AN INTRODUCTION TO THE STABLE TRACE FORMULA

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The goal of this introduction is to motivate the problem of stabilization of the trace formula and to present the approach to stabilization of the elliptic terms carried out by Langlands and Kottwitz in the 1980s. The first two sections are devoted to describing the trace formula and presenting two typical applications. In this series of books we studiously avoid all reference to the analytic problems resolved by Arthur in incorporating parabolic terms into the invariant trace formula. The reader who wishes to learn more about this material should consult Arthur's article [A], which we have also used as a reference for introductory material. In this introduction, we restrict our attention to automorphic representations of groups that are anisotropic (or at least anisotropic modulo center); then all terms on the geometric side of the trace formula are necessarily elliptic. Other chapters in this volume will make use of Arthur's simple trace formula, in which only elliptic terms even when the group is not anisotropic; the description of stabilization works identically in this situation.

The third section explains the problem of stabilization arises naturally when one wants to compare trace formulas for different groups. The remaining sections present the various steps involved in stabilization, mainly following Kottwitz' articles on the subject, as reinterpreted by Ngô in his seminar talks in 2003. At various points in his articles, Kottwitz' invokes conjectures in order to proceed with stabilization of the two sides of the trace formula. The stabilization of the elliptic part of the geometric side, involving orbital integrals, depends crucially on the Fundamental Lemma, as we explain in §6. Twenty years later, this is now a theorem, so Kottwitz' stabilization of the geometric side of the trace formula for anisotropic groups is now complete. The stabilization of the spectral side depends on the partition of the irreducible admissible representations of a p-adic group into L-packets (more generally, Arthur packets) of a specific form. There has been considerable recent progress on this question but much remains to be done. In this introduction we concentrate on the geometric side. Some of the subsequent articles will consider the spectral side of the trace formula for unitary groups, with applications in Book 2 to the cohomology of related Shimura varieties.

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Conventions for harmonic analysis on reductive groups over local fields and adèle groups, and for automorphic representations, are as in Arthur's article [A].

1. Automorphic representations and the trace formula

Here and in what follows, E denotes a number field and G denotes a connected reductive group over E, with center Z_G. We choose a maximal compact subgroup $K_\infty \subset G_\infty := G(E \otimes_{\mathbb{Q}} \mathbb{R})$. This choice is not always innocent, but little or nothing in the present volume will depend on that choice. We also let $\mathfrak{g}$ denote the complexified Lie algebra of $G(\mathbb{R})$, $\mathbf{A}_E$ the adèle ring of E, and $\mathbf{A}^f{}_E$ the ring of finite adèles of E. Thus $G(\mathbf{A}_E) = G_\infty \times G(\mathbf{A}^f{}_E)$, and any Haar measure dg on $G(\mathbf{A}_E)$ decomposes correspondingly as a product $dg_\infty dg^f$. Let $C_c^\infty(G(\mathbf{A}_E))$ (resp. $C_c^\infty(G_\infty)$, resp. $C_c^\infty(G(\mathbf{A}^f{}_E))$) denote the space of compactly supported functions on $G(\mathbf{A}_E)$ that are C^∞ in the archimedean variables and locally constant in the non-archimedean variables. Likewise, for any place v of E, we define $C_c^\infty(G(E_v))$. The map

$$(\phi_\infty, \phi_f) \mapsto [(g_\infty, g_f) \mapsto \phi_\infty(g_\infty) \cdot \phi_f(g_f)$$

defines an isomorphism

$$C_c^\infty(G_\infty) \otimes C_c^\infty(G(\mathbf{A}^f{}_E)) \xrightarrow{\sim} C_c^\infty(G(\mathbf{A}_E)).$$

These spaces of *test functions* are algebras under convolution (cf. [K7, 1.2]).

Our main class of examples will be unitary groups. Let $\mathcal{K}/E$ be a quadratic extension, and let V be an n-dimensional $\mathcal{K}$-vector space with a non-degenerate hermitian form $<,>_V$ relative to the extension $\mathcal{K}/E$; the group $G = U(V)$ of symmetries of this hermitian form is naturally a group scheme over E, whose A-valued points for any E-algebra A is given by

$$G(A) = \{g \in GL(V \otimes_E A) \mid < gv, gv' >_V = < v, v' >_V, \forall v, v' \in V \otimes_E A\}$$

If v is a place of E that splits as a product $w \cdot w'$ in $\mathcal{K}$, then

$$V_v = V \otimes_E E_v = V_w \oplus V_{w'}$$

where $V_w = V \otimes_{\mathcal{K}} \mathcal{K}_w$. Then V_w and $V_{w'}$ are maximal isotropic subspaces of V_v, and the hermitian form defines an isomorphism $V_{w'} \xrightarrow{\sim} V_w^\vee$. Then $G(E_v) \subset GL(V_w \oplus V_{w'})$ fixes both V_w and $V_{w'}$ and its action on the former defines a canonical isomorphism

$$G(E_v) \xrightarrow{\sim} GL(V_w).$$

The center Z_G is the diagonal subgroup $U(1)_{\mathcal{K}/E}$, whose group of A-valued points is the subgroup of $(\mathcal{K} \otimes_E A)^\times$ of elements of norm 1 down to $A^\times$; in particular, Z_G is anisotropic. We can also consider $G = GL(n)_E$, with $Z_G = GL(1)_E$, embedded diagonally in G.

Suppose G is anisotropic; for example $G = U(V)$, where E has a real place that becomes complex in $\mathcal{K}$, such that $G(E_v)$ is isomorphic to the compact unitary group $U(n)$. Then the adèlic quotient $G(E)\backslash G(\mathbf{A}_E)$ is *compact*. Right-translation defines a unitary action of $G(\mathbf{A}_E)$ on the Hilbert space $L_2(G(E)\backslash G(\mathbf{A}_E))$. This action gives rise to an action of $C_c^\infty(G(\mathbf{A}_E))$ by integration: for $f \in L_2(G(E)\backslash G(\mathbf{A}_E))$, $\phi \in C_c^\infty(G(\mathbf{A}_E))$, define

$$R(\phi)f(g) = \int_{G(\mathbf{A}_E)} \phi(gh)f(h)dh.$$

The map $\phi \mapsto R(\phi)$ defines an algebra homomorphism from the algebra $C_c^\infty(G(\mathbf{A}_E))$ of *test functions* to the C^* algebra of bounded operators on $L_2(G(E)\backslash G(\mathbf{A}_E))$. In fact, $R(\phi)$ is a *compact* operator for all ϕ (cf. [A], p.8). It follows formally [A, p. 9] that

Proposition 1.1. *When G is anisotropic, $L_2(G(E)\backslash G(\mathbf{A}_E))$ decomposes as a countable Hilbert space direct sum*

$$L^2(G(E)\backslash G(\mathbf{A}_E)) = \bigoplus_\pi m_\pi \pi,$$

where π runs over the set of equivalence classes of irreducible unitary representations of $G(\mathbf{A}_E)$ (or equivalently, of $C_c^\infty(G(\mathbf{A}_E))$).

The operators $R(\phi)$ are of trace class, [A, pp. 14-15]. The multiplicities m_π can be determined, in principle, by calculating the distribution $\phi \mapsto trR(\phi)$, as follows. The irreducible representation π of $G(\mathbf{A}_E)$ can be written, non-canonically, as a restricted tensor product [cf. A, (2.2)]

$$\pi \xrightarrow{\sim} \bar{\pi}_\infty \otimes \bar{\pi}_f = \bigotimes_v{}' \bar{\pi}_v$$

where $\bar{\pi}_\infty$, resp. $\bar{\pi}_f$, resp. $\bar{\pi}_v$, is an irreducible unitary representation of G_∞, resp. $G(\mathbf{A}^f{}_E)$, resp. $G(E_v)$]. Let $K^f \subset G(\mathbf{A}^f{}_E)$ be a compact open subgroup, which we assume to be of the form $K^f = \prod_v K_v$, where v runs over non-archimedean places of E and each K_v is a compact open subgroup of $G(E_v)$, maximal compact for almost all v.

Fact 1.2. *For almost all v, π_v is* unramified*, i.e.the space $\bar{\pi}_v^{K_v}$ of K_v-invariant vectors in $\bar{\pi}_v$ is one-dimensional.*

We let $\pi_v \subset \bar{\pi}_v$ be the subspace of K_v-finite vectors, and we also write π for the restricted tensor product of the π_v. For v-non-archimedean, π_v is a *smooth* representation – every vector is fixed by an open compact subgroup of K_v – and is moreover always an *admissible* representation – for every open subgroup $U \subset K_v$, the subspace π_v^U is finite-dimensional.

The algebra $C_c^\infty(G(\mathbf{A}_E))$ can be written as a restricted tensor product of local algebras $C_c^\infty(G(E_v))$; every ϕ can be written as a finite sum of *factorizable* test functions: infinite tensor products $\otimes\phi_v$ where, for almost all non-archimedean v, ϕ_v is the characteristic function 1_{K_v} of the chosen K_v. We decompose $dg^f = \prod dg_v$, where dg_v is a Haar measure on $G(E_v)$. We can arrange that, for almost all non-archimedean v,

$$\int_{G(E_v)} 1_{K_v} dg_v = 1.$$

For non-archimedean v, the action of $C_c^\infty(G(E_v))$ on π_v, defined by

$$\pi_v(\phi_v)(w) = \int_{G(E_v)} \pi_v(g)(w)\phi_v(g)dg_v, w \in \pi_v, \phi_v \in C_c^\infty(G(E_v))$$

is a finite sum with finite-dimensional image. In particular, $\pi_v(\phi_v)$ is a trace class operator. The same is true for archimedean v, provided on takes ϕ_v to be left and right K_v-finite. We assume this to be the case henceforward. The *character* of π_v is the distribution $\phi_v \mapsto tr\ \pi_v(\phi_v)$. It is invariant under conjugation by $G(E_v)$, or simply an *invariant distribution.*

When π_v is unramified $tr\ \pi_v(1_{K_v}) = \int_{G(E_v)} 1_{K_v} dg_v = 1$ for almost all v. Thus when $\phi = \otimes\phi_v$, the infinite product

$$tr\ \pi(\phi) = \prod_v tr\ \pi_v(\phi_v)$$

is actually finite and well-defined, and extends linearly to all K_∞-finite elements in $C_c^\infty(G(\mathbf{A}_E))$.

Fact 1.3. *(a) The distribution $\phi \mapsto tr\ \pi$ determines π up to isomorphism.*

(b) The sum $\sum m_\pi tr\ \pi$ of distributions converges to the distribution $tr\ R$.

(c) (linear independence of characters) *The distribution $tr\ R$ determines the m_π uniquely.*

We emphasize that the distributions $tr\ \pi$ and $tr\ R$ depend on the choice of Haar measure dg.

What we have described so far is a straightforward generalization of the theory of characters of finite-dimensional representations of finite groups, though the proofs are in general not so straightforward. In the case of finite groups, the class function analogous to $tr\ R$ is the character of the regular representation on a coset space, and thus can be written as the sum of characteristic functions of conjugacy classes with explicit coefficients. The corresponding assertion for the automorphic trace trR is the *Selberg trace formula.* The characteristic functions of conjugacy classes are replaced by *orbital integrals.* Suppose $\gamma \in G(\mathbf{A})$ and $\phi \in C_c^\infty(G(E_v))$ is a K_∞-finite test function. Let $H_\gamma \subset G(\mathbf{A})$ denote the centralizer of γ. Suppose for the moment that H_γ has a Haar measure dg_γ, and let $d\dot{g} = dg/dg_\gamma$ denote the quotient measure on $G_\gamma\backslash G(\mathbf{A})$. We define the orbital integral

$$O_\gamma(\phi) = \Phi_{G(\mathbf{A})}(\gamma, \phi) = \int_{H_\gamma\backslash G(\mathbf{A})} \phi(x^{-1}\gamma x) d\dot{x}. \tag{1.4}$$

The two expressions on the left correspond to two systems of notation used in the literature.

Convergence of the integral (1.4) is a hypothesis that is automatically satisfied in favorable circumstances, for example when $\gamma \in G(E)$ for anisotropic G. In that case H_γ is the group of $\mathbf{A}_E$-points of the centralizer G_γ of γ, an E-rational algebraic subgroup of G. Since G is anisotropic over E, and $\gamma \in G(E)$ is necessarily semi-simple, and G_γ is thus reductive. It is often convenient to assume the derived subgroup $G^{der} \subset G$ is *simply connected*; then G_γ is even a connected reductive group. With or without this hypothesis, the map $\phi \mapsto O_\gamma(\phi)$ defines a $(G(\mathbf{A}_E)$-$)$invariant distribution on $C_c^\infty(G(\mathbf{A}_E))$. It depends both on the choice of measure dg_γ on $G_\gamma(\mathbf{A}_E)$ as well as on dg. On the other hand, dg_γ defines a volume

$$v_\gamma = \int_{G_\gamma(E)\backslash G_\gamma(\mathbf{A}_E)} dg_\gamma.$$

Theorem 1.5 (Selberg trace formula, anisotropic case). *There is an identity of distributions:*

$$trR(\phi) = \sum_{\gamma} v_{\gamma} O_{\gamma}(\phi)$$

where γ runs over $G(E)$-conjugacy classes in $G(E)$ and, for any fixed choice of ϕ, the sum of orbital integrals is finite. In other words

$$\sum_{\gamma} v_{\gamma} O_{\gamma}(\phi) = \sum_{\pi} m_{\pi} tr\ \pi(\phi).$$

The left-hand side is called the *geometric side* of the trace formula, the right-hand side is the *spectral side.*

Remark. We write $\phi = \phi_{\infty} \otimes \phi_f$. The function ϕ_f is biinvariant under some open compact subgroup $K_f \subset G(\mathbf{A}^f)$, and thus the sum on the right hand side only involves representations with K_f-fixed vectors. In practice, we will ϕ_{∞} such that $Tr(\pi(\phi_{\infty})$ is non-zero only when π is a cohomological representation with fixed infinitesimal character. It then (almost) follows that the spectral side is finite for the given ϕ. This is not quite true if G is anisotropic modulo its center but if Z_G has a non-trivial split component. There are various ways to modify the trace formula to avoid this problem, some of which are illustrated in the articles of Labesse and Clozel-Moeglin. We will ignore the question in this introduction, which is mainly concerned with local properties of the geometric side.

The finiteness of the geometric side in the anisotropic case is a consequence of reduction theory.

The coefficients v_{γ} and m_{π} are global and cannot be simplified further, although when dg_{γ} is chosen to be Tamagawa measure v_{γ} can be expressed in terms of cohomological data. When dg_{γ} and dg are Tamagawa measures, we write $\tau(G_{\gamma})$ in place of v_{γ}; thus

$$\sum_{\gamma} \tau(G_{\gamma}) O_{\gamma}(\phi) = \sum_{\pi} m_{\pi} tr\ \pi(\phi). \tag{1.6}$$

The terms $tr\ \pi(\phi)$ and $O_{\gamma}(\phi)$, on the other hand, are local: if ϕ is factorizable, then so are the distributions. We have already seen this for $tr\ \pi$, and for the orbital integrals it is even easier:

$$O_{\gamma}(\phi) = \prod_{v} O_{\gamma}(\phi_v)$$

where γ is viewed as an element of $G(E_v)$ and

$$O_{\gamma}(\phi_v) = \int_{G_{\gamma}(E_v)\backslash G(E_v)} \phi_v(x^{-1}\gamma x) d\dot{x}_v,$$

with $d\dot{x} = \prod d\dot{x}_v$ a factorization of the quotient measure.

When G/Z_G is anisotropic but Z_G is not, there are various ways to modify these constructions to compensate for the fact that $Z_G(E)\backslash Z_G(\mathbf{A}_E)$ has infinite volume.

When G is itself not anisotropic, Arthur, extending Selberg's constructions for $GL(2)$ and some other rank one groups, generalized Theorem 1.5 by generalizing both sides of the equality. Although $L_2(G(E)\backslash G(\mathbf{A}_E))$ is no longer a direct sum of irreducible automorphic representations, it does decompose as a direct integral of unitary representations. In particular, the subspace $L_2^d(G(E)\backslash G(\mathbf{A}_E))$ defined as the direct sum of all irreducible direct summands of $L_2(G(E)\backslash G(\mathbf{A}_E))$ – the *discrete spectrum* of $G(E)\backslash G(\mathbf{A}_E))$ has a character that can be placed on the spectral side of the Arthur-Selberg trace formula. It is unfortunately not alone, and needs to be accompanied by quite complicated integral terms. On the geometric side, the problem is that it is no longer true that every $\gamma \in G(E)$ is semisimple, and this leads to the appearance of complicated terms on this side as well. Under favorable choices of test functions, all these bad terms disappear and we are left with a formula very similar to Theorem 1.5, called the *simple trace formula.* For the heuristic purposes of this introduction, it will be enough to restrict our attention to the anisotropic case.

2. Simple applications of the trace formula

It should be taken as axiomatic that the spectral side of the trace formula is intrinsically interesting, whereas the geometric side lends itself to calculation. Thus it is in principle possible to obtain information about the spectral side by calculating the geometric side for carefully chosen test functions ϕ. In what follows we write G_v in place of $G(E_v)$.

Example 2.1: Limit multiplicities and pseudocoefficients. The first example develops a theme that goes back to Langlands' early work on the dimensions of spaces of automorphic forms. Here we follow Clozel's approach in [C1], though only in the simplest cases. As before, we assume G is anisotropic, and to simplify the discussion we assume G semisimple as well. Let S be a finite set of places of E, including all archimedean places, and for each $v \in S$ choose an irreducible admissible representation π_v^0 of G_v. Suppose for each v we can choose a test function ϕ_v such that

$$tr\ \pi_v^0(\phi_v) = 1,\ tr\ \pi_v(\phi_v) = 0, \pi_v \neq \pi_v^0, \tag{2.1.1}$$

where π_v runs over all irreducible admissible representations of G_v. For nonarchimedean places $w \notin S$ we let $\phi_w = 1_{K_w}$, so that

$$tr\ \pi_w(\phi_w) = vol(K_w) \dim \pi_w^{K_w}, \forall \pi_w. \tag{2.1.2}$$

Let $K^S = \prod_{w\notin S} K_w$, $\pi_S^0 = \otimes_{v\in S}\pi_v^0$. With this choice of test functions the spectral side of the trace formula would then be

$$\sum_{\pi=\pi_S^0\otimes\pi^S} m_\pi vol(K^S) \dim(\pi^S)^{K^S}. \tag{2.1.3}$$

In other words, this choice of ϕ counts the automorphic forms of type π_S at S and of level K^S away from S. For example, suppose S equals the set of archimedean places, G_v is the automorphism group of a hermitian symmetric domain D_v for every archimedean v, and π_v^0 is the representation corresponding to holomorphic modular forms of some fixed weight on D_v. Then $m_(\pi)$ counts the holomorphic modular forms of given weight and level K^S.

Functions with property (2.1.1) can be found when v is nonarchimedean, π_v^0 is supercuspidal, and the center $Z_G(E_v)$ is compact. Indeed, supercuspidal representations π_v^0 are characterized by the condition that their *matrix coefficients*

$$\phi_{e,e^\vee}(g) = e^\vee(\pi_v^0(g)e), e \in \pi_v^0, e^\vee \in (\pi_v^0)^\vee$$

are compactly supported modulo $Z_G(E_v)$, which we are assuming finite. Thus under these hypotheses, such a $\phi_{e,e^\vee} \in C_c^\infty(G_v)$. On the other hand, just as for finite groups, the matrix coefficient of a supercuspidal representation π_v^0 acts as zero, hence *a fortiori* has trace zero, on any representation other than π_v^0, but for appropriate choices of $e, e^\vee$ has non-zero trace on π_v^0. Thus, up to multiplication by a scalar, matrix coefficients of supercuspidal representations satisfy (2.1.1). The same is true when v is archimedean and G_v is compact.

Assume for the moment, then, that G_v is compact for all archimedean v, and that π_v^0 is supercuspidal for finite v in S. When $K^S = K_f$ is sufficiently small an argument based on the discreteness of $G(E)$ in $G(\mathbf{A})$ implies that the only conjugacy class that meets the support of ϕ is the class of the identity element e [C1, Lemma 5]. Thus the geometric side of the trace formula is reduced to the single term $v_e O_e(\phi) = v_e\phi(e)$. We thus obtain

(2.1.4) $$\sum_\pi vol(K^S)\dim(\pi^S)^{K^S} \text{ is constant for sufficiently small } K^S.$$

When the compactness and supercuspidality hypotheses are not satisfied, there are in general no test functions satisfying (2.1.1). Matrix coefficients of discrete series representations can be used, provided they are integrable, but they are not compactly supported, and a different version of the trace formula is needed. However, one can make do with a weaker hypothesis

(2.1.1(bis)) $$tr\ \pi_v^0(\phi_v) = 1,\ tr\ \pi_v(\phi_v) = 0, \pi_v \neq \pi_v^0, \pi_v \textbf{ tempered.}$$

Test functions satisfying (2.1.1)(bis) are called *pseudocoefficients* and are known to exist for all square integrable representations; this is due to Clozel and Delorme for real groups and to Bernstein, Deligne, and Kazhdan for p-adic groups. On the other hand, it can be shown that the non-tempered representations that have non-zero trace for a given pseudocoefficient have strictly smaller limit multiplicities than discrete series representations. (The relevant arguments are in [C1].) This can be made precise, but the upshot is that (2.1.4) remains true when π_S is merely discrete series, with the word "constant" replaced by "asymptotically constant." See [C1] for details.

Example 2.2: Jacquet-Langlands transfer.

We present the Jacquet-Langlands correspondence in the simplest possible setting. Let E be a totally real field of even degree. Then up to isomorphism there exists a unique quaternion algebra D over E such that for all archimedean places v D_v is ramified, i.e., isomorphic to the Hamiltonion quaternions, and such that for all non-archimedean v $D_v \simeq M(2, E_v)$. Let D' be a second quaternion algebra over E that is ramified at all archimedean places but not isomorphic to D. Then there exists a non-empty finite set S of non-archimedean places of E such that D'_v

is a division algebra if and only if $v \in S$ or v is archimedean; moreover, S has even cardinality. We let $G = D^{\times}$, $G' = D'^{,\times}$, viewed as group schemes over $Spec(E)$. Thus $G_v \xrightarrow{\sim} GL(2, E_v)$ for all finite v, and $G'_v \xrightarrow{\sim} G_v$ for $v \notin S$, but G'_v is an inner form of $GL(2, E_v)$, anistropic modulo its center $E_v^{\times}$, for $v \in S$.

For $v \notin S$ we may identify

$$C_c^{\infty}(G_v) \xrightarrow{\sim} C_c^{\infty}(G'_v), \tag{2.2.1}$$

The isomorphism (2.2.1) is based on an isomorphism of groups and is unique up to inner automorphism. In particular, (2.2.1) determines an isomorphism of spaces of invariant distributions (such as orbital integrals) which is canonical up to the action of the center.

We let $N : G \to GL(1)$ denote the reduced norm, and use the same notation for G'. For v archimedean all irreducible representations of $G_v \xrightarrow{\sim} G'_v$ are finite-dimensional. The space of C^{∞} K_v-finite functions is spanned, up to the action of the center, by the matrix coefficients of these finite-dimensional representations, and we would like to use these as test functions. However, these matrix coefficients are not compactly supported. There are three ways to get around this problem: (a) work with the class of test functions that transform under a fixed character of the center, and with the version of the trace formula adapted to this situation; (b) truncate the matrix coefficient ϕ_v^1 by multiplying it by $c_V(N(g))$, where $V \subset \mathbb{R}^{\times}$ is a compact subset and c_V is the characteristic function of V; (c) ignore it. In other situations we will be attempted to follow the strategy (c), but here we can use (b), it being understood that as V varies among compact sets and ϕ_v^1 varies among matrix coefficients of finite-dimensional representations, the functions $\phi_{V,v}^1 = c_V \circ N \cdot \phi_v^1$ separate representations. To simplify further, we will choose as test functions for G_v as well as for G'_v the (truncations of the) function $\phi_v^1 \equiv 1$, the matrix coefficient of the trivial representation, for all archimedean v.

For $v \in S$ the groups G_v and G'_v are not isomorphic. However, the conjugacy classes in G'_v can be identified with a subset of the conjugacy classes of G_v. Every element $\gamma' \in G'_v$ generates the subalgebra $E_v[\gamma'] \subset D'_v$. Since D'_v is a division algebra, $E_v[\gamma']$ is a field, which equals E_v if $\gamma' \in Z_{G'_v}$; otherwise, $[E_v[\gamma'] : E_v] = 2$. The conjugacy class of γ' in D'_v is completely determined by the minimal polynomial of γ' over E_v, which is irreducible of degree 1 or 2. Given such a minimal polynomial $P_{\gamma'}$, we can associate to it a conjugacy class $[\gamma] \subset G_v = GL(2, E_v)$. If $\gamma \in [\gamma]$, then its minimal polynomial equals $P_{\gamma'}$, and indeed $E_v[\gamma] \xrightarrow{\sim} E_v[\gamma']$. Such conjugacy classes in G_v are called *ellliptic*; they are characterized by the fact that their minimal polynomials are irreducible, or equivalently by the fact that $E_v[\gamma] \subset M(2, E_v)$ is a field. We say that the conjugacy classes $[\gamma'] \subset G'_v$ and $[\gamma] \subset G_v$ are *associated*, and write $\gamma \leftrightarrow \gamma'$. A conjugacy class in G_v has an associated conjugacy class in G'_v if and only if it is elliptic.

The keys to comparing automorphic representations of G and of G' are contained in the following list of facts:

Fact 2.2.2 (Existence of transfer). *To any function $\phi'_v \in C_c^{\infty}(G'_v)$ we can associate a function $\phi_v \in C_c^{\infty}(G_v)$ with the properties*

$$O_{\gamma}(\phi_v) = O_{\gamma'}(\phi'_v), \gamma \leftrightarrow \gamma' \ \textit{regular} \tag{2.2.2.1}$$

$$O_{\gamma}(\phi_v) = 0 \ \textit{if} \ \gamma \in G_v \ \textit{is semisimple non-elliptic.} \tag{2.2.2.2}$$

The function ϕ_v is called a transfer *of ϕ'_v, and we write $\phi_v \leftrightarrow \phi'_v$.*

Fact 2.2.3 (Density of orbital integrals). *The function ϕ_v of (2.2.2) is not unique but the properties (2.2.2.1) and (2.2.2.2) suffice to determine tr $\pi(\phi_v)$ for any irreducible admissible representation π of G_v.*

Implicit in (2.2.2) is the possibility to associate Haar measures on G'_v and G_v. Since the centralizers of associated elements are isomorphic – $(G'_v)_\gamma \xrightarrow{\sim} (G_v)_\gamma$ – this suffices to pin down the quotient measures. We will have more to say about associated measures in subsequent sections.

Fact 2.2.4 (local Jacquet-Langlands correspondence). *To any irreducible admissible representation π'_v of G'_v, there exists an irreducible admissible representation π_v of G_v, defined up to isomorphism, with the following properties.*

(a) *If $\dim \pi'_v > 1$, then π_v is supercuspidal; if $\pi'_v = \chi \circ N$ for some character $\chi : E_v^\times \to \mathbb{C}^\times$, then $\pi_v \xrightarrow{\sim} St \otimes \chi \circ \det$, where St is the Steinberg representation of $GL(2, E_v)$; We write $\pi_v = JL(\pi'_v)$.*

(b) *Let $\phi'_v \in C_c^\infty(G'_v)$ with transfer $\phi_v \in C_c^\infty(G_v)$. Then*

$$tr\ JL(\pi'_v)(\phi_v) = -tr\ \pi'_v(\phi'_v);$$

$tr\ \pi_v(\phi_v) = 0$ if π_v is neither one-dimensional nor of the form $JL(\pi'_v)$.

(c) *The space of distributions tr $JL(\pi'_v)$, as π'_v varies over all irreducible admissible representations of G'_v, separates the transfers to G_v of functions in $C_c^\infty(G'_v)$. More precisely, suppose $\{\pi'_{i,v}, i \in I\}$ is a finite set and $a_i \in \mathbb{C}$ for all $i \in I$. Suppose*

$$\sum a_i tr\ JL(\pi'_{i,v})(\phi_v) = 0$$

for all transfers ϕ_v from $C_c^\infty(G'_v)$. Then $a_i = 0$ for all i.

This fact is assumed for simplicity of exposition. In fact, it was obtained by Jacquet and Langlands as a consequence of the constructions we are about to present. Note again that it is assumed implicitly that measures are associated. Condition (c) is a simple consequence of the facts already presented; one can even let ϕ_v run over the space of all test functions on G_v whose orbital integrals vanish off the elliptic set, since these spaces are the same.

We need one more fact:

Fact 2.2.5 (global transfer of conjugacy classes). *Let $(\gamma_v) \in G(\mathbf{A}_E)$ be a conjugacy class. Suppose there exists $\gamma' \in G'(E)$ such that, for all $v \notin S$, γ' is conjugate to γ_v in $G_v \simeq G'_v$ and such that, for $v \in S$, $\gamma_v \leftrightarrow \gamma'$. Then there exists $\gamma \in G(E)$ such that γ is conjugate to γ_v for all v.*

Conversely, suppose $\gamma \in G(E)$, and suppose γ is elliptic at all $v \in S$. Then there exists $\gamma' \in G'(E)$ such that γ' is conjugate to γ at all $v \notin S$, and $\gamma \leftrightarrow \gamma'$ at all $v \in S$.

We write $\gamma \leftrightarrow \gamma'$. Then $\tau(G_\gamma) = \tau(G_{\gamma'})$ (equality of Tamagawa measure).

In the present case most of this is easy: γ' generates a quadratic extension $E[\gamma']$ of E that embeds locally in D' everywhere. Since there is no additional local obstruction to embedding in D, $E[\gamma']$ also embeds globally in D, and we let γ denote the image of γ' under any such embedding; it is well-defined up to conjugacy. Conversely, if γ is as in the second paragraph, the local hypotheses at v guarantee that $E[\gamma]$ embeds locally in D' everywhere, hence there is a global embedding, which defines γ' up to conjugacy. For groups other than multiplicative groups of division algebras, this is a much more serious issue.

The equality of Tamagawa measures is a special case of a theorem due in complete generality to Kottwitz. At this point it suffices to remark that when Tamagawa measures are taken on both sides, then it can be assumed that local measures are associated as implicit in (2.2.2) and (2.2.4)

Now let $\phi' \in C_c^\infty(G'(\mathbf{A}_E))$ be a factorizable test function, with $\phi'_v = \phi^1_{V,v} = c_V \circ N \cdot 1$ for every archimedean v. Define a factorizable function $\phi \in C_c^\infty(G(\mathbf{A}_E))$ by

$$\phi_v = \phi'_v, v \notin S \text{ (via (2.2.1))}; \quad \phi_v \leftrightarrow \phi'_v, v \in S.$$

We write $\phi \leftrightarrow \phi'$.

Theorem 2.2.6 (global Jacquet-Langlands correspondence). *Let π' be an automorphic representation of G', and suppose $\dim \pi' > 1$, π'_v trivial for all archimedean v. Then there exists an automorphic representation $\pi = JL(\pi')$ of G, with*

$$\pi_v \xrightarrow{\sim} \pi'_v, v \notin S; \quad \pi_v = JL(\pi'_v), v \in S.$$

Moreover, $m(JL(\pi')) = m(\pi')$.

Proof. We first apply the Selberg trace formula to G', with test functions ϕ' as above. Write $\phi' = \phi'_S \otimes \phi'^{,S}$. Automorphic representations of G and G' can be factorized analogously. The spectral side of the trace formula is then

(2.2.6.1)
$$\sum_{\dim \pi' > 1, \pi'_\infty = 1} m(\pi') tr\ \pi'_S(\phi'_S) tr\ \pi'^{,S}(\phi'^{,S}) + \sum_{\dim \pi' = 1, \pi'_\infty = 1} m(\pi') tr\ \pi'_S(\phi'_S) tr\ \pi'^{,S}(\phi'^{,S}).$$

The restriction to π'_∞ is imposed by our choice of ϕ'_v at archimedean v, and is in fact completely unnecessary. Similarly, for $\phi \leftrightarrow \phi'$, the spectral side of the trace formula for G is

(2.2.6.2)
$$\sum_{\dim \pi > 1, \pi_\infty = 1} m(\pi) tr\ \pi_S(\phi_S) tr\ \pi^S(\phi^S) + \sum_{\dim \pi = 1, \pi_\infty = 1} m(\pi) tr\ \pi_S(\phi_S) tr\ \pi^S(\phi^S).$$

The geometric side of the trace formula for G' is

$$\sum_{\gamma'} v_{\gamma'} O_{\gamma'}(\phi'_S) \cdot O_{\gamma'}(\phi'^{,S}). \tag{2.2.6.3}$$

Likewise, the geometric side of the trace formula for G is

$$\sum_{\gamma} v_\gamma O_\gamma(\phi_S) \cdot O_\gamma(\phi^S), \tag{2.2.6.4}$$

where the sum is taken over all conjugacy classes γ in $G(E)$. Now G is anisotropic over E, so every $\gamma \in G(E)$ is semisimple, hence is still semisimple in G_v for every

v. But (2.2.2.2) implies that $O_\gamma(\phi_S) = 0$ unless γ is elliptic at all $v \in S$. Thus (2.2.5) implies that (2.2.6.4) equals

$$\begin{aligned}
&\sum_{\gamma', \gamma \leftrightarrow \gamma'} v_\gamma O_\gamma(\phi_S) \cdot O_\gamma(\phi^S) \\
&= \sum_{\gamma'} v_\gamma O_{\gamma'}(\phi'_S) \cdot O_{\gamma'}(\phi'^{,S}) \qquad (2.2.6.5) \\
&= \sum_{\gamma'} v_{\gamma'} O_{\gamma'}(\phi'_S) \cdot O_{\gamma'}(\phi'^{,S})
\end{aligned}$$

The second equality is a consequence of the last assertion in (2.2.5).

Now the last line is equal to (2.2.6.3). Thus the Selberg trace formula implies equality of (2.2.6.1) and (2.2.6.2). The one dimensional automorphic representations of G' and G correspond in an obvious way, and we thus obtain

(2.2.6.6)
$$\sum_{\dim \pi' > 1, \pi'_\infty = 1} m(\pi') tr\ \pi'_S(\phi'_S) tr\ \pi'^{,S}(\phi'^{,S}) = \sum_{\dim \pi > 1, \pi_\infty = 1} m(\pi) tr\ \pi_S(\phi_S) tr\ \pi^S(\phi^S).$$

Apart from the (unnecessary) restriction on π'_∞, the representations as well as the test functions on the left-hand side are completely general. By (2.2.4) (b) the right-hand side is the sum over π with π_v in the image of the local Jacquet-Langlands correspondence for all $v \in S$. The theorem is then a simple consequence of (2.2.4)(c) and general results on linear independence of characters of $G(\mathbf{A}_E^S)$, and we leave the details to the reader [until the next draft!!].

3. Stable conjugacy

Example 2.2 is perhaps the most elementary use of the trace formula to transfer automorphic representations from one group to another. In this case G' and G are inner forms of each other, but similar arguments are used when $G' = GL(n, E)$ and $G = GL(n, E')$, with E'/E a cyclic extension (cyclic base change, due to Langlands for $n = 2$ and Arthur-Clozel in general) or when $G' = GL(n, E)$, E is a cyclic extension of F of degree d, and $G = GL(nd, F)$ (automorphic induction, due to Arthur-Clozel, and revisited by Henniart-Herb). Each such case follows the general pattern of Example 2.2, especially the transfer of local and global conjugacy classes, and the transfer of orbital integrals. There are local difficulties, especially in obtaining the correct form of base change to substitute for (2.2.2.1) and (2.2.2.2), but the main difficulties are global and analytic and arise from the presence of non-elliptic and non-cuspidal terms in the general trace formula for isotropic groups.

A quite different sort of difficulty arises when G and G' are not inner forms of $GL(n)$. The main relevant property of $GL(n)$ is concealed in the discussion of transfer of associated conjugacy classes preceding (2.2.2). Semi-simple conjugacy classes in inner forms of $GL(n)$ are completely characterized by their characteristic polynomials. Similar characterizations are possible for other groups, but only over an algebraically closed field. In other words, two semi-simple elements of $G(F)$, where F is either a global or a local field, can belong to the same $G(\bar{F})$-conjugacy class but to distinct $G(F)$-conjugacy classes. It is often possible to associate the former for different groups, but there is no natural way to associate the latter. However, the geometric expansion of the trace formula is a sum of integrals over $G(F)$-conjugacy classes. The discrepancy between $G(F)$-conjugacy and $G(\bar{F})$-conjugacy, first studied when $G = SL(2)$ by Labesse and Langlands, is the origin of the stable trace formula.

Strictly speaking, the above remarks only apply when the derived subgroup $G^{der} \subset G$ is assumed simply connected, and we will make this assumption henceforward. We then say that two semi-simple elements $\gamma_1, \gamma_2 \in G(F)$ are *stably conjugate* if they belong to the same $G(\bar{F})$-conjugacy class; the intersections with $G(F)$ of semi-simple $G(\bar{F})$-conjugacy classes are called *stable conjugacy classes.* The *stable orbital integral* of a test function f attached to the semi-simple element $\gamma \in G(F)$ is the sum, weighted by sign factors to be described below, of the orbital integrals of f over the conjugacy classes in the stable conjugacy class associated to γ. For F local as well as global, we will soon see that this sum is finite, and its terms can be parametrized in terms of Galois cohomology. A *stable distribution* on $G(F)$, or $G(\mathbf{A}_F)$, is a distribution that is invariant over $G(\bar{F})$ (resp. $G(\bar{\mathbf{A}}_F)$) and not only over $G(F)$ (resp. $G(\mathbf{A}_F)$). The trace formula would be *stable* if it were a sum of stable distributions.

For some very special groups G, stable conjugacy and conjugacy coincide. We have already seen this for inner forms of $GL(n)$. It is also effectively the case globally for certain inner forms of unitary groups. This is the basis of the results of Kottwitz on the zeta functions of "simple Shimura varieties" and of Clozel and Labesse on stable base change from unitary groups to $GL(n)$. The work of Harris-Taylor on Galois representations at places of bad reduction of Shimura varieties was carried out in this setting, as was Harris' proof of a version of the Jacquet-Langlands correspondence between two such unitary groups. Certain test functions are also insensitive to the difference between stable conjugacy and $G(F)$-conjugacy. This was first observed by Kottwitz in his proof of Weil's conjecture on Tamagawa numbers, and was developed elsewhere, most recently in Labesse's book on stable base change [Lab] and in an article of Harris-Labesse [HL].

For general G and for general test functions these simplifications are not possible. Hence the problem arises of *stabilizing* the trace formula. The trace formula is not itself a sum of stable distributions. The error, however, can conjecturally be expressed in terms of stable distributions on other groups: these are the *endoscopic groups.* The stable trace formula is this conjectural expression. Several decades of work by a number of people have reduced the stabilization of the trace formula to a collection of conjectures, known collectively and informally as the "fundamental lemma," concerning a specific class orbital integrals on p-adic groups. The breakthrough in the proof of the fundamental lemma was finally made by Laumon and Ngô in the spring of 2004, and this is the motivation for the present collection of books.

The subsequent sections of the introduction roughly follow the notes of the lectures of Ngô at IHES in the spring of 2003. Ngô's notes sketch proofs of a number of the theorems stated here, and they are available for consultation at various locations on the internet, notable at http://www.ihes.fr/IHES/Scientifique/Seminaires/seminaire-LL.html. Complete proofs are of course in the articles quoted below.

The first step, due to Langlands and Kottwitz, is to determine the set of conjugacy classes in a stable conjugacy class. When F is a non-archimedean local field this is not bad at all. Let $\gamma \in G(F)$ be a semi-simple element, and let $I = I_\gamma \subset G$

denote its centralizer (also denoted G_γ). Since γ is semi-simple, I is a reductive group, and since G^{der} is simply connected, I is *connected.* This is the main reason we assume G^{der} simply connected; the general case can also be handled by reduction to this case.

Suppose γ' is stably conjugate to γ, i.e. there exists $g \in G(\bar{F})$ such that $g^{-1}\gamma g = \gamma'$. Let $\Gamma = Gal(\bar{F}/F)$. For any Γ-module M, we write $H^\bullet(F, M)$ for $H^\bullet(\Gamma, M)$. For any $\sigma \in \Gamma$,

$$g^{-1}\gamma g = \gamma' = \sigma(\gamma') = \sigma(g^{-1})\gamma\sigma(g),$$

hence $\sigma \mapsto g\sigma(g)^{-1}$ defines a 1-cocycle

$$c_\sigma \in H^1(F, I) := H^1(F, I(\bar{F})).$$

It is easy to see that the cohomology class of c_σ is independent of the choice of g, and that the conjugacy class of γ' depends only on this cohomology class. If γ is a regular element of G then I is a torus, and thus $H^1(F, I)$ is obviously an abelian group. On the other hand, the image of c_σ in $H^1(F, G)$ is obviously a coboundary; thus

$$c_\sigma \in \ker[H^1(F, I) \to H^1(F, G)] \tag{3.1}$$

In fact, because G^{der} is simply connected, it is a result of Kneser that $H^1(F, G^{der}) = 1$, and after some work it follows that $H^1(F, G)$ can be interpreted in terms of $H^1(F, D)$, where $D = G/G^{der}$ is a torus; in particular, the kernel in (3.1) is an abelian subgroup of $H^1(F, I)$.

We write $inv(\gamma, \gamma')$ for the class in $\ker[H^1(F, I) \to H^1(F, G)]$ corresponding to the conjugacy class of γ'.

After a bit more work, Kottwitz showed in [K3] that (3.1) is an abelian group even if γ is not regular, and indeed even if G^{der} is not simply connected. This has been reformulated systematically by Labesse in terms of a generalization of what Borovoi called the *abelianized Galois cohomology* of I and G. Labesse defined in a functorial way a set $H^0(F, I\backslash G)$ that fits into a short exact sequence:

$$0 \to I(F)\backslash G(F) \to H^0(F, I\backslash G) \to \ker[H^1(F, I) \to H^1(F, G)] \to 0. \tag{3.2}$$

The term $I(F)\backslash G(F)$ is naturally the $G(F)$-conjugacy class of γ, whereas the kernel on the right is a finite abelian group that parametrizes the conjugacy classes in the stable conjugacy class. Thus the term in the middle is the stable conjugacy class. We write $\gamma' \overset{st}{\sim} \gamma$ if γ' and γ are stably conjugate.

Fact 3.3. *For any $\gamma' \overset{st}{\sim} \gamma$, the centralizer $I_{\gamma'}$ is the inner form of I_γ defined as the image of the cocycle c_σ defined by γ' under the natural map $H^1(F, I_\gamma) \to H^1(F, I_{\gamma,ad})$. There is a map (the Kottwitz sign):*

$$e : \{\text{ Inner forms of } I_\gamma\} = H^1(F, I_{\gamma,ad}) \to \{\pm 1\}$$

The stable orbital integral $SO_\gamma(f)$ of $f \in C_c^\infty(G(F))$ is the sum

$$\sum_{\gamma' \overset{st}{\sim} \gamma} e(I_{\gamma'})O_{\gamma'}(f) = \int_{H^0(F, I\backslash G)} e(x) f(x^{-1}\gamma x) d\dot{x}.$$

The right-hand side is here treated as a suggestive expression whose precise definition can be found on p. 67 of [Lab].

Stable conjugacy over global fields has an analogous but more complicated treatment, to which we turn after introducing the L-group.

4. L-groups and stable conjugacy

We let F be either a local field or a number field, and let Γ denote either $Gal(\bar{F}/F)$ or the Weil group $W_{\bar{F}/F}$. Let G be a connected reductive group over F. Langlands' theory of the L-group begins with the observation that a connected reductive algebraic group $\mathcal{G}$ over an algebraically closed field (of characteristic zero, for simplicity) can be reconstructed up to canonical isomorphism in terms of the *based root datum* $\Psi_0(\mathcal{G}) = (X^*, \Delta^*, X_*, \Delta_*)$. Here T is a maximal torus of $\mathcal{G}$, $X^* = X^*(T)$, resp. X_* is its group of characters (resp. cocharacters), $B \supset T$ is a Borel subgroup, Δ^* is the set of positive simple roots of T in B, Δ_* the set of positive simple coroots. These data depend on a number of choices, but for any two choices the corresponding data are *canonically* isomorphic, hence the notation $\Psi_0(\mathcal{G})$ is justified.

The quadruple $(X^*, \Delta^*, X_*, \Delta_*)$ satisfies a collection of axioms including as a subset the axioms for a root datum satisfied by the first two items. It suffices to mention that any quadruple Ψ of the same type, with X^* and X_* finitely generated free abelian groups, and $\Delta^* \subset X^*$ and $\Delta_* \subset X_*$, that satisfy these axioms comes from a connected reductive group. The dual based root datum $\hat{\Psi} = (X_*, \Delta_*, X^*, \Delta^*)$, obtained by switching the first two items with the last two, also satisfies these axioms, and therefore defines up to canonical isomorphism a connected reductive algebraic group $\hat{\mathcal{G}}$ over $\mathbb{C}$ with $\Psi_0(\hat{\mathcal{G}}) \xrightarrow{\sim} \hat{\Psi}$.

We write $\hat{G}$ for $\hat{G_{\bar{F}}}$. The F-rational structure on $G_{\bar{F}}$ translates to an action of Γ on $\Psi_0(G_{\bar{F}})$, preserving the natural pairing between characters and cocharacters, and thus, almost, to an action of Γ on $\hat{G}$. To define such an action unambiguously one needs to choose a *splitting* ("épinglage") of $\hat{G}$, i.e., a triple $(\hat{T}, \hat{B}, \{X_\alpha, \alpha \in \hat{\Delta}))$, where $\hat{T}$ is a maximal torus of $\hat{G}$, $\hat{B} \supset \hat{T}$ a Borel subgroup, $\hat{\Delta}$ the set of positive simple roots of $\hat{T}$ in $\hat{B}$, and X_α is a non-zero element of the root space $\hat{\mathfrak{g}}_\alpha \subset Lie(\hat{B})$ for every $\alpha \in \hat{\Delta}$. An action of Γ on $\hat{G}$ that fixes some splitting is called an L-action, and an L-group ${}^LG$ of G is the semi-direct product of $\hat{G}$ with Γ with respect to a fixed L-action.

If $\mathcal{G}_1 \to \mathcal{G}_2$ is an F-homomorphism whose image is a normal subgroup, then the dual construction yields a homomorphism $\hat{\mathcal{G}}_2 \to \hat{\mathcal{G}}_1$ that is canonical up to $(\hat{\mathcal{G}}_1)^\Gamma$-conjugacy.

In the following section, we will begin with an object on the L-group side and use it to reconstruct endoscopic groups of H.

This construction has a number of subtle and unexpected properties that ought ideally to be mentioned here but are probably too difficult to motivate at this stage. The skeptical reader is advised that motivation will come through experience working with these notions (as I keep telling myself). It should not be hard to see that the L-groups of inner forms are isomorphic. For classical groups the L-groups are easy to calculate explicitly.

Example 4.1. Unitary groups. Let Φ_n be the matrix whose ij entry is $(-1)^{i+1}\delta_{i,n-j+1}$:

$$\Phi_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & -1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ (-1)^{n-1} & 0 & \dots & 0 & 0 \end{pmatrix}$$

Let $\mathcal{K}/E$ be a quadratic extension as above, and define an action of $Gal(\mathcal{K}/E)$ on

$\hat{G} = GL(n, \mathbb{C})$ by letting the non-trivial element c act as

$$c(g) = \Phi_n {}^t g^{-1} \Phi_n^{-1}.$$

Conjugation by Φ_n is necessary in order to preserve the standard splitting of $\hat{G}$ defined by the standard upper triangular Borel subgroup with its standard diagonal torus, and the standard basis of the simple root subspaces in $Lie(\hat{G})$ given by the matrices $X_{i,i+1}$, $i = 1, \ldots, n-1$, with entry 1 in the $(i, i+1)$ place and zero elsewhere: one checks that $c(X_{i,i+1}) = X_{n-i,n-i+1}$ for all i. Without this condition on the splitting, the Langlands parametrization of representations (local or automorphic) of G is just wrong.

An action of $Gal(\bar{E}/E)$, or of the Weil group W_E, on $\hat{G}$ is defined by projection onto $Gal(\mathcal{K}/E)$. The L-group in the Weil (resp. Galois) normalization is the semi-drect product of $\hat{G}$) with W_E (resp. $Gal(\bar{E}/E)$) with respect to this action.

The endoscopic groups that contribute to the stable trace formula are naturally defined in terms of the L-group. The first indication of this relation is given by the following determination, due to Langlands and Kottwitz [L,K2], of the cohomology group parametrizing the stable conjugacy class of a semi-simple element. We let $Z(\hat{G})$ denote the center of $\hat{G}$ and ^{D} denote Pontryagin dual.

Proposition 4.2 [K2]. *For any local field F, there is a canonical map*

$$H^1(F, G) \to \pi_0(Z(\hat{G})^\Gamma)^D,$$

functorial in G, that is an isomorphism if F is p-adic. In particular, if $\gamma \in G(F)$ is semi-simple, $I = I_\gamma$, then the stable conjugacy class of γ pairs canonically with

$$\pi_0(Z(\hat{I})^\Gamma / Z(\hat{G})^\Gamma) = \pi_0(Z(\hat{I})^\Gamma Z(\hat{G})/Z(\hat{G}))$$

and this pairing is a duality when F is p-adic.

Let F be any local field. If $\gamma \in G(F)$ is semi-simple, $\gamma' \overset{st}{\sim} \gamma$, we define

$$s \mapsto < inv(\gamma, \gamma'), s >$$

for the character $\pi_0(Z(\hat{I})^\Gamma / Z(\hat{G})^\Gamma) \to \mathbb{C}^\times$ defined by the pairing in (4.2). The determination of the set of conjugacy classes in a stable conjugacy class (in the p-adic case) in terms of the Langlands dual proceeds by replacing the cohomology of the reductive group G by the cohomology of a complex of tori. Proposition 4.2 is then derived as a consequence of, and generalization of, local Tate-Nakayama duality for cohomology of tori.

Global Tate-Nakayama duality for tori associates all the local duality maps by means of a reciprocity law. The generalization to arbitrary reductive groups is due to Langlands and Kottwitz. We begin by generalizing the notion of elliptic elements to arbitrary reductive groups.

Definition 4.3. *Let F be either a p-adic field or a number field. Let $\gamma_0 \in G(F)$ be a semi-simple element, with centralizer I. The element γ_0 is* **elliptic** *if $Z(I)^0/Z(G)^0$ is anisotropic. Equivalently, γ_0 is elliptic if it belongs to a maximal torus of G contained in no proper parabolic subgroup.*

Following the method pioneered by Harish-Chandra, harmonic analysis on p-adic and adelic groups is reduced by a procedure known as descent to the study of elliptic elements. When F is a number field we will only be considering anisotropic groups, all of whose elements are elliptic, and descent will not be an issue.

Definition 4.4. *Let F be a number field. Let $\gamma_0 \in G(F)$ be a semi-simple elliptic element, with centralizer I. Define*

$$\mathfrak{K}(\gamma_0) = \{c \in \pi_0((Z(\hat{I})/Z(\hat{G}))^\Gamma) \mid \forall v \; c \in \pi_0(Z(\hat{I})^{\Gamma_v} Z(\hat{G})/Z(\hat{G}))\} \tag{4.4.1}$$

where v runs over places of F, and Γ_v is a decomposition group at v.

Theorem 4.5 [K3, Theorem 6.6]. *Let $\gamma_0 \in G(F)$ be a semi-simple elliptic element, Let $\gamma = (\gamma_v) \in G(\mathbf{A}_F)$ and suppose γ_v is stably conjugate to γ_0 for all v. Then there exists $\gamma' \in G(F)$ in the $G(\mathbf{A}_F)$-conjugacy class of γ if and only if*

$$\sum_v inv(\gamma_0, \gamma_v) \mid_{\mathfrak{K}(\gamma_0)} = 0. \tag{4.5.1}$$

Kottwitz' statement has a slightly different form but is equivalent to this one. The proof of this theorem makes use of the Hasse principle for simply connected semi-simple groups. (When Kottwitz' article was written this had not yet been established for E_8, so he includes the hypothesis, no longer necessary, that G contain no E_8 factors.) The sum of local invariants is finite because at almost all places, γ_v and γ_0 belong to the same hyperspecial maximal compact subgroup. Excluding a finite set of places defined in terms of the divisors of $1 - \alpha(\gamma_0)$, where α runs through the roots of G not orthogonal to γ_0, it then follows that γ_v and γ_0 are in the same conjugacy class [K3, Prop. 7.1].

It is now relatively easy to complete the first stage of the stabilization of the trace formula. Details are contained in Ngô's notes as well as in the original sources. Theorem 4.5 provides a necessary and sufficient condition for the intersection of a $G(\mathbf{A}_F)$-conjugacy class with the $G(\mathbf{A}_F)$-stable conjugacy class of γ_0 to contain a $G(F)$-conjugacy class. The next proposition completes this condition by determining the number of $G(F)$-conjugacy classes in the $G(\mathbf{A}_F)$-conjugacy class of γ.

Proposition 4.6 [K3, §9]. *Let γ_0 and γ be as in Theorem 4.5, and suppose γ satisfies the reciprocity condition (4.5.1). Then the $G(\mathbf{A}_F)$-conjugacy class of γ contains*

$$|\mathfrak{K}(\gamma_0)|\tau(G)\tau(G_{\gamma_0})^{-1}$$

$G(F)$-conjugacy classes.

The geometric side of the trace formula can thus be simplified. First, let E_0 denote a set of representatives in $G(F)$ of the set of stable conjugacy classes in $G(F)$. Write $T_e(\phi)$ for the geometric side of the trace formula, i.e, for the sum of terms on the left of (1.6). These can be grouped according to stable conjugacy:

$$T_e(\phi) = \sum_\gamma \tau(G_\gamma) O_\gamma(\phi) = \sum_{\gamma_0 \in E_0} \tau(G_{\gamma_0}) \sum_{\gamma \overset{st}{\sim} \gamma_0} O_\gamma(\phi).$$

Here we have used the important theorem of Kottwitz that

$$\tau(G_\gamma) = \tau(G_{\gamma'}) \text{ if } \gamma \overset{st}{\sim} \gamma'. \tag{4.7}$$

Now $O_\gamma(\phi)$ is purely local: it only depends on the $G(\mathbf{A}_F)$-conjugacy class of γ. Proposition 4.6 allows us to group together the terms in the inner sum in terms of adelic conjugacy:

$$T_e(\phi) = \tau(G) \sum_{\gamma_0 \in E_0} |\mathfrak{K}(\gamma_0)| \sum_{\gamma \overset{st_{\mathbf{A}}}{\sim} \gamma_0} O_\gamma(\phi) \tag{4.7}$$

where $\gamma \overset{st_{\mathbf{A}}}{\sim} \gamma_0$ means $\gamma \in G(F)$ is stably conjugate to γ_0 but is taken only up to $G(\mathbf{A}_F)$-conjugacy.

The inner sum in (4.7) is a sum of local terms, indexed by a condition that is part local and part global. Theorem 4.5 allows us to separate these conditions by Fourier analysis on the group $\mathfrak{K}(\gamma_0)$:

$$\begin{aligned} T_e(\phi) &= \tau(G) \sum_{\gamma_0 \in E_0} \sum_{\kappa \in \mathfrak{K}(\gamma_0)} \prod_v \sum_{\gamma_v \overset{st}{\sim} \gamma_0} < \kappa, inv(\gamma_0, \gamma_v) > O_{\gamma_v}(\phi_v) \\ &= \tau(G) \sum_{\gamma_0 \in E_0} \sum_{\kappa \in \mathfrak{K}(\gamma_0)} \prod_v O^\kappa_{\gamma_0}(\phi_v). \end{aligned} \tag{4.8}$$

The expression $O^\kappa_{\gamma_0}(\phi_v)$, the κ-orbital integral, is defined locally by

$$O^\kappa_{\gamma_v}(\phi_v) = \sum_{\gamma_v \overset{st}{\sim} \gamma_0} e(I_{\gamma_v}) < \kappa, inv(\gamma_0, \gamma_v) > O_{\gamma_v}(\phi_v). \tag{4.9}$$

Thus when $\kappa = 1$ is the trivial character, we recover the stable orbital integral:

$$O^1_{\gamma_0}(\phi_v) = SO_{\gamma_0}(\phi_v).$$

The Kottwitz signs in (4.9) disappear in the product because all terms come from global groups and the product over all places of Kottwitz signs for a global group is $+1$.

In certain rare situations all the O^κ vanish except for $\kappa = 1$, and then (4.8) is a sum of stable distributions. In the case studied by Kottwitz in [K5], and taken up again in [HT], $\mathfrak{K}(\gamma_0)$ is always trivial, thus there are no non-trivial κ-orbital integrals. This is also possible for more general groups with appropriate choices of test functions.

Remark 4.10. Note that the abelian group $\mathfrak{K}(\gamma_0)$ is attached to the centralizer I rather than to the element γ_0. In the notation of [Lab], recalled in the appendix to this section,

$$\mathfrak{K}(\gamma_0) = \mathfrak{K}(I, G; F) = H^0_{ab}(\mathbf{A}_F/F, I\backslash G)^D.$$

Here and in the remainder of this section one could have followed Labesse and expressed all calculations in terms of abelianized Galois cohomology, without reference to the L-group. For example, Labesse's formalism expresses the global κ-orbital integral concisely as

$$\int_{H^0(\mathbf{A}_F, I\backslash G)} < \kappa, \dot{x} > \phi(x^{-1}\gamma_0 x) d\dot{x}.$$

The expression (4.4.1) will be necessary in subsequent sections in order to express the stabilization in terms of endoscopic groups.

Example 4.11: Stable conjugacy and κ in the case of unitary groups. We let $G = U(V)$ as in §1, and we suppose G to be anisotropic. Then any maximal torus of G is anisotropic. Let L_i be an extension of E of degree n_i, $\mathcal{K}_i = L_i \otimes_E \mathcal{K}$, and assume $\mathcal{K}_i$ is a field for all i. Let $T_i \subset R_{\mathcal{K}_i/E}\mathbb{G}_m$ be the kernel of the norm map $R_{\mathcal{K}_i/E}\mathbb{G}_m \to R_{L_i/E}\mathbb{G}_m$. Any maximal torus of G is isomorphic to one of the form $\prod_i T_i$, with T_i as above and $\sum_i n_i = n$. Conversely, any such torus embeds in the quasi-split inner form of G, and embeds in G provided certain local and global obstructions vanish.

Suppose $\gamma \in G(F)$ is a regular (semi-simple) element, with centralizer I isomorphic to $\prod_{i=1}^r T_i$ as above. Then $Z(\hat{I}) = \hat{I} = \prod \hat{T}_i$. The non-trivial element of the Galois group $Gal(\mathcal{K}/E) = Gal(\mathcal{K}/L_i)$ acts on each $\hat{T}_i$ by $x \mapsto x^{-1}$. Thus

$$\pi_0(Z(\hat{I})^\Gamma) = Z(\hat{I})^\Gamma = \prod_{i=1}^r \mathbb{Z}/2\mathbb{Z} = (\mathbb{Z}/2\mathbb{Z})^r.$$

It follows that

$$\mathfrak{K}_0(I) = \pi_0(Z(\hat{I})^\Gamma/Z(\hat{G})^\Gamma) \xrightarrow{\sim} (\mathbb{Z}/2\mathbb{Z})^{r-1}.$$

Whether or not γ is regular, $\mathfrak{K}_0(G_\gamma)$ is of the form $(\mathbb{Z}/2\mathbb{Z})^{r-1}$ for some $r \leq n$.

Remark 4.12. In the literature there are frequent references to "strongly regular" semisimple elements. In general groups, a semisimple element is **regular** if its centralizer is a diagonalizable subgroup, and **strongly regular** if it is a torus. In other words, an element is strongly regular if and only if it is regular and its centralizer is connected. In this introduction we are exclusively concerned with G for which G^{der} is simply connected, thus every regular element is strongly regular.

Appendix to §4: Labesse's cohomological formalism for endoscopy

In this introduction we have mainly adopted the notation of Langlands and Kottwitz for the Galois cohomology groups relevant to stable conjugacy, because most of the literature on the subject is written using this notation. Kottwitz' constructions have been given a more functorial treatment by Borovoi in [B]; this work was extended by Milne in [M]. Building on this approach, Labesse [Lab] found a unified cohomological formalism that applies to twisted as well as standard endoscopy; it is flexible as well as intuitively satisfying. Moreover, Labesse's cohomological objects can be defined without reference to Tate-Nakayama duality, though the latter remains the best tool for calculation. In this appendix, we state some of the main results of [Lab]. As an application, we use Labesse's formalism to classify unitary groups over local and global fields, a calculation that will be needed in the body of this text.

Inner forms of unitary groups.

Then the abelianized Galois cohomology in the sense of Borovoi [B] and [Lab] is easy to compute for U_n^* and $U_{n,ad}^*$. The abelianized cohomology of U_n^* is the cohomology of its cocenter U_1^*. The abelianized cohomology of $U_{n,ad}^*$ is, by definition, the hypercohomology of the crossed module $SU_n^* \to U_{n,ad}^*$. This complex is quasi-isomorphic to the complex of tori $U_1^* \xrightarrow{n} U_1^*$ and hence, up to a shift by 1, to the diagonalisable group $D_n = \ker[U_1^* \xrightarrow{n} U_1^*]$. In particular there are isomorphisms

$$\mathbf{H}^i_{ab}(F, U_n^*) \xrightarrow{\sim} \mathbf{H}^i(F, U_1^*) \quad \text{and} \quad \mathbf{H}^i_{ab}(F, U_{n,ad}^*) \xrightarrow{\sim} \mathbf{H}^{i+1}(F, D_n) .$$

Lemma 4.A.1.

(i) *We have*

$$\mathbf{H}^1_{ab}(\bullet, U^*_n) = \mathbb{Z}/2\mathbb{Z} \qquad \textit{and} \qquad \mathbf{H}^2_{ab}(\bullet, U^*_n) = 1$$

where $\bullet = F$ *(resp.* $\bullet = \mathbb{A}_F/F$*) when* E/F *is a quadratic extension of local (resp. global) fields. Moreover, if* F *is global, we have* $\ker^i(F, U^*_1) = 1$ *for* $i \geq 0$.

(ii) *We have*

$$\mathbf{H}^1_{ab}(\bullet, U^*_{n,ad}) = \mathbf{H}^2(\bullet, D_n) = \begin{array}{cc} 1 & \textit{if } n \textit{ is odd} \\ \mathbb{Z}/2\mathbb{Z} & \textit{if } n \textit{ is even} \end{array}$$

where $\bullet = F$ *(resp.* $\bullet = \mathbb{A}_F/F$*) when* E/F *is a quadratic extension of local (resp. global) fields. This is also the case if* $F = \mathbb{R}$ *and* $E = \mathbb{R} \oplus \mathbb{R}$ *. When* F *is a non archimedean local field and* $E = F \oplus F$ *then*

$$\mathbf{H}^1_{ab}(F, U^*_{n,ad}) = \mathbf{H}^2(F, D_n) = \mathbb{Z}/n\mathbb{Z} \ .$$

When $F = \mathbb{C}$ *then* $\mathbf{H}^2(F, D_n) = 1$ *.*

(iii) *When* E/F *is a quadratic extension of global fields the map*

$$\mathbf{H}^2(F_v, D_n) \to \mathbf{H}^2(\mathbb{A}_F/F, D_n)$$

is surjective, unless maybe when $F_v = \mathbb{C}$ *. Moreover* $\ker^i(F, D_n) = 1$ *for* $i \geq 2$ *.*

A. ssertions (i) and (ii) follow easily from the Tate-Nakayama isomorphism and from the above remarks. We still have to prove (iii). The co-localization map can be computed using Poitou-Tate duality [Ltam, Corollaire 2.2]; it is known that $\ker^i(F, D_n) = 1$ for $i \geq 3$ [Ltam, Corollaire 2.4]. We are left to prove that $\ker^2(F, D_n) = 1$. Using that

$$D_n = \ker[U^*_1 \xrightarrow{n} U^*_1]$$

we obtain, when n is odd, a commutative and exact diagram

$$\begin{array}{ccccccc} & & & & 1 & & 1 \\ & & & & \downarrow & & \downarrow \\ 1 & \to & \ker^2(F, D_n) & \to & \mathbf{H}^2(F, D_n) & \to & \mathbf{H}^2(\mathbb{A}_F, D_n) \\ & & & & \downarrow & & \downarrow \\ & & 1 & \to & \mathbf{H}^2(F, U^*_1) & \to & \mathbf{H}^2(\mathbb{A}_F, U^*_1) \end{array}$$

and the conclusion is clear in this case. When n is even we get a commutative and exact diagram

$$\begin{array}{ccccccc} & & & & 1 & & 1 \\ & & & & \downarrow & & \downarrow \\ & & 1 & \to & \mathbf{H}^1(F, U^*_1) & \to & \mathbf{H}^1(\mathbb{A}_F, U^*_1) \\ & & & & \downarrow & & \downarrow \\ 1 & \to & \ker^2(F, D_n) & \to & \mathbf{H}^2(F, D_n) & \to & \mathbf{H}^2(\mathbb{A}_F, D_n) \\ & & & & \downarrow & & \downarrow \\ & & 1 & \to & \mathbf{H}^2(F, U^*_1) & \to & \mathbf{H}^2(\mathbb{A}_F, U^*_1) \end{array}$$

and again the conclusion is easy.

Inner forms of U_n^* (resp. GU_n^*) will be denoted U_n (resp. GU_n) or simply U (resp. GU). The set of their isomorphism classes is in bijection with $\mathbf{H}^1(F, U_{n,ad}^*)$. An inner form of U_n^* (resp. GU_n^*) defines a class, called its invariant, in $\mathbf{H}^1_{ab}(F, U_{n,ad}^*)$ via the abelianization map

$$\mathbf{H}^1(F, U_{n,ad}^*) \to \mathbf{H}^1_{ab}(F, U_{n,ad}^*) \ .$$

There is also a natural map

$$\mathbf{H}^1(F, U_{n,ad}^*) \to \mathbf{H}^1(F, G_{n,ad}^*)$$

and hence an inner form U_n of U_n^* defines an inner form $Res_{E/F}B^\times$ of G_n^* where B is a simple algebra over E and we may regard U_n as the unitary group of the simple E-algebra B with an involution of the second kind, denoted $b \mapsto b^\dagger$. Then $U_n = \{x | xx^\dagger = 1\}$ and the F_0-similitude group GU_n is the subgroup of $Res_{E/F_0}B^\times$ such that, for any F_0-algebra R,

$$GU_n(R) = \{x \in (B \otimes_{F_0} R)^\times \mid x \cdot x^\dagger = \nu(x) \in R^\times\} \ .$$

We shall now recall the classification of inner forms of unitary groups over local and global fields (see also [C2]).

If F is local non archimedean, the invariant suffices to determine the inner form, since the abelianization map

$$\mathbf{H}^1(F, U_{n,ad}^*) \to \mathbf{H}^1_{ab}(F, U_{n,ad}^*)$$

is an isomorphism in this case. When $F = \mathbb{R}$ and $E = \mathbb{C}$ the set $\mathbf{H}^1(F, U_{n,ad}^*)$ is in natural bijection with signatures (p, q) with $p + q = n$ and $p \geq q$. The invariant is $[\frac{p-q}{2}]$ modulo 2 if n is even. The description in other cases is left to the reader.

Now let E/F be a quadratic extension of global fields. We want to describe the local-global obstructions i.e. the description of the image of the map $\mathbf{H}^1(F, U_{n,ad}^*) \to \mathbf{H}^1(\mathbb{A}_F, U_{n,ad}^*)$. This will be an immediate consequence of the next lemma.

Lemma 4.A.2. *The adjoint unitary group satisfies the Hasse principle i.e.*

$$\ker^1(F, U_{n,ad}^*) = 1 \ .$$

We have

$$\mathbf{H}^1_{ab}(\mathbb{A}_F/F, U_{n,ad}^*) = \begin{matrix} 1 & \textit{if } n \textit{ is odd} \\ \mathbb{Z}/2\mathbb{Z} & \textit{if } n \textit{ is even} \end{matrix}$$

and the natural map

$$\mathbf{H}^1_{ab}(F_v, U_{n,ad}^*) \to \mathbf{H}^1_{ab}(\mathbb{A}_F/F, U_{n,ad}^*)$$

is always surjective.

W. e first observe that, according to [Lab, Corollaire 1.6.11],

$$\ker^1(F, U_{n,ad}^*) = \ker^1_{ab}(F, U_{n,ad}^*) \ .$$

The assertions then follow from 4.A.1.

Proposition 4.A.3. *Let F be global field. There is at most one global inner form of U_n^* when the local ones are prescribed. There is no local-global obstruction for inner forms when n is odd. There is a parity condition when n is even: the sum of images of local invariants must zero in $\mathbb{Z}/2\mathbb{Z}$.*

T. he exact sequence

$$\ker^1(F,G) \to \mathbf{H}^1(F,G) \to \mathbf{H}^1(\mathbb{A}_F,G) \to \mathbf{H}^1_{ab}(\mathbb{A}_F/F,G) \to \ker^2_{ab}(F,G)$$

valid for any reductive group (see [Lab, Proposition 1.6.12]) reads in our case, in view of 4.A.1 and 4.A.2,

$$1 \to \mathbf{H}^1(F,U^*_{n,ad}) \to \mathbf{H}^1(\mathbb{A}_F,U^*_{n,ad}) \to \mathbf{H}^1_{ab}(\mathbb{A}_F/F,U^*_{n,ad}) \to 1 \ .$$

The Hasse principle implies that the local inner forms determine uniquely the global one. The local-global obstruction can be represented by a cohomology class in

$$\mathbf{H}^1_{ab}(\mathbb{A}_F/F,U^*_{n,ad})$$

and the proposition follows from 4.A.2.

5. Endoscopic groups

The goal of stabilization is to rewrite the expression (4.8) as a sum of stable trace formulas for groups over F, without reference to the finite groups $\mathfrak{K}(\gamma_0)$. The new sets of groups over F, called *endoscopic groups*, are derived directly from the elements $\kappa \in \mathfrak{K}(\gamma_0)$. There are two obvious obstacles to this program. In the first place, the conjugacy classes γ_0 belong to $G(F)$. In the second place, the test functions ϕ are defined on $G(\mathbf{A}_F)$. We have already seen in example 2.2 how to overcome these obstacles in a special case. Transfer of conjugacy classes to endoscopic groups was solved in the 1980s by Kottwitz, following the first steps of Langlands. It is fair to say that the endoscopic groups were devised in order to make this possible, given the expression (4.8). The difficulty has thus been shifted to the problem of transferring test functions to endoscopic groups. This is the main topic of the present volume and will only be discussed briefly in this introduction.

Version (4.8) of the trace formula writes it as a sum over pairs (γ_0,κ), where γ_0 is a stable elliptic conjugacy class in $G(F)$ and κ is an element of the group (4.4.1) defined in terms of the dual group $\hat{G}$. The goal is to rewrite it as a sum over pairs (γ^H,H) where H runs through a family of connected reductive groups over F and γ^H is a stable elliptic conjugacy class in $H(F)$ whose contribution to the sum is its stable orbital integral. This turns out to be too naive, and the object H needs to be endowed with additional structure in order to qualify as an endoscopic group.

Recall that κ is an element of the center of the dual group of the centralizer I of the elliptic element γ_0. With a great deal of hindsight, this might suggest that one look for H containing γ^H among groups whose L-groups contain $Z(\hat{I})$ as central subgroups. Here is a precise definition:

Definition 5.1. *An* **endoscopic triple** *for G is a datum of the form (H,s,ξ) where*

(i) *$s \in \hat{G}$ is a semi-simple element;*
(ii) *H is a quasi-split connected reductive group over F;*
(iii) *$\xi : \hat{H} \to \hat{G}$ is an injective homomorphism identifying $\xi(\hat{H}) = Z_{\hat{G}}(s)^0$.*

These data satisfy the following conditions:

(a) ξ *is* Γ*-equivariant up to* $\hat{G}$*-conjugation; i.e., for all* $\sigma \in \Gamma$*, there exists* $x_\sigma \in \hat{G}$ *such that*

$$\sigma_{\hat{G}} \circ \xi \sigma_{\hat{H}}^{-1} = ad(x_\sigma) \circ \xi$$

where $\sigma_{\hat{G}}$ *and* $\sigma_{\hat{H}}$ *are the respective* L*-actions of* Γ *on the dual groups;*

(b) *The image of* s *in* $Z(\hat{H})/Z(\hat{G})$ *is* Γ*-invariant and its image in* $H^1(F, Z(\hat{G}))$ *under the connecting homomorphism associated to the short exact sequence of* Γ*-modules:*

$$1 \to Z(\hat{G}) \to Z(\hat{H}) \to Z(\hat{H})/Z(\hat{G}) \to 1$$

is everywhere locally trivial (i.e., belongs to the Tate-Shafarevich group $ker^1(F, Z(\hat{G}))$*).*

Condition (b) should be recognized as a variant of the condition satisfied by κ in (4.4.1), see below. Denote by $\mathfrak{A}(H/F)$ the subgroup of $\pi_0(Z(\hat{H})/Z(\hat{G})^\Gamma)$ satisfying this condition. Condition (a) implies that ξ extends to an L-homomorphism, but this extension is not unique. The group H will be called an **endoscopic group**, though it should always be kept in mind that the full triple is needed for the sake of classification.

Definition 5.2. *An isomorphism between two endoscopic triples* (H, s, ξ) *and* (H', s', ξ') *for* G *is an* F*-isomorphism* $\alpha : H \to H'$ *such that*

(i) $\xi \circ \hat{\alpha} : \hat{H}' \to \hat{G}$ *and* ξ' *are* $\hat{G}$*-conjugate;*
(ii) *The isomorphism* $\mathfrak{A}(H'/F) \xrightarrow{\sim} \mathfrak{A}(H/F)$ *determined by (i) takes* s' *to* s*.*

In particular, the image $H^{ad}(F)$ of $H(F)$ in $Aut(H)$ is contained in the group $Aut((H, s, \xi))$ of automorphisms of the endoscopic triple. We let

$$\Lambda_H = Aut((H, s, \xi))/H^{ad}(F). \tag{5.3}$$

This is obviously a finite group, and we let λ_H denote its order.

Definition 5.4. *The endoscopic triple* (H, s, ξ) *is* **elliptic** *if* $\xi(Z(\hat{H})^\Gamma) \cdot Z(\hat{G})/Z(\hat{G})$ *is a finite group.*

We note that Definitions 5.1, 5.2, and 5.4 all make sense when F is a local field as well as a global field; we just have to replace “locally trivial” by “trivial” in 5.1(b).

5.5. Example: elliptic endoscopic triples for unitary groups.

Let $G = U(V)$, as in the previous examples. The following proposition is proved in Rogawski's book [R]:

Proposition 5.5.1 [R, 4.6.1]. *Let* (H, s, ξ) *be an elliptic endoscopic triple for* G*. Then* H *is isomorphic to* $U(a) \times U(b)$ *where* a *and* b *are non-negative integers such that* $a + b = n$*. The triple is determined by* $\{a, b\}$ *up to isomorphism. Furthermore,* Λ_H *is a group of order* 2 *if* $a = b$ *and is trivial otherwise.*

Sketch of proof. Since $\hat{G}^{der}$ is simply connected, ξ identifies $\hat{H}$ with the centralizer of the semisimple element $s \in \hat{G}$. Thus $\hat{H} = \prod_{i=1}^{t} GL(n_i)$ for some partition

$n = \sum n_i$. A simple reduction using property (b) of (5.1) shows we can assume $s \in Z(\hat{H})^\Gamma$, and that the action of Γ on H factors through $Gal(\mathcal{K}/E)$. Now Definition 5.4 implies that the identity component $Z(\hat{H})^{\Gamma,0}$ is contained in $Z(\hat{G})^\Gamma = \{\pm 1\}$, hence that $Z(\hat{H})^\Gamma$ is itself a finite group. This in turn implies that the non-trivial element $c \in Gal(\mathcal{K}/E)$ fixes the individual blocks $GL(n_i) \subset \hat{H}$ and acts non-trivially on the center $GL(1) \subset GL(n_i)$ for each i. In other words, $c(z_i) = z_i^{-1}$ for $z_i \in Z(GL(n_i))$. Thus $s = (z_1, \ldots, z_t)$ with each $z_i = \pm 1$. Since $\hat{H}$ is the centralizer of s, it follows that $t \leq 2$. Similar arguments prove the remaining assertions.

Application of the stable trace formula to unitary groups is complicated by the fact that the isomorphism between H, whose L-group is canonically the connected centralizer of s in $\hat{G}$, and a fixed product $U(a) \times U(b)$ of unitary groups, depends on an additional arbitrary choice. In the following discussion we let H denote the quasi-split E-group $U(a)^* \times U(b)^*$ (here the superscript * is standard notation for the quasi-split inner form, unique up to isomorphism). The transfer from H to G of parameters, and of automorphic representations, depends on the choice of an L-homomorphism

$$\xi : {}^L H \to {}^L G \tag{5.5.2}$$

which we normalize following Rogawski, p. 68 of [Montreal]. Let $\eta_{\mathcal{K}/E}$ denote the quadratic character of $\mathbf{A}_E^\times$ corresponding to the quadratic extension $\mathcal{K}$. Fix Hecke characters μ_a and μ_b of $\mathcal{K}$ which extend the characters $\eta_{\mathcal{K}/E}^a$ and $\eta_{\mathcal{K}/E}^b$ of the idèles of E. On $\hat{H} = {}^L H^0 = GL(a) \times GL(b)$, we have

$$\xi(h_a, h_b) = \begin{pmatrix} h_a & 0 \\ 0 & h_b \end{pmatrix} \in \hat{G} \subset {}^L G, \ \ h_a \in GL(a), h_b \in GL(b) \tag{5.5.3}$$

The Hecke characters μ_i can be viewed as a character of $W_\mathcal{K}$ by class field theory, and for $w \in W_\mathcal{K}$ we let

$$\xi(w) = \begin{pmatrix} \mu_b(w) I_a & 0 \\ 0 & \mu_a(w) I_b \end{pmatrix} \times w \in \hat{G} \ltimes W_\mathcal{K} \tag{5.5.4}$$

Finally, if $w_\sigma \in W_E$ is a representative of the non-trivial coset of $W_\mathcal{K}$, we set

$$\xi(w_\sigma) = \begin{pmatrix} \Phi_a & 0 \\ 0 & \Phi_b \end{pmatrix} \cdot \Phi_n^{-1} \times w_\sigma. \tag{5.5.5}$$

Here Φ_m, $m = a, b, n$, is defined as in (4.1).

One verifies that the formulas (5.5.3-5.5.5) define an L-homomorphism ξ, which we can denote ξ_{μ_a,μ_b} to emphasize the choices. The insertion of these characters is required in order to make the homomorphisms well-defined with respect to the Φ_m, and we have seen in (4.1) that these latter are required by the need to preserve the standard splittings.

5.6. Classification of elliptic endoscopic triples for other classical groups.

These are worked out in various places, for example in Waldspurger's Asterisque volume. Details will be provided in later versions.

Let (H, s, ξ) be an elliptic endoscopic triple for G. and let $\gamma_H \in H$ be an elliptic semi-simple element, T_H a maximal elliptic torus of the connected centralizer $I_H = I^0_{\gamma_H}$ of γ_H in H. There is a canonical G-conjugacy class of embeddings $j : T_H \to G$, defined over $\bar{F}$. Choose such a j and let $\gamma = j(\gamma_H)$. The stable conjugacy class of γ is independent of the choice of j and depends only on the stable conjugacy class of γ_H. We say $\gamma_0 \in G(F)$ is associated to γ_H, or "comes from " γ_H, or (more commonly) γ_H is **an image** of γ_0, if γ_0 is in the stable conjugacy class of $j(\gamma_H)$. If G is quasi-split, then a theorem of Steinberg and Kottwitz [K1] guarantees there is always such a γ_0. In general, there are both local and global obstructions. An example of local obstruction: at a real place v of F, γ_H has eigenvalues of absolute value > 1, but G_v is compact. The global obstruction belongs to a certain H^2.

The image $j(T_H) \subset G_{\bar{F}}$ is a maximal torus $T \subset G$. Thus roots of T in G define characters of T_H. We say γ_H is **(G, H)-regular**. if $\alpha(\gamma_H) \neq 1$ for any root of T in G that does not restrict to a root of H. This condition is independent of the choice of T_H and j.

We write $I_0 = G_{\gamma_0}$ if γ_0 comes from γ_H. If γ_H is (G, H)-regular, then j extends to an isomorphism $I_H \xrightarrow{\sim} I$ of centralizers, unique up to conjugacy by T. The element $s \in Z(\hat{H})$ can be viewed as an element of $Z(\hat{I_H})$ and thus as an element of $Z(\hat{I_0})$. Condition 5.1(b) then implies that the corresponding element of $\pi_0((Z(\hat{I})/Z(\hat{G}))^\Gamma)$ belongs to $|\mathfrak{K}(I_0)|$. We denote this element κ.

To the quadruple (H, s, ξ, γ_H), with γ_H (G, H)-regular, we have thus associated a pair (γ_0, κ), with γ_0 a stable conjugacy class in $G(F)$ and $\kappa \in \mathfrak{K}(I_0)$, provided G is quasi-split. If not, the pair (γ_0, κ) may or may not exist. We let

$$\pi : \{ \text{ quadruples } (H, s, \xi, \gamma_H)\} \to \{ \text{ pairs } (\gamma_0, \kappa)\}$$

be the partially defined map. The double sum over pairs (γ_0, κ) is transformed into a double sum over stable conjugacy classes in elliptic endoscopic groups by means of the following proposition:

Lemma 5.7. *The (partially defined) map π is surjective, and the fiber above any pair (γ_0, κ) in the image is either empty or contains λ_H quadruples.*

Following Langlands, define

$$i(G, H) = \tau(G)\tau(H)^{-1}\lambda_H.$$

Let $\mathfrak{E}$ denote a set of representatives of equivalence classes of of elliptic endoscopic triples. For any $(H, s, \xi) \in \mathfrak{E}$, let $E_{H,0}$ denote a set of representatives of (G, H)-regular stable conjugacy classes in $H(F)$. The above constructions allow us to rewrite the double sum over $E_0 \times \mathfrak{K}(I_0)$ in (4.8) as a double sum over $\mathfrak{E} \times E_{H,0}$:

$$T_e(\phi) = \sum_{(H,s,\xi)\in\mathfrak{E}} i(G, H)\tau(H) \sum_{\gamma_H\in E_{H,0}} O^\kappa_{\gamma_0}(\phi), \tag{5.8}$$

where $(\gamma_0, \kappa) = \pi((H, s, \xi), \gamma_H)$. This is obviously useless, because the term $SO^\kappa_{\gamma_0}(\phi)$ is still defined in terms of G rather than H. The next section explains how to rewrite $O^\kappa_{\gamma_0}(\phi)$ as a stable orbital integral on H.

6. Transfer and the fundamental lemma

The expression of $SO^{\kappa}_{\gamma_0}(\phi)$ in terms of stable orbital integrals on endoscopic groups is given explicitly by the following conjecture when γ_0 is a regular elliptic element.

Conjecture 6.1 (Langlands-Shelstad). *Let F be a local field. Let (H, s, ξ) be an elliptic endoscopic triple for G, Let $\phi \in C_c^\infty(G(F))$. Then there is a function $\phi^H \in C_c^\infty(H(F))$, well-defined as a functional on stably invariant distributions on H, such that, for if γ_H a stable semi-simple conjugacy class in H, $(\gamma_0, \kappa) = \pi((H, s, \xi), \gamma_H)$, with γ_0 is regular in G, we have*

$$SO_{\gamma_H}(\phi^H) = \sum_{\gamma \overset{st}{\sim} \gamma_0} \Delta(\gamma_H, \gamma) O_\gamma(\phi) \tag{6.1.1}$$

Here $\Delta(\gamma_H, \gamma)$ is the **transfer factor** *to be defined in §7.*

This conjecture – which is now a theorem, as we will see below – has been stated in various versions in the literature, and one of the objectives in Chapter II of this book is to explain the relations between these versions. The analogous result for real groups is due to Shelstad, and is explained in Chapter II.A. We immediately derive the expected conclusion. Let

$$ST^*_e(\phi^H) = \tau(H) \sum_{\gamma_H \in E_{H,0}} SO_{\gamma_H}(\phi^H)$$

where the sum runs over (G, H)-regular elliptic γ_H.

Theorem 6.2 (Kottwitz). *Assume Conjecture 6.1 (in Kottwitz' version, see Remark 6.3 (b)). Then (5.8) can be written*

$$T_e(\phi) = \sum_{(H,s,\xi) \in \mathfrak{E}} i(G, H) ST^*_e(\phi^H)$$

for a certain function $\phi^H \in C_c^\infty(H(\mathbf{A}_F))$, defined as the tensor product of local transfers of ϕ.

Remark 6.3. (a) Conjecture 6.1 is purely local, whereas Theorem 6.2 is global. The global test function ϕ can be considered factorizable as $\otimes_v \phi_v$, and then for almost all v ϕ_v is necessarily the characteristic function of a hyperspecial maximal compact subgroup. In order for the right-hand side of the expression in (6.2) to make sense, each ϕ^H has to have the same property. Thus Conjecture 6.1 needs to be supplemented by the Fundamental Lemma (Conjecture 6.4), to be stated below.

(b) Kottwitz' version (Conjecture 5.5 of [K3]) is an extension of Conjecture 6.1 to (G, H)-regular γ_H. Conjecture 6.1 as stated only suffices for an identity between the regular terms on both sides. Kottwitz' version is derived by Langlands and Shelstad from Conjecture 6.1 [LS2,Lemma 2.4.A].

(c) Theorem 6.2 is derived from the transfer conjectures by identifying the right-hand side of (6.1.1) with the κ-orbital integral of ϕ over γ. This depends on formal properties of the transfer factors, which will be explained in §7, at which point it will be explained in greater detail how one passes from (6.1.1) to (6.2).

The main theorem of [W1] is that the Langlands-Shelstad transfer conjecture is a consequence of the following special case, now a theorem thanks primarily to Ngô:

Theorem 6.4 (Fundamental Lemma). *Let G be an unramified group over the local field F, and let (H, s, ξ) be as in conjecture 6.1, with H also unramified. Let $K \subset G(F)$ (resp. $K_H \subset H(F)$ be a hyperspecial maximal compact subgroup of G (resp. H), $\phi = 1_K$ the characteristic function of K. Then there is a constant c, depending only on the measures of K and K_H, such that (6.1.1) holds with $\phi^H = c \cdot 1_{K_H}$ with 1_{K_H} the characteristic function of K_H.*

As mentioned in (6.3)(a), something of this sort is necessary even in order to formulate the stabilization globally. Waldspurger's method actually does not require the full strength of the Fundamental Lemma; in order to deduce the transfer conjecture, it suffices to know Theorem 6.4 locally in almost all residue characteristics. (See Chapter II.B. for an account of this and related work.) Moreover, Hales showed in [H2] that it suffices to know the fundamental lemma for unit elements in the Hecke algebra, for sufficiently large residue characteristics; cf. §9 for a complete statement. Thanks to Laumon-Ngô [LN] and Waldspurger [W3], this has now been established for unitary groups. The results of Laumon and Ngô are presented in Chapter II.C following the more recent work of Ngô [N1, N2] which proves the fundamental lemma for general reductive groups over local fields in positive characteristic. The results of [W3], that show that the Fundamental Lemma depends only on the residue field, are treated in Chapter II.D.. An alternative approach to the independence of the characteristic is presented in Chapter II.E.

More fundamentally, perhaps, Conjecture 6.4 is necessary in order to characterize the image of endoscopic transfer at (almost all) unramified places. This topic is treated in §9 and in more detail in Chapter III.C.

7. Formal properties of transfer factors

Transfer factors are combinatorially extremely involved, but the most important terms are roots of unity and the main complication is in the correct definition of signs. We begin by assuming G to be quasi-split; the endoscopic groups H are by definition quasi-split. Transfer factors can be derived for general G from those for quasi-split G. In the present section F is a non-archimedean local field and (H, s, ξ) is an elliptic endoscopic triple for G.

Let $\mathfrak{h}$ and $\mathfrak{g}$ denote the F-Lie algebras of H and G, respectively. If G is quasi-split, it admits a Γ-stable splitting (épinglage) $(T, B, \{X_\alpha, \alpha \in \Delta\})$. The data are as in §4; the Borel subgroup B and its maximal torus T are defined over F, and the root data X_α are Γ-stable in the sense that $\sigma(X_\alpha) = X_{\sigma(\alpha)}$ if $\sigma \in \Gamma$. Let $\mathfrak{b}$ be the Lie algebra of B. Let $X_+ = \sum_{\alpha\in\Delta} X_\alpha$. Then $X_+ \in \mathfrak{g}(F)$. For each $\alpha \in \Delta$ one can define $sl(2)$ triples $(X_\alpha, H_\alpha, X_{-\alpha})$ so that

$$[H_\alpha, X_{\pm\alpha}] = \pm 2X_{\pm\alpha}, [X_\alpha, X_{-\alpha}] = H_\alpha.$$

Then let $X_- = \sum_{\alpha\in\Delta} X_{-\alpha}$; X_- is also in $\mathfrak{g}(F)$.

We begin by observing that stable conjugacy can be defined for semisimple elements of $\mathfrak{g}$ just as for G, and are classified in the same way in terms of the cohomology of centralizers in G. Moreover, if $X, X' \in \mathfrak{g}$ are stably conjugate, then one can define an invariant

$$inv(X, X') \in \ker[H^1(F, I) \to H^1(F, G)]$$

if $I \subset G$ is the stabilizer of X.

Transfer factors for Lie algebras:

$$\Delta_{G,H}(\bullet, \bullet) : \mathfrak{h}^{G-reg}(F) \times \mathfrak{g}^{reg}(F) \to \mathbb{C}$$

were defined by Waldspurger. These factors are simpler than the corresponding factors for groups, and satisfy the relation

$$\Delta_{G,H}(X_H, X) = \Delta'(exp(X_H), exp(X)) \tag{7.1}$$

when X_H and X are sufficiently close to 0 in the corresponding Lie algebras. Here *exp* is the exponential function in the Lie algebra and Δ' is the Langlands-Shelstad transfer factor with the volume term removed (this will be explained below).

Theorem 7.2 (Kottwitz, [K6]). *Assume G is quasi-split.*

(a) *(Kostant) Any stable semisimple conjugacy class in $\mathfrak{g}$ contains an element in $X_- + \mathfrak{b}$; i.e. of the form $X_- + Y$ where $Y \in \mathfrak{b}$.*
(b) *Let $X_H \in H$ be a semisimple element that transfers to a regular stable conjugacy class $[X_G]$ in $\mathfrak{g}$. Let $X_G^{Kostant} \in \mathfrak{g}(F) \cap (X_- + \mathfrak{b})$ be stably conjugate to X_G. Then $\Delta_{G,H}(X_H, X_G^{Kostant}) = 1$.*
(c) *More generally*

$$\Delta_{G,H}(X_H, X_G) =< inv(X_G, X_G^{Kostant}), s >$$

(See Ngô's notes on transfer factors, §5.)

Remark 7.3. Let $\mathfrak{g}$ be a reductive Lie algebra over $\bar{F}$ with maximal torus $\mathfrak{t}$ and Weyl group W. The scheme $Spec(\mathfrak{t}^W)$ is isomorphic to the scheme-theoretic quotient of $\mathfrak{g}$ by the adjoint action of G and thus gives a coarse parametrization of conjugacy classes in $\mathfrak{g}$. The Kostant map defined above on semisimple conjugacy classes extends to a map from $Spec(\mathfrak{t}^W)$ to $\mathfrak{g}$ with image contained in the open subscheme of *regular* elements (including regular unipotent elements as well as regular semisimple elements). For example, when $\mathfrak{g} = \mathfrak{gl}_n$, $Spec(\mathfrak{t}^W)$ is the affine space of (coefficients of) characteristic polynomials, so the Kostant section gives a way of picking out an element with given characteristic polynomial, once a splitting has been fixed. The Kostant section is central in the geometric approach to endoscopy and should be recalled when reading the chapters on the fundamental lemma.

The general Langlands-Shelstad transfer factor $\Delta(\gamma_H, \gamma)$ is given as a product of five terms, denoted Δ_I, Δ_{II}, $\Delta_1 = \Delta_{III_1}$, $\Delta_2 = \Delta_{III_2}$, and Δ_{IV}. Various of these factors are only defined in terms of additional arbitrary choices, most of which disappear in the product. The term Δ_I also depends on the choice of splitting. When G is not quasi-split, the term Δ_1 is only defined up to an arbitrary constant. The choice of constants at places of a fixed number field is normalized globally by the condition that almost all of the constants can be taken to equal 1 and the product of the local constants is forced to equal 1. This is the "global hypothesis", cf. (7.3.7) below. For this reason the literature sometimes describes the local transfer factors as being defined only up to a constant multiple, the quotient

$$\Delta(\gamma_H, \gamma; \gamma'_H, \gamma') = \Delta(\gamma_H, \gamma)/\Delta(\gamma'_H, \gamma')$$

being independent of all choices. Note that the simplified factor in (7.2) also depends on the choice of splitting.

All terms in the product are roots of unity, with the exception of Δ_{IV}, which is a power of the order of the residue field. Waldspurger's transfer factor for Lie algebras omits the volume factor Δ_{IV}, which reappears elsewhere, and on elements close to the origin Δ_2 is trivial. The main terms are in any case the remaining terms. These are described briefly below. Here it suffices to recall some of the main formal properties satisfied by the transfer factors.

7.3. Formal properties of transfer factors. The field F is assumed to be local until (7.3.7), where it is a global field. The G is assumed to be a non-archimedean connected reductive group, except in (7.3.4) and (7.3.5) where it can be a real algebraic group.

Property 7.3.1 (change of inner form). Up to now we have been supposing G quasi-split. We drop that hypothesis. This means, in particular, that a given γ_H may not correspond to a stable class in G containing a point in $G(F)$. If it does, the transfer factor $\Delta(\gamma_H, \gamma)$ is defined for any $\gamma \in G(F)$ in the stable class. Let G^* be the quasi-split inner form of G, $\gamma^* \in G^*(F)$ an element of the stable class. If γ_H is sufficiently close to the identity then one can use the Kostant section of the Lie algebra to define a base point; one can also find a base point when the groups are unramified (see (7.3.3)). In general there is no natural choice. On the other hand, G^* is also an endoscopic group of G, and γ is an image in $G(F)$ of γ^*, so there is also a transfer factor $\Delta(\gamma^*, \gamma)$. The relation is described in [LS1]. Let

$$\Delta_{G|G^*}(\gamma_H, \gamma, \gamma^*) = \Delta(\gamma_H, \gamma)/\Delta(\gamma_H, \gamma^*)$$

and let $\gamma'_H \in H(F), \gamma' \in G(F), \gamma'^{,*} \in G^*(F)$ be another triple with the same property. Then

$$\Delta_{G|G^*}(\gamma_H, \gamma, \gamma^*)/\Delta_{G|G^*}(\gamma'_H, \gamma', \gamma'^{,*}) = \lambda_H(\gamma, \gamma^*; \gamma', \gamma'^{,*})$$

where the term on the right hand side is given by a cohomological invariant whose expression can be found in [LS1, 4.2].

Property 7.3.2 (cocycle property).

Suppose γ and γ' are stably conjugate in $G(F)$ with image γ_H in $H(F)$ The relation (7.2)(c) generalizes as follows:

$$\Delta(\gamma_H, \gamma) =< inv(\gamma, \gamma'), s > \Delta(\gamma_H, \gamma').$$

There seem to be different sign conventions in the literature for the definition of the invariant, or maybe I have just got the order wrong. This property allows us to rewrite (6.1.1) as

$$\begin{aligned} SO_{\gamma_H}(\phi^H) &= \sum_{\gamma \overset{st}{\sim} \gamma_0} < inv(\gamma, \gamma'), s > \Delta(\gamma_H, \gamma_0) O_\gamma(\phi) \\ &= c \sum < inv(\gamma, \gamma'), \kappa > O_\gamma(\phi) \end{aligned}$$

with $c = \Delta(\gamma_H, \gamma_0)$. This is the same as $cO^{\kappa}_{\gamma_0}(\phi)$ (when γ is regular its centralizer is a torus, so the Kottwitz signs are trivial) and the product over the different places of the constants c turns out to equal 1.

Property 7.3.3 (transfer factors for unramified groups). In [H], Hales provides explicit formulas for transfer factors when G and H are both unramified. This is possible because the a-data and χ-data of Langlands-Shelstad can be chosen more or less canonically in this case. The formulas are given by a series of reductions, including (7.3.6) below, so in fact there is no simple way to write them down.

Property 7.3.4 (transfer factors for real groups).

Conjugacy classes can be described more explicitly in real groups than in p-adic groups, and the transfer factors are also more explicit. It is more natural to discuss these factors in the chapter on endoscopy for real groups.

Property 7.3.5 (descent to centralizers of semisimple elements). This is the topic of the article [LS2]. Let F be a local field, $\epsilon \in G(F)$ a semisimple element, $\epsilon_H \in H(F)$ an image of ϵ. Let G_ϵ, H_{ϵ_H} denote their respective centralizers. We assume ϵ_H is chosen so that H_{ϵ_H} is quasi-split, so that it is an endoscopic group for G_ϵ. Let $\gamma, \gamma' \in G_\epsilon(F)$ be semisimple elements that are regular in $G(F)$, and let γ_H, γ'_H be their images in $H(F)$, which can be chosen in $H_{\epsilon_H}(F)$. Thus there are two transfer factors: $\Delta(\gamma_H, \gamma; \gamma'_H, \gamma')$, taken relative to G and H, and $\Delta_\epsilon(\gamma_H, \gamma; \gamma'_H, \gamma')$, taken relative to G_ϵ and H_{ϵ_H}. Let $\Theta = \Delta/\Delta_\epsilon$.

Theorem [LS2,1.6.A]. *As γ_H and γ'_H tend to ϵ_H and γ, γ' tend to ϵ,*

$$\lim \Theta(\gamma_H, \gamma; \gamma'_H, \gamma') = 1.$$

This is the property that allows the reduction of the transfer conjecture to the analogous conjecture for Lie algebras.

Property 7.3.6 (descent to Levi subgroups). Here G is quasi-split and unramified and M is a Levi subgroup of a parabolic subgroup $P \subset G$. Then M is also quasi-split and unramified. There is an endoscopic group H_M associated to M.

Lemma [H,9.2]. *Let $\gamma \in M(F)$, $\gamma_H \in H_M(F)$. Then*

$$\Delta^G(\gamma_H, \gamma) = \Delta^M(\gamma_H, \gamma) \cdot \prod_\alpha |(\alpha(\gamma) - 1)|^{\frac{1}{2}}$$

where α ranges over roots of G outside M and H.

Hales verifies that all factors of both sides agree except for the volume factor Δ_{IV}, where the difference is obviously compensated by the product indicated.

Property 7.3.7 (global reciprocity). Suppose F is a number field, $\{v\}$ the set of places of F. Let $\gamma_H \in H(F)$, up to stable conjugacy, $\gamma = (\gamma_v) \in G(\mathbf{A}_F)$, and suppose γ_H, viewed as an element of $H(F_v)$, is an image of γ_v for all v. The normalizations of the local transfer factors, denoted $\Delta^v(\bullet, \bullet)$, can be chosen so that $\Delta^v(\gamma_H, \gamma_v) = 1$ for almost all v and

$$\prod_v \Delta^v(\gamma_H, \gamma_v) = \prod_v < inv(\gamma_0, \gamma_v), \kappa >$$

if $(\gamma_0, \kappa) = \pi(\gamma_H, s, \xi) \in G(F) \times \mathfrak{K}_0(I_{\gamma_0})$.

Property 7.3.8 (extension to (G,H)-regular classes). As mentioned in Remark (6.3)(b), Kottwitz stabilization of the elliptic part of the trace formula requires matching of orbital integrals for all (G,H)-regular classes, and not only for G-regular classes. This in turn requires extending the definition of transfer factors to pairs (γ_H, γ) where γ_H is (G,H)-regular. This is carried out in [LS2,2.4], using the descent Property 7.3.5. Formula (6.1.1) is replaced by

$$SO_{\gamma_H}(\phi^H) = \sum_{\gamma \overset{st}{\sim} \gamma_0} e(I_\gamma)\Delta(\gamma_H, \gamma)O_\gamma(\phi)$$

([LS2], (2.4.1)) and the cocycle property (7.3.2) remains valid in this more general setting, as required for comparison with (4.9).

8. Explicit expressions for transfer factors for unitary Lie algebras and groups

The regular semisimple conjugacy classes in classical Lie algebras over local fields can be described in an elementary way in terms of elements of field extensions and linear algebra. In §10 of [W2], Waldspurger provides explicit formulas for transfer factors for unitary, orthogonal, and symplectic Lie algebras. Here we present his formulas in the unitary case.

Let $G = U(V, q_V)$, where q_V is the hermitian form on the n-dimensional vector space V over $\mathcal{K}$. In Waldspurger's normalization q_V is anti-linear in the first variable and linear in the second. The parametrization of conjugacy classes in $\mathfrak{g} = Lie(G)$ parallels the discussion of (4.11). Let I be a finite set. For each $i \in I$ let E_i be an extension of E of degree n_i, $\mathcal{K}_i = E_i \otimes_E \mathcal{K}$; choose $a_i, b_i \in \mathcal{K}_i^\times$. Let c denote the non-trivial element of $Gal(\mathcal{K}/E)$, $c_i = 1 \otimes c$ acting on $\mathcal{K}_i$. In order to follow Waldspurger we temporarily allow $\mathcal{K}_i$ to be a direct sum of two fields, in which case c_i exchanges the two factors. We assume

(1) For all i, $\mathcal{K}_i = \mathcal{K}[a_i]$;
(2) For all $i \neq j \in I$, there is no $\mathcal{K}$-linear morphism $\mathcal{K}_i \to \mathcal{K}_j$ taking a_i to a_j (i.e., the minimal polynomials of a_i and a_j have no common factors);
(3) For all i, $c_i(a_i) = -a_i$, $c_i(b_i) = b_i$;
(4) $\sum_i [\mathcal{K}_i : \mathcal{K}] = \sum_i n_i = n$

Set $W = \oplus_i \mathcal{K}_i$ and define a hermitian form q_W on the $\mathcal{K}$-vector space W by

$$q_W(\sum_i w_i, \sum_i w_i') = \sum_i n_i^{-1} trace_{\mathcal{K}_i/\mathcal{K}}(c_i(w_i)w_i' b_i). \tag{8.1}$$

Define an element $X_{(a_i),W} \in End(W)$ by

$$X_{(a_i),W}(\sum_i w_i) = \sum a_i w_i.$$

Then $X_{(a_i),W}$ is a regular semisimple element of the Lie algebra of $U(W, q_W)$. Now suppose there is an isomorphism $(W, q_W) \xrightarrow{\sim} (V, q_V)$. Fixing such an isomorphism, $X_{(a_i),W}$ defines an element $X_{(a_i)} \in \mathfrak{g}^{reg}$ whose conjugacy class is denoted $\mathcal{O}(I, (a_i), (b_i))$. Two such conjugacy classes $\mathcal{O}(I, (a_i), (b_i))$ and $\mathcal{O}(I', (a_i'), (b_i'))$ coincide if and only if there are a bijection $I \xrightarrow{\sim} I'$ denoted $i \mapsto i'$ and isomorphisms $\sigma_i : \mathcal{K}_{i'} \xrightarrow{\sim} \mathcal{K}_i$ taking a_i' to a_i and such that

$$\varepsilon_{\mathcal{K}_i/E_i}(\sigma_i(b_i')) = \varepsilon_{\mathcal{K}_i/E_i}(b_i) \text{ for all } i \tag{8.2}$$

where ε_* is the quadratic character associated to the extension (so there is no condition if $\mathcal{K}_i$ is not a field). On the other hand, $\mathcal{O}(I,(a_i),(b_i))$ and $\mathcal{O}(I',(a'_i),(b'_i))$ are in the same stable conjugacy class if $I \xrightarrow{\sim} I'$ and the σ_i exist as indicated, but (8.1.2) is dropped. The isomorphism $(W,q_W) \xrightarrow{\sim} (V,q_V)$ imposes one constraint on the signs $\varepsilon(b_i)$. Thus the set of conjugacy classes in the stable conjugacy class is in bijection with $(\mathbb{Z}/2\mathbb{Z})^{|I^*|-1}$ where $I^* \subset I$ is the subset for which $\mathcal{K}_i$ is a field, the term -1 corresponding to the constraint given by the isometry class of (V,q_V).

As in (4.11), $I^* = I$ if and only if $\mathcal{O}(I,(a_i),(b_i))$ is elliptic. An elliptic endoscopic datum (H,s,ξ) is defined by a partition $I = I_1 \coprod I_2$. Let $I^* = I_1^* \coprod I_2^*$ be the corresponding partition. Let $\daleth \in \mathcal{K}$ be an element such that $c(\daleth) = (-1)^{n+1}\daleth$, and assume the matrix of q_V is $2\daleth\Phi_n$ with Φ_n as in (4.1).

Let P denote the characteristic polynomial of $X = X_{(a_i)}$, viewed as an element of $End_{\mathcal{K}}(V)$, P' its derivative. For $i \in I$, set

$$C_i = \daleth[\mathcal{K}_i : E]b_i^{-1}P'(a_i). \tag{8.3}$$

One checks that $C_i \in E_i$ for all i. Now let $Y \in \mathfrak{h}^{reg}$ and suppose the stable classes of X and Y correspond. Let

$$D_G(X) = |\prod_{\alpha} d\alpha(X)|_E^{\frac{1}{2}}$$

where α runs over the roots of G and define $D_H(Y)$ likewise. The following proposition is a special case of [W2,X.8]:

Proposition 8.4. *Under these hypotheses*

$$\Delta_{G,H}(Y,X) = D_G(X)D_H(Y)^{-1}\prod_{i\in I_2^*}\varepsilon_{\mathcal{K}_i/E_i}(C_i).$$

The transfer factor in the above proposition evidently includes the volume term Δ_{IV}. However, the term $\Delta_{III,2}$ is trivial for Lie algebras and Waldspurger indicates that he has made implicit choices that trivialize the term Δ_{II} as well as the significant term $\Delta_{III,1}$. So only Δ_I (a sign) and Δ_{IV} (a power of the residue characteristic) are present.

8.5. On global reciprocity. Labesse's Chapter III.D includes an explicit determination of the local and global transfer factors, up to sign, in the case of interest in the final part of the book. More precisely, by restricting to the values of the transfer factor on the simplest elliptic torus common to a unitary group and a specific endoscopic group, he determines the normalization of the local transfer factors compatible with our choice of endoscopic data (in particular, with the embedding ξ) and with the global property. This provides a link between the determination of local transfer at unramified places in [M.III.C], which correspond to Waldspurger's calculations recalled above, and at archimedean places in [C.III.A].

9. Endoscopic transfer of representations

9.1. Formalism for transfer of stable distributions.

We start from the stable trace formula in the form of Theorem 6.2:

$$T_e(\phi) = \sum_{(H,s,\xi)\in\mathfrak{E}} i(G,H)ST_e^*(\phi^H)$$

and assume the subscript $_e$ and asterisks can be removed:

$$T(\phi) = \sum_{(H,s,\xi)\in\mathfrak{E}} i(G,H)ST_H(\phi^H) \tag{9.1.1}$$

The subscripts H have been added for clarity. If G is anisotropic, as we assume in the applications, there is no difference between $T_e(\phi)$ and $T(\phi)$. In the applications we will also choose the local test functions ϕ_∞ so that a similar equality holds on the right-hand side; see Labesse's chapter IV.A, where this choice is made and justified.

For each triple (H, s, ξ), the map

$$\phi \mapsto ST(\phi^H)$$

defines an invariant distribution on $C_c^\infty(G(\mathbf{A}_E))$. Although ϕ^H is not uniquely determined, the distribution is well defined by the properties of the transfer map. More generally, let t be a *stable* (stably invariant) distribution on H. Then the map

$$\phi \mapsto \xi_*(t)(\phi) := t(\phi^H) \tag{9.1.2}$$

is a well-defined invariant distribution on $C_c^\infty(G(\mathbf{A}_E))$. The notation ξ_* may not be optimal, but I have not seen alternative notation for this map of distributions and the present choice, in emphasizing the choice of ξ_*, is useful for our applications to unitary groups, where the exact normalization of ξ, as in (5.5), plays an important role in the explicit formulas.

Formula (9.1.1) then becomes an equality of distributions.

$$T(\phi) = \sum_{(H,s,\xi)\in\mathfrak{E}} i(G,H)\xi_*(\phi) \tag{9.1.3}$$

Endoscopic transfer then comes down to the explicit calculation of the distributions $\xi_*(ST_H)(\phi)$. Since G is anisotropic, we can write

$$T(\phi) = T_d(\phi) = \sum_{\pi\subset\mathcal{A}_d(G)} m_\pi Tr\ \pi(\phi) \tag{9.1.4}$$

as in §1. Here $\mathcal{A}_d(G)$ is the sum of automorphic representations occurring in the discrete spectrum L_2^d, defined as at the end of §1. Since G is anisotropic, $L_2^d = L_2$, but we also assume we are in the situation of the *simple trace formula*, as in IV.A, so that the terms on the right-hand side of (9.1.1) can also be written

$$ST_H(\phi^H) = \sum_{[\tau]\subset\mathcal{A}_d(H)} m_{[\tau]} Tr[\tau](\phi^H) \tag{9.1.5}$$

Here $[\tau]$ is ad hoc notation for a stable character of H, a finite linear combination $[\tau] = \sum a_i \tau_i$ of characters of irreducible representations which is stably invariant, and then

$$Tr[\tau] = \sum a_i Tr\ \tau_i.$$

The existence of such a decomposition is still conjectural, but it is known in the applications to be considered in IV.B, and we can admit it as a hypothesis. More refined decompositions were considered in [LL] and a general expression was conjectured in [K2], but this is not what we choose to emphasize here. The upshot is that

$$(9.1.6) \qquad \sum_{\pi \subset \mathcal{A}_d(G)} m_\pi Tr\ \pi(\phi) = \sum_{(H,s,\xi)\in\mathfrak{E}} i(G,H) \sum_{[\tau]\subset\mathcal{A}_d(H)} m_{[\tau]}\xi_*(Tr\ [\tau])(\phi).$$

By linear independence of characters (1.3(c)), in order to determine the m_π it suffices to know the $m_{[\tau]}$ and to calculate the maps ξ_* explicitly. The stable character $[\tau]$ is a (restricted) tensor product $\otimes'_v[\tau]_v$ over places v of E, so the latter reduces to the local problem of calculating the $\xi_{*,v}[\tau]_v$ for all v. For archimedean v this problem was posed and solved by Shelstad for tempered representations and, more generally, by Adams, Barbasch, and Vogan; these results are reviewed in Renard's chapter II.A and Adams' chapter III.B. When $[tau]_v$ is an unramified (spherical) representation, there is a natural conjectural expression for $\xi_{*,v}[\tau]_v$ in terms of Satake parameters. In [H2] Hales showed how to reduce this conjecture to the special case treated by the fundamental lemma; thus this is now also a theorem thanks to [LN, N1, N2]. In §9.2 we describe these formulas when G is a unitary group, with the elliptic endoscopic groups parametrized as in (5.5) in terms of auxiliary choices of Hecke characters.

Remark 9.1.7. The terms on the right hand side of (9.1.6) are not linearly independent. Cancellation occurs between the terms occurring in the expansions of different (H,s,ξ) and these cancellations often have interesting arithmetic applications. Moreover, it should be noted that the coefficients $i(G,H)$ are in general fractions and not integers. This forces some of the distributions $m_{[\tau]}\xi_*(Tr\ [\tau])$ to be *fractional* linear combinations of irreducible characters, frequently with negative coefficients. Of course, the final coefficients m_π are necessarily non-negative integers.

9.2. Functoriality and Arthur parameters.
Here and below we write $\xi_*([\tau]) = \xi_*(Tr\ [\tau])$ for brevity. The basic stable discrete automorphic distributions on H are supposed to be associated to admissible maps (Arthur parameters)

$$\Phi : \mathcal{L}_E \times SU(2) \to {}^L H$$

Here $\mathcal{L}_E$ is the hypothetical Langlands group and the factor $SU(2)$ measures the failure of the packet $\Pi(\Phi)$ of discrete automorphic representations of H indexed by Φ to be tempered. We only consider Φ trivial on the factor $SU(2)$ and replace the map of the Langlands group by the collection Φ_v of restrictions of Φ, as v varies over places of E, to the local Weil-Deligne group $WD_v = WD(E_v)$. Composition of Φ with the L-homomorphism $\xi : {}^L H \to {}^L G$ defines an Arthur parameter $\tilde{\xi}_*(\Phi)$ for G. If Φ corresponds to the stable character $[\tau]$, then the corresponding packet $\Pi(\tilde{\xi}_*(\Phi))$ of admissible irreducible representations of $G(\mathbf{A})$ is conjecturally related to the transfer $\xi_*([\tau])$, as discussed above. However, $\Pi(\tilde{\xi}_*(\Phi))$ is no longer a stable character, and the characters of the various members of the collection $\Pi(\tilde{\xi}_*(\Phi))$ are expected to occur in $\xi_*([\tau])$ with multiplicities determined by an explicit formula depending on the interaction of the datum s with the parameter $\tilde{\xi}_*(\Phi)$; this is the content of Arthur's multiplicity conjectures. Since these conjectures can only be stated in terms of local L-packets, since the complete analysis of local L-packets for unitary groups is not yet available as of this writing, and since in any case the structure these L-packets, constructed by Moeglin, is too intricate for the purposes of this introduction, I will limit my explicit description of the endoscopic transfer to the simplest possible cases.

9.3. Explicit formulas for endoscopic transfer.

We work in the setting of unitary groups, with the elliptic endoscopic groups described in (5.5). Let $n = a + b$ be a partition of n, with $a \geq b$, and let $H_{a,b} = U(a)^* \times U(b)^*$ (quasi-split inner forms); thus $H_{n,0} = G^*$ is the quasi-split inner form of G. Choose characters μ_a, μ_b as in (5.5) and define the L-homomorphism

$$\xi = \xi_{\mu_a,\mu_b} : {}^L H_{a,b} \to {}^L G$$

by the formulas (5.5.3-5.5.5). Let v be a place of E and consider a stable character $[\tau]_v$ of $H_{a,b}(E_v)$. Our goal is to describe the character $\xi_{\mu_a,\mu_b,v,*}([\tau]_v)$ in a few simple cases. Note that

Lemma 9.3.1. *For all places v, the characters $\mu_{a,v}$ and $\mu_{b,v}$ of $(\mathcal{K} \otimes_E E_v)^\times$ are unitary.*

It suffices to show that μ_a and μ_b are unitary characters; but this is clear because they restrict trivially to the kernel in $\mathbf{A}_E^\times$ of $\eta_{\mathcal{K}/F}$, and this kernel is cocompact in the idèles of $\mathcal{K}$.

9.3.2 Split places.

If v splits in $\mathcal{K}/E$ then $H_{a,b}(E_v) \simeq GL(a,E_v) \times GL(b,E_v)$ and $G(E_v) \simeq GL(n,E_v)$. Consider the standard maximal parabolic $P_{a,b} \subset G(E_v)$ corresponding to the partition $n = a + b$, and let $M_{a,b} \subset P_{a,b}$ denote its standard Levi subgroup. Then $H_{a,b}(E_v)$ is clearly isomorphic to $M_{a,b}$, so we can identify their corresponding sets of irreducible admissible representations

$$i : \mathcal{A}(H_{a,b}(E_v)) \xrightarrow{\sim} \mathcal{A}(M_{a,b}).$$

The set of such identifications is a homogeneous space under the action of the group $X(M_{a,b})$ of continuous characters of $M_{a,b}$, which can in turn be identified with the group $X(E_v^\times)^2$ of ordered pairs of characters of E_v, via composition with the determinant $M_{a,b} \to E_v^\times \times E_v^\times$. Every irreducible representation $\tau_v = (\tau_{a,v}, \tau_{b,v})$ of $H_{a,b}(E_v)$ is stable and therefore defines a stable character $[\tau_v]$. We assume τ_v is a *unitary* representation (in practice it will in fact be tempered). If we identify $M_{a,b}$ with $GL(a,E_v) \times GL(b,E_v)$, then let

$$i(\xi_{\mu_a,\mu_b})_*([\tau_v]) = (\tau_{a,v} \otimes \mu_b \circ \det, \tau_{b,v} \otimes \mu_a \circ \det) \tag{9.3.2.1}$$

and define

$$\xi_{\mu_a,\mu_b,v,*}([\tau_v]) = I_{P_{a,b}}^{GL(n,E_v)} i(\xi_{\mu_a,\mu_b})_*([\tau_v]) \tag{9.3.2.2}$$

where the representation of $M_{a,b}$ in (9.3.2.1) has been inflated to a representation of $P_{a,b}$. In particular $\xi_{\mu_a,\mu_b,v,*}([\tau_v])$ is an irreducible admissible representation of $G(E_v)$, and we have the following identity of standard L-functions

$$L(s,\xi_{\mu_a,\mu_b,v,*}([\tau_v])) = L(s,\tau_{a,v}\otimes\mu_b\circ\det)L(s,\tau_{b,v}\otimes\mu_a\circ\det) \tag{9.3.2.3}$$

and the analogous identity for standard ε-factors.

9.3.3 Unramified representations.

Suppose v is inert (and unramified) in $\mathcal{K}/E$, and suppose the unitary group G is unramified at v. Thus $G(E_v)$ as well as $H_{a,b}(E_v)$ are quasi split and have hyperspecial maximal compact subgroups K_v and $K_{a,b}$, respectively. Irreducible admissible representations τ_v of $H_{a,b}(E_v)$ with $K_{a,b}$-fixed vectors are in one-to-one correspondence with characters of the unramified Hecke algebra $\mathcal{H}(H_{a,b}) = C_c^\infty(K_{a,b}\backslash H_{a,b}(E_v)/K_{a,b})$ of $K_{a,b}$-bi-invariant test functions on $H_{a,b}(E_v)$. These latter, in turn, are in one-to-one correspondence with unramified admissible homomorphisms (Satake parameters)

$$\Phi : W_{E_v} \to {}^LH_{a,b}. \tag{9.3.3.1}$$

Here *unramified* means that the composition of Φ with projection on $\hat{H}_{a,b}$ is trivial on the inertia subgroup, hence factors through the quotient $E_v^\times/\mathcal{O}_{E_v}^\times$ of W_{E_v}; *admissible* means that the composition of Φ with the natural map ${}^LH_{a,b} \to W_{E_v}$ is the identity map. Let

$$\lambda_{a,b}(\Phi) : \mathcal{H}(H_{a,b}) \to \mathbb{C} \tag{9.3.3.2}$$

denote the corresponding character of the unramified Hecke algebra. Of course the Satake parameter Φ is also an Arthur parameter, in the sense of (9.2), and the associated spherical representation τ_v is denoted $\Pi(\Phi)$, following the notation of (9.2). This is consistent:

Proposition 9.3.3.3. *Any such spherical* $\Pi(\Phi)$ *has a stable character, hence defines a stable distribution* $[\Pi(\Phi)]$.

For details, see Minguez' chapter II.C.

The same Satake parametrization is valid for the quasi-split unitary group $G(E_v)$. Define $\tilde{\xi}_{\mu_a,\mu_b,*}(\Phi) : W_{E_v} \to {}^LG$ as in (9.2). This is now a Satake parameter for G, hence defines a spherical representation $\Pi(\xi_{\mu_a,\mu_b,*}(\Phi))$. Define the unramified Hecke algebra $\mathcal{H}(G)$ with respect to K_v, as above, and let

$$\lambda_n(\tilde{\xi}_{\mu_a,\mu_b,*}(\Phi)) : \mathcal{H}(G) \to \mathbb{C}$$

be the character of $\mathcal{H}(G)$ corresponding to the indicated Satake parameter. The map $\tilde{\xi}_{\mu_a,\mu_b,*}$ on Satake parameters is dual to a homomorphism

$$\tilde{\xi}^*_{\mu_a,\mu_b} : \mathcal{H}(G) \to \mathcal{H}(H_{a,b}). \tag{9.3.3.4}$$

The following theorem is a reformulation in abstract terms of Theorem 4.3 of [M.III.C].

Theorem 9.3.3.5. *We have the equality*

$$\xi_{\mu_a,\mu_b,*}([\Pi(\Phi)]) = \Pi(\tilde{\xi}_{\mu_a,\mu_b,*}(\Phi)),$$

where $\xi_{\mu_a,\mu_b,*}$ *is the transfer map on distributions defined by (9.1.2).*

This theorem is the basic fact about endoscopic transfer, specialized to the case of unitary groups. The analogous theorem is valid for elliptic endoscopic transfer of unramified representations in general.

Sketch of proof. It follows from the definitions that the theorem is equivalent to the following assertion:

$$\forall\phi \in \mathcal{H}(G), \phi^{H_{a,b}} = \tilde{\xi}^*_{\mu_a,\mu_b}(\phi). \tag{9.3.3.6}$$

Allowing for differences of notation, this is what is called in [H2] (and elsewhere) the *fundamental lemma for standard endoscopy*, in the case under consideration. The article [H2] of Hales reduces this equality to the special case in which ϕ is the unit element 1_{K_v} and therefore the right-hand side of (9.3.3.6) is just $1_{K_{a,b}}$. In other words, Hales reduces (9.3.3.6) to Conjecture 6.4; indeed he obtains the equality assuming Conjecture 6.4 has been established for sufficiently large residue characteristic. The theorem thus follows by combining [H2] with the results of [LN] and [W3].

The analogue of the relation (9.3.2.3) holds for the Langlands L-functions attached to the standard representations of ${}^LH_{a,b}$ and ${}^LG$, more or less tautologically.

9.3.4 Global endoscopic transfer. Now suppose that $\mathcal{K}/E$ and the group G are unramified at all finite places and consider the restriction of the transfer maps $\phi \mapsto \phi^{H_{a,b}}$ to ϕ which are biinvariant under an appropriately chosen compact open subgroup $K_f \subset G(\mathbf{A}^f)$ containing K_v for all inert places v. The transfers are then defined on automorphic representations τ with fixed vectors under the corresponding compact open subgroups of $H_{a,b}$; in particular, each such τ is spherical at every inert place v. Under these hypotheses (corresponding to the "simplifying hypotheses" to be considered in Book 3), any stable L-packet of $H_{a,b}$ is of the form

$$[\tau] = [\tau]_\infty \otimes \tau_f$$

where τ_f is a (single) irreducible representation of $H_{a,b}(\mathbf{A}^f)$. Then

$$\xi_{\mu_a,\mu_b,*}([\tau]) = \xi_{\mu_a,\mu_b,\infty,*}([\tau]_\infty) \otimes \bigotimes_{v\nmid\infty}{}' \xi_{\mu_a,\mu_b,v,*}(\tau_v),$$

where the restricted tensor product is taken over finite places. The terms $\xi_{\mu_a,\mu_b,v,*}(\tau_v)$ are explicitly computed by the formulas in (9.3.2) and (9.3.3). The calculation of $\xi_{\mu_a,\mu_b,\infty,*}([\tau]_\infty)$ is the topic of the chapters of Renard and Adams.

In the applications, especially in Part IV, $[\tau]_\infty$ is a discrete series L-packet, and $\xi_{\mu_a,\mu_b,\infty,*}([\tau]_\infty)$ is a sum of discrete series characters of $G(E_\infty)$, weighted by signs determined long ago by Shelstad. More details are provided in [C.III.A, CHL.IV.B].

9.4 Twisted endoscopy and base change for unitary groups.

Notation in this section differs from that of the previous section. Here the unitary group is denoted $\mathbf{U}^*$, following [L.IV.A], and the letter G is reserved for the group appearing in the twisted trace formula. From the third paragraph G is assumed to be a general linear group.

In the twisted trace formula, the group G is replaced by a disconnected group $G \rtimes < \theta >$ whose component group is cyclic and generated by an element θ (usually) of prime order, or alternately by the component $\tilde{G} = G \rtimes \theta$ of this group whose image generates the component group. The spectral side of the twisted trace formula roughly counts automorphic representations of the identity component G invariant under the action of the component group; the geometric side takes the twisting into account in a natural way. Twisted endoscopy [KS,Lab2] is the theory that leads to an expression analogous to Theorem 6.2, or (9.1.1), for the geometric side of the twisted trace formula. The endoscopic groups of $\tilde{G}$ are defined by a procedure analogous to that discussed above.

A complete treatment of twisted endoscopy would require (at least) another chapter, and is beyond the scope of these books. However, the topic inevitably arises in the global applications in Part IV, especially in [L.IV.A], and an important aspect of the local theory is addressed in [W.II.A.2]. In the case of concern to us here, $G = GL(n)_E$, where E/F is a quadratic extension of number fields, with F totally real and E totally imaginary, $< \theta >= Gal(E/F)$, and the action of the non-trivial element $c \in Gal(E/F)$ on $GL(n)_E$ is given by

$$\theta(g) = J_n{}^t(\bar{g})^{-1} J_n^{-1}, \tag{9.4.1}$$

where J_n is the anti-diagonal matrix with alternating -1s and 1s, cf. [L.IV.A]; here $\bar{g}$ is the result of applying c to the coefficients of g. Conjugation of the element $(\delta, \theta) \in \tilde{G}$ by en element $g \in G$ takes the following form:

$$g(\delta, \theta)g^{-1} = (g\delta\theta(g)^{-1}, \theta) \tag{9.4.2}$$

where the first component on the right is the θ-conjugate of δ.

The elliptic part of the geometric side of the twisted trace formula is an expression in θ-conjugacy classes in G, which are identified via (9.4.2) with conjugacy classes in $\tilde{G}(\mathbf{A}) = G(\mathbf{A}) \rtimes \theta$. Let ϕ be a test function on $\tilde{G}(\mathbf{A})$. The twisted trace formula is thus the equality of spectral and geometric expressions for

$$T^{\tilde{G}}(\phi) = tr\ R_{disc}(\phi) = J(\tilde{G}) \sum_{\delta} \tau(I_\delta) O_\delta(\phi) = \sum_{\tilde{\pi}} tr\tilde{\pi}(\phi) \tag{9.4.3}$$

where we are adopting the notation of [L.IV.A, (3.2)]; here δ runs over conjugacy classes in $\tilde{G}(E)$, or θ-conjugacy classes in $G(E)$, and I_δ is what was denoted G_γ above. The $\tilde{\pi}$ are indexed by automorphic representations of G that are isomorphic to their θ-conjugates, but the action of $\tilde{G}$ needs to be made more precisely, and that is done in [L.IV.A]. The multiplicities $m(\pi)$ are absent because of strong multiplicity one for $GL(n)$.

On the other hand, it is important to stress that $G = GL(n)$ is not anisotropic, and (9.4.3) is unrealistically simple. The choice of ϕ_∞ made in [L.IV.A], where it is moreover assumed that $[F : \mathbb{Q}] \geq 2$, eliminates the difficult terms of the geometric side, and the sum really runs over elliptic δ, as well as most of the continuous spectrum. However, there remain isolated terms in the continuous spectrum of G that contribute non-trivially to the spectral side. These questions are treated carefully in [L.IV.A] and [CHL.IV.B], and we need not dwell on them here.

The stabilization of the twisted trace formula rewrites the geometric side of (9.4.3) as a sum over (twisted) endoscopic groups for $\tilde{G}$. These twisted endoscopic groups are listed in [L.IV.A, §4.2] Here the choice of archimedean test functions in [L.IV.A] leads to a real simplification; only one twisted endoscopic group contributes to the result, namely the quasi-split unitary group of size n, denoted $\mathbf{U}^*$ in [L.IV.A]. The key formula is then Theorem 4.13 of [L.IV.A], which is here stated in simplified form:

$$T^{\tilde{G}}(\phi) = ST^{\mathbf{U}^*}(f) \tag{9.4.4}$$

when f and ϕ are associated functions in the sense of having matching orbital integrals for a pair $(\delta, \gamma) \in G(E) \times \mathbf{U}^*(F)$ where γ is a norm of δ; we make this more explicit in (9.4.5), below. The orbital integral is strictly speaking defined not for δ but rather for (δ, θ), but we ignore this distinction. The condition that f and ϕ have matching orbital integrals is exactly as in standard endoscopy, for example as in (6.1.1).

The fundamental lemma in this situation was proved for unit elements by Kottwitz and for general Hecke functions by Clozel and Labesse. At unramified places there is a well-defined homomorphism $\phi \mapsto f$ from the spherical Hecke algebra of G to that of $\mathbf{U}^*$, dual to the map of Satake parameters denoted $\widetilde{BC}$ in [M.III.C, Theorem 4.1]. As in (9.3.3), let

$$\Phi : W_{F_v} \to {}^L\mathbf{U}^*$$

be a local Langlands parameter. Then the restriction of Φ to W_{E_v} is mapped to the L-group of G, and the base change corresponds to this restriction. A general Φ is associated to an L-packet, but if Φ is unramified then the restriction is precisely the map of Satake parameters. Details are provided in [M.III.C].

9.4.5. Norms. The norm is defined carefully in [L.IV.A, 4.4]; roughly it means that γ is conjugate to $\delta\theta(\delta)$ in $GL(n)(\bar{E})$. The illuminating case $n = 1$ is discussed in [W.II.B.2], for example at the end of his section 5.2. There $\delta \in G(E_v) = E_v^\times$ and $\gamma = \delta \cdot \theta(\delta) = \delta \cdot c(\delta)^{-1}$. For general n a semisimple conjugacy class δ is characterized by its collection of eigenvalues λ_i, in some algebraic extension of E_v. Then the typical eigenvalue of the corresponding γ is of the form $\lambda_i \cdot c(\lambda_j)^{-1}$ for an appropriate extension of c. See §5 of [W.II.B.2] for a more detailed account of the relation taking stable conjugacy into account.

References

References of the form [X.Y.Z.n], where X denotes one or more initials, Y is a Roman numeral, Z a letter, and the optional n a small integer, are to chapters of this book, with X the initials of the authors, Y the section heading, Z the chapter heading, and n the possible subchapter (only present when Y = II). Thus [W.II.B.2] refers to Waldspurger's chapter.

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