INTRODUCTION TO "THE STABLE TRACE FORMULA, SHIMURA VARIETIES, AND ARITHMETIC APPLICATIONS"

Michael Harris

UFR de Mathématiques
Université Paris 7
2 Pl. Jussieu
75251 Paris cedex 05, FRANCE

The present volume is the first in a projected series of three or four collections of mainly expository articles on the arithmetic theory of automorphic forms. The books are primarily intended for two groups of readers. The first group is interested in the structure of automorphic forms on reductive groups over number fields, and specifically in qualitative information about the multiplicities of automorphic representations. The second group is interested in the problem of classifying ℓ-adic representations of Galois groups of number fields. Langlands' conjectures elaborate on the notion that these two problems overlap to a considerable degree. The goal of this series of books is to gather into one place much of the evidence that this is the case, and to present it clearly and succinctly enough so that both groups of readers are not only convinced by the evidence but can pass with minimal effort between the two points of view.

More than a decade's worth of progress toward the stabilization of the Arthur-Selberg trace formula, culminating in the recent proof by Laumon and Ngô[1] of the Langlands-Shelstad conjecture for unitary groups over function fields, better known as the *fundamental lemma*, has made this series timely. The 1980s saw the formulation of increasingly explicit conjectures, primarily by Langlands, Shelstad, Arthur, and Kottwitz, concerning both the ultimate form of the stable trace formula and its consequences for multiplicities of automorphic representations and the structure of Galois representations occurring in the ℓ-adic cohomology of Shimura varieties. The test case of the group $U(3)$ was treated comprehensively in Rogawski's book on the subject, and the applications to Galois representations were derived in the volume edited by Langlands and Ramakrishnan. Although progress on these conjectures subsequently stalled, their statements became widely known. New applications to the arithmetic of Galois representations were derived, assuming the truth of these conjectures. Simultaneously, work of Waldspurger and Arthur succeeded in reducing these conjectures to variants of the fundamental lemma. The latter thus became the bottleneck limiting progress on a host of arithmetic questions.

Institut des Mathématiques de Jussieu, U.M.R. 7586 du CNRS. Membre, Institut Universitaire de France.

[1]References are to be found in the bibliographies of individual chapters

The breakthrough article of Laumon and Ngô, building on earlier work of Goresky, Kottwitz, and MacPherson, and of Laumon, removed that obstacle in the crucial case of unitary groups over local fields of positive characteristic. Subsequent work of Ngô extended the geometric techniques of his work with Laumon and succeeded in proving the fundamental lemma for all groups, again over local fields. At roughly the same time, Waldspurger proved that the fundamental lemma depends only on the residue field, thus deriving the fundamental lemma for p-adic fields from the results of Laumon and Ngô. More recently, Cluckers, Hales, and Loeser obtained similar results on independence of characteristic using methods of motivic integration and model theory. All these results are only valid when the residue characteristic is sufficiently large, but earlier work of Hales showed that this implies the general case.

In the broadest possible terms, the Arthur-Selberg trace formula, and its stable variant, is an identity between expressions of two sorts. We fix a reductive group G over a number field F. The *spectral side* is an expression for the traces of a family of operators that determine the representation of $G(\mathbf{A}_F)$ on the discrete spectrum of $L_2(G(F)\backslash G(\mathbf{A}_F))$.[2] The spectral side, in other words, contains the information of ultimate interest about spaces of automorphic forms. It can be viewed as a sum (or more generally an integral) of characters (more generally weighted characters) of irreducible representations of $G(\mathbf{A}_F)$. The *geometric side* is a sum of orbital integrals and their generalizations that can be understood in terms of harmonic analysis on $G(F_v)$, as v runs over the completions of F. It can be viewed as a sum indexed by conjugacy classes in $G(F)$. The principle is that the geometric side lends itself to comparisons between different groups, but only after the sums over conjugacy classes are replaced by sums over *stable* conjugacy classes, a notion discussed in detail in the first section of this book. Such comparisons are then used to compare the spectral sides for different groups, which could not otherwise be compared directly.

It may be helpful to think of the stable trace formula not as a single equality but rather as a collection of techniques for generating formulas that can be used for a variety of purposes: to compare the automorphic representations of different groups, in the spirit of Langlands' functoriality conjectures; to obtain qualitative information about representations of local (generally p-adic) groups from global constructions; and in certain cases to relate automorphic representations to Galois representations. The purpose of these books, then, is to present the stable trace formula in a manner that should be accessible not only to specialists but to anyone who may benefit from application of these techniques.

Before going on, I should list some of the things these books are not. It is certainly NOT a detailed introduction to the general trace formula. The stable trace formula presents two quite distinct sorts of difficulties. Analytic problems, some of them very deep, arise as soon as one attempts to understand the discrete automorphic spectrum of a non-compact (isotropic) reductive group. The resolution of these problems was carried out in a long series of articles, mainly by Jim Arthur. We cannot hope to improve on Arthur's recent expository presentation of this material, to which we refer the reader.

[2]This is not strictly accurate if the center of G contains a torus split over F, but there are several standard substitutes for $L_2(G(F)\backslash G(\mathbf{A}_F))$ that will be considered in the text and we ignore this issue here.

On the other hand, the difficulties connected with stabilization of the trace formula already appear in their full complexity for anistropic groups. The simple version of the trace formula, derived by Arthur from his general trace formula for appropriate choices of data, can also be stabilized, and the result is practically identical to that obtained for anisotropic groups. For the applications we have in mind, these cases will suffice. In particular, we will only be considering orbital integrals (not weighted orbital integrals) and characters (not weighted characters). Thus this book also does NOT contain a complete treatment of the stable trace formula. This was carried out in general in a more recent series of articles, again by Jim Arthur, and is presented in his forthcoming book on functorial transfer between classical groups and $GL(n)$.

To summarize the two preceding paragraphs, it will be enough for us to present the stabilization of the *elliptic terms* of the geometric side of the trace formula. After the pioneering article of Labesse-Langlands on the stable trace formula for $SL(2)$, the stabilization of the elliptic terms was initiated by Langlands and pursued by Kottwitz in a series of articles written during the 1980s. A different point of view, making systematic use of non-abelian cohomology, was developed by Labesse and is used either implicitly or explicitly in a number of chapters. The two introductory chapters by Harris and Labesse present this material from somewhat different perspectives, described below. Here it should only be mentioned that these chapters do NOT contain complete proofs of the intermediate steps in stabilization of even the elliptic terms of the trace formula; but it does provide detailed references for those who want to read the proofs.

Most importantly, it should be stressed this book does NOT pretend to provide a general treatment of Langlands' functoriality conjectures, even for the special groups considered, even in the special situation of endoscopy that has motivated much of the work on the trace formula over the past three decades. For the most part we restrict our attention to the stabilization of the trace formula for unitary groups of vector spaces over CM fields. In the first place, the initial breakthrough of Laumon-Ngô was the proof of the fundamental lemma for unitary groups, and even now that the fundamental lemma is known in general, unitary groups are somewhat easier to understand because of their intimate relation to general linear groups. In particular, the geometric constructions of Laumon and Ngô, involving the Hitchin fibration, can be expressed in the intuitively satisfying terms of matrices and their characteristic polynomials. On the other hand, the Shimura varieties attached to unitary groups are the most fruitful source for construction of compatible systems of ℓ-adic Galois representations, again because of the close connection to general linear groups via stable base change. If one is merely interested in the construction of Galois representations attached to certain classes of automorphic representations of $GL(n)$, it is unnecessary to concern oneself with the fine points of representation theory of general reductive groups over p-adic fields; in particular, we can and do restrict our attention to automorphic representations of unitary groups for which the problem of local L-packets does not arise. Moreover, we can arrange matters so that only *tempered* automorphic representations appear, thus avoiding the subtle analytic problems connected with the general case. This suffices in the first instance for the construction of Galois representations, but non-tempered automorphic representations are needed for certain applications, and we expect to return to this topic in later books. On the other hand, as Moeglin realized, the simple stable trace formula considered in the first book is sufficiently flexible to allow her to construct and classify local L-packets for classical groups; in particular her construction for local unitary groups requires nothing more than the theory developed here. Thus, while this book does NOT work out the general theory of endoscopic transfer even for unitary groups, it does establish the theory in sufficient generality for several applications of importance in arithmetic as well as representation theory.

It deserves to be mentioned separately that the only automorphic representations considered in this book are those of cohomological type at archimedean places. This is in part because the test functions used to identify cohomological representations can be used to simplify the trace formula, and even to simplify the stabilization of the twisted trace formula for $GL(n)$ for the base change from unitary groups. This theory is worked out in Labesse's chapter IV.A. The techniques now exist to extend base change relating general automorphic representations of $U(n)$ to automorphic representations of $GL(n)$, as in Rogawski's book, which develops the case $n = 3$. The present series of books, intended as an introduction to the stable trace formula, are not the appropriate place for such an ambitious undertaking. Again, only cohomological representations can be associated to cohomology of the Shimura varieties attached to $U(n)$, the only systematic source of the Galois representations attached to automorphic representations. Thus this book does NOT shed any light whatsoever on questions of functoriality connected with Maass forms and their generalizations in higher dimension.

Finally, as Langlands has stressed increasingly in recent years, endoscopy, including the twisted endoscopy of base change and transfer between classical groups and $GL(n)$, only accounts for very special cases of functoriality. This book does NOT look beyond endoscopy; for that, readers are strongly encouraged to read Langlands' recent articles on the subject.

The remainder of this introduction will use the tables of contents of the first book in the series as a pretext for reviewing in detail the developments described in the preceding paragraphs . Most of the material in the first book should be familiar to the first group of intended readers, less so to the second group. For the material in the second book, whose table of contents should be available in the near future, the proportions should be reversed.

Book 1. The stable trace formula

Section I. Introduction to the stable trace formula.

As mentioned above, this portion of the book consists of two chapters by Harris and Labesse. The bulk of Harris's chapter is a review of Kottwitz' theory. The results are stated for the most part without proof, since Kottwitz' proofs are extremely clear. The main purpose of this chapter is to assemble all this material in a single place. In this Harris has followed the unpublished notes of the IHES seminar of 2003-2004 on the stable trace formula, especially the lectures of Ngô. Labesse's chapter contains a more sustained treatment of the first non-trivial case, that of $SL(2)$.

Section II.A, B. Endoscopy: the local theory.

In Kottwitz' articles of the 1980s, and in Arthur's later work on the full stabilization of the trace formula, the fundamental lemma is assumed as a working hypothesis. Thanks especially to the work of Laumon and Ngô, the fundamental lemma is now a theorem. The second portion of the first book provides a detailed introduction to the fundamental lemma and to the problem of endoscopic transfer in general. This breaks up logically into three chapters, treating separate aspects of the theory of endoscopic transfer. Sections II.A and B. present the local theory of endoscopy. The theory for real groups was developed in a series of fundamental papers by Shelstad in the 1970s. Renard's chapter II.A reviews this material in the light of subsequent work on representation theory, and on the systematic definitions of transfer factors due to Langlands, Shelstad, and Kottwitz.

The basic technical problem of endoscopy is to show how to relate the test functions that enter into the trace formula for different groups (a group and its endoscopic groups). We have already seen that this *transfer problem* was solved for real groups by Shelstad. For p-adic groups the fundamental lemma, which gives an explicit formula for transfer in the simplest case, plays a special role. In the 1980s, Clozel and Labesse independently showed how to use the global trace formula to reduce the general transfer problem for stable base change of Hecke functions to the special case of the fundamental lemma, proved in this case by Kottwitz. This method was adapted to endoscopy by Hales and especially by Waldspurger, who reduced the general endoscopic transfer problem to the fundamental lemma for endoscopy. These results are reviewed in Chaudouard's chapter II.B.1. More recently, Waldspurger has shown how to reduce transfer for twisted endoscopy, as required in Arthur's work on classical groups, to what he calls the "non-standard fundamental lemma." Waldspurger's chapter II.B.2 provides an introduction to this work.

I.C. The fundamental lemma, geometric techniques.

The work of Laumon and Ngô has introduced an unfamiliar array of concepts into the heart of the theory of automorphic forms. As Ngô's most recent article makes clear, the central concept is that of the Hitchin fibration on the moduli stack of vector bundles with additional structure on a fixed algebraic curve. As Langlands wrote recently, a key insight is that the formalism of stacks allows provides a geometric meaning to global orbital integrals. These orbital integrals can be interpreted in particular in terms of certain perverse sheaves on the Hitchin fibration. For local fields of sufficiently large positive characteristic, endoscopy is obtained in the papers of Laumon and Ngô by analyzing the functorial behavior of these perverse sheaves, using the basic properties of perversity to reduce to the situation where explicit calculation is possible.

Section II.C. presents this material in four chapters. The first two chapters, due to Dat, introduces the general framework; the remaining chapters of Ngô and Harris shed light on crucial steps in Ngô's argument.

II.D, E. The fundamental lemma, independence of characteristic.

The geometric techniques of Laumon and Ngô only apply to local fields of positive characteristic, whereas the global applications of this series of books are primarily to number fields. The bridge is provided by a theorem of Waldspurger that shows that the fundamental lemma in (sufficiently large) mixed characteristic can be derived from the corresponding statement for local fields of positive characteristic. Earlier work of Hales shows that this suffices for the general case.

Waldspurger's theorem, presented in Lemaire's chapter II.D, is proved using a detailed analysis of the fine structure of the lattice of open compact subgroups of a reductive group over a local field, based in turn on earlier results of DeBacker and Kim-Murnaghan. A very different proof of independence of characteristic has been subsequently obtained by Cluckers, Hales, and Loeser. This proof is based on a *transfer principle* in model theory that uses formal properties of integrals on groups over local fields to transfer statements from one family of local fields to another. The basic framework is the theory of motivic integration, as developed in the work of Denef-Loeser and extended by Cluckers-Loeser. This perspective makes use of notions of mathematical logic that are undoubtedly unfamiliar to most specialists in arithmetic and automorphic forms, including the editors of this volume. Once these notions have been introduced, however, the technique proves to be highly adaptable. The chapter II.E. of Cluckers, Hales, and Loeser reviews the basic constructions quickly and then explains how they can be applied to three different sorts of fundamental lemmas.

III. Local representation theory of unitary groups..

In the earlier chapters, special attention has been given to unitary groups, and their relations to $GL(n)$, as an example to illustrate the general principles of endoscopy. In the remainder of the book we restrict our attention to unitary groups. Section III is a survey of basic results in the theory of representations of unitary groups over local fields, as it intervenes in the spectral and geometric sides of the simple trace formula. Most of this material is standard but some is new, and in any case it has not previously been assembled in one place.

Clozel's Chapter III.A is a review of the local theory of representations of real unitary groups, base change to $GL(n, \mathbb{C})$, and works out Shelstad's theory of endoscopy explicitly in this setting. It can also serve as an introduction to the local Langlands parametrization in a concrete situation where much of the subtlety of the general case is already apparent. Adams' Chapter III.B treats the same material from a different perspective, namely that of the author's book with Barbasch and Vogan on stable distributions on real reductive groups. While Adams only considers discrete series representations of unitary groups, whose theory had already been developed by Shelstad in the 1970s, the formalism he presents is in some ways more intuitive. In particular, the signs in endoscopic transfer of discrete series representations can be given a uniform treatment when all (strong) inner forms of unitary groups are considered simultaneously.

Chapter III.C, written by Minguez, develops the analogous theory for unitary groups over p-adic fields. For the purposes of the applications to construction of Galois representations, only the most elementary aspects of the theory, such as the explicit base change of spherical representations of quasi-split unitary groups, need to be treated. However, the literature on representations of p-adic unitary groups is scattered, and it seemed useful to present more of the theory than is strictly necessary for immediate applications. There is also an introduction to Moeglin's recent work on L-packets for of p-adic unitary groups.

IV. Base change and endoscopic transfer for unitary groups.

The two chapters of this section contain the only genuinely new results of this book. Labesse's chapter IV.A develops the theory of base change for cohomological automorphic representations of inner forms of the quasi-split group $U(n)$, attached to a quadratic extension E/F, to the corresponding $GL(n)$, extending his earlier work with Clozel and Harris. In Labesse's article, it is assumed that F is totally real and of degree at least 2, and that E is totally imaginary and unramified over F at all finite primes. It is moreover assumed that the automorphic representations Π to which one applies base change are spherical at all finite places that remain inert in E/F. For some applications, it is also assumed that the infinitesimal parameter of Π at any archimedean place is sufficiently regular. These hypotheses are restrictive and certainly unnecessary for stable base change, and indeed no such hypotheses are made in Rogawski's book on $U(3)$. However, the comparison of stable trace formulas simplifies considerably under these hypotheses, which suffice for applications to construction of Galois representations as well as for all purely local applications.

Chapter IV.B, by Clozel and Harris, analyzes the endoscopic transfer in the simplest possible case, from the endoscopic group $U(n)\times U(1)$ to certain inner forms of $U(n+1)$. Again, simplifying assumptions are made on the local behavior at finite primes (to avoid complications from L-packets) and on infinitesimal characters at archimedean primes. The results in this section, inspired by an article of Blasius and Rogawski, are used to construct n-dimensional Galois representations in the cohomology of Shimura varieties obtained as quotients of the unit ball in $\mathbb{C}^n$.

Book 2. Cohomology of Shimura varieties and Galois representations (in preparation)