\mu_v \circ \det)(h_n) f^{P_{n,1}}(h_n, h_1), \tag{3.2.4}$$

cf. [M.III.C, Corollary 4.7] for a more general formula.

3.3. Local endoscopic transfer at real places.

Now let v be a real place. The analogue of transfer formula (3.2.2) is in this case due to Shelstad, and is recalled in [C.III.A]. We apply it when f_v is a pseudocoefficient for a member of the discrete series L-packet attached to the local representation $W(\Pi)_v$. Let $\phi_G : W_{\mathbb{R}} \to {}^L G$ be the Langlands parameter of this L-packet. As in [C.III.A], §3, let ϕ_H be a discrete series parameter for H_v with $\phi_G = \xi \circ \phi_H$. We assume ϕ_H dominant (formula (3.14) in [C.III.A]) and ϕ_G *regular.* Then

$$(\Theta_{\phi_H}, f_v^H) = \sum_{\pi \in \Pi_G(\phi_G)} \Delta_v(\phi_H, \pi)(\Theta_\pi, f_v) \tag{3.3.1}$$

where the transfer factors $\Delta_v(\phi_H, \pi)$ are signs calculated explicitly by Clozel, following Shelstad and Kottwitz, in [C.III.A., Theorem 3.4].

We only need to determine the local transfer factors explicitly in two situations. Notation and terminology is as in [C.III.A].

Compact places. Assume $v \neq v_0$, so that G_v is compact. Then $\Pi_{G_v}(\phi_G)$ is a singleton, which we denote simply by π_v. Let $\omega \in \Omega$ be the unique element such that $\omega^{-1} r$ is dominant, with r the parameter associated to ϕ_G in [C.III.A, before (3.14)]. Then

$$\Delta_v(\phi_H, \pi_v) = \det \omega. \tag{3.3.2}$$

Indeed, Theorem 3.4 in [C.III.A] yields

$$\Delta_v(\phi_H, \pi_v) = < a_{\omega_\sigma}, s > \det \omega$$

where $s = \begin{pmatrix} I_n & \\ & -1 \end{pmatrix} \in \hat{G}$. Here $\sigma \in \Omega/\Omega_{\mathbb{R}}$, the quotient of the complex Weyl group by the real Weyl group, so we can take $\sigma = 1$. At any rate, G_v being compact, the cocycle a_ω is trivial [C.III.A, before (3.12)]. Thus

$$\Delta_v(\phi_H, \pi_v) := \delta_v = \det \omega. \tag{3.3.3}$$

Note that ω depends on v.

The non-compact place. We assume $v = v_0$ and choose ϕ_H – i.e., we choose χ, because the representation of U^* will be fixed – so that ϕ_G is in dominant position. Then with our notation, which is that of Theorem 3.4 in [C.III.A], $\omega = 1$. The transfer factor is determined as follows. Recall that $G_v = U(1, n)$, $\Omega = \mathfrak{S}_{n+1}$,

$\Omega_{\mathbb{R}} = \mathfrak{S}_n$ (acting on the last n letters). The discrete series L-packet of G_v associated to ϕ comprises the representations

$$\pi(\phi, \sigma^{-1}B_0)$$

[C.III.A, after (3.11)], $\sigma \in \Omega/\Omega_{\mathbb{R}} \simeq \{1, 2, \dots, n+1\}$, the bijection being given by $\sigma \mapsto \sigma \cdot 1$. We identify σ with $\sigma \cdot 1$ and write $\pi(\phi, i)$ for $\pi(\phi, \sigma^{-1}B_0)$. Then, for a suitable choice of holomorphic structure[5] on the symmetric space of G:

$$\pi(\phi, 1) \text{ is a holomorphic discrete series} \tag{3.3.4}$$

$$\pi(\phi, n+1) \text{ is an antiholomorphic discrete series} \tag{3.3.5}$$

$$\Pi(\phi, \frac{n+2}{2}) \text{ has a Whittaker model .} \tag{3.3.6}$$

Since $s = \begin{pmatrix} I_n & \\ & -1 \end{pmatrix}$, formula (3.13) of [C.III.A] now yields

$$\Delta_v(\phi_H, \Pi(\phi, i)) = < a_\sigma, s > = \{ \begin{matrix} 1 & i \leq n \\ -1 & i = n+1 \end{matrix} .$$

Thus

Lemma 3.3.7. *The discrete series L-packet $\Pi_G(\phi_G)$ is partitioned into two subsets Π^+ and Π^- such that*

1. *Π^+ has n elements, Π^- has 1 element;*
2. *$\Delta_v(\phi_H, \pi) = 1$ ($\pi \in \Pi^+$), -1 ($\pi \in \Pi^-$).*
3. *Π^- is the unique antiholomorphic discrete series.*

In conclusion, we have

Corollary 3.3.8. *Let $\delta \in \pm 1$. Fix an archimedean L-packet τ_∞ of the first factor U^*_∞ of H_∞. There is a choice of local characters χ_v for $v \in \Sigma_F$, or equivalently a character χ_∞ of the second factor $U(1)_\infty$ of H_∞, so that, letting $\phi_H(\tau_\infty, \chi_\infty)$ be the corresponding Langlands parameter, the product over the archimedean places*

$$\Delta_{v_0}(\phi_H(\tau_\infty, \chi_\infty), \pi_{v_0}) \cdot \prod_{v \neq v_0} \Delta_v(\phi_H(\tau_\infty, \chi_\infty), \pi_v) = \delta \Leftrightarrow \pi_{v_0} \in \Pi^+.$$

Here for $v \neq v_0$ we write $\Pi_{G_v}(\phi_G) = \{\pi_v\}$.

Proof. As in the proof of Lemma 3.3.7, we choose χ_{v_0} such that $\xi \circ \phi_H(\tau_{v_0}, \chi_{v_0})$ is a dominant discrete series parameter for G. Then

$$\delta_{v_0} = \Delta_{v_0}(\phi_H(\tau_{v_0}, \chi_{v_0}), \pi_{v_0}) = 1$$

[5] In the applications to Galois representations, this will be specified more precisely in Book 2.

for $\pi_{v_0} \in \Pi^+$ (cf. Lemma 3.3.7). We fix a $v_1 \neq v_0$ in Σ_F and for $v \notin \{v_0, v_1\}$, we choose χ_v arbitrarily again, and let $\delta_v = \Delta_v(\phi_H(\tau_v, \chi_v), \pi_v)$. Let $\varepsilon = \prod_{v \neq v_1} \delta_v$. Thus we need to show that, for an appropriate choice of χ_{v_1}, we have

$$\Delta_{v_1}(\phi_H(\tau_{v_1}, \chi_{v_1}), \pi_{v_1}) = \varepsilon \cdot \delta.$$

The left-hand side is the determinant of the permutation ω such that $\omega^{-1} r(\xi \circ \phi_H(\tau_{v_1}, \chi_{v_1}))$ is dominant. Recall that there is a Grössencharacter μ in the definition of ξ, cf. (3.1.2). Let $p_\tau = (p'_1, \dots, p'_n)$ be the (integral) parameter of τ_{v_1}, m that of μ_{v_1}, p' that of χ_{v_1}. Then

$$r = (m + p'_1, \dots, m + p'_n, p') := (p_1, \dots, p_{n+1}).$$

Note that $m \equiv \frac{1}{2} \pmod 1$, cf. [C.III.A, §3.1]. For instance, we could take $m = \frac{1}{2}$. There are now two cases to consider.

Assume first that $\varepsilon\delta = 1$. It will suffice to choose $p' = p_{n+1}$ such that r is dominant. The parameter $(p_1, \dots, p_n)$ being so by assumption, it will be enough to take p' very negative.

Assume $\varepsilon\delta = -1$. We seek ω with sign -1 such that

$$p_{\omega(1)} > \cdots > p_{\omega(n+1)}.$$

We know that $p_1 > \cdots > p_n$; moreover, the *weight* being regular we have $p_i \geq p_{i+1} + 2$ for all i. We can thus choose $p' = p_{n+1}$ such that $p_i > p_{n+1} > p_{i+1}$ for i such that the corresponding shuffle permutation has sign -1. This proves the Corollary.

Remark. The proof shows that the assumed regularity of the highest weight is too strong a condition. For instance, if $p_i \geq p_{i+1} + 2$ for two successive values $(i, i+1)$, the argument applies. Nevertheless, the regularity is useful elsewhere, for example in the proof of multiplicity one in [L.IV.A §10].

4. Global calculation

4.1. We let f be a test function on $G(\mathbf{A})$ for which the simple stable trace formula is valid as in [L.IV.A, Théorème 8.4]. Recall that this is true if $[F : \mathbb{Q}] \geq 2$ and if f is a very cuspidal coefficient of discrete series at the real primes. This is an identity

$$T^G_{disc}(f) = \sum \iota(G, H) ST^H_{disc}(f^H) \tag{4.1.1}$$

where H runs over the endoscopic groups for G. Since G is anisotropic,

$$T^G_{disc}(f) = T^G(f) = \sum_\pi \text{trace } \pi(f) \tag{4.1.2}$$

where $\{\pi\}$ decomposes $L^2(G(F)\backslash G(\mathbf{A}_F))$. By our choice of f_∞, f^H_∞ can be taken to be a linear combination of pseudo-coefficients of discrete series (cf. Shelstad [Shel1], Clozel-Delorme [CD]). Since f^H_∞ intervenes only through its stable orbital integrals, this linear combination may be chosen stable. Then f^H_∞ is, up to a scalar, an Euler-Poincaré function ([CD, Théorème 3], [L, Proposition 9]). Fix H. Let φ^H be a function on $H(\mathbf{A}_\mathcal{K})$ corresponding to f^H by base change. This is legitimate in our situation because base change for Hecke algebras is surjective, cf. [M.III.C, Cor. 4.2]. At the archimedian places φ^H is a twisted Euler-Poincaré function, as in [CL]. By [L.IV.A, §4], we may suppose that φ^H_∞ and f^H_∞ are very cuspidal. Theorem 4.4 of [L.IV.A] guarantees that

$$ST^H_{disc}(f) = T^{L^*}_{disc}(\varphi^H \times \theta). \tag{4.1.3}$$

The notation is essentially that of Labesse: L^* is the "espace tordu" associated to $(H(\mathbf{A}_\mathcal{K}), \theta)$; cf. [*loc. cit*, 2.1], where θ is the Galois automorphism. We have written $\varphi^H \times \theta$ for the function on the non-identity component L^* obtained from φ^H by translation by θ.

A description of the terms of the twisted trace $T^{L^*}_{disc}$ will be given below.

With this choice of f_∞, we now have:

Lemma 4.2. *We can choose our test function f_f on $G(\mathbf{A}^f)$ such that*

(1) *For all finite v, f_v is biinvariant under the subgroup K'_v of §3.2;*
(2) *For all finite v, $Tr(f_v; \pi_v(\tau_v, \chi_v)) \neq 0$;*
(3) $ST^H(f^H) = 0$ *unless* $H = U(n+1)^*$ *or* $H = U(n)^* \times U(1)$

Proof. Note that by the choice of f_∞, (4.1.2) and all terms (4.1.4) now range over a finite set of representations once the level of f_f is fixed.

Consider a term of (4.1.1) indexed by an endoscopic group $H = U^*(a) \times U^*(b)$, $a + b = n + 1$. By Labesse's result

$$ST^H(f^H) = T^{L^*_{a,b}}(\varphi^H) \tag{4.2.1}$$

Again we have adapted Labesse's notation from [L.IV.A, §2.2]. Thus $L^*_{a,b}$ is the "espace tordu" $GL(a)_\mathcal{K} \times GL(b)_\mathcal{K} \rtimes \theta$ with the Galois conjugation θ acting on both factors. However, unlike Labesse, we denote by φ^H a function on the adelic group $H_\mathcal{K}(\mathbf{A}) = GL(a)(\mathbf{A}_\mathcal{K}) \times GL(b)(\mathbf{A}_\mathcal{K})$. Then we have further [L.IV.A, Prop. 3.7]

$$T^{L^*_{a,b}}(\varphi^H) = \sum_\Pi c_\Pi \text{ trace } (\Pi(\varphi^H) I_\theta). \tag{4.2.2}$$

On the right-hand side, Π runs over a "discrete" set of representations of $GL(a, \mathbf{A}_\mathcal{K}) \times GL(b, \mathbf{A}_\mathcal{K})$ such that Π and $\Pi \circ \theta$ are isomorphic, θ being "complex conjugation" with respect to H_F; I_θ is the naturally defined, global intertwining operator $\Pi \simeq \Pi \circ \theta$; and $c_\Pi \neq 0$. Let us recall what these representations are. We may as well consider only one factor $GL(a, \mathbf{A}_\mathcal{K})$; change notation and write Π for a representation of this group. The condition on Π is that

(1) (tempered case) Π should be of the form $\Pi_1 \boxplus \cdots \boxplus \Pi_r$, Π_i a cuspidal representation of $GL(a_i, \mathbf{A}_\mathcal{K})$, $a_1 + \ldots a_r = a$, $\Pi_i \simeq \Pi_i^\theta$, and $\Pi_i \neq \Pi_j$. Here $\boxplus$ denotes block parabolic induction,
(2) (general case) More generally, Π has a similar decomposition, but with Π_i the Speh representation $Sp(P_i, b_i)$ where $a_i = c_i b_i$ and P_i is cuspidal for $GL(c_i, \mathbf{A}_\mathcal{K})$. (Cf. [A, Theorem 26.2] or [MW]).

For a justification, cf. [L.IV.A, Cor. 9.2 and Lemme 5.1]). The condition $\Pi_i \neq \Pi_j$ is not explicitly mentioned there but is automatic for representations such that trace $\Pi_\infty(\varphi^H_\infty) \neq 0$, as follows from the regularity of discrete series parameters (cf. [C.III.A, (3.6)]). As we will see, the case (2) is irrelevant for us.

Now fix the level of f_f, for instance $K' = \prod K'_v$. Consider our representation $\pi_f(\tau,\chi) = \otimes_v \pi_v(\tau_v,\chi_v)$. For $H = U^*(n)\times U(1)$, (4.2.2) contains a term such that, for $v \notin S$, the associated character of $\mathcal{H}_v(H)$ is, composed with the endoscopic morphism $\mathcal{H}_v(G) \to \mathcal{H}_v(H)$ of [H.A.1, (9.3.3.3)], the character of $\mathcal{H}_v(G)$ given by π_v. On the right-hand side, it is simply the twisted trace associated to $(\Pi \otimes \mu \circ \det) \otimes \chi$ (§1). For $H = U^*(n+1)$, there is a similar term of the form (1), which is now composite, associated to an induced representation $\Pi \otimes \mu \circ \det \boxplus \chi$.

We can certainly choose f verifying (1), (2) of Lemma 4.2. We may vary f at the primes $v \notin S$, preserving (2).

Now consider the terms (4.2.2) for $H = U^*(a)\times U^*(b)$, and a or $b > 1$. By our choice of f, thus of f^H, φ^H is a twisted Euler-Poincaré function [CL,Theorem 5.1]. In particular, for fixed level, the sum in (4.2.2) is again finite. By a theorem of Jacquet-Shalika [JS], the characters of

$$\mathcal{H}^S_{\mathcal{K}} = \otimes'_{w\notin S}\mathcal{H}(GL(n,\mathcal{K}_w), GL(n,\mathcal{O}_w))$$

associated to *all* terms in (4.2.2) for *all* H with $(a,b) \neq (n,1)$ or $(n+1,0)$ are linearly independent. (For instance, the non-tempered terms of type (2) above have different Hecke eigenvalues by the Jacquet-Shalika estimate for the Hecke eigenvalues of cusp forms; if a or $b > 1$ terms of type (1) do not contain a cuspidal factor of dimension n and we apply [JS, II, Theorem 4.4]).

A fortiori the terms (4.2.1) in $ST^H(f^H)$ are linearly independent, as characters of the unramified Hecke algebra of G, via the endoscopic morphisms $\mathcal{H}(G_v) \to \mathcal{H}(H_v)$, from the term (2) of the lemma, if we consider a large set of $v \notin S$. This proves the Lemma.

The same proof shows the stronger statement:

Corollary 4.3. *Let $\mathcal{W}$ be a finite set of finite-dimensional representations of $G(\mathbb{R})$. We can choose f_f as in Lemma 4.2 to satisfy the additional conditions*

(4) *Suppose π is an automorphic representation of G of level $\prod_v K'_v$, the product taken over all finite places of F, such that π_∞ is cohomological relative to some $W \in \mathcal{W}$. Then $Tr(f_f,\pi_f) = 0$ unless $\pi_f = \otimes'_v \pi_v(\tau_v,\chi_v)$, in which case $Tr(f_f,\pi_f) = 1$.*

The hypotheses on π imply that π belongs to a finite set of automorphic representations. The assertion now follows from the standard argument of separation by Hecke eigenvalues that is needed in every application of the trace formula.

Henceforward, we assume f_f to be chosen as in 4.2 and 4.3. We write $\pi_f(\tau_f,\chi_f) = \otimes'_v \pi_v(\tau_v,\chi_v)$. The next two subsections determine the contributions $ST^H(f^H)$ for $H = U(n+1)^*$ and $H = U(n)^* \times U(1)$, respectively, when f_∞ is chosen to be a (very cuspidal) pseudocoefficient of a fixed discrete series representation. These contributions will be controlled by base change, using Labesse's results in [L.IV.A].

4.4. Calculation of $ST^H(f^H)$, $H = U(n+1)^*$.

Let $H = U(n+1)^*$, so that $\iota(G,H) = 1$. By the Jacquet-Shalika classification, the representation $\pi_f(\tau_f,\chi_f)$ cannot base change to the finite part of a cuspidal automorphic representation of $GL(n+1,\mathcal{K})$. It does, however, base change to the (finite part of) an Eisenstein representation on $GL(n+1,\mathcal{K})$ induced from the standard maximal parabolic subgroup P_n already encountered in (3.2). In what follows we write P instead of P_n. The components τ and χ are respectively θ_n-stable and θ_1-stable, where θ_i is the outer automorphism of $GL(i,\mathcal{K})$ considered in [L.IV.A]. Thus the corresponding automorphic representation, occurring in the space of Eisenstein series on $GL(n+1,\mathcal{K})$, say $\Pi(\tau,\chi)$, is θ_{n+1}-stable. Moreover, although it belongs to a one-dimensional complex continuous family of automorphic representations, $\Pi(\tau,\chi)$ is even θ_{n+1}-discrete in the sense that it is the only fixed point in the family under θ_{n+1}. It will, then, contribute a discrete term (4.4.3) to the left hand side of (4.4.1).

Now let f_∞ be a very cuspidal pseudocoefficient of a discrete series representation $\pi_\infty \in \Pi_G(\phi_G)$ in the L-packet with parameter $\phi_G = \xi \circ \phi_H$, as in (3.3). We assume the strong regularity hypothesis 1.3. By Théorème 8.2 of [L.IV.A], we have

$$T^{L^*}_{disc}(\varphi^H \times \theta) = ST^H(f^H) \tag{4.4.1}$$

for $f^H = f^H_\infty \otimes f^H_f$ and φ^H, f^H associated. The left hand side is the twisted trace as in [L.IV.A], L^* being the component of $1 \rtimes \theta$ in $GL(n+1,\mathcal{K}) \rtimes \{1,\theta\}$. The function φ^H_∞, being associated to f^H_∞, must be such that

$$\text{trace } (\Pi_\infty(\varphi^H \times \theta)) = 1, \tag{4.4.2}$$

the 1 on the right-hand side being the trace of f^H_∞ in the L-packet for $H(F_\infty)$ associated to Π_∞. Here Π_∞ has been extended to the twisted space using the Whittaker normalization: $\Pi_\infty(\theta)$ is the Whittaker-normalized intertwining operator for $\theta = \theta_{n+1}$, as in [C.III.A,§2] (see the remarks at the end of [§2 of that chapter]). *Up to a scalar*, ϕ_∞ is a Lefschetz function for Π_∞.

Remark. In (4.4.1) one would like to replace H by G, but the stable trace is only defined for quasi-split groups. Since $H(\mathbf{A}^f) = G(\mathbf{A}^f)$, the function $f^H_f = f_f$ is in fact a function on $G(\mathbf{A}^f)$. On the other hand, f_∞ and f^H_∞ are associated, hence the 1 on the right-hand side of (4.4.2) can be interpreted as the trace of f_∞ in the L-packet for $G(F_\infty)$ associated to Π_∞. Thus in fact one could make sense of the right-hand side of (4.4.1) in terms of G directly.

The purpose of this section is to determine the contribution of $ST^H(f)$ to the formula (4.1.1). In the calculation that follows, we identify this contribution with the left-hand side of (4.4.1).

Using the finiteness arguments in the proof of Lemma 4.2 and the fact that ϕ_∞ traces only in cohomological representations, we can now refine our choice of f^H (and associated φ^H) so that only one class of terms survives in the expression of [L.IV.A, (5.2)] for (4.4.1), namely

$$\begin{aligned} T^{L^*}_{disc}(\varphi^H \times \theta) &= \sum_{M \equiv GL(1)\times GL(n)} |W(M)|^{-1} T^L_{M,disc}(\varphi^H \times \theta) \\ &= \frac{1}{2}\sum_{\Pi} \text{trace } I_P(\Pi,s)(\varphi^H \times \theta) \\ &= \frac{1}{2}\text{trace } I_P(\Pi(\tau,\chi),s)(\varphi^H \times \theta) \end{aligned} \tag{4.4.3}$$

where $I_P(\Pi,s)(\varphi^H)$ is the operator defined in [L.IV.§5]; the argument used in the proof of Lemma 4.2 guarantees that we can choose f^H so that only the term $\Pi = \Pi(\tau,\chi)$ survives. Recall that Labesse writes φ for our $\varphi^H \times \theta$.

Proposition 4.4.4. *Let f_∞ be a very cuspidal pseudocoefficient of a discrete series representation $\pi_\infty \in \Pi_G(\phi_G)$ as above, and let f_f^H satisfy the conditions of (4.2) and (4.3). Then, if f^H is a stable transfer of f on $H = G^* = U(n+1)^*$,*

$$ST^H_{disc}(f^H) = \frac{1}{2}.$$

Proof. Using

$$T^{L^*}_{disc}(\varphi^H \times \theta) = ST^H(f^H)$$

it suffices to show that (4.4.3) is equal to $\frac{1}{2}$ when φ^H admits f^H as stable transfer. One has to compute

$$\text{trace } I_P(\Pi, s)(\varphi^H \times \theta)$$

Implicit in the definition there is a globally defined intertwining operator

$$I_\theta = I_P(\Pi, s)(\theta)$$

using the Arthur normalization (cf. [L.IV.A, §3.4]). Now

$$\Pi(\tau, \chi) = ind_{P(\mathbf{A}_\mathcal{K})}^{GL(n+1,\mathbf{A}_\mathcal{K})}(\tau_\mathcal{K} \otimes \chi_\mathcal{K}) := I_P^G(\tau_\mathcal{K} \otimes \chi_\mathcal{K})$$

where $\tau_\mathcal{K}$ is the representation Π of §1 – the base change to $GL(n, \mathbf{A}_\mathcal{K})$ of the L-packet τ of (3.1) – and $\chi_\mathcal{K} = \chi \circ N_{\mathcal{K}^\times/U(1)}$ is the base change of χ (denoted $\tilde{\chi}$ above), the notation $N_{\mathcal{K}^\times/U(1)}$ being self-explanatory.

The term (4.4.3) comes from Arthur's twisted trace formula, and in particular the implicit normalization of the intertwining operator is defined by Arthur. It is the product of two operators ([L.IV.A, §3.2], where they are denoted $M_{Q|s(Q)}(0)$ and $\rho_Q(s, 0, y)$).

The automorphism θ_{n+1} acts on $GL(n+1, \mathbf{A}_\mathcal{K})$, and $\theta_n \times \theta_1$ acts on the $GL(n)$-component of the Levi subgroup $GL(n) \times GL(1)$. There is a tautological intertwining operator

$$T_\theta : I_P^G(\tau_\mathcal{K} \otimes \chi_\mathcal{K}) \to I^G_{{}^\theta P}({}^\theta\tau_\mathcal{K} \otimes {}^\theta\chi_\mathcal{K}) \tag{4.4.5}$$

where ${}^\theta P$ is now of type $(1, n)$. The groups P and ${}^\theta P$ contain the unipotent radical of the Borel subgroup, and T_θ sends the natural Whittaker functional W' (on the right-hand side of (4.4.3), with respect to a choice of additive character) to its analogue W on the left-hand side.

We now have to send $I^G_{{}^\theta P}({}^\theta\tau_\mathcal{K} \otimes {}^\theta\chi_\mathcal{K})$ back to $I_P^G(\tau_\mathcal{K} \otimes \chi_\mathcal{K})$, and this is done by the standard unnormalized intertwining operator (integration on a suitable unipotent subgroup), composed with θ_M. Note that

$$\theta_M(\tau_\mathcal{K} \otimes \chi_\mathcal{K}) \simeq \tau_\mathcal{K} \otimes \chi_\mathcal{K}.$$

Further, θ_M fixes a Whittaker functional on the space of the inducing representation and therefore also on the space of the induced representation.

Let us denote by

$$A : I^G_{{}^\theta P}({}^\theta\tau_\mathcal{K} \otimes {}^\theta\chi_\mathcal{K}) \to I_P^G(\tau_\mathcal{K} \otimes \chi_\mathcal{K})$$

the unnormalized intertwining operator. It is defined, a priori, for a complex parameter σ such that $Re(\sigma) > 0$. We take σ real and consider $A(\sigma)$. Following Langlands and Shahidi [Sh1] we normalize $A(\sigma)$ in the form

$$A(\sigma) = n(\Pi, \sigma) N(\sigma),$$

where

$$n(\Pi,\sigma) = \frac{L(\tau_{\mathcal{K}} \otimes \chi_{\mathcal{K}}^{\vee}, \sigma)}{\varepsilon(\tau_{\mathcal{K}} \otimes \chi_{\mathcal{K}}^{\vee}, \sigma) L(\tau_{\mathcal{K}} \otimes \chi_{\mathcal{K}}^{\vee}, \sigma+1)}, \tag{4.4.6}$$

the L and ε factors being the Rankin-Selberg L-function and ε-factor (in this case for $GL(n) \times GL(1)$) as in [JS], including the archimedean factors. The functional equation

$$L(\tau_{\mathcal{K}} \otimes \chi_{\mathcal{K}}, \sigma) = \varepsilon(\tau_{\mathcal{K}} \otimes \chi_{\mathcal{K}}, \sigma) L(\tau_{\mathcal{K}}^{\vee} \otimes \chi_{\mathcal{K}}^{\vee}, 1-\sigma)$$

here gives the following result.

$$n(\Pi,\sigma) = \frac{L(\tau_{\mathcal{K}}^{\vee} \otimes \chi_{\mathcal{K}}, 1-\sigma)}{L(\tau_{\mathcal{K}} \otimes \chi_{\mathcal{K}}^{\vee}, 1+\sigma)}.$$

But $\tau_{\mathcal{K}}$ and $\chi_{\mathcal{K}}$ are unitary, thus isomorphic to the duals of their complex conjugates (complex conjugation acting on the *coefficients*), and σ is real. We thus obtain

$$n(\Pi,\sigma) = \frac{L(\tau_{\mathcal{K}} \otimes \chi_{\mathcal{K}}^{\vee}, 1-\sigma)}{L(\tau_{\mathcal{K}} \otimes \chi_{\mathcal{K}}^{\vee}, 1+\sigma)}.$$

Arthur's formula involves the value of this function at $\sigma = 0$, and thus

$$n(\Pi, 0) = 1.$$

We still need to calculate the effect of $N(0)$ on the Whittaker functionals. The normalized operator is a product of local operators. At the finite places where our data are unramified, we know that $N(0)$ acts trivially on the vector fixed by the hyperspecial maximal compact subgroup (Lemma 6.1 in [KS]; note that the special assumptions in that paper are removed in [Sh2]) and that it acts by 1 on the Whittaker functional. At split places, a simple calculation shows that this is still true. (One could also use the argument that follows, completing [Sh1, loc. cit.], by the fact that it is now known that his local coefficients $c_{\psi}(s, \Pi)$ admit analogous formulas to those that we use below at real places.)

Now consider a real place v. If f is a function in the local induced representation

$$I_{\theta P}^{G}({}^{\theta}\tau_w \otimes {}^{\theta}\chi_w) = I_{\theta P}^{G}(\tau_w \otimes \chi_w)$$

for w dividing v, and if the natural Whittaker functions (for which the previous arguments, relative to induction, apply) on the two induced representations are denoted Wh, Shahidi [Sh1, p. 298] shows the formula

$$< Wh, f >= c(\sigma, \tau \otimes \chi) < Wh, A(\sigma) f >$$

where we have dropped the subscripts on τ and χ and the factor $c(\sigma, \tau\otimes\chi)$, which depends *locally* on an additive character ψ of $\mathcal{K}_w$ – is given by

$$c(\sigma,\tau\otimes\chi) = c_\psi(\sigma,\tau\otimes\chi) = \varepsilon(\tau\otimes\chi^\vee,\psi,\sigma)\times\frac{L(\tau^\vee\otimes\chi,1-\sigma)}{L(\tau\otimes\chi^\vee,\sigma)}.$$

We now return to the formula (4.4.6) for the normalizing factor. We have (*locally*) $N(\sigma) = n(\sigma)A(\sigma)$, thus

$$< Wh, N(\sigma)f >= n(\sigma)c(\sigma,\tau\otimes\chi) < Wh, f > .$$

The scalar product is thus

$$\varepsilon(\tau\otimes\chi^\vee,\psi,\sigma)\times\frac{L(\tau^\vee\otimes\chi,1-\sigma)}{L(\tau\otimes\chi^\vee,\sigma)}\cdot\frac{L(\tau\otimes\chi^\vee,\sigma)}{\varepsilon(\tau\otimes\chi^\vee,\psi,\sigma)L(\tau\otimes\chi^\vee,\sigma+1)}$$

At $\sigma = 0$, this equals

$$\frac{L(\tau^\vee\otimes\chi,1)}{L(\tau\otimes\chi^\vee,1)}$$

But τ and χ are (locally) unitary, thus $\tau^\vee = \bar\tau$, $\chi^\vee = \bar\chi$, and since $\sigma = 1$ is real we have

$$L(\tau^\vee\otimes\chi,1) = L(\bar\tau\otimes\bar\chi^\vee,1) = L(\tau\otimes\chi^\vee,1).$$

Now, by the choice of our functions locally everywhere at finite places and thanks to the compatibility of the Whittaker normalization with the spectral transfer [C.III.A, §2] we have

$$\text{trace } \pi_v(\varphi_v^H\times\theta) = \text{trace } \tau_v(f_v)$$

if π_v is the base change of τ_v (known to exist for unramified representations and at split or archimedean places) extended to the twisted space using the Whittaker normalisation [C.III.A,§2]. In particular, with I_θ as defined by Arthur,

$$\text{trace } (I_P(\Pi,s)(\varphi^H\times\theta)) = 1$$

if $\Pi = \Pi(\tau,\chi)$.

Note that this theorem does not depend on the choice of χ, nor does it depend on the choice of π in the L-packet $\Pi_G(\phi_G)$. This will not be the case in the following section.

4.5. Calculation of $ST^H(f^H)$, $H = U(n)^*\times U(1)$.

Now let $H = U(n)^*\times U(1)$. Then $\iota(U^*, H) = \frac{1}{2}$ [L.IV.A, §3]. We have the analogue of (4.4.1):

$$T_{disc}^{L_H^*}(\varphi^H\times\theta) = ST_{disc}^H(f^H) \tag{4.5.1}$$

where L_H^* is to $\mathcal{G}_n\times\mathcal{G}_1$ as L^* above was to $\mathcal{G}_{n+1}$, and ϕ^H and f^H are matching functions. Moreover, in this case the proof of Lemma 4.2 shows that the analogue of (4.4.3) is simply

$$T^{L^*}_{disc}(\varphi^H \times \theta) = T^{L^*}_{L,disc}(\varphi^H \times \theta). \tag{4.5.2}$$

For the precise meaning of the notation in (4.5.2), see [L.IV.A,§2]. In words, the only $\theta_n \times \theta_1$-discrete representations that contribute are the θ-stable representations in the discrete spectrum for $GL(n, \mathcal{K}) \times GL(1, \mathcal{K})$. Hypothesis 1.3 guarantees that these representations are in fact cuspidal, and we use this fact without comment in what follows.

To determine the contribution of $\iota(U^*, H)ST^H(f^H) = \frac{1}{2}ST^H(f^H)$ to (4.1), it suffices to determine f^H. This has been done in (3.2) and (3.3), and we just collect the answers. The result depends this time on the choice of $\pi \in \Pi_G(\phi_G)$. Refer to Corollary 3.3.8, let $\delta = 1$ and choose χ_∞ so that

$$\Delta_{v_0}(\phi_H(\tau_\infty, \chi_\infty), \pi_{v_0}) \cdot \prod_{\sigma \neq v_0} \Delta_\sigma(\phi_H(\tau_\infty, \chi_\infty), \pi_\sigma) = \delta \Leftrightarrow \pi_{v_0} \in \Pi^+. \tag{4.5.3}$$

Then

Proposition 4.5.4. *With this choice of χ_∞, let f_∞ be a (very cuspidal) pseudo-coefficient of a discrete series representation $\pi_\infty = \otimes_{\sigma \in \Sigma_F} \pi_\sigma$. Let f_f be as above. Then*

$$\iota(U^*, H)ST^H(f^H) = \{ \begin{array}{l} \frac{1}{2} \textit{ if } \pi_{v_0} \in \Pi^+ \\ -\frac{1}{2} \textit{ if } \pi_{v_0} \in \Pi^- \end{array}$$

Proof. By the previous remarks, and by refining the choice of f_f as in the proof of Lemma 4.2, we see that there is only one term in the left-hand side of (4.5.1). With notation that may be clearer than in [L.IV.A] we get

$$\text{trace } (\Pi'(\varphi^H \times \theta)) = ST^H_{disc}(f^H) \tag{4.5.4.1}$$

where $\Pi' = \Pi \otimes \chi$, and where $I_\theta = \Pi'(\theta)$ (acting on the space of automorphic forms on $GL(n, \mathbf{A}_\mathcal{K}) \times \mathbf{A}^\times_\mathcal{K}$) stabilizes a Whittaker vector [Cl,§2], and ϕ^H, f^H are associated. The right-hand side is a finite sum, in fact equal to $T^H_{disc}(f^H)$ (proof of Lemma 4.2 again); on all representations of $H(\mathbf{A})$ involved, the trace of f_f is 1. Thus (4.5.4.1) is an identity

$$\text{trace } (\Pi'(\varphi^H \times \theta)) = \sum_{\pi_H} \text{trace } \pi_{H,\infty}(f^H_\infty). \tag{4.5.4.2}$$

Assume $\pi_{v_0} \in \Pi^+$ and $f_\infty = \otimes_v f_v$ with f_{v_0} a pseudocoefficient of π_{v_0}. According to the results in [C.III.A,§3] recalled in this chapter, the stable trace, trace $\Pi_{H,\infty}(f^H_\infty)$ is equal to 1 when $\Pi_{H,\infty}$ is the L-packet of discrete series $\Pi_H(\phi_H)$ defined by the Langlands parameter ϕ_H. It is 0 for other discrete series L-packets (this was not stated in [C.III.A] but follows easily from Shelstad's construction [Shel1] of f^H, or from a simple argument with infinitesimal characters). Assume moreover the Strong Regularity Hypothesis 1.3. Then f^H has non-zero trace only in discrete series representations. Now by [C.III.A, §2], since I_θ is Whittaker normalized, we have

$$\text{trace } (\Pi'(\varphi^H \times \theta)) = \sum_{\pi'_\infty \in \Pi_H(\phi_H)} \text{trace } \pi'_\infty(f^H_\infty). \tag{4.5.4.3}$$

(We extend the results, trivially, to take care of the one-dimensional factor.)

Comparing (4.5.4.2) and (4.5.4.3), and keeping in mind that the right-hand side of (4.5.3) contributes only representations with the specified Hecke eigenvalues at (almost all) finite primes, we obtain the first part of the Proposition. If π_{v_0} belongs to (the singleton) Π^-, the same argument applies, now with trace $\Pi_{H,\infty}(f_\infty^H) = -1$, yielding the requisite equality in that case.

Combining the last two propositions, we conclude:

Theorem 4.6. *Let Π be a cuspidal automorphic representation of $\mathcal{G} = \mathcal{G}_n$ satisfying Hypotheses 1.1, 1.2, and (strong regularity) 1.3. Define $G = U(V_1)$ as in Lemma 1.4. Then there exists a Hecke character χ of $U(1)$ such that the representation $\pi_\infty \otimes \pi_f(\tau_f, \chi_f)$ occurs with multiplicity one (resp. multiplicity zero) in the automorphic spectrum of G for any $\pi_\infty \in \Pi^+ \subset \Pi_G(\phi_G)$ (resp. $\pi_\infty \notin \Pi^+$, in particular $\pi_\infty \in \Pi^-$). Here the partition $\Pi_G(\phi_G) = \Pi^+ \coprod \Pi^-$ is as in Lemma 3.3.6, and depends on ϕ_G, hence on χ_∞.*

The arguments given above apply without change if the group G is replaced by any anisotropic unitary group in $n+1$ variables which is quasi-split at all finite places and either $U(n,1)$ or compact at each archimedean place. The article of Bellaïche and Chenevier about the determination of the sign of the Galois representations constructed in these books requires the following additional case.

Assume now that G is the unitary group associated to a totally definite hermitian space of odd dimension $n+1$ over K/F. The letter χ still denotes a character of $U(1)$ such that χ_v is unramified if v is finite and inert. We have $\chi_v(z) = z^{-a_v}$ for some $a_v \in \mathbb{Z}$ if v is archimedean. Let Π be a cuspidal automorphic representation of $\mathcal{G}_n$ satisfying Hypotheses 1.1, 1.2, and 1.3. The unitary group G being compact at each real place, the Langlands parameter ϕ_G defined in 3.3 defines a unique (finite dimensional) irreducible representation π_∞ of $\prod_{v \text{ real}} G(F_v)$.

Theorem 4.7. *If a_v is big enough for each real place v, the representation $\pi_\infty \otimes \pi_f(\tau_f, \chi_f)$ occurs in the automorphic spectrum of G (and with multiplicity one).*

Indeed, the analysis of §4 applies verbatim to this case, the only difference occuring in §4.5. There is no choice now for the element $\pi \in \Pi_G(\phi_G) = \{\pi_\infty\}$ and Proposition 4.5.4 has to be replaced by the following observation. Assume that for each real place v of F, the integer a_v is big enough so that the endoscopic L-parameter ϕ_G at v is dominant "in the natural order" (the permutation ω_v of 3.3 is then the identity), then $\iota(U^*, H)ST^H(f^H) = \frac{1}{2}$. Indeed, by the proof of Prop. 4.5.4 and the character identity (3.3.1), whose right-hand side reduces to the term π_∞ now, it is enough to show that $\Delta_v(\phi_H, (\pi_\infty)_v) = 1$ for each v. But this follows from (3.3.2) as ω_v is the identity, and concludes the proof. Note that the conclusion of the statement would hold as well under the "opposite" assumption "$-a_v$ is big enough at each real place v", as then ω_v would be a $n+1$-cycle, whose signature is 1 again as n is even.

5. A remark on transfer factors

Identities such as (4.1.1) are true on the nose, not up to a constant, and therefore rely on a precise definition of the correspondence $f \mapsto f^H$. This has been defined by Langlands and Shelstad [LS]. Locally, the transfer factors may be defined up to a constant; globally, they must satisfy a product formula proved in §6 of [LS].

The transfer factors we use must therefore be those of Langlands and Shelstad, and it is those that are described by Renard in [R.II.1]. However, in [C.III.A] and in the endoscopic transfer of §3, we have used Kottwitz' simple description of the factors $\Delta_v(\gamma_H, \gamma)$, denoted $\Delta_0(\gamma_H, \gamma)$ in [C.III.1, Définition 3.1]. (We choose once and for all Δ_0 as our "Kottwitz-Shelstad" transfer factor.)

On the other hand, the global transfer factor is also determined by the choice of endoscopic embedding, denoted ξ in (3.2.3), and this choice is reflected in the determination of the non-archimedean component $\pi_f(\tau_f, \chi_f)$ in the statement of Theorem 4.6.

It is therefore incumbent upon us to verify, on the one hand, that the factors $\Delta^{LS}(\gamma_H, \gamma)$ and $\Delta^{KS}(\gamma_H, \gamma)$ of Langlands-Shelstad, resp. Kottwitz-Shelstad, coincide; on the other, to verify that they are compatible with the local factors at finite places corresponding to our choice of endoscopic embedding. In the present section we content ourselves with the following. We are working at a single real prime, and the data are as in [C.III.A]. We note that Shelstad [Shel2] has now given a careful discussion of the real transfer factors in relation in [LS]. Referring to that paper, however, would have required a comparison of the factors of [Shel2] with those of Kottwitz, i.e. those of [Shel1]. We leave this task to the reader.

Proposition 5.1. *There exists a sign $\varepsilon = \pm 1$, depending on G and H, such that*

$$\Delta^{LS}(\gamma_H, \gamma) = \varepsilon \Delta^{KS}(\gamma_H, \gamma).$$

Remark. The compatibility of this choice of archimedean transfer factor with our chosen endoscopic embedding is verified in [L.III.D, §7].

Proof. The factor on the left is, we repeat, the one given by Renard . We work out his constructions explicitly in our context, following [R.II.1, §5.5-5.9]. The factor Δ^{LS} is given as a product of the five terms

$$\Delta_{disc},\ \Delta_I,\ \Delta_{II},\ \Delta_{III_1},\ \text{and}\ \Delta_{III_2}.$$

We are in the situation of [C.III.1, §3.2], so $T_H = T_G$ is the diagonal unitary torus in $GL(n, \mathbb{C})$ (note that n is actually $n+1$ in this chapter. . .) and the factor Δ_0 is evaluated in Définition 3.1 at a pair ($\gamma_H = \gamma_H = \gamma$).

The first factor Δ_{disc} is not included by Renard because he uses Harish-Chandra's normalization of the orbital integrals [R.II.1, §3.2]. Compensating for this, we have

$$\Delta_{disc}(\gamma, \gamma) = \frac{D_G(\gamma)^{\frac{1}{2}}}{D_H(\gamma)^{\frac{1}{2}}} \tag{5.2}$$

where $D_G(\gamma) = |\det(1-\gamma) \mid \mathfrak{g}/\mathfrak{t}| = \prod_\alpha |1 - \gamma^\alpha|$, the product over all the roots of (G, T).

The second factor Δ_I is a sign depending only on (G, T, H) [R.II.1, §5.6]. The third factor, in the notation of [C.III.1], is

$$\Delta_{II}(\gamma, \gamma) = \prod_{\alpha \in B, \alpha \notin B^H} \frac{\gamma^\alpha - 1}{|\gamma^\alpha - 1|}. \tag{5.3}$$

The fourth factor is apparently complicated but we can make it trivial. It depends on the datum of $\gamma = \gamma_H$ and of another analogous pair, say (γ', γ'_H). Globally

[LS, p. 267] γ'_H will be taken in $H(F)$ and $\gamma' \in G(\mathbf{A}_F)$, associated, at all primes, to γ'_H. We may, be weak approximation, take (γ', γ'_H) of the same form as γ, γ_H. Then the elements u_σ and v_σ ($\sigma \in Gal(\mathbb{C}/\mathbb{R})$) of [R.II.1, §5.8] are equal and

$$\underline{inv}(\gamma_H, \gamma; \gamma'_H, \gamma') = 1.$$

The fifth factor is obtained as in [C.III.1, §3.2] by comparing η_0^G and $\xi_0\eta_0^H$ and is equal to $\chi_0(\gamma)$ [R.II.1, §5.9]:

$$\Delta_{III_1}(\gamma, \gamma) = \chi_0(\gamma). \tag{5.4}$$

Finally, we get from (5.2), (5.3), and (5.4)

$$\begin{aligned}\Delta^{LS}(\gamma,\gamma) &= \prod_{\alpha\in B, \alpha\notin B^H} (\gamma^\alpha - 1)\chi_0(\gamma)\varepsilon \\ &= (-1)^{ab}(-1)^{q(G)+q(H)}\varepsilon\Delta^{KS}(\gamma,\gamma)\end{aligned}$$

where the sign ε comes from Δ_I. This implies the Proposition.

We now seem to have a problem with the proof of Theorem 4.6, which relied on Kottwitz' factors. Suppose, however, that the product of transfer factors at archimedian primes is the opposite of the ones we used in §4. The comparison in §4.4 does not change (the transfer factor is 1 for the inner form). In §4.5, we may now use Corollary 3.3.8 with $\delta = -1$ (this entails taking a different value of χ_v at a compact archimedian place). Now Proposition 4.5.4, with a different choice of χ_∞, remains true, and Theorem 4.6 follows.

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