CONSTRUCTION OF AUTOMORPHIC GALOIS REPRESENTATIONS

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1. A patching lemma

Let F be a number field, p a prime, S a finite set of places of F, and $\mathcal{I}$ a set of cyclic Galois extensions K/F of prime degree q_K. For all $K \in \mathcal{I}$, we assume we are given an n-dimensional semisimple continuous representation $\rho_K : \Gamma_K \rightarrow GL(n, \overline{\mathbb{Q}}_p)$, with the following two properties

(a) For any $K \in \mathcal{I}$, $\sigma \in Gal(K/F)$, $\rho_K^\sigma \xrightarrow{\sim} \rho_K$;
(b) if $K, K' \in \mathcal{I}$, then

$$\rho_K \mid_{\Gamma_{K\cdot K'}} \xrightarrow{\sim} \rho_{K'} \mid_{\Gamma_{K\cdot K'}} .$$

We assume $\mathcal{I}$ is S-*general* in the sense that, for any $v \notin S$ and any finite extension M/F, there is $K \in \mathcal{I}$ in which v splits completely which is linearly disjoint from M.

Proposition 1.1. *Under the above hypotheses, there is a semisimple representation ρ, unique up to isomorphism, such that $\rho : \Gamma_F \rightarrow GL(n, \overline{\mathbb{Q}}_p)$ such that $\rho \mid_{\Gamma_K} \xrightarrow{\sim} \rho_K$ for all $K \in \mathcal{I}$ except for possibly one.*

Proof. The proof is as in [HT], pp. 229-232, or [BR], which treat the case $q_K = 2$ for all K. Fix $K_0 \in \mathcal{I}$ and let $\rho_0 = \rho_{K_0}$, $\Gamma_0 = \Gamma_{K_0}$. Let T be the set of irreducible constituents $\tau \subset \rho_0$, counted with multiplicity. Let $r = \oplus_{\tau \in T} \tau$, H the Zariski closure of the image of r, $H^0 \subset H$ its identity component, $M \supset K_0$ the fixed field of H^0.

By hypothesis (1), $G_0 = Gal(K_0/F)$, a cyclic group of order $q = q_{K_0}$, acts on T. Let $C \subset T$ (resp. $P \subset T$) be the set of non-trivial orbits (resp. fixed points). For each $c \in C$, choose an element $\tau_c \in c$, and let $C_0 = \{\tau_c \mid c \in C\}$. For $\tau \in P$, (resp. $\tau \in C_0$) we let $\tilde{\tau}$ denote an extension of τ to a representation of Γ_F (resp. we set

$$\tilde{\tau} = Ind_{\Gamma_0}^{\Gamma_F} \tau.$$

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For $\tau \in C_0$, $\tilde{\tau}$ is independent of the choice of τ_c, whereas for $\tau \in P$, $\tilde{\tau}$ is unique up to twisting by a character factoring through G_0. We let $X(G_0)$ denote the set of characters of G_0.

Let N/F be a finite Galois extension disjoint from M over F. As in [HT, p. 230], we see that

(1.1.1) Every $\tau\mid_{\Gamma_{N\cdot K_0}}$ is irreducible, $\tau \in P \coprod C_0$;
(1.1.2) For $\tau \in P$, $\tilde{\tau}\mid_{\Gamma_N}$ is irreducible;
(1.1.3) If $\tau\mid_{\Gamma_{N\cdot K_0}} \simeq \tau'\mid_{\Gamma_{N\cdot K_0}}$ then $\tau = \tau'$;
(1.1.4) If $\tau, \tau' \in P$, $\eta \in X(G_0)$, then $\tilde{\tau}\mid_{\Gamma_N} \simeq \tilde{\tau'} \otimes \eta\mid_{\Gamma_N}$ implies $\tau = \tau'$ and $\eta = 1$.

For any $K \in \mathcal{I}$ disjoint from M over F, it follows as on the top of p. 231 of [HT] that there exists $\eta_{\tilde{\tau},K} \in X(G_0)$ for all $\tau \in P$ such that

(1.1.5.) $$\rho_K = \oplus_{\tau \in C_0} \tilde{\tau}\mid_{\Gamma_K} \bigoplus \oplus_{\tau \in P} \tilde{\tau} \otimes \eta_{\tilde{\tau},K}\mid_{\Gamma_K}$$

Fix a $K_1 \in \mathcal{I}$ disjoint from M over F, and set

(1.1.6) $$\rho = \oplus_{\tau \in C_0} \tilde{\tau} \bigoplus \oplus_{\tau \in P} \tilde{\tau} \otimes \eta_{\tilde{\tau},K_1},$$

one finds as in [loc. cit.] that ρ is well defined and satisfies $\rho\mid_{\Gamma_K} \xrightarrow{\sim} \rho_K$ for all $K \in \mathcal{I}$ disjoint from M over F, in particular for all $K \neq K_0$. This completes the proof.

We will need a variant. We let $\mathcal{I}$ be a collection of solvable Galois extensions of F and, for each $K \in \mathcal{I}$, with $B = Gal(K/F)$, given a filtration

$$B = B_0 \supset B_1 \supset \cdots \supset B_N = \{1\}$$

with each B_j normal in B and B_j/B_{j+1} a cyclic group of prime order q_j, we set $K_j = K^{B_j}$. For each such K_j, let $\mathcal{I}_{K_j} subset \mathcal{I}$ of extensions containing K_j. We call $\mathcal{I}_{K_j}$ S-general if the collection of cyclic extensions of K_j of prime order contained in some $K \in \mathcal{I}_{K_j}$ is S-general, in the sense above, and we say $\mathcal{I}$ is S-general if $\mathcal{I}_{K_j}$ is S-general for every K_j obtained as above. By induction, we have the following corollary:

Corollary 1.2. *Let $\mathcal{I}$ be an S general collection of solvable Galois extensions of F. Assume that, for all $K \in \mathcal{I}$ we have an n-dimensional continuous representation $\rho_K : \Gamma_K \rightarrow GL(n, \overline{\mathbb{Q}}_p)$ satisfying conditions (a) and (b) above. Then there is a semisimple representation ρ, unique up to isomorphism, such that $\rho : \Gamma_F \rightarrow GL(n, \overline{\mathbb{Q}}_p)$ such that $\rho\mid_{\Gamma_K} \xrightarrow{\sim} \rho_K$ for all $K \in \mathcal{I}$ except for possibly one.*

Remark 1.3. It is well known that $\mathcal{I}$ is S-general if it satisfies the following alternative hypothesis: for any finite set Σ of places of F disjoint from S, there is $K \in \mathcal{I}$ in which every $v \in \Sigma$ splits completely. Indeed, let M/F be a finite extension, which we may assume Galois, and let $M_i \subset M$ be the set of subfields of M Galois over F with $Gal(M_i/F)$ a simple group. For each M_i pick a place v_i of F, not in S, which does not split completely in M_i, and let Σ be the set of these v_i. Then any K in which each v_i splits completely is necessarily linearly disjoint from M.

2. Hypotheses

Let F be a totally real field, $\mathcal{K}/F$ a totally imaginary quadratic extension, $d = [F:\mathbb{Q}]$, $c \in Gal(\mathcal{K}/F)$ the non-trivial Galois automorphism. Let n be a positive integer and $\mathcal{G} = \mathcal{G}_n$ be the algebraic group $R_{\mathcal{K}/\mathbb{Q}}GL(n)_{\mathcal{K}}$. Let $\mathfrak{g} = Lie(\mathcal{G}(\mathbb{R}))$, $K_\infty \subset \mathcal{G}(\mathbb{R})$ the product of a maximal compact subgroup with the center $Z_{\mathcal{G}}(\mathbb{R})$. We consider cuspidal automorphic representations Π of $\mathcal{G}$ satisfying the following two hypotheses:

General Hypotheses 2.1. *Writing* $\Pi = \Pi_\infty \otimes \Pi_f$, *where* Π_∞ *is an admissible* $(\mathfrak{g}, K_\infty)$*-module, we have*

(i) *(Regularity) There is a finite-dimensional irreducible representation* $W(\Pi) = W_\infty$ *of* $\mathcal{G}(\mathbb{R})$ *such that*

$$H^*(\mathfrak{g}, K_\infty; \Pi_\infty \otimes W_\infty) \neq 0.$$

(ii) *(Polarization) The contragredient* $\Pi^\vee$ *of* Π *satisfies*

$$\Pi^\vee \xrightarrow{\sim} \Pi \circ c.$$

We next make the following temporary hypotheses:

Special Hypotheses 2.2.

(2.2.1) $\mathcal{K}/F$ *is unramified at all finite places (in particular* $d > 1$*).*
(2.2.2) Π_v *is spherical (unramified) at all non-split non-archimedean places* v *of* $\mathcal{K}$.
(2.2.3) *The degree* $d = [F:\mathbb{Q}]$ *is even.*
(2.2.4) *All primes of small residue characteristic relative to* n *are split in* $\mathcal{K}/F$.

The irreducible representation $W(\Pi)$ factors over the set Σ of real embeddings of of F

$$W(\Pi) = \otimes_{\sigma\in\Sigma} W_\sigma,$$

where W_σ is an irreducible representation of $\mathcal{G}(\mathcal{K}\otimes_{F,\sigma}\mathbb{R}) \xrightarrow{\sim} GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$. The highest weight of W_σ is denoted $\mu(\sigma)$; it can be identified in the usual way with a pair of Young diagrams, one for each extension of σ to a complex embedding of $\mathcal{K}$, and thus with a pair of non-increasing n-tuples of non-negative integers $(\mu(\tilde{\sigma}, \mu(\tilde{\sigma}^c))$, where we write

$$\mu(\tilde{\sigma}) = (\mu_1(\tilde{\sigma}) \geq \mu_2(\tilde{\sigma}) \geq \ldots \mu_n(\tilde{\sigma})).$$

Moreover, the polarization condition implies that one of the Young diagrams is dual to the other, or equivalently that

$$\mu_i(\tilde{\sigma}^c) = -\mu_{n-i-1}(\tilde{\sigma}).$$

Special Hypothesis 2.3. *For at least one* $\sigma \in \Sigma$, *the highest weight* $\mu(\sigma)$ *is sufficiently far from the walls; in practice, it suffices to assume* $\mu(\sigma)$ *is regular, i.e.* $\mu_i(\tilde{\sigma}) \neq \mu_j(\tilde{\sigma})$ *if* $i \neq j$.

Let K be a p-adic field, WD_K its Weil-Deligne group. Let $\mathcal{A}(n,K)$ denote the set of equivalence classes of irreducible admissible representations of $GL(n,K)$, and let

$\mathcal{G}(n,K)$ denote the set of equivalence classes of n-dimensional Frobenius semisimple representations of WD_K. We denote by

$$\mathcal{L} : \mathcal{A}(n,K) \rightarrow \mathcal{G}(n,K)$$

the local Langlands correspondence, normalized to coincide with local class field theory when $n = 1$ in such a way that a uniformizer of $K^\times$ is sent to a geometric Frobenius.

The following result is expected to be proved in Books 1 and 2.

Expected Theorem 2.4. *(i) Suppose n is odd and Π satisfies Hypotheses (2.1) and (2.2). Then there is a number field $E(\Pi)$ and a compatible system $\rho_{\lambda,\Pi} : \Gamma_{\mathcal{K}} \rightarrow GL(n, E(\Pi)_\lambda)$ of λ-adic representations, where λ runs through the finite places of $E(\Pi)$, such that*

(a) *For all finite primes v of $\mathcal{K}$ of residue characteristic prime to $N_{E(\Pi)/\mathbb{Q}}(\lambda)$,*

$$\rho_{\lambda,\Pi}^{F-ss} \mid_{\Gamma_v} \xrightarrow{\sim} \mathcal{L}(\Pi_v \otimes | \bullet |_v^{\frac{1-n}{2}}).$$

Here the superscript $^{F-ss}$ denotes Frobenius semisimplification.

(b) *For all finite primes v of $\mathcal{K}$ dividing $N_{E(\Pi)/\mathbb{Q}}(\lambda)$, $\rho_{\lambda,\Pi} \mid_{\Gamma_v}$ is de Rham, and its Hodge-Tate numbers have multiplicity at most one (i.e., $\rho_{\lambda,\Pi}$ is Hodge-Tate regular and are determined by Π_∞, or equivalently by $W(\Pi)$, in accordance with the recipe given in (2.5), below.*

(c) *Let v be a finite prime of $\mathcal{K}$ dividing $N_{E(\Pi)/\mathbb{Q}}(\lambda)$. Suppose Π_v has a vector fixed by a hyperspecial maximal compact subgroup of $GL(n, \mathcal{K}_v)$. Then $\rho_{\lambda,\Pi} \mid_{\Gamma_v}$ is crystalline.*

(d) *Let v be a finite prime of $\mathcal{K}$ dividing $N_{E(\Pi)/\mathbb{Q}}(\lambda)$. Suppose Π_v has a vector fixed by an Iwahori subgroup of $GL(n, \mathcal{K}_v)$. Then $\rho_{\lambda,\Pi} \mid_{\Gamma_v}$ is semistable.*

(ii) If n is even, the same conclusions hold as in (i), provided Π in addition satisfies Hypothesis 2.3.

When Π satisfies the additional hypothesis that Π_{v_0} is square-integrable for some finite place v_0, then this theorem is mostly proved in [HT], extending an earlier theorem due to Clozel and Kottwitz [C,K2] obtaining (a) at most places where Π is unramified. The theorem is completed in [TY]. What we here call $\rho_{\lambda,\Pi}$ is the representation denoted $R_\ell(\Pi^\vee)$ in [HT].

(2.5) Hodge-Tate numbers of automorphic Galois representations.

Fix a prime λ of the coefficient field $E(\pi)$, say of residue characteristic p. The automorphic Galois representation $\rho_{\lambda,\Pi}$ constructed in Book 2 is obtained in the cohomology of a geometric p-adic local system $\tilde{W}_p(\Pi)$ on a Shimura variety, obtained in a standard way from the finite-dimensional representation $W(\Pi)$ introduced above. It is therefore of geometric type, in the sense of Fontaine and Mazur: each $\rho_{\lambda,\Pi}$ is unramified outside a finite set of places of $\mathcal{K}$, and at every place dividing the residue characteristic of λ, $\rho_{\lambda,\Pi}$ is de Rham. The latter fact is a consequence of the comparison theorems of p-adic Hodge theory, and in particular the Hodge-Tate numbers can be read off from the Hodge numbers of the de Rham cohomology of the flat vector bundle $\tilde{W}(\Pi)$ associated to $W(\Pi)$. The comparison of $\tilde{W}_p(\Pi)$ and $\tilde{W}(\Pi)$, and therefore the determination of the Hodge-Tate numbers from the highest weights $\mu(\sigma)$ of W_σ, presupposes a dictionary relating complex and p-adic places of $\mathcal{K}$. In [HT] this is given by an isomorphism $\iota : \bar{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. For what follows it suffices to identify the algebraic closure of $\mathbb{Q}$ in $\bar{\mathbb{Q}}_p$ with the field of algebraic numbers in $\mathbb{C}$. Then the p-adic embeddings of $\overline{\mathbb{Q}}$, and in particular of $\mathcal{K}$, are identified with the complex embeddings; if s is an embedding of $\mathcal{K}$ in $\bar{\mathbb{Q}}_p$, we write ιs for the corresponding complex embedding.

Let s be an embedding of $\mathcal{K}$ in $\bar{\mathbb{Q}}_p$, and let $D_{dR,s}$ denote Fontaine's functor from representations of $\Gamma_s = Gal(\bar{\mathbb{Q}}_p/\overline{s(\mathcal{K})})$ to filtered $\bar{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} E(\pi)_\lambda$-modules:

$$D_{dR,v}(R) = (R \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_v}.$$

The Hodge-Tate numbers of R (with respect to v) are the j such that $gr^j D_{dR,s}(R) \neq (0)$. Then in the situation of Theorem 2.4, the Hodge-Tate numbers of $\rho_{\lambda,\Pi}$ with respect to s are the j of the form

$$j = i - \mu_{n-i}(\iota(s)^c), \ i = 0, \ldots, n-1. \tag{2.6}$$

This is to be compared to part 4 of Theorem VII.1.9 of [HT]; the replacement of $\iota(s)$ by $\iota(s)^c$ corresponds to our replacement of Π by $\Pi^\vee$ in the definition of $\rho_{\lambda,\Pi}$.

Suppose n is even but Π does not satisfy Hypothesis 2.3. Then we expect to prove the following theorem in Books 1 and 2.

Expected Theorem 2.7. *Suppose Π satisfies Hypotheses (2.1) and (2.2). Then there is a number field $E(\Pi)$ and a compatible system $\rho_{\lambda,\Pi} : \Gamma_{\mathcal{K}} \to GL(\frac{n(n-1)}{2}, E(\Pi)_\lambda)$ of λ-adic representations, where λ runs through the finite places of $E(\Pi)$, such that*

(a) *For almost all finite primes v of $\mathcal{K}$ of residue characteristic prime to $N_{E(\Pi)/\mathbb{Q}}(\lambda)$ at which Π_v is unramified, $\rho_{\lambda,\Pi}$ is an unramified representation, and*

$$\rho^{ss}_{\lambda,\Pi} \mid_{\Gamma_v} \xrightarrow{\sim} \wedge^2 \mathcal{L}(\Pi_v)(2-n).$$

(b) *For all finite primes v of $\mathcal{K}$ dividing $N_{E(\Pi)/\mathbb{Q}}(\lambda)$, $\rho_{\lambda,\Pi} \mid_{\Gamma_v}$ is de Rham.*

(c) *Let v be a finite prime of $\mathcal{K}$ dividing $N_{E(\Pi)/\mathbb{Q}}(\lambda)$. Suppose Π_v has a vector fixed by a hyperspecial maximal compact subgroup of $GL(n, \mathcal{K}_v)$. Then $\rho_{\lambda,\Pi} \mid_{\Gamma_v}$ is crystalline.*

This theorem is equally valid for odd and even n, but is a consequence of Expected Theorem 2.4 when n is odd.

Expected Theorems 2.4 and 2.7 are the Galois counterparts of expected theorems about stable base change and endoscopic transfer for unitary groups that will be recalled in §5. The latter theorems are purely analytic consequences of the stable trace formula, and include some of the main expected results of Book 1, whereas the consequences for Galois representations, which include special cases of the conjectures formulated by Kottwitz in [K1], should be derived in Book 2.

The remainder of this note explains how to extend the conclusions of of Expected Theorem 2.4, with the exception of (i)(d), in the absence of Hypotheses 2.2 and 2.3.

3. Eigenvarieties and p-adic families of Galois representations

This is a report on what is expected in the first part of Book 3, following [BC], where some of the following results are presented conditionally, primarily assuming the results recalled in §6.

Let $\mathcal{K}/F$ be a CM quadratic extension of a totally real field, satisfying Special Hypotheses (2.2.1), (2.2.3), and (2.2.4). Hypotheses (2.2.1) and (2.2.3) imply by a standard Galois cohomological argument (cf. Book 1) that

Lemma 3.1. *There exists a hermitian space $V_0/\mathcal{K}$ relative to the extension $\mathcal{K}/F$ such that the unitary group $G_0 = U(V_0)$ satisfies*

(i) *For all finite places v, $G_0(F_v)$ is quasi-split and splits over an unramified extension of F_v; in particular, $G_0(F_v)$ contains a hyperspecial maximal compact subgroup.*
(ii) *For all real places v, $G_0(F_v)$ is compact.*

Moreover, G_0 is unique up to isomorphism.

We let complex conjugation $c \in Gal(\mathcal{K}/F)$ act on $GL(n, \mathbf{A}_{\mathcal{K}})$ relative to the unitary group G_0. We choose a c-invariant open compact subgroup $K = \prod_w K_w \subset GL(n, \mathbf{A}^f_{\mathcal{K}})$ such that $K_w \xrightarrow{\sim} GL(n, \mathcal{O}_v)$ if w is not split over F. Let $U = \prod_v U_v \subset G_0(\mathbf{A}^f_F)$ be a corresponding open compact subgroup. At inert places U_v is hyperspecial maximal compact. At split v, $v_{\mathcal{K}} = w \cdot w^c$, the identification

$$G_0(F_v) \xrightarrow{\sim} GL(n, \mathcal{K}_w)$$

identifies U_v with K_w. Our hypothesis that K is c-invariant ensures that this does not depend on the choice of w dividing v. Let S be the set of places v of F where U_v is not hyperspecial maximal compact.

Special Hypotheses 3.2. *Let v be a place of F dividing the rational prime p.*

(3.2.1) *v splits in $\mathcal{K}$*
(3.2.2) *The subgroup U_v contains an Iwahori subgroup of $GL(n, \mathcal{O}_v)$.*

For every $v \in S$, $G_0(F_v) \xrightarrow{\sim} GL(n, F_v)$. We fix a semisimple type for $GL(n, F_v)$, in the sense of Bushnell-Kutzko, and denote it τ_v. In fact, there is an open compact subgroup $K_v \subset GL(n, F_v)$ and an irreducible representation τ_v of K_v such that the set of irreducible admissible representations π_v of $G_0(F_v)$ containing τ_v form an inertial equivalence class, denoted $I(\tau_v)$. Let E be an extension of $\mathbb{Q}_p$ containing the fields of definition of all the τ_v, $v \in S$. The following Theorem summarizes some of the results of [BC §7]. The results are dispersed in several sections of [*loc. cit.*]; future drafts will include more detailed references.

Theorem 3.3 [BC]. *There exists a reduced and separated rigid analytic space $X_U = X_U(\{\tau_v, v \in S\})/E$, two collections of analytic functions*

(a) *$\kappa_{1,s}, \ldots, \kappa_{n,s} \in \mathcal{O}(X_U)$, where s runs over embeddings of $\mathcal{K}$ in $\bar{\mathbb{Q}}_p$;*
(b) *$F_w(Y) \in \mathcal{O}(X_U)[Y]$, where w runs over finite places of $\mathcal{K}$ dividing $v \notin S$ and each F_w is a polynomial of degree n with coefficients in $\mathcal{O}(X_U)$.*

an n-dimensional pseudorepresentation

$$T : Gal(\overline{\mathbb{Q}}/\mathcal{K}) \to \mathcal{O}(X_U),$$

and a pair of Zariski dense subsets $Z_{reg} \subset Z \subset X_U$ such that

(i) *The points of Z are parametrized by cuspidal automorphic representations Π satisfying General Hypotheses 2.1 and Special Hypothesis (2.2.2), such that $\Pi^K \neq 0$, or equivalently by automorphic representations π of G_0, such that $\pi^U \neq 0$.*

(ii) *For any $v \notin S$, w a place of $\mathcal{K}$ dividing v, and $x \in X_U$, the n-dimensional representation $\rho_{T,x} : Gal(\overline{\mathbb{Q}}/\mathcal{K}) \rightarrow GL(n, \mathcal{O}_{\mathbb{C}_p})$ associated to the specialization of T at x has the property that*

$$\det(I - \rho_{T,x}(Frob_w)Y) = F_w(Y);$$

(iii) *For any embedding $s : \mathcal{K} \rightarrow \bar{\mathbb{Q}}_p$ and $x \in X$, the restriction of $\rho_{T,x}$ to Γ_s (notation as in (2.5)) has Hodge-Tate-Sen weights $\kappa_{i,s}$, $i = 1, \ldots, n$;*

(iv) *For $x \in Z$ and $s : \mathcal{K} \rightarrow \bar{\mathbb{Q}}_p$, the restriction of $\rho_{T,x}$ to Γ_s is Hodge-Tate and has Hodge-Tate numbers given in terms of Π_∞ by the formula (2.6).*

(v) *For $x \in Z$ and w as in (ii),*

$$F_w(Y) = P_{\Pi_w}(Y)$$

is the normalized Hecke polynomial associated to the unramified representation Π_w.

(vi) *For $x \in Z$ and $v \in S$, the corresponding Π has the property that Π_v is in the intertial equivalence class $I(\tau_v)$.*

(vii) *For any $x \in X_U$ and v in S prime to ℓ, the restriction to the local inertia group I_v of n-dimensional representation $\rho_{T,x}$ is the representation corresponding to the inertial equivalence class $I(\tau_v)$.*

(viii) *For $x \in Z_{reg}$, with corresponding representation Π, the the restriction of $\rho_{T,x}$ to Γ_s is de Rham.*

If n is odd, then $Z_{reg} = Z$. If n is even, then Z_{reg} is parametrized by those Π satisfying Special Hypotheses 2.3.

The analytic space $X_U(\{\tau_v, v \in S\})$ is the eigenvariety corresponding to the ramification information $(S, \{\tau_v, v \in S\})$. The points in Z are usually called "classical points" of the eigenvariety. Special Hypotheses 3.2 correspond to the condition "finite slope" for the eigencurve of Coleman-Mazur.

Remark 3.4. *In the final version, there should also be analytic functions at primes $v \in S$, v not dividing ℓ, that correspond to Frobenius eigenvalues after base change to eliminate wild inertia. At v dividing ℓ, there should also be analytic functions corresponding to eigenvalues of crystalline Frobenius.*

4. Removal of Special Hypotheses

We admit Expected Theorems 2.4 and 2.7 for the duration of this section.

4.1. Removal of Special Hypotheses 2.2.

Let now $\mathcal{K}/F$ be any CM quadratic extension of a totally real field, and let Π be an automorphic representation of $GL(n, \mathcal{K})$ satisfying Hypothesis 2.1.

Proposition 4.1.1. *There is a finite set S of places of F and an S-general collection $\mathcal{I}$ of totally real quadratic extensions F_i/F such that, for each $F_i \in \mathcal{I}$, letting*

$\mathcal{K}_i = F_i \cdot \mathcal{K}$, Π_i the base change of Π to $\mathcal{K}_i$, the triple $(F_i, \mathcal{K}_i, \Pi_i)$ satisfies Special Hypotheses 2.2. Moreover, we can assume that, for every $v \in S$ and every $F_i \in \mathcal{I}$, either v splits in $\mathcal{K}/F$ or the unique extension of v to $\mathcal{K}$, denoted $v_{\mathcal{K}}$, splits in $\mathcal{K}_i$.

Proof. Let S be the set of primes of v at which (2.2.1), (2.2.2), or (2.2.4) fails: either v ramifies in $\mathcal{K}/F$, or v stays prime in $\mathcal{K}$ and the corresponding component $\Pi_{v_{\mathcal{K}}}$ is ramified, or the residue characteristic of v is small. We take $\mathcal{I}$ to be the set of totally real quadratic extensions F_i/F with the property that, for all $v \in S$, $F_{i,v} \xrightarrow{\sim} \mathcal{K}_v$. It is obvious that this set has the properties claimed.

Theorem 4.1.2. *Let Π be an automorphic representation of $GL(n, \mathcal{K})$ satisfying Hypothesis 2.1 and Special Hypothesis 2.3 if n is even. Assume Expected Theorem 2.4. Then the conclusions of Expected Theorem 2.4 also hold for Π.*

If n is even and satisfies Hypothesis 2.1, then the conclusions of Expected Theorem 2.7 also hold for Π.

Proof. The first part is deduced from Expected Theorem 2.4 and Proposition 1.1 exactly as in [HT], pp. 229-232. We omit the details, since the more complicated case of a general solvable extension will be treated in the next section. The second assertion is deduced from Expected Theorem 2.7 and Proposition 1.1 in the same way.

4.2. Removal of Special Hypotheses 2.3.

Since Special Hypotheses 2.3 are only relevant to even n, we assume n to be even. Moreover, the case $n = 2$ is already understood. Thus we assume $n \geq 4$.

We state a variant of Lemma 4.1.2 of [CHT].

Lemma 4.2.1 [CHT]. *Let F be a totally real field, v a place of F, $w \neq v$ a second finite place, M/F any finite extension. Let L be a finite Galois extension of F_v. There exists a totally real solvable Galois extension F'/F in which w splits completely, linearly disjoint from M, such that, for every place v' of F' dividing v, the extension $F'_{v'}/F_v$ is isomorphic to L/F_v.*

In the statement of [*loc. cit.*], we take $S = \{v, w, \} \cup S_\infty$, where S_∞ is the set of real places of F, $D = M$, $L = E'_v$, and $F_w = E'_w$.

Corollary 4.2.2. *Let Π be an automorphic representation of $\mathcal{G}$ satisfying General Hypotheses 2.1 and let M/F be any finite extension not containing $\mathcal{K}$. Let w be a place of F and let S be the set of places at which Π is ramified. There is a totally real solvable Galois extension F'/F in which w splits completely, linearly disjoint from M, such that, letting $\mathcal{K}' = \mathcal{K} \cdot F'$ the base change $\Pi_{\mathcal{K}'}$ of Π to $GL(n, \mathcal{K}')$ has the following property: for every prime $v \in S$ not dividing w and every prime v' of $\mathcal{K}'$ dividing v, the local component $\Pi_{\mathcal{K}', v'}$ has an Iwahori-fixed vector.*

Proof. By induction on the number of places in S we may assume S is the set of (one or two) primes above a single place v of F, $v \neq w$. Passing to a quadratic extension if necessary, as in (4.1), we may assume v splits in $\mathcal{K}$ as $w \cdot w^c$. We may thus identify Π_w with an irreducible admissible representation of $GL(n, F_v)$. It follows from the local Langlands correspondence, and indeed from the numerical correspondance proved by Henniart, that there exists a finite Galois extension L/F_v, necessarily solvable, such that the base change $\Pi_{w,L}$ of Π_w to $GL(n, L)$ has an Iwahori-fixed vector. Equivalently, letting (s, N) be the representation of the Weil-Deligne group of F_v corresponding to Π_w – s is a Frobenius semisimple representation of the Weil group of F_v and N is a nilpotent endomorphism satisfying the usual commutation rules – the restriction of s to the Weil group of L is unramified. We now apply Lemma 4.2.1 to this triple (L, w, M).

It follows that, letting S be as in the statement of Corollary 4.2.2, the collection $\mathcal{I}$ of solvable extensions F'/F for which $\Pi_{\mathcal{K}'}$ has an Iwahori-fixed vector locally above all places in S is S-general.

Now by Expected Theorem 3.3, Π corresponds to a (classical) point x in the subset $Z \subset X_U$. We let $\rho_{\Pi,\ell}$ denote the Galois representation $\rho_{T,x}$ (we take $\ell = p$ in Theorem 3.3).

Proposition 4.2.3. *Suppose* Π *satisfies General Hypotheses 2.1 and Special Hypothesis 3.2, in the sense that, for all* v *dividing* ℓ, Π_v *has an Iwahori fixed vector. Then the Galois representation* $\rho_{\Pi,\ell} : \Gamma_{\mathcal{K}} \to GL(n, \bar{\mathbb{Q}}_\ell)$ *satisfies the conclusions of Expected Theorem 2.4 (i), (a) and (b) for* Π.

Proof. Condition (a) follows from Theorem 3.3 (ii) and (v). It follows from (iv) of Theorem 3.3 that $\rho_{\Pi,\ell}$ is at least Hodge-Tate, with the right weights, at primes dividing ℓ. It remains to prove that $\rho_{\Pi,\ell}$ is at least de Rham. But now part (b) of Expected Theorem 2.7, which holds for Π thanks to Theorem 4.1.2, together with condition (a) and Chebotarev density, implies at least that $\wedge^2\rho_{\Pi_\ell}$ is de Rham. Now since $n \geq 4$, the map from $\wedge^2 : GL(n) \to GL(\frac{n(n-1)}{2})$ is an isogeny. A theorem of Wintenberger [Wi] asserts that if L is an ℓ-adic and $\rho : \Gamma_L \to GL(n, \bar{\mathbb{Q}}_\ell)$ is a Hodge-Tate representation whose image under an isogeny is de Rham, then ρ is itself de Rham. This completes the proof.

Theorem 4.2.4. *Fix a prime* ℓ. *Let* Π *be an automorphic representation of* $GL(n, \mathcal{K})$ *satisfying General Hypotheses 2.1. Then the conclusions of Expected Theorem 2.4 (i) (a) and (b) hold for* Π.

Proof. It follows from the preceding proposition that, in Expected Theorem 2.4(ii), we can replace the condition "Π in addition satisfies Special Hypothesis 2.3" by the condition "Π satisfies Special Hypothesis 3.2" in the sense . But it follows immediately from the remark following the proof of Corollary 4.2.2 that we can use Corollary 1.2 to reduce to the case of Π satisfying Special Hypothesis 3.2. Note that by the p-adic monodromy theorem the property of being de Rham is preserved under finite base change, as are Hodge-Tate numbers.

Remark 4.2.5. The theorem of Wintenberger invoked in the proof of Proposition 4.2.3, together with (c) of Expected Theorem 2.7, implies in this case that, if Π_v has a vector fixed by a hyperspecial maximal compact subgroup of $GL(n, \mathcal{K}_v)$, for some v dividing ℓ, then the local representation at v is crystalline up to twisting by a quadratic character. It should be possible to eliminate this twist, but I haven't yet thought about the problem. If Π_v has a vector fixed by an Iwahori subgroup, then the local representation at v ought to be semistable; this should follow from the corresponding fact in the setting of Expected Theorem 2.7, but the latter seems to require new information on models of Shimura varieties for $U(2, n-2)$ at Iwahori level. As Haines has pointed out, Faltings has proved that the local models have the required properties, but this is not quite enough to conclude.

The Galois representations constructed above by ℓ-adic continuity form a compatible system in the weak sense that their characteristic polynomials of Frobenius coincide for all ℓ. Since the coefficients of the characteristic polynomials all belong to the coefficient field $E(\Pi)$ of the cohomological representation Π, by (ii) and (v) of Theorem 3.3, it follows that these Galois representations are compatible in the strong sense of having coefficients in the same number field. I omit the formulation of the theorem in the present draft.

5. Stable base change, endoscopic transfer, and Expected Theorems 2.4, 2.7, and 3.3

We consider automorphic representations Π satisfying General Hypotheses 2.1 and Special Hypotheses 2.2. Each of the Expected Theorems 2.4, and 2.7 corresponds to a pair of theorems, the first about stable base change and endoscopic transfer, the second about the cohomology of certain Shimura varieties. In this section I limit my attention to the first sort of theorem, which can be stated simply without introduction of additional notation. The following lemma is proved by the standard Galois cohomological argument already mentioned in §3:

Lemma 5.1. *(i) Suppose n is odd. Under Special Hypotheses 2.2 there exists a hermitian space $V_1/\mathcal{K}$ relative to the extension $\mathcal{K}/F$ such that, letting $G_1 = U(V_1)$,*

(5.1.1) *For all finite places v of F, $G_1(F_v)$ is quasi-split and splits over an unramified extension of F_v; in particular, $G_1(F_v)$ contains a hyperspecial maximal compact subgroup.*

(5.1.2) *For all real places v of F , with the exception of one place v_0, $G_1(F_v)$ is compact; $G_1(F_{v_0}) \xrightarrow{\sim} U(1, n-1)$.*

(ii) Suppose n is even. Under Special Hypotheses 2.2 there exists a hermitian space $V_2/\mathcal{K}$ relative to the extension $\mathcal{K}/F$ such that, letting $G_1 = U(V_1)$,

(5.1.3) *For all finite places v of F, $G_0(F_v)$ is quasi-split and splits over an unramified extension of F_v; in particular, $G_0(F_v)$ contains a hyperspecial maximal compact subgroup.*

(5.1.4) *For all real places v, with the exception of one place v_0, $G_2(F_v)$ is compact; $G_2(F_{v_0}) \xrightarrow{\sim} U(2, n-2)$.*

For any finite prime v of F, let G_v denote a quasisplit unitary group over v, relative to the extension $\mathcal{K}/F$. With w a prime of $\mathcal{K}$ dividing v, we can construct a formal descent π_v of Π_w as follows:

5.2 v split. Then $G_v \xrightarrow{\sim} GL(n, F_v) \xrightarrow{\sim} GL(n, \mathcal{K}_w)$. We let $\pi_v = \Pi_w$ with respect to this isomorphism.

5.3 v inert. Then Π_w is unramified by (2.2.2) and invariant under the outer automorphism corresponding to descent to G_v. We let π_v be the spherical representation of G_v whose formal base change, defined in terms of the Satake isomorphism, is isomorphic to Π_w. The existence and properties of this formal base change will need to be recalled in Book 1.

We define a representation $\pi^f = \otimes'_v \pi_v$ of $G_*(\mathbf{A}^f)$, $* = 0, 1, 2$, the restricted tensor product taken over all finite places of F.

The following theorem corresponds to stable base change (twisted endoscopic transfer) for cuspidal automorphic representations.

Expected Theorem 5.2. *Let Π satisfy General Hypotheses 2.1 and Special Hypotheses 2.2. Let $G_* = G_1$ if n is odd, $G_* = G_0$ or G_2 if n is even. Let $\mathcal{A}(G_*) = \mathcal{A}^0(G_*)$ be the space of automorphic forms on G_*.*

(a) If n is odd, there are exactly n distinct representations π_i of $G_{1,\infty}$ such that $\pi_i \otimes \pi^f$ occurs in $\mathcal{A}(G_1)$, each with multiplicity one.

(b) If n is even, there is exactly 1 representation π_∞ of $G_{0,\infty}$ (resp. exactly $\frac{n(n-1)}{2}$ distinct representations π_j of $G_{2,\infty}$) such that $\pi_j \otimes \pi^f$ occurs in $\mathcal{A}(G_*)$, each with multiplicity one.

This automorphic input suffices for Expected Theorem 2.4 (i), Expected Theorem 2.7, and Expected Theorem 3.3.

Now let n be even again, and consider $H = U(n) \times U(1)$ as endscopic group for G_1, defined with respect to the odd number $n+1$. Let Π satisfy Hypotheses 2.1 and 2.2 as always, and let χ be a Hecke character of $U(1)$ unramified outside places that split in $\mathcal{K}/F$. For every finite place v of F, one can define the local endoscopic transfer $\pi_v(\Pi_v, \chi_v) = \pi_v(\Pi, \chi)$ as an explicit representation of $G_1(F_v)$; it is a single representation which is spherical if v is inert in $\mathcal{K}$. The explicit formula will be included in a later draft. Define $\pi^f(\Pi, \chi) = \otimes'_v \pi_v(\Pi, \chi)$.

Expected Theorem 5.3. *Assume n is even and Π satisfies General Hypotheses 2.1 and Special Hypotheses 2.2 and 2.3. Then there is a character $\chi_\infty of U(1)(\mathbb{R})$ such that, for any Hecke character χ of $U(1)$ with infinity type χ_∞, there are exactly n distinct representations π_i of $G_{1,\infty}$ such that $\pi_i \otimes \pi^f(\Pi, \chi)$ occurs in $\mathcal{A}(G_1)$, each with multiplicity one.*

This is the automorphic input for Expected Theorem 2.4 (ii).

Of course the representations π_i in 5.2 and 5.3 are discrete series representations, and can be defined in terms of the coefficients W_∞ of 2.1 (i) in the case of Expected Theorem 5.2, where the set of π_i fills out the complete discrete series L-packet. In Expected Theorem 5.3, the discrete series L-packet is defined by χ_∞ as well as W_∞, and has $n+1$ members; only n of them occur with non-zero multiplicity. The parameters for the relevant discrete series representations will be provided explicitly in a later draft.

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