# INTRODUCTION TO ENDOSCOPY
Snowbird Lectures, June 2006

Jean-Pierre Labesse
Institut Mathématique de Luminy, UMR 6206
Marseille, France

Revised version
August 2007

## Contents

# 1 Introduction

## 1.1 What is endoscopy about?

Many questions about non commutative Lie groups boil down to questions in invariant harmonic analysis: the study of distributions on the group that are invariant by conjugacy. The fundamental objects of invariant harmonic analysis are orbital integrals and characters. For various reasons (among which is the trace formula), one calls "geometric" the objects related to conjugacy classes and orbital integrals and "spectral" the objects related to representations and characters.

In the Langlands program a cruder form of conjugacy called **stable conjugacy** plays a role. The study of Langlands functoriality often leads to correspondences that are defined only up to stable conjugacy. Endoscopy is the name given to a series of techniques aimed to investigate the difference between ordinary and stable conjugacy. The word "endoscopy" has been coined to express that we want to see ordinary conjugacy **inside** stable conjugacy. The aim is to recover orbital integrals and characters from of their stable avatars on a family of auxiliary groups called endoscopic groups.

One also expects a parametrization of irreducible admissible representations in term of the so-called dual picture where one considers Langlands parameters that are classes of admissible homorphisms of the Weil group $W_F$ (see 2.6), or more generally the Langlands group $\mathcal{L}_F$, into the L-group. Moreover endoscopy should be encoded in this dual picture. Such a dual parametrization is conjectural in general but has been completely established for real groups by Langlands and Shelstad.

## 1.2 The contents of these lectures

In this series of lectures we shall introduce the basic notions of local endoscopy: stable conjugacy, $\kappa$-orbital integrals, endoscopic groups, endoscopic transfer of orbital integrals and its dual for characters with an emphasis on the case of real groups, following the work of Diana Shelstad.

We start in section 2 with a brief review of some basic objects: we recall the definition of pseudo-coefficients, then we discuss stable conjugacy, $L$-packets, Weil groups, L-groups and Langlands parameters. In section 3 we review the classification and the Langlands parameters for admissible representations of $GL(2,\mathbb{R})$ in order to fix the notation to be used for this basic example. In section 4 we describe the endoscopy for $SL(2,\mathbb{R})$ which is the simplest non trivial example. We describe in this case the stable conjugacy, the $L$-packets as well as the endoscopic transfer of functions and of characters. In section 5 we briefly describe the endoscopy for the real unitary group $U(2,1)$. In section 6 we introduce some simple objects from Galois cohomology in order to define stable conjugacy, $\kappa$-orbital integrals and endoscopic groups in general. In section 7 we discuss the endoscopic transfer of pseudo-coefficients of discrete series. We give a simple direct proof of it. In the last section we discuss some extensions of the preceding material.

# 2 Some basic definitions

The Langlands theory and in particular endoscopy deals with connected reductive algebraic groups defined over local or global fields.

One should be warned that even if $G$ is defined over $\mathbb{R}$ and is connected as an algebraic group it may happen that $G(\mathbb{R})$ is disconnected as a topological group. For example the algebraic group $GL(2)$ is connected while $GL(2,\mathbb{R})$ has two connected components.

## 2.1 Orbital integrals

Let $G$ be a connected reductive algebraic group defined over a local field $F$. Let $\gamma \in G(F)$ be a strongly regular semisimple element: this means that its centralizer $G_\gamma$ is a (maximal) torus. Let $f \in \mathcal{C}_c^\infty(G(F)$, the orbital integral of $f$ on the orbit of $\gamma$ is given by

$$\mathcal{O}_\gamma(f) = \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)\, d\dot{x}$$

where $d\dot{x}$ is a $G(F)$-invariant measure on the quotient $G_\gamma(F)\backslash G(F)$.

## 2.2 Pseudo-coefficients for discrete series

Let $F$ be a local field. Assume that the center of $G(F)$ is compact. Let $\pi$ be a discrete series representation of $G(F)$ we say that a function $f \in \mathcal{C}_c^\infty(G(\mathbb{R}))$ is a (normalized) pseudo-coefficient for $\pi$ if for any **tempered** irreducible representation $\pi'$ we have

$$\text{trace } \pi'(f) = \begin{cases} 1 & \text{if } \pi' \simeq \pi \\ 0 & \text{otherwise} \end{cases}$$

Observe that it would be more canonical to consider $f\, d\mu$ where $d\mu$ is a Haar measure in $G(\mathbb{R})$ since the choice of the Haar measure enters in the definition of trace $\pi(f)$.

A suitably normalized diagonal matrix coefficient of $\pi$ (or rather its complex conjugate) would satisfy these requirements but for one condition: matrix coefficients are not compactly supported unless the group is compact. The interest of using pseudo-coefficients instead of matrix coefficients is that since they are compactly supported trace $\pi(f)$ makes sense even if $\pi$ is not tempered and they can be used in the trace formula. This is important to observe since non tempered representations show up in the spectral decomposition of the space of square integrable automorphic forms.

When $F = \mathbb{R}$ the existence of pseudo-coefficients is an easy consequence of the existence of multipliers due to Arthur [A] and Delorme [D]. We refer the reader to [CD] or [Lab2] for the construction. In [Lab2] more general functions, called index functions are constructed. They allow in particular to deal with certain linear combinations of limits of discrete series although individual ones do not have pseudo-coefficients. For $p$-adic fields the existence of pseudo-coefficients is due to Kazhdan [Ka].

We shall denote by $f_\pi$ a pseudo-coefficient for $\pi$, although it is highly non unique. But as regards invariant harmonic analysis this plays no role. In particular the orbital integrals are independent of the choice of the pseudo-coefficient; they are also independent of the choice of the Haar measure on $G(F)$ but one has to use the canonical measure on the compact torus $G_\gamma(F)$. The orbital integrals of $f_\pi$ are easily computed for $\gamma$ regular semisimple:

$$\mathcal{O}_\gamma(f_\pi) = \begin{cases} \Theta_\pi(\gamma^{-1}) & \text{if } \gamma \text{ is elliptic} \\ 0 & \text{otherwise} \end{cases}$$

where $\Theta_\pi$ is the character of $\pi$.

## 2.3 Stable conjugacy

Let $G$ be a connected reductive group over a local field $F$. Recall that one says that $\gamma$ and $\gamma'$ in $G(F)$ are conjugate if there exists $x \in G(F)$ such that $\gamma' = x\gamma x^{-1}$. Roughly speaking, stable conjugacy amounts to conjugacy over the algebraic closure $\overline{F}$. At least for strongly regular semisimple elements, one says that $\gamma$ and $\gamma'$ in $G(F)$ are stably conjugate if there is an $x \in G(\overline{F})$ such that $\gamma' = x\gamma x^{-1}$. This definition is also valid when $\gamma \in G(F)$ is any semisimple element if $G$ has a simply connected derived group. In general the definition is slightly more technical and will be given in 6.3.

Let $f \in \mathcal{C}_c^\infty(G(F))$, the stable orbital integral for $\gamma \in G(F)$ strongly regular is by definition

$$SO_\gamma(f) = \sum_{\gamma' \in S(\gamma)} \mathcal{O}_{\gamma'}(f)$$

where $S(\gamma)$ is a set of representatives for conjugacy classes in the stable conjugacy class of $\gamma$. Implicit here is the choice of a natural measure on the various conjugacy classes: one uses that the various centralizers $G_{\gamma'}$ are canonically isomorphic.

Stably invariant harmonic analysis is the study of stable distributions: the distributions that are, roughly speaking, constant on stable conjugacy classes. This is the closure of the space spanned by stable orbital integrals.

The miracle is that one can recover ordinary orbital integrals as a linear combination of stable orbital integrals on endoscopic groups (see 6.5). This supposes to be able to transfer stable orbital integrals on endoscopic groups to $\kappa$-integrals on the group $G$ itself. The transfer involves the so-called "transfer factors" of Langlands and Shelstad. The existence of the transfer is due to Shelstad when $F = \mathbb{R}$ (see 6.7 and the paper of Shelstad in this volume). For arbitrary local fields it has been obtained only recently thanks to Ngô and Waldspurger.

## 2.4 $L$-packets

To define $L$-packets for representations of $G(F)$ we consider first the case of discrete series. Consider a discrete series representation $\pi$ and let us denote by $f_\pi$ a pseudo-coefficient. Two discrete series representations $\pi$ and $\pi'$ for $G(F)$ will be said to belong to the same $L$-packet if for any strongly regular semisimple $\gamma$ one has

$$S\mathcal{O}_\gamma(f_\pi) = c(\pi, \pi') S\mathcal{O}_\gamma(f_{\pi'})$$

where $c(\pi, \pi')$ is some non zero constant.

Over the reals, an $L$-packet of discrete series of a reductive Lie group is the set of all (equivalence classes of) discrete series with the same infinitesimal character and then $c(\pi, \pi') \equiv 1$. The discrete series $L$-packets are in bijection with the (equivalence classes of) finite dimensional representations of the compact inner form.

One can construct all tempered irreducible representations in term of the discrete series representations of Levi subgroups using unitary parabolic induction and by taking subrepresentations. Using this it is now easy to define

$L$-packets for tempered representations. Two tempered irreducible representations $\pi$ and $\pi'$ are said to be in the same $L$-packet if, up to equivalence, $\pi$ and $\pi'$ are subrepresentations of parabolically induced representations from discrete series $\sigma$ and $\sigma'$ in the same $L$-packet for some Levi subgroup. One should get a stable distributions by taking a suitable linear combination of characters in an $L$-packet (namely the sum when $F = \mathbb{R}$).

The notion of $L$-packets of representations is the stable avatar for irreducible tempered representations. The second expected miracle is that one should be able express tempered characters in terms of stable characters on endoscopic groups.

### 2.5 Arthur packets

Using parabolic induction from quasi-tempered[1] irreducible representations of Levi subgroups and by taking appropriate subquotients (the Langlands quotients) one can construct all admissible irreducible representations of $G(F)$ and thus one could extend the notion of $L$-packets to this more general case. But, in general, no non trivial linear combination of character in the $L$-packet will yield a stable distribution. For the non tempered case one needs the notion of Arthur packets; they are bigger packets than $L$-packets and the linear combination to be taken is more subtle even when $F = \mathbb{R}$. This will be examined in Vogan's lectures.

### 2.6 The Weil and the Langlands groups

Let $F$ be a local field. One denotes by $W_F$ the Weil group of $F$ defined as follows. The Weil group of $\mathbb{C}$ is simply $\mathbb{C}^\times$:

$$W_{\mathbb{C}} = \mathbb{C}^\times \ .$$

The Weil group for $\mathbb{R}$ can be defined as the subgroup matrices in $SU(2)$ generated by

$$\left\{ \mathbf{z} = \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} \ , \ z \in \mathbb{C}^\times \right\} \quad \text{and} \quad w_\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The element $w_\sigma$ acts by conjugacy as the non trivial element in the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\mathbb{C}^\times$, in other words there is an exact sequence

$$1 \to W_{\mathbb{C}} \to \mathbb{W}_{\mathbb{R}} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$$

[1] i.e. tempered up to a twist by a quasi-character (a non necessarily unitary character).

and a section of the map $\mathbb{W}_{\mathbb{R}} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is defined by

$$\sigma \mapsto w_\sigma$$

Observe that $w_\sigma^2 = -\mathbf{1}$ and hence the above extension of $W_{\mathbb{C}} = \mathbb{C}^\times$ by $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is the non trivial one.

When $F$ is a $p$-adic field the Weil group $W_F$ can be viewed as a subgroup of $\mathrm{Gal}(\overline{F}/F)$. It sits in the following diagram with exact lines

$$\begin{array}{ccccccccc} 1 & \to & I_F & \to & W_F & \to & \mathbb{Z} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & I_F & \to & \mathrm{Gal}(\overline{F}/F) & \to & \overline{\mathbb{Z}} & \to & 1 \end{array}$$

where $I_F$ is the inertia group and $\overline{\mathbb{Z}}$ is the completion of $\mathbb{Z}$ for the topology of finite index subgroups. The group $\mathbb{Z}$ in the first line is endowed with the discrete topology. The vertical arrows are continuous inclusions with dense image, the first one being a topological isomorphism.

If $K/F$ is a finite Galois extension one has an exact sequence

$$1 \to W_K^{ab} \to W_F \to \mathrm{Gal}(K/F) \to 1$$

where $W_K^{ab}$ is the abelianized quotient of $W_K$.

The Langlands group $\mathcal{L}_F$ is defined to be

$$\mathcal{L}_F = W_F$$

when $F = \mathbb{C}$ or $\mathbb{R}$. When $F$ is $p$-adic one must take

$$\mathcal{L}_F = W_F \times SL(2, \mathbb{C}) \ .$$

## 2.7 L-groups and Langlands parameters

Let $F$ be a local field and $G$ a connected reductive group over $F$. Denote by $G^*$ the quasisplit inner form. Choose a Borel pair $(B, T)$ in $G^*$. We get a based root datum:

$$(X, \Delta, \check{X}, \check{\Delta})$$

Denote by $\check{G}$ the **complex** reductive Lie group defined by the dual based root datum

$$(\check{X}, \check{\Delta}, X, \Delta)$$

For example

$$\text{if}\quad G = GL(n) \quad \text{then} \quad \check{G} = GL(n,\mathbb{C})$$
$$\text{if}\quad G = SL(n) \quad \text{then} \quad \check{G} = PGL(n,\mathbb{C})$$
$$\text{if}\quad G = SP(2n) \quad \text{then} \quad \check{G} = SO(2n+1,\mathbb{C})$$

There is a canonical bijection between conjugacy classes of parabolic subgroups of $G^*$ and of $\check{G}$. Those coming from $G$ form a subset.

The Galois group $\mathrm{Gal}(\overline{F}/F)$ acts on the based root datum and hence it acts on $\check{G}$ via **holomorphic automorphisms** assumed to preserve a splitting. If $G$ is split this action is trivial. Now $W_F$ acts via its natural map to $\mathrm{Gal}(\overline{F}/F)$ and one denotes by ${}^LG$ the semidirect product $\check{G} \rtimes W_F$.

Let us give a non trivial example: if $F = \mathbb{R}$ and $G(\mathbb{R}) = U(2,1)$ then $\check{G} = GL(3,\mathbb{C})$ and the holomorphic action of the non trivial element $\sigma$ in the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\check{G}$ is given by

$$\check{\sigma} : g \mapsto J^t g^{-1} J^{-1} \quad \text{with} \quad J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

A Langlands parameter is a $\check{G}$-conjugacy class of homomorphisms

$$\varphi \ : \ \mathcal{L}_F \ \to \ {}^LG$$

holomorphic on the $SL(2)$ factor in the $p$-adic case, such that the compositum with the natural projection of $\mathcal{L}_F \to W_F$

$$\mathcal{L}_F \ \to \ {}^LG \ \to \ W_F$$

is natural projection of $\mathcal{L}_F$ onto $W_F$. One asks moreover that the images of elements in $W_F$ are semisimple. A parameter is said to be relevant (for $G$) if the image of $\varphi$ in $\check{G}$ does not lie in a parabolic subgroup unless it comes from $G$.

One of the goals of the theory is to show that one has a natural parametrization of irreducible admissible representations of $G(F)$ when $F$ is a local field (and of automorphic representations when $F$ is a global field) using the Langlands parameters: to each Langlands parameter there should be associated an $L$-packet $\Pi(\varphi)$ of representations of $G(F)$. More precisely, relevant Langlands parameters should parametrize $L$-packets and the constituents of an $L$-packet should be attached to representations of some finite group defined by the parameter.

When $F = \mathbb{R}$ Langlands has established such a correspondence for an arbitrary $G$ in a paper that remained unpublished for a long time but is now available in print [Lan2]. The correspondence between constituents of $L$-packets and representations of a finite group defined by the parameter is due to Shelstad (see 6.7.4). To obtain a nice formulation it is better to use K-groups (see 8.1). We explain these correspondences for $GL(2,\mathbb{R})$ and $SL(2,\mathbb{R})$ in 3.2 and 4.3.

For non archimedean local fields little is known except for $GL(n)$, but rapid progress are to be expected at least for classical groups.

# 3 GL(2)

## 3.1 Representations of $GL(2,\mathbb{R})$

By representations we understand admissible representations. We shall use Jacquet-Langlands notation. The classification is as follows: all admissible irreducible representations of $GL(2,\mathbb{R})$ are subquotients of principal series

$$\rho(\mu_1,\mu_2)$$

where $\mu_i$ are characters of $\mathbb{R}^\times$. The principal series representations are induced by characters from the Borel subgroup: $\rho(\mu_1,\mu_2)$ is the right regular representation in the space of smooth functions such that

$$f(\begin{pmatrix}\alpha & x\\ 0 & \beta\end{pmatrix} g) = \mu_1(\alpha)\mu_2(\beta)\left|\frac{\alpha}{\beta}\right|^{1/2} f(g)\ .$$

We shall restrict ourselves to representations that induce unitary characters on the center and hence we assume that the product $\mu_1\mu_2$ is unitary. We have three types of subquotients according to the value of

$$\mu = \mu_1\mu_2^{-1}$$

1 – Irreducible principal series

$$\pi(\mu_1,\mu_2)$$

where

$$\mu \neq x^n \,.\, \mathrm{sign}\,(x)$$

for some $n \in \mathbb{Z} - \{0\}$. These representations are unitarizable if $\mu$ is unitary or if $\mu = |x|^s$ with $s$ real and $-1 < s < 1$.

2 – Finite dimensional subquotients

$$\pi(\mu_1, \mu_2)$$

when $\mu = x^n \,.\, \mathrm{sign}\,(x)$ with $n \in \mathbb{Z} - \{0\}$. It is unitarizable if $n = \pm 1$.

3 – Discrete series subquotients

$$\sigma(\mu_1, \mu_2)$$

when $\mu = x^n \,.\, \mathrm{sign}\,(x)$ with $n \in \mathbb{Z} - \{0\}$. These representations are unitarizable.

The various representations are equivalent under permutation of the $\mu_i$: $\pi(\mu_1, \mu_2) \simeq \pi(\mu_1, \mu_2)$ etc...

## 3.2 Langlands parameters for $GL(2, \mathbb{R})$

A Langlands parameter for $GL(2, \mathbb{R})$ is a conjugacy classes of homomorphisms of $\mathbb{W}_\mathbb{R}$ in $GL(2, \mathbb{C})$ with semisimple images.

For $z = \rho e^{i\theta}$ let $\chi_{s,n}(z) = \rho^s e^{in\theta}$ then, up to conjugacy, the admissible maps $\varphi$ are of the following form:

1 – for some $s_i \in \mathbb{C}$ and $m_i \in \mathbb{Z}_2$

$$\varphi_{s_1,m_1,s_2,m_2}(\mathbf{z}) = \begin{pmatrix} \chi_{s_1,0}(z) & 0 \\ 0 & \chi_{s_2,0}(z) \end{pmatrix}$$

with

$$\varphi_{s_1,m_1,s_2,m_2}(w_\sigma) = \begin{pmatrix} (-1)^{m_1} & 0 \\ 0 & (-1)^{m_2} \end{pmatrix}$$

Up to conjugacy $\varphi_{s_1,m_1,s_2,m_2} \simeq \varphi_{s_2,m_2,s_1,m_1}$.

2 – for some $s \in \mathbb{C}$ and $n \in \mathbb{Z}$

$$\varphi_{s,n}(\mathbf{z}) = \begin{pmatrix} \chi_{s,n}(z) & 0 \\ 0 & \chi_{s,-n}(z) \end{pmatrix}$$

with

$$\varphi_{s,n}(w_\sigma) = \begin{pmatrix} 0 & (-1)^n \\ 1 & 0 \end{pmatrix}$$

Up to conjugacy $\varphi_{s,n} \simeq \varphi_{s,-n}$.

The intersection of the two sets of conjugacy classes of maps is the class of parameters of the form

$$\varphi_{s,0} \simeq \varphi_{s,1,s,0} \simeq \varphi_{s,0,s,1}$$

Let us denote by $\varepsilon$ the homomorphism from $\mathbb{W}_{\mathbb{R}}$ to $\mathbb{C}^{\times}$ defined by

$$\varepsilon(\mathbf{z}) = 1 \quad \text{and} \quad \varepsilon(w_\sigma) = -1$$

**Lemma 3.2.1** *If $\varphi$ is a Langlands parameter, then*

$$\varphi \otimes \varepsilon \simeq \varphi$$

*if and only if $\varphi$ is in the class of $\varphi_{s,n}$ for some $s$ and some $n$.*

*Proof*: First it is clear that

$$\varphi_{s,n} \otimes \varepsilon = \alpha\, \varphi_{s,n}\, \alpha^{-1} \quad \text{where} \quad \alpha = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Conversely we have to observe that

$$\varphi_{s_1,m_1,s_2,m_2} \not\simeq \varphi_{s_1,m_1+1,s_2,m_2+1}$$

unless $s_1 = s_2$ and $m_1 + m_2 \equiv 1$

$\square$

Recall the Langlands correspondence for $GL(1,\mathbb{R})$ : a character $\mu$ of $GL(1,\mathbb{R}) = \mathbb{R}^{\times}$ of the form

$$\mu(x) = |x|^s \operatorname{sign}(x)^m$$

corresponds to a character of the Weil group

$$\mathbf{z} \mapsto \chi_{s,0} \quad \text{and} \quad w_\sigma \mapsto (-1)^m$$

(this is a particular case of Tate-Nakayama duality; see 6.5)

The correspondence between irreducible representations and Langlands parameters for $GL(2,\mathbb{R})$ is obtained as follows. We have a natural bijection between equivalence classes of admissible irreducible representations of $GL(2,\mathbb{R})$ and conjugacy classes of admissible homomorphisms of $\mathbb{W}_\mathbb{R}$ in $GL(2,\mathbb{C})$ as follows:

$$\pi(\mu_1,\mu_2)\mapsto\varphi_{s_1,m_1,s_2,m_2}\quad\text{with}\quad\mu_i(x)=|x|^{s_i}\,\mathrm{sign}\,(x)^{m_i}$$

and

$$\sigma(\mu_1,\mu_2)\mapsto\varphi_{s,n}\quad\text{with}\quad\mu_1\mu_2(x)=|x|^{2s}\,\mathrm{sign}\,(x)^{n+1}$$

where we assume moreover that

$$\mu_1\mu_2^{-1}(x)=x^n\,\mathrm{sign}\,(x)$$

A Langlands parameter corresponds to a tempered representation if the image of the map is bounded i.e. if the $s_i$ are imaginary.

## 3.3 Endoscopy for $GL(2,F)$

Inside $GL(n,F)$ conjugacy over $F$ and over $\overline{F}$ coincide. Hence there is no non-trivial endoscopy for $GL(n)$ in the sense that conjugacy and stable conjugacy coincide and $L$-packets are singletons. Nevertheless, if we consider an inner form $G$ of $G^*=GL(n)$, then there is are correspondences for conjugacy classes and for representations between $G$ and $G^*$.

Recall that $G$ is the multiplicative group of a quaternion algebra. There is a bijection between elliptic conjugacy classes in $G(F)$ and $G^*(F)$: this correspondence matches elements with the same eigenvalues in $\overline{F}$. The bijection $\pi\mapsto\pi^*$ between discrete series representations of $G(F)$ and of $G^*(F)$ is characterized by the equality of character distributions up to a sign:

$$\Theta_\pi(\gamma)=-\Theta_{\pi^*}(\gamma^*)$$

whenever $\gamma\in G(F)$ and $\gamma^*\in G^*(F)$ are regular elliptic elements with the same eigenvalues in $\overline{F}$.

These correspondences, often called Jacquet-Langlands correspondences, are nowadays considered as an instance of endoscopy.

# 4 $SL(2)$

Besides Jacquet-Langlands correspondence for $GL(2)$, endoscopy for $SL(2)$ is the simplest instance of it and in fact it is in this case that it was first studied.

As far as I know endoscopy was discovered by Langlands while he was trying to understand the Zeta function of some simple Shimura varieties: those attached to inner forms of groups between $GL(2)$ and $SL(2)$ over a totally real field.

## 4.1 Endoscopy for $SL(2,\mathbb{R})$

Endoscopy arises because the conjugacy over $F$ and over $\overline{F}$ (the algebraic closure of $F$) may be different. For example, for $\theta \notin \mathbb{Z}\pi$, the two rotations

$$r(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and

$$r(-\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

are not conjugate in $SL(2,\mathbb{R})$ although they are conjugate inside $SL(2,\mathbb{C})$ and also inside $GL(2,\mathbb{R})$ by

$$w = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad \text{and} \quad \alpha = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The union of the sets of conjugates of elements in the pair $(r(\theta), r(-\theta))$ is said to be a **stable conjugacy class**. Similarly the unipotent elements

$$u_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad u_0^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

are conjugate in $SL(2,\mathbb{C})$ but not in $SL(2,\mathbb{R})$. There are two conjugacy classes but only one stable conjugacy class of non trivial unipotent elements.

Endoscopy on the spectral side is also easily described for $SL(2,\mathbb{R})$: discrete series and limits of discrete series representations come by pairs called $L$-packets.

In all above examples the objects inside pairs are exchanged by conjugacy under $w = i\alpha$ an element in the normalizer of $SO(2)$ in $SL(2,\mathbb{C})$. We observe that if $\sigma$ is the non trivial element of the Galois group, then

$$a_\sigma = w\sigma(w)^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

generates a subgroup of order 2 that can be identified with $H^1(\mathbb{C}/\mathbb{R}, SO(2))$. The characters of this 2-group are called endoscopic character. The endoscopic groups for $SL(2,\mathbb{R})$ correspond to these two characters. The endoscopic group corresponding to the trivial endoscopic character is $SL(2,\mathbb{R})$ himself while the endoscopic group corresponding to the non trivial one is

$$T(\mathbb{R}) = SO(2,\mathbb{R})$$

the compact torus.

## 4.2 Representations of $SL(2,\mathbb{R})$

The representations of $SL(2,\mathbb{R})$ and their $L$-packets are easily understood in terms of those for $GL(2,\mathbb{R})$ recalled in 3.1. In fact it is an easy exercise to show that any irreducible representation of $SL(2,\mathbb{R})$ occurs in the restriction of an irreducible representation of $GL(2,\mathbb{R})$. Those restrictions either remain irreducible (which is the case for principal series for generic values of the parameter) or split into two irreducible components whose union is an $L$-packet for $SL(2,\mathbb{R})$.

Two representations $\pi$ and $\pi'$ are in the same $L$-packet if and only if, up to equivalence, they are conjugated by $\alpha$:

$$\pi' \simeq \pi \circ \mathrm{Ad}(\alpha) \quad \text{where} \quad \alpha = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

We have the following classification:

1 – Irreducible principal series: they are the $\pi(\mu)$ obtained by restriction of $\pi(\mu_1, \mu_2)$ with $\mu \neq x^n \,.\, \mathrm{sign}\,(x)$ with $n \in \mathbb{Z}$.

2 – Finite dimensional representations of dimension $n$: they are the $\pi(\mu)$ obtained by restriction of $\pi(\mu_1, \mu_2)$ with $\mu = x^n \,.\, \mathrm{sign}\,(x)$ and $n \neq 0$.

3 – Discrete series $L$-packets

$$\sigma(\mu) = \{D^+_{|n|}, D^-_{|n|}\}$$

are obtained by restriction of $\sigma(\mu_1, \mu_2)$ with

$$\mu = x^n \,.\, \mathrm{sign}\,(x)$$

and $n \in \mathbb{Z} - 0$.

4 – Limits of discrete series $L$-packet

$$\sigma(\mu) = \{D_0^+, D_0^-\}$$

is obtained by restriction of $\pi(\mu_1, \mu_2)$ with

$$\mu = \text{sign}\,(x)$$

The $L$-packets of representations that are indexed by characters $\mu$ and $\mu^{-1}$ are equivalent.

The minimal $K$-type for $D_n^{\pm}$ with $n \in \mathbb{N}$ is $\pm(n+1)$ i.e.

$$r(\theta) \mapsto \exp\left(\pm i\,(n+1)\,\theta\right)$$

if

$$r(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

The character of $D_n^+$ on $K = SO(2,\mathbb{R})$ is given by

$$\Theta_n^+(r(\theta)) = \frac{e^{i(n+1)\theta}}{1 - e^{2i\theta}} = \frac{-e^{in\theta}}{e^{i\theta} - e^{-i\theta}}$$

while the character of $D_n^-$ is the complex conjugate:

$$\Theta_n^-(r(\theta)) = \frac{e^{-i(n+1)\theta}}{1 - e^{-2i\theta}} = \frac{e^{-in\theta}}{e^{i\theta} - e^{-i\theta}}$$

## 4.3 Langlands parameters for $SL(2,\mathbb{R})$

From the bijection between equivalence classes of representations and conjugacy classes of Langlands parameters for $GL(2,\mathbb{R})$ recalled in 3.2 one derives a bijection between equivalence classes of $L$-packets of admissible irreducible representations of $SL(2,\mathbb{R})$ and conjugacy classes of admissible homomorphisms of $\mathbb{W}_{\mathbb{R}}$ in $PGL(2,\mathbb{C})$. We need the

**Lemma 4.3.1** *Any projective representation of* $\mathbb{W}_{\mathbb{R}}$ *lifts to a representation.*

*Proof*: This is a particular case of a result of [Lab1]. This particular case can also be found in [Lan2]. We shall prove it only for two dimensional representations. Consider a two-dimensional projective parameter. The image of

$\mathbb{C}^\times$ is inside a torus. Now this image is either trivial or is a maximal torus in $PGL(2,\mathbb{C})$ and hence of dimension 1. If this image is trivial the lemma is easy to prove. If this image is a torus the map restricted to $\mathbb{C}^\times$ is given by a non trivial character $\chi_{s,n}$. If $n = 0$ th existence of a lift is easy. If $n \neq 0$ then and the image of $w_\sigma$ must lie in the normalizer of the torus and acts non trivially and hence its square is central i.e. projectively trivial and $s = 0$. Since $w_\sigma^2 = -1$ this shows that $n$ is even and hence $\varphi_{0,n/2}$ is a lift.

□

The correspondence is a follows:

1 – the parameter for $\pi(\mu)$ is the conjugacy class of the projective parameter $\varphi_{s,m}$ defined by $\varphi_{s,m,0,0}$ with

$$\mu(x) = |x|^s \operatorname{sign}(x)^m$$

2 – the parameter for $D_n^\pm$ is the conjugacy class of the projective parameter $\varphi_n$ defined by $\varphi_{0,n}$.

We have seen in lemma 3.2.1 that

$$\varphi_{0,n} \otimes \varepsilon = \alpha \varphi_{0,n} \alpha^{-1} \quad \text{where} \quad \alpha = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

But $\varepsilon$ has a central image and hence the projective parameters defined by $\varphi_{0,n}$ and $\varphi_{0,n} \otimes \varepsilon$ are equal. This shows that the projective image of $\alpha$ belongs to the centralizer of the projective image of $\varphi_{0,n}$.

Let $\varphi_n$ be the projective parameter defined by $\varphi_{0,n}$ and let $S_{\varphi_n}$ denote the centralizer of the image of $\varphi_n$ and $\mathfrak{S}_{\varphi_n}$ the quotient of $S_{\varphi_n}$ by its connected component $S^0_{\varphi_n}$ times the center $Z_{\check{G}}$ of $\check{G}$ (trivial here). When $n \neq 0$ we have

$$\mathfrak{S}_{\varphi_n} = S_{\varphi_n} \simeq \{1, \alpha\}$$

When $n = 0$ the group $S^0_{\varphi_0}$ is a torus but again $\mathfrak{S}_{\varphi_0}$ is generated by the image of $\alpha$.

## 4.4 Character identities

We observe that the sum

$$S\Theta_n = \Theta_n^+ + \Theta_n^-$$

verifies

$$S\Theta_n(r(\theta)) = -\frac{e^{in\theta} - e^{-in\theta}}{e^{i\theta} - e^{-i\theta}}$$

and hence is invariant under stable conjugacy; we say that $S\Theta_n$ is a stable character, while

$$\Delta(r(\theta))(\Theta_n^+ - \Theta_n^-)(r(\theta)) = e^{in\theta} + e^{-in\theta}$$

where

$$\Delta(r(\theta)) = -2i \sin\theta$$

is the sum of two characters of $T(\mathbb{R})$. This is the spectral endoscopic transfer for $SL(2,\mathbb{R})$. For further generalization we observe that

$$\Delta(r(\theta)) = -e^{i\theta}(1 - e^{-2i\theta}) = -i \text{ sign}\,(\sin(\theta))|e^{i\theta} - e^{-i\theta}|$$

The group $G^* = SL(2)$ over $\mathbb{R}$ has an inner form $G$ such that

$$G(\mathbb{R}) = SU(2)$$

Observe that $SL(2,\mathbb{R})$ and $SU(2)$ have in common the compact Cartan subgroup

$$T(\mathbb{R}) = SO(2)$$

The conjugacy classes in $SU(2)$ are classified by the pairs of eigenvalues $\{e^{i\theta}, e^{-i\theta}\}$. Hence they are in bijection with elliptic stable conjugacy classes in $SL(2,\mathbb{R})$. This correspondence is dual to the correspondence between representation $F_n$ of dimension $n$ of $SU(2)$ and $L$-packets of discrete series $D_n^{\pm}$ and there is the character identity

$$\text{trace } F_n(r(\theta)) = -S\Theta_n(r(\theta))$$

## 4.5 Asymptotic behaviour of orbital integrals and geometric transfer

Let $f$ be a smooth and compactly supported function on $G = SL(2,\mathbb{R})$. We are to study its orbital integrals. For $\gamma \in G$ let us denote by $G_\gamma$ its centralizer in $G$. The orbital integral is

$$\mathcal{O}_\gamma(f) = \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)\, d\dot{x}$$

Observe it depends on the choice of Haar measures on $G$ and $G_\gamma$. We shall study the asymptotic behaviour of $\mathcal{O}_\gamma(f)$ when $\gamma$ is diagonal and $\gamma \to 1$ and when $\gamma = r(\theta)$ and $\theta \to 0$. We may and will assume that $f$ is $K$-central i.e. $f(kx) = f(xk)$.

Consider the first case where $\gamma$ is diagonal:

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

with $ab = 1$. Then

$$\mathcal{O}_\gamma(f) = \int_U f(u^{-1}\gamma u)\, du$$

for a standard choice of Haar measures, where $U$ is the group of unipotent matrices and hence

$$\mathcal{O}_\gamma(f) = \int_{\mathbb{R}} f \begin{pmatrix} a & (a-b)x \\ 0 & b \end{pmatrix} dx$$

which yields the

**Lemma 4.5.1** *Let* $\Delta(\gamma) = |a - b|$ *then*

$$h(\gamma) = \Delta(\gamma)\mathcal{O}_\gamma(f)$$

*extends to a smooth function on the group* $A(\mathbb{R})$ *of diagonal real matrices.*

This is the simplest example of endoscopic geometric transfer for a non elliptic endoscopic group.

When $\gamma = r(\theta)$ the orbital integral can be computed using the Cartan decomposition $G = KAK$. Here we may take $K = T(\mathbb{R})$ the group of rotations. We have

$$\mathcal{O}_{r(\theta)}(f) = c\, F(\sin\theta)$$

with $c$ a constant that depends on the choice of Haar measures and

$$F(\lambda) = \int_1^\infty f \begin{pmatrix} a(\lambda) & t\lambda \\ -t^{-1}\lambda & a(\lambda) \end{pmatrix} \left|t - t^{-1}\right| \frac{dt}{t}$$

where

$$a(\lambda) = \sqrt{1-\lambda^2}$$

Observe that since $f$ is $K$-central we have for any $a$, $b$ and $c$

$$f \begin{pmatrix} a & b \\ c & a \end{pmatrix} = f \begin{pmatrix} a & -c \\ -b & a \end{pmatrix}$$

and hence
$$F(\lambda) = \int_0^\infty \varepsilon(t-1) f \begin{pmatrix} a(\lambda) & t\lambda \\ -t^{-1}\lambda & a(\lambda) \end{pmatrix} dt$$
with $\varepsilon(x) = \mathrm{sign}(x)$. To study the asymptotic its behaviour when $\lambda \to 0$ we consider
$$A(\lambda) = \int_0^\infty \varepsilon(t-1) f \begin{pmatrix} a(\lambda) & \lambda t \\ 0 & a(\lambda) \end{pmatrix} dt$$
By Taylor-Lagrange formula
$$F(\lambda) = A(\lambda) + \lambda B(\lambda)$$
where
$$B(\lambda) = \int_0^\infty \varepsilon(t-1) g \begin{pmatrix} a(\lambda) & t\lambda \\ -t^{-1}\lambda & a(\lambda) \end{pmatrix} \frac{dt}{t}$$
for some smooth function $g$ compactly supported in the upper right variable and that has a $O(u)^{-1}$ decay in the lower left variable $u$ so that the integral is absolutely convergent.

Observe that
$$A(\lambda) = |\lambda|^{-1} \int_0^\infty f \begin{pmatrix} 1 & \varepsilon(\lambda)u \\ 0 & 1 \end{pmatrix} du - 2f(\mathbf{1}) + o(\lambda)$$
Since $B(\lambda)$ is at most of logarithmic growth we see that the even functions
$$G(\lambda) = |\lambda|(F(\lambda) + F(-\lambda))$$
and
$$H(\lambda) = \lambda(F(\lambda) - F(-\lambda))$$
extend to continuous functions at the $\lambda = 0$.

We need more precise informations on the asymptotic behaviour. Hence we have to look more carefully to the error term $B(\lambda)$ which, we recall, equals:
$$\int_0^\infty \varepsilon(t-1) g \begin{pmatrix} a(\lambda) & t\lambda \\ -t^{-1}\lambda & a(\lambda) \end{pmatrix} \frac{dt}{t}$$
But this is the difference of two terms whose leading terms are equivalent to $\ln(|\lambda|^{-1})g(\mathbf{1})$ up to continuous terms. Hence $B$ is continuous. Generalizing this process one gets asymptotic expansions of the following form:
$$G(\lambda) = \sum_{n=0}^{N} \left(a_n\, |\lambda|^{-1} + b_n\right) \lambda^{2n} + o(\lambda^{2N})$$

and

$$H(\lambda) = \sum_{n=0}^{N} h_n \, \lambda^{2n} + o(\lambda^{2N})$$

and hence $H(\lambda)$ is smooth. We have proved the following

**Lemma 4.5.2** *There is a smooth fonction $h$ on $T(\mathbb{R})$ such that:*

$$h(\gamma) = \Delta(\gamma) \left( O_\gamma(f) - O_{w(\gamma)}(f) \right)$$

*for $\gamma = r(\theta) \in T(\mathbb{R})$ and*

$$\Delta(r(\theta)) = -2i \sin\theta$$

This lemma establishes the simplest case of a non trivial geometric endoscopic transfer for an elliptic endoscopic group. A variant of this lemma establishes the transfer between orbital integrals weighted by the character of order two

$$\omega = \text{sign} \circ \det$$

for functions on $GL(2, \mathbb{R})$:

$$\mathcal{O}_\gamma^\omega(f) = \int_{G_\gamma \backslash G} \omega(x) f(x^{-1}\gamma x) \, d\dot{x}$$

and functions on the elliptic maximal torus

$$\widetilde{T}(\mathbb{R}) \simeq \mathbb{C}^\times = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in GL(2, \mathbb{R}) \right\}$$

**Lemma 4.5.3** *There is a smooth fonction $h$ on $\widetilde{T}(\mathbb{R})$ such that:*

$$h(\gamma) = \Delta(\gamma) O_\gamma^\omega(f)$$

*for $\gamma = \rho \, r(\theta) \in \widetilde{T}(\mathbb{R})$ and*

$$\Delta(r(\theta)) = -2i \sin\theta$$

An other simple example is given by the transfer between inner forms. Consider the compact group $G = SU(2,\mathbb{R})$. This is an inner form of $G^* = SL(2,\mathbb{R})$. In fact one has $g \in SU(2,\mathbb{R})$ iff

$$g \in SL(2,\mathbb{C}) \quad \text{and} \quad g = J\sigma(g)J^{-1}$$

with

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $\sigma(g)$ is the complex conjugate of $g$.

There is an injection of the set of conjugacy classes inside $G$ into the set of stable conjugacy classes in $G^*$ induced by the following correspondence: given $\gamma \in SU(2,\mathbb{R})$ we denote by $\tilde{\gamma}$ any semisimple element $\tilde{\gamma} \in SL(2,\mathbb{R})$ with the same eigenvalues.

**Lemma 4.5.4** *Given $f \in \mathcal{C}_c^\infty(SU(2,\mathbb{R}))$ there is a function $\tilde{f} \in \mathcal{C}_c^\infty(SL(2,\mathbb{R}))$ such that for $\gamma \in SU(2,\mathbb{R})$ non central*

$$SO_{\tilde{\gamma}}(\tilde{f}) = -O_\gamma(f)$$

*where*

$$SO_\gamma(\tilde{f}) = \sum SO_{w(\tilde{\gamma})}(\tilde{f})$$

*and the sum is over the complex Weyl group for $SO(2) \subset SL(2)$ and*

$$SO_{\tilde{\gamma}}(\tilde{f}) = O_{\tilde{\gamma}}(\tilde{f}) = 0$$

*if $\tilde{\gamma} \in SL(2,\mathbb{R})$ has distinct real eigenvalues.*

An easy proof is obtained using pseudo-coefficients. This will be explained in a fairly general case in 7.2.

In the case of $GL(2)$ a similar result is used for all local fields in Jacquet-Langlands. This lemma is a particular case of the general result due to Shelstad [She1] on the transfer between inner forms of real groups.

# 5 $U(2,1)$

## 5.1 Endoscopy for $U(2,1)$

Consider the quasisplit unitary group in three variables $G(\mathbb{R}) = U(2,1)$. A maximal compact torus $T(\mathbb{R})$ in $G(\mathbb{R})$ is such that

$$T(\mathbb{R}) \simeq U(1)^3$$

and can be represented by matrices

$$g(u,v,w) = \begin{pmatrix} u\cos\theta & 0 & iu\sin\theta \\ 0 & v & 0 \\ iu\sin\theta & 0 & u\cos\theta \end{pmatrix}$$

where $u$ and $v$ are complex numbers with $|u| = |v| = 1$ and $w = r(\theta)$. More precisely

$$(u,v,w) \mapsto g(u,v,w)$$

defines a twofold cover $T_1$ of $T$. The root system can be represented by the $\alpha_{i,j} = \phi_i - \phi_j$ with

$$e^{i\phi_1} = ue^{i\theta} \quad e^{i\phi_2} = v \quad e^{i\phi_3} = ue^{-i\theta}$$

We take as positive roots those with $i < j$. We observe that the half sum of the roots verifies $\rho = \alpha_{1,3}$.

The complex Weyl group is isomorphic to $\mathfrak{S}_3$ while the real Weyl group is isomorphic to $\mathfrak{S}_2$. The set of conjugacy classes inside a strongly regular stable elliptic conjugacy class is in bijection with the pointed set

$$\mathfrak{D}(\mathbb{R},T,G) = \mathfrak{S}_3/\mathfrak{S}_2$$

(see 6.4) that can be viewed as a sub-pointed-set of the group

$$\mathfrak{E}(\mathbb{R},T,G) = (\mathbb{Z}_2)^2$$

We shall denote by $\mathfrak{K}(\mathbb{R},T,G)$ its Pontryagin dual.

Choose as a maximal compact subgroup $K$ in $G(\mathbb{R})$

$$K = \left\{ k = \frac{1}{2}\begin{pmatrix} \lambda+\nu & -\sqrt{2}\sigma & \lambda-\nu \\ \sqrt{2}\tau & 2\mu & -\sqrt{2}\tau \\ \lambda-\nu & \sqrt{2}\sigma & \lambda+\nu \end{pmatrix} \right\}$$

Observe that $k$ is a conjugate of

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & \tau \\ 0 & \sigma & \nu \end{pmatrix} \in U(1)\times U(2) \subset U(3)$$

The compact group $K$ has been so chosen that it contains $T(\mathbb{R})$. The Weyl group of $K$ is generated by $w_0$ the symmetry with respect to $\rho$.

## 5.2 Discrete series and transfer for $U(2,1)$

Now, consider a dominant weight $\lambda$. One attaches to it a discrete series representation $\pi_\mu$ with

$$\mu = \lambda + \rho$$

The $L$-packet containing $\pi_\mu$ contains also the various $\pi_{w\mu}$. Moreover $\pi_{w\mu}$ is equivalent to $\pi_{w_0 w\mu}$ and hence we may parametrize the elements in the $L$-packet by the elements in the Weyl group such that $\operatorname{sign}(w) = 1$. For such a $w$ the character of $\pi_{w\mu}$ is given by

$$\Theta_{w\mu}(\gamma) = \frac{\gamma^{w(\mu)} - \gamma^{w_0 w(\mu)}}{\gamma^\rho \Delta_B(\gamma)}$$

with

$$\Delta_B(\gamma) = \prod_{\alpha>0}(1 - \gamma^{-\alpha})$$

Consider $\kappa \neq 1$ in $\mathfrak{K}(\mathbb{R}, T, G)$ such that $\overline{\kappa}(\check{\alpha}_{1,3}) = 1$. Such a $\kappa$ is unique: in fact one has necessarily

$$\overline{\kappa}(\check{\alpha}_{1,2}) = \overline{\kappa}(\check{\alpha}_{2,3}) = -1$$

The endoscopic group $H$ one associates to $\kappa$ is isomorphic to

$$U(1,1) \times U(1)$$

the positive root of $T$ in $H$ (for a compatible order) being

$$\alpha_{1,3} = \rho$$

The group $H$ can be embedded in $G$ as the subgroup of matrices in $G$ of the form

$$g(u,v,w) = \begin{pmatrix} ua & 0 & iub \\ 0 & v & 0 \\ -iuc & 0 & ud \end{pmatrix}$$

with

$$w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad ad - bc = 1$$

It will be useful to consider also the twofold cover

$$H_1 = U(1) \times U(1) \times SL(2)$$

(with maximal torus $T_1$) defined by

$$(u, v, w) \mapsto g(u, v, w)$$

Let $f_\mu$ be a pseudo-coefficient for $\pi_\mu$ then the $\kappa$-orbital integral of a $\gamma$ regular in $T(\mathbb{R})$ is given by

$$\mathcal{O}^\kappa_\gamma(f_\mu) = \sum_{\mathrm{sign}\,(w)=1} \kappa(w)\Theta^G_\mu(\gamma_w^{-1})$$

but using the bijection

$$\Omega_G/\Omega_K \simeq \Omega_K\backslash\Omega_G$$

this can be rewritten

$$\mathcal{O}^\kappa_\gamma(f_\mu) = \sum_{\mathrm{sign}\,(w)=1} \kappa(w)\Theta_{w\mu}(\gamma^{-1})$$

The transfer factor $\Delta(\gamma, \gamma_H)$ is given by

$$(-1)^{q(G)+q(H)}\chi_{G,H}(\gamma)\Delta_B(\gamma^{-1})\,.\,\Delta_{B_H}(\gamma_H^{-1})^{-1}$$

for some character $\chi_{G,H}$ defined as follows. It would be canonical to take

$$\chi_{G,H}(\gamma^{-1}) = \gamma^{\rho-\rho_H}$$

but this is does not make sense since $\rho_H$ defines a character of the twofold cover $T_1(\mathbb{R})$ but not of $T(\mathbb{R})$. A way out is to consider a weight $\xi$ that defines a character of the cocenter of the twofold cover $H_1(\mathbb{R})$ of $H(\mathbb{R})$ so that

$$\chi^{-1}_{G,H} = e^{\rho-\rho_H+\xi}$$

descends to a character of $T(\mathbb{R})$. With such a choice we get when $\mathrm{sign}\,(w) = 1$

$$\Delta(\gamma^{-1}, \gamma_H^{-1})\Theta^G_{w\mu}(\gamma) = -\frac{\gamma_H^{w(\mu)+\xi} - \gamma_H^{w_0 w(\mu)+\xi}}{\gamma^{\rho_H}\Delta_{B_H}(\gamma_H)}$$

We observe that $\kappa(w) = -1$ if $\mathrm{sign}\,(w) = 1$ and $w \neq 1$ and that $w\mu$ is positive or negative with respect to $B_H$ according to the sign of $\kappa(w)$. Hence

$$\Delta(\gamma, \gamma_H)\Theta^G_{w\mu}(\gamma^{-1}) = \kappa(w)^{-1} S\Theta^H_\nu(\gamma_H^{-1})$$

where $S\Theta_\nu^H$ is the character of the $L$-packet $\Sigma_\nu$ of discrete series for $H$ defined by the parameter

$$\nu = w(\mu) + \xi$$

Altogether we get

$$\Delta(\gamma, \gamma_H)\mathcal{O}_\gamma^\kappa(f_\mu) = \sum_\nu S\Theta_\nu^H(\gamma_H^{-1})$$

the sum being over the $\nu = w(\mu) + \xi$ with $\mathrm{sign}\,(w) = 1$ and this can be rewritten

$$\Delta(\gamma, \gamma_H)\mathcal{O}_\gamma^\kappa(f_\mu) = \sum_\nu S\mathcal{O}_{\gamma_H}(g_\nu)$$

where $g_\nu$ is a pseudo-coefficient for any one of the discrete series of $H(\mathbb{R})$ in the $L$-packet $\Sigma_\nu$ attached to $\nu$. In other words we have

$$f_\mu^H = \sum_\nu g_\nu$$

Notice that the set of parameters $\nu$ depends on the choice of $\xi$. Alternatively we could use the canonical choice

$$\chi_{G,H}(\gamma^{-1}) = \gamma^{\rho - \rho_H}$$

if we replace $H$ by $H_1$ and then the set of parameters would be canonical. This second solution is not the classical one but it seems after all more natural and at any rate considering central extensions $H_1$ instead of $H$ cannot be avoided in general.

Now more generally we get

$$f_{w\mu}^H = \sum_\nu a(w, \nu) g_\nu$$

with

$$a(w_1, w_2\mu) = \kappa(w_2)\kappa(w_2 w_1)^{-1} \ .$$

Since pseudo-coefficients give a dual basis of the set of characters, in the space they generate, we get

$$\text{trace } \Sigma_\nu(f^H) = \sum_w a(w, \nu) \text{ trace } \pi_{w\mu}(f)$$

at least when $f$ is a linear combination of pseudo-coefficients. Now, observe that

$$a(w_1, w_2\mu) = \kappa(w_2)\kappa(w_2w_1)^{-1} = \kappa^{w_2}(w_1)^{-1}$$

Let us denote by $\kappa_\nu$ the character

$$x \mapsto \kappa^w(x)^{-1}$$

when $\nu = w(\mu) + \xi$. Now, using the bijection

$$\Pi_\mu \simeq \mathfrak{D}(\mathbb{R}, T, G)$$

we get a pairing denoted $< , >$ between $\Pi_\mu$ and $\mathfrak{K}(\mathbb{R}, T, G)$ and the transfer equation can be written

$$\text{trace } \Sigma_\nu(f^H) = \sum_{\pi \in \Pi_\mu} < \kappa_\nu, \pi > \ \text{trace } \pi(f)$$

The above formula for the dual transfer can be rewritten

$$\text{trace } \Sigma_s(f^H) = \sum_{\pi \in \Pi_\Sigma} < s, \pi > \ \text{trace } \pi(f)$$

where the pairing $< s, \pi >$ is the Shelstad pairing between

$$\mathbf{S}_\varphi \simeq \mathfrak{K}(\mathbb{R}, T, G)$$

and $\Pi_\mu$. Observe that $\Pi_\mu$ is of order 3 and is identified by this pairing to a subset of the group of characters of $\mathfrak{K}(\mathbb{R}, T, G)$; the missing character corresponds to a representation of the compact inner form that has to be added to get a K-group (see 8.1).

## 5.3 The dual picture for $U(2, 1)$

Let us now describe the dual picture. Observe that $\overline{\kappa}$ is the image, via Tate-Nakayama duality (see 6.5), of

$$s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \check{T} \subset \check{G}$$

and hence

$$\check{H} = \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}$$

The holomorphic Galois action in ${}^LG$ is given by

$$g \mapsto J^t g^{-1} J^{-1}$$

with

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

while in ${}^LH$ it is $h \mapsto J_H {}^t h^{-1} J_H^{-1}$ with

$$J_H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

As a consequence of the difference between $J$ and $J_H$ there is no canonical admissible embedding of ${}^LH$ in ${}^LG$.

# 6 Galois cohomology and Endoscopy

## 6.1 Non abelian hypercohomology

Consider first a set $X$ with an action of a group $\Gamma$. By definition $H^0(\Gamma, X)$ is the fixed point set $X^\Gamma$: the set of $x \in X$ such that

$$x = \sigma(x) \quad \text{for all} \quad \sigma \in \Gamma$$

Now let $G$ be an algebraic group with an action of $\Gamma$ compatible with the group structure. Then $H^0(\Gamma, G)$ is a group. Moreover we may define a set $H^1(\Gamma, G)$ as the quotient of the set $Z^1(\Gamma, G)$ of 1-cocycles i.e. maps

$$\sigma \in \Gamma \mapsto a_\sigma \in G$$

such that

$$a_\sigma \sigma(a_\tau) a_{\sigma\tau}^{-1} = 1$$

by the equivalence relation

$$a'_\sigma \simeq a_\sigma \Longleftrightarrow a'_\sigma = b\, a_\sigma\, \sigma(b)^{-1}$$

for some $b \in G$. The set $H^1(\Gamma, G)$ has no natural group structure. But the trivial class makes it a pointed set. Finally, if $A$ is an abelian group, we have the familiar cohomology groups in all degrees

$$H^i(\Gamma, A)$$

This can be slightly generalized as follows. One can define a hyper-cohomological theory in degrees $\leq 0$ for complexes of the form

$$[G \Rightarrow X]$$

where $G$ is a group acting (on the left) on a set $X$ with a group $\Gamma$ acting on the complex. The cohomology set

$$H^0(\Gamma, G \Rightarrow X)$$

it the quotient of the set of pairs $(x, a)$ with $x \in X$ and $a$ a map from $\Gamma$ to $X$ with

$$x = a_\sigma \,.\, \sigma(x) \quad \text{and} \quad a_\sigma \sigma(a_\tau) a_{\sigma\tau}^{-1} = 1$$

by the equivalence relation

$$(x', a') \simeq (x, a) \Longleftrightarrow x' = bx \quad \text{and} \quad a'_\sigma = b\, a_\sigma\, \sigma(b)^{-1}$$

There is also an hyper-cohomological theory in degrees $\leq 1$ for complexes of groups of the form

$$[G' \to G]$$

that are "left crossed modules". This means that $G$ and $G'$ are groups and we are given not only a homomorphism $G' \to G$, which yields an action of $G'$ on $G$ by left translations, but also an action of $G$ on $G'$ and that those two actions are compatible with adjoint actions. This implies in particular that

$$\ker[G' \to G]$$

is an abelian group.

A morphism of crossed modules

$$[G' \to G] \to [H' \to H]$$

is a pair of maps $G' \to H'$ and $G \to H$ that intertwines the various actions, in particular we have a commutative diagram

$$\begin{array}{ccc} G' & \to & G \\ \downarrow & & \downarrow \\ H' & \to & H \end{array}$$

Such a morphism is a quasi-isomorphism if it induces isomorphisms for kernels and cokernels.

If we have an action of a group $\Gamma$, compatible with the structure of crossed module, one has a cohomology theory

$$H^i(\Gamma, G' \to G)$$

in degrees $i \leq 1$. The $H^1$ is defined as a quotient of the set of 1-hyper-cocycles $(a, b)$ with

$$a_\sigma \sigma(a_\tau) a_{\sigma\tau}^{-1} = \rho(b_{\sigma,\tau})$$

where $\rho$ is the homomorphism $G' \to G$ and

$$a_\sigma * \sigma(b_{\tau,\mu}) \,.\, b_{\sigma,\tau\mu} = b_{\sigma,\tau} \,.\, b_{\sigma\tau,\mu}$$

where $*$ denotes the action of $G$ on $G'$. The equivalence relation is

$$a'_\sigma = \alpha\, \rho(\beta_\sigma) a_\sigma\, \sigma(\alpha)^{-1}$$

and

$$b'_{\sigma,\tau} = \alpha * (\beta_\sigma(a_\sigma * \sigma(\beta_\tau)) b_{\sigma,\tau} \beta_{\sigma\tau}^{-1})$$

There is a natural group structure on $H^0$ but again $H^1$ is simply a pointed set while

$$H^{-1}(\Gamma, G' \to G) := H^0(\Gamma, \ker[G' \to G])$$

is an abelian group. These cohomological objects are functorial for the category of crossed modules and, as for usual hypercohomology theories, they are invariant under quasi-isomorphisms. Cohomology in higher degrees cannot be defined (in a functorial way) without further structures.

We observe that $G$ can be seen as the crossed module

$$[1 \to G]$$

and we have

$$H^i(\Gamma, G) = H^i(\Gamma, 1 \to G)$$

There is an exact sequence

$$1 \to H^{-1}(\Gamma, G' \to G) \to H^0(\Gamma, G') \to H^0(\Gamma, G) \to$$

$$H^0(\Gamma, G' \to G) \to H^1(\Gamma, G') \to H^1(\Gamma, G) \to H^1(\Gamma, G' \to G)$$

with

$$H^{-1}(\Gamma, G' \to G) := H^0(\Gamma, \ker \rho)$$

The exactness has to be understood for pointed sets. In general the exact sequence cannot be continued since the $H^2$ does not make sense for a non commutative group.

If we denote by $K$ the kernel of $[G' \to G]$ and by $L$ the cokernel we have another exact sequence:

$$1 \to H^0(\Gamma, K) \to H^{-1}(\Gamma, G' \to G) \to H^{-1}(\Gamma, L) \to$$

$$H^1(\Gamma, K) \to H^0(\Gamma, G' \to G) \to H^0(\Gamma, L) \to H^2(\Gamma, K)$$
$$\to H^1(\Gamma, G' \to G) \to H^1(\Gamma, L) \to H^3(\Gamma, K)$$

where $H^{-1}(\Gamma, H) = 1$. Observe that since $K$ is abelian the $H^i(\Gamma, K)$ are defined for all $i$.

If moreover the complex $G' \to G$ is quasi-isomorphic to a complex of abelian groups $B \to A$ (in a way compatible with $\Gamma$-actions) then one can define hyper-cohomology groups

$$H^i(\Gamma, G' \to G) := H^i(\Gamma, B \to A)$$

in all degrees and the above exact sequence extends indefinitely.

## 6.2 Galois cohomology and abelianized cohomology

Let $F$ be a field and let $\overline{F}$ be an separable closure. Consider an algebraic variety $X$ defined over $F$ then the set $X(F)$ of points with value in $F$ is the fixed point set of $\mathrm{Gal}(\overline{F}/F)$ on $X(\overline{F})$. This is the 0-th cohomology set of the Galois group with values in $X(\overline{F})$:

$$X(F) = H^0(\mathrm{Gal}(\overline{F}/F), X(\overline{F}))$$

We shall also use the standard notation for Galois cohomology

$$H^*(F, X) := H^*(\mathrm{Gal}(\overline{F}/F), X(\overline{F}))$$

whenever defined. If $G$ is an algebraic group over $F$ we have also the set

$$H^1(F, X) := H^1(\mathrm{Gal}(\overline{F}/F), X(\overline{F}))$$

Finally, if $A$ is an abelian algebraic group we have

$$H^i(F, X) := H^i(\mathrm{Gal}(\overline{F}/F), X(\overline{F}))$$

for all $i$.

Let $G$ be a connected reductive group over a field $F$ and let $G_{SC}$ be the simply connected cover of its derived subgoup. We observe that complexes

$$[G_{SC} \to G]$$

are crossed modules for the obvious adjoint actions and hence we have at hand

$$H^*(F, G_{SC} \to G)$$

in degrees $\leq 1$. But in fact such a complex is quasi-isomorphic to a complex of abelian groups:

$$[Z_{sc} \to Z]$$

where $Z$ is the center of $G$ and $Z_{sc}$ is the center of $G_{SC}$. Since hypercohomology is invariant by quasi-isomorphisms this allows to define abelian groups

$$H^*(F, G_{SC} \to G) = H^*(F, Z_{sc} \to Z)$$

in all degrees. Another quasi-isomophic complex of abelian groups is useful: let $T$ be a torus in $G$ and let $T_{sc}$ its preimage in $G_{SC}$ then

$$H^*(F, G_{SC} \to G) = H^*(F, T_{sc} \to T)$$

We shall use the following compact notation

$$H^*_{ab}(F, G) := H^*(F, G_{SC} \to G)$$

**Theorem 6.2.1** *Let $G$ be a connected reductive group over a field $F$ and let $G_{SC}$ be the simply connected cover of its derived subgoup. There is a family of abelian group $H^*_{ab}(F, G)$ with natural maps*

$$H^i(F, G) \to H^i_{ab}(F, G) \quad \textit{for} \quad i \leq 1$$

*giving rise to a long exact sequence*

$$\to H^0_{ab}(F,G) \to H^1(F,G_{SC}) \to H^1(F,G) \to H^1_{ab}(F,G)$$

*Moreover, when $F$ is a local field*

$$H^1(F,G) \to H^1_{ab}(F,G)$$

*is surjective and even bijective if $F$ is non-archimedean.*

*Proof*: . The surjectivity follows from the existence of fundamental tori and that such tori have a vanishing $H^2$. The injectivity follows from a theorem due to Kneser quoted below.

□

**Theorem 6.2.2** *When $F$ is a non-archimedean local field then*

$$H^1(F,G_{SC}) = 1$$

We observe that using the universal property of simply connected spaces one can show that the abelianization map

$$H^1(F,G) \to H^1_{ab}(F,G)$$

is functorial in $G$ (cf.[Lab3] lemme I.6.3)

Before the introduction of abelianized cohomology by Borovoi [Bo] (see also [Lab3]), Kottwitz had found a substitute for the abelianization map but that was more subtle to define, not obviously functorial and was restricted to reductive groups over local fields. Given $G$ reductive over some field $F$ consider the abelian group

$$\pi_0(Z(\check{G})^\Gamma)$$

where $\check{G}$ is the complex dual group, $Z(\check{G})$ its center, $\Gamma$ the Galois group $\mathrm{Gal}(\overline{F}/F)$ and $\pi_0$ the group of connected components. Now, using Tate-Nakayama duality, Kottwitz has shown that when $F$ is a local field there exists a canonical map

$$H^1(F,G) \to \pi_0(Z(\check{G})^\Gamma)^D$$

where the exponent $D$ denotes the Pontryagin duality. Using abelianized cohomology one recovers Kottwitz's map using that

$$[\check{G} \to (G_{SC})\check{}\,]$$

is quasi-isomorphic to

$$[Z(\check{G}) \to 1]$$

and by Tate-Nakayama duality one gets an injective homomorphism

$$H^1_{ab}(F, G) \to \pi_0(Z(\check{G})^\Gamma)^D$$

which is bijective when $F$ is non archimedean ([Lab3] Proposition 1.7.3).

We still have to introduce the Kottwitz signs [Ko1]. Given a reductive group $G$ consider its quasisplit inner form $G^*$ and $G^*_{ad}$ the adjoint group. Let $a(G)$ be the cohomology class in $H^1(F, G^*_{ad})$ defining $G$ as an inner form of $G^*$. There is an isomorphism from

$$H^*_{ab}(F, G^*_{ad}) \to H^2(F, Z(G^*_{SC}))$$

and by composition we get a class

$$\overline{a}(G) \in H^2(F, Z(G^*_{SC}))$$

Now the half sum of positive roots (for some order) defines a map from $Z(G^*_{SC})$ to the group of elements of order 2 in the multiplicative group and hence if $F$ is local we get a number $e(G) \in \{\pm 1\}$ called the Kottwitz sign. It can be shown that when $F = \mathbb{R}$ one has

$$e(G) = (-1)^{q(G^*)-q(G)}$$

where $q(G)$ is half the dimension of the symmetric space attached to $G$.

## 6.3 Stable conjugacy and $\kappa$-orbital integrals

We shall now define stable conjugacy in general. Let $G$ be a connected reductive group over some field $F$. For simplicity we assume $F$ is a field of characteristic zero.

We denote by $G_{SC}$ the simply connected cover of its derived subgroup. Given $\gamma \in G(F)$ we denote by $I_\gamma$ the subgroup of $G$ generated by $Z(G)$ (the center of $G$) and the image in $G$ of the centralizer of $\gamma$ in $G_{SC}$. We call $I_\gamma$ the stable centralizer of $\gamma$.

When $\gamma$ is strongly regular semisimple i.e. if the centralizer is a torus, then $I_\gamma$ is this torus. If $\gamma$ is regular semisimple i.e. if the centralizer has a torus as connected component, then again $I_\gamma$ is this torus. But in general $I_\gamma$ is not a torus and can even be disconnected as seen in the example of $n_0 \in SL(2,\mathbb{R})$ whose centralizer is $\pm N$ with $N$ the group of upper-triangular unipotent matrices. Nevertheless we have a

**Lemma 6.3.1** *Let $\gamma$ be semisimple in $G(F)$. Then $I_\gamma$ is a connected reductive subgroup, namely the connected component of the centralizer.*

*Proof*: The centralizer in $G_{SC}$ of a semisimple element $\gamma$ is a connected reductive group; this is due to Steinberg. Its image in $G$ is again reductive and connected. It contains a maximal torus $T$, but any $T$ contains $Z(G)$. □

Consider $\gamma$ and $\gamma'$ in $G(F)$. We shall say that $\gamma$ and $\gamma'$ are stably conjugate if there is an $x \in G(\overline{F})$ such that

$$(i) \qquad x^{-1}\gamma\, x = \gamma'$$

and

$$(ii) \qquad a_\sigma = x\sigma(x)^{-1} \in I_\gamma \quad \text{for all} \quad \sigma \in \mathrm{Gal}(\overline{F}/F)$$

where $\mathrm{Gal}(\overline{F}/F)$ is the Galois group of $\overline{F}/F$. In other words we assume that $a_\sigma$ is a 1-cochain with values in $I_\gamma$.

The same relations hold if we replace $x$ by $y = tx$ with $t \in I_\gamma$. The second condition is automatically satisfied if the first is, whenever $G = G_{SC}$: in fact $a_\sigma$ always belongs to the centralizer of $\gamma$.

The pair $(x, a_\sigma)$ defines a 0-cocycle with values in the quotient set

$$I_\gamma \backslash G$$

and hence a class in the Galois cohomology set associated to this set

$$H^0(\mathrm{Gal}(\overline{F}/F), I_\gamma \backslash G)$$

also denoted

$$H^0(F, I_\gamma \backslash G) \quad \text{or} \quad (I_\gamma \backslash G)(F)$$

This is the set of rational points of the quotient $I_\gamma\backslash G$. This cohomology set is the quotient of the set of 0-hypercocycles as above by the equivalence relation:

$$(x, a_\sigma) \simeq (y, b_\sigma)$$

whenever there exists $t \in I_\gamma$ such that

$$y = tx \quad \text{and} \quad b_\sigma = t a_\sigma \sigma(t)^{-1}$$

We observe that there is an exact sequence of pointed sets

$$H^0(F, I_\gamma\backslash G) \to H^1(F, I_\gamma) \to H^1(F, G)$$

induced by the map $(x, a_\sigma) \mapsto a_\sigma$ and the inclusion $I_\gamma \to G$. One denotes by $\mathfrak{D}(F, I_\gamma\backslash G)$ the image of $H^0(F, I_\gamma\backslash G)$ into $H^1(F, I_\gamma)$ or equivalently the kernel of the next map

$$\mathfrak{D}(F, I_\gamma\backslash G) = \ker\left[H^1(F, I_\gamma) \to H^1(F, G)\right]$$

With this we have a small exact sequence of pointed sets

$$1 \to I_\gamma(F)\backslash G(F) \to (I_\gamma\backslash G)(F) \to \mathfrak{D}(F, I_\gamma\backslash G) \to 1$$

To continue, let us assume first that $\gamma$ is regular semisimple and hence $I_\gamma$ is a maximal torus $T$ in $G$ defined over $F$. We shall now introduce the abelian groups $\mathfrak{E}(F, T\backslash G)$. We first recall that

$$\mathfrak{D}(F, T\backslash G) = \ker\left[H^1(F, T) \to H^1(F, G)\right]$$

the group $\mathfrak{E}(F, T\backslash G)$ is an abelianized version of it namely

$$\mathfrak{E}(F, T\backslash G) = \ker\left[H^1(F, T) \to H^1_{ab}(F, G)\right]$$

There is a natural injective map

$$\mathfrak{D}(F, T\backslash G) \to \mathfrak{E}(F, T\backslash G)$$

Observe that by composition we get a map

$$(T\backslash G)(F) \to \mathfrak{E}(F, T\backslash G)$$

This can also be seen as follows: we have

$$\mathfrak{E}(F, T\backslash G) = \text{ Im } \left[H^1(F, T_{sc}) \to H^1(F, T)\right]$$

and that there is a natural map

$$(T\backslash G)(F) \to H^1(F, T_{sc})$$

since $H^1(F, T_{sc})$ can be identified with an abelianized avatar of

$$H^0(F, T\backslash G) = (T\backslash G)(F)$$

Assume from now on that $F$ is a local field. When $F$ is non archimedean the map

$$\mathfrak{D}(F, T\backslash G) \to \mathfrak{E}(F, T\backslash G)$$

is bijective since

$$H^1(F, G) \to H^1_{ab}(F, G)$$

is bijective in this case. Over the reals, as we shall see in the next section, the set $\mathfrak{D}(\mathbb{R}, T\backslash G)$ is a subset but not usually a subgroup of $\mathfrak{E}(\mathbb{R}, T\backslash G)$.

We shall denote by $\mathfrak{K}(F, T\backslash G)$ the Pontryagin dual of the finite abelian group $\mathfrak{E}(F, T\backslash G)$. We call the elements of $\mathfrak{K}(F, T\backslash G)$ endoscopic characters. Let $\kappa \in \mathfrak{K}(F, T\backslash G)$ it defines using the map

$$(T\backslash G)(F) \to \mathfrak{E}(F, T\backslash G)$$

a function, again denoted $\kappa$ on $(T\backslash G)(F)$. The function $\kappa$ has the following multiplicative property: assume that $x$ and $z = xy$ in $G$ define elements in $(T\backslash G)(F)$. This means that $x\sigma(x)^{-1} \in T$ and $z\sigma(z)^{-1} \in T$ for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$ then

$$y\sigma(y)^{-1} \in T_x := x^{-1}Tx \quad \text{and} \quad \kappa(xy) = \kappa^x(y)\kappa(x)$$

where $\kappa^x$ is the character of $\mathfrak{E}(F, T_x\backslash G)$ obtained from $\kappa$ via the $F$-isomorphism

$$T \to x^{-1}Tx$$

A $\kappa$-orbital integral for $\gamma$ regular in $T(\mathbb{R})$ is defined by

$$\mathcal{O}^{\kappa}_{\gamma}(f) = \int_{(T\backslash G)(F)} \kappa(x) f(x^{-1}\gamma\, x)\, d\dot{x}$$

Implicit in the definition is a compatible choice of Haar measures on the various stable conjugates $x^{-1}Tx$ of $T$ for $x \in (T\backslash G)(F)$ obtained by transporting invariant differential forms of maximal degree.

The stable orbital integrals are the $\mathcal{O}^1_\gamma(f)$ i.e. $\kappa$-orbital integrals when $\kappa$ is trivial. They are often denoted $S\mathcal{O}_\gamma(f)$.

Now consider the case of an arbitrary semisimple element $\gamma$ and let $I = I_\gamma$ its stable centralizer. We put again

$$\mathfrak{D}(F, I\backslash G) = \ker\left[H^1(F, I) \to H^1(F, G)\right]$$

it has an abelianized avatar

$$\mathfrak{E}(F, I\backslash G) = \ker\left[H^1_{ab}(F, I) \to H^1_{ab}(F, G)\right]$$

with a map

$$\mathfrak{D}(F, I\backslash G) \to \mathfrak{E}(F, I\backslash G)$$

which need not be injective nor surjective in general. Now given $\kappa$ a character of $\mathfrak{E}(F, I\backslash G)$ the $\kappa$-orbital integral is defined as follows

$$\mathcal{O}^\kappa_\gamma(f) = \int_{(I\backslash G)(F)} e(I_x)\kappa(x) f(x^{-1}\gamma\, x)\, d\dot{x}$$

where $e(I_x)$ is the Kottwitz sign of the group

$$I_x := x^{-1} I\, x$$

the stable centralizer of $x^{-1}\gamma\, x$.

## 6.4 Stable conjugacy and compact Cartan subgroups over $\mathbb{R}$

We shall now consider the special case where $F = \mathbb{R}$ and $\gamma$ is elliptic and regular in $G$ which means that

$$T = I_\gamma$$

is an elliptic torus i.e. such that $T_{sc}$ is an $\mathbb{R}$-anisotropic torus.

**Lemma 6.4.1** *Let $T$ be an elliptic torus in a real reductive group $G$. For any $n$ in $N_G(T)$ the automorphism $w = \mathrm{Ad}(n)$ restricted to $T$ is defined over $\mathbb{R}$.*

*Proof*: Any automorphism $w$ of $T$ is defined by a $\mathbb{Z}$-linear automorphism $w^*$ of $X^*(T_{ad})$; it suffices to prove that $w^*$ is real. But $\sigma$ the non trivial element in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts as -1 on $X^*(T_{ad})$ and hence commutes with $w^*$.

$\square$

We denote by $\Omega_{\mathbb{C}}(G,T)$, or simply $\Omega_G$, the complex Weyl group, which is the group generated by the automorphisms of $T$ induced by $\mathrm{Ad}(n)$ with $n \in N_G(T)$ and $\Omega_{\mathbb{R}}(G,T)$, or $\Omega_K$, the subgroup generated by the automorphisms of $T$ induced by $\mathrm{Ad}(n)$ with

$$n \in G(\mathbb{R}) \cap N_G(T) = K \cap N_G(T)$$

where $K$ is the maximal compact subgroup of $G(\mathbb{R})$ containing $T(\mathbb{R})$.

**Proposition 6.4.2** *Let $T$ be an elliptic torus in $G$ over $\mathbb{R}$. There is a natural bijection*

$$\mathfrak{D}(\mathbb{R}, T\backslash G) \to \Omega_{\mathbb{C}}(G,T)/\Omega_{\mathbb{R}}(G,T)$$

*Proof*: Consider $\gamma$ strongly regular in $T$ and in particular $T = I_\gamma$. Then, we observe that if $\gamma'$ is stably conjugate to $\gamma$ then, $I_{\gamma'}$ is again an elliptic torus; but all elliptic tori are conjugate in $G(\mathbb{R})$. Hence, up to ordinary conjugacy we may replace $\gamma'$ by $\gamma'' \in T$ and now there is $n$ in the normalizer $N_G(T)$ of $T$ in $G$ such that

$$n^{-1}\gamma\, n = \gamma''$$

The image of $n$ in the Weyl group is uniquely defined by the pair $(\gamma, \gamma'')$ since $\gamma$ is strongly regular but the element $\gamma''$ is defined by $\gamma'$ only up to the action of the real Weyl group. This yields an injective map

$$\mathfrak{D}(\mathbb{R}, T\backslash G) \to \Omega_{\mathbb{C}}(G,T)/\Omega_{\mathbb{R}}(G,T)$$

Now the map is surjective thanks to lemma 6.4.1.

□

More generally we observe that all conjugacy classes inside the stable conjugacy class of an elliptic element in $T$, intersect $T$.

**Remark** – The set $\mathfrak{D}(\mathbb{R}, T\backslash G)$ is not usually a group. In fact $\Omega_{\mathbb{R}}(G,T)$ is not an invariant subgroup of $\Omega_{\mathbb{C}}(G,T)$ in general although $\mathfrak{D}(\mathbb{R}, T\backslash G)$ can be embedded naturally in the abelian group $\mathfrak{E}(\mathbb{R}, T\backslash G)$. For example consider $G = U(2,1)$, then

$$\Omega_{\mathbb{C}}(G,T) = \mathfrak{S}_3 \quad \text{while} \quad \Omega_{\mathbb{R}}(G,T) = \mathfrak{S}_2$$

In this case $\mathfrak{D}(\mathbb{R}, T\backslash G)$ has three elements and the group $\mathfrak{E}(\mathbb{R}, T\backslash G)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

## 6.5 Endoscopic groups

We have to recall the Tate-Nakayama isomophism for tori. Let $S$ be a torus defined over $F$ and split over a Galois extension $K$; we denote by $\check{X}(S)$ the $\mathbb{Z}$-free module of finite rank of its cocharacters.

**Theorem 6.5.1** *Assume that $F$ is a local field. There is a canonical isomorphism between Tate cohomology groups*

$$\hat{H}^i(K/F, \check{X}(S)) \to \hat{H}^{i+2}(K/F, S(K))$$

In particular there is a canonical isomorphism

$$\hat{H}^{-1}(K/F, \check{X}(S)) \to H^1(K/F, S(K)) = H^1(F, S)$$

where, by definition

$$\hat{H}^{-1}(\Gamma, X) := \hat{H}_0(\Gamma, X)) = X^{N_\Gamma}/I_\Gamma X$$

where $X^{N_\Gamma}$ is the kernel of the norm endomorphism of $X$:

$$N_\Gamma : x \mapsto \sum_{\sigma\in\Gamma} \sigma(x)$$

and $I_\Gamma$ is the augmentation ideal in the group algebra $\mathbb{Z}[\Gamma]$

$$I_\Gamma = \left\{ \tau = \sum_{\sigma\in\Gamma} n_\sigma\, \sigma \;\middle|\; \sum_{\sigma\in\Gamma} n_\sigma = 0 \right\}$$

Altogether we have the

**Proposition 6.5.2** *There is a canonical isomorphism*

$$\check{X}(S)^{N_\Gamma}/I_\Gamma \check{X}(S) \to H^1(F, S)$$

*with* $\Gamma = \mathrm{Gal}(K/F)$.

Now there is a surjective map

$$H^1(F, T_{sc}) \to \mathfrak{E}(F, T\backslash G)$$

and if moreover we assume $T$ elliptic then $T_{sc}$ is anisotropic and hence we have a bijection

$$\check{X}(T_{sc})^{N_\Gamma} \to \check{X}(T_{sc})$$

The above proposition yields a surjective map

$$\check{X}(T_{sc}) \to \mathfrak{E}(F, T\backslash G)$$

and hence it $T$ is elliptic any character $\kappa$ of $\mathfrak{E}(F, T\backslash G)$ defines a character $\overline{\kappa}$ of $\check{X}(T_{sc})$.

Let $R_G$ and $\check{R}_G$ denote the set of roots and coroots of $T$ in $G$. We observe that the coroots are homomorphisms of the multiplicative group $\mathbb{G}_m$ into $T$ that factor through $T_{sc}$ and hence

$$\check{R}_G \subset \check{X}(T_{sc})$$

Let us denote by $\check{R}_{\overline{\kappa}}$ the subset of coroots $\check{\alpha} \in \check{R}$ such that

$$\overline{\kappa}(\check{\alpha}) = 1$$

**Lemma 6.5.3** *This is a (co)root system*

*Proof*: In fact $\check{R}_{\overline{\kappa}}$ is the root system for the connected centralizer $\check{H}$ of $s \in \check{T} \subset \check{G}$ whose image in $\check{X}(T_{sc})$ by duality is $\overline{\kappa}$.

□

Let $X = X(T)$ and $\check{X} = \check{X}(T)$, there is a root datum

$$(X, R_{\overline{\kappa}}, \check{X}, \check{R}_{\overline{\kappa}})$$

which inherits, from the Galois action on $T$, a natural Galois action and we get a quasi-split reductive group denoted $H_{\overline{\kappa}}$ (or simply $H$) with maximal torus $T_H \simeq T$ and coroot system

$$\check{R}_H \simeq \check{R}_{\overline{\kappa}}$$

This is the elliptic endoscopic group attached to $\kappa$.

More generally, if we do not assume $T$ elliptic, one can associate, as above, an endoscopic group $H$ to any characters $\overline{\kappa}$ of $\check{X}(T_{sc})/I_\Gamma\check{X}(T_{sc})$. Such $\overline{\kappa}$'s naturally occur through local-global constructions for the stabilization of the trace formula. Now $\overline{\kappa}$ defines, by restriction to $\check{X}(T_{sc})^{N_\Gamma}$, a character $\kappa$ of

$$H^1(F, T_{sc})$$

which descends to a character of $\mathfrak{E}(F, T\backslash G)$ if $\kappa$ is trivial on the kernel of the map

$$H^1(F, T_{sc}) \to H^1(F, T)$$

and in all cases $\kappa$ defines a function on $(T\backslash G)(F)$ via the natural map

$$(T\backslash G)(F) \to H^1(F, T_{sc})$$

It gives rise, when $\kappa$ is non trivial on the above kernel, to the variant of ordinary endoscopy that deals with orbital integrals weighted by a character $\omega$ as in 4.5.3 above.

We observe that $H$ and $G$ share a torus and that the Weyl group $\Omega(T_H, H)$ is canonically isomorphic to a subgroup of $\Omega(T, G)$. Nevertheless there may not exist any homomorphism from $H$ to $G$ that extends the isomorphism $T_H \simeq T$.

## 6.6 The dual picture

Consider $T$, $\overline{\kappa}$ and $H$ as above. Recall that the dual groups $\check{G}$ and $\check{H}$ are equipped with a "splitting" (an épinglage) and a Galois action that preserves the splitting. The Galois action on the characters of $T$ defines by duality an holomorphic action of Galois on $\check{T}$ (chosen as the torus in the splitting of $\check{G}$). For $\sigma$ in the Galois group let us denote by $\check{\sigma}_T$ the corresponding automorphism of $\check{T}$ and by $\check{\sigma}_G$ the corresponding automorphism of $\check{G}$; then for $t \in \check{T}$ one has

$$\check{\sigma}_T(t) = n_\sigma \check{\sigma}_G(t)\, n_\sigma^{-1}$$

with $n_\sigma$ in the normalizer of $\check{T}$ in $\check{G}$. Thanks to a vanishing property of inflation from Galois groups to Weil groups this Galois action that can be lifted to an homomorphism $\xi$ of $W_F$ into ${}^LG$

$$w \mapsto \xi(w) = n_w \rtimes w$$

with

$$\check{\sigma}_T(t) = \xi(w_\sigma) t\, \xi(w_\sigma)^{-1}$$

whenever $w_\sigma$ projects to $\sigma$, allowing to embedd ${}^LT$ in ${}^LG$. This means that $n_\sigma$ and $\xi(w_\sigma)$ have the same image in the Weyl group for $\check{T}$ in $\check{G}$.

Consider an element $s \in \check{T}$ giving $\overline{\kappa}$ by duality; $s$ is defined only up to an element in the center of $\check{G}$ and its class is Galois-invariant. The connected centralizer of $s$ in $\check{G}$ can be identified with $\check{H}$. The action of the Weil group $W_F$ on $\check{T}$ induces an action on $\check{H}$ which coincides, up to inner automorphisms, with the action of $W_F$ on $\check{H}$ in ${}^LH$. Now consider the subgroup $\mathcal{H}$ of ${}^LG$

generated by $\check{H}$ and the image of the Weil group by the above homomorphism $\xi$. The group $\mathcal{H}$ is independent of the choice of $s$ and $\xi$ and is a split extension:

$$1 \to \check{H} \to \mathcal{H} \to W_F \to 1$$

(as a split extension it depends on the choice of $\xi$).

An admissible embedding of ${}^LH$ in ${}^LG$ is an $L$-homomorphism

$$\eta : {}^LH \to {}^LG$$

that extends the natural inclusion

$$\check{H} \to \check{G}$$

(by $L$-homomorphism we understand an homomorphism whose restriction to $\check{H}$ is holomorphic and induces the identity on $\mathbb{W}_{\mathbb{R}}$). The existence of $\eta$ is tantamount to the existence of an L-isomorphism

$$\mathcal{H} \to {}^LH$$

which is the identity on $\check{H}$.

But in general $\mathcal{H}$ and ${}^LH$ are not isomorphic and hence there is no admissible embedding (see [Sh3]) and even if they are isomorphic there may not exist a canonical one (an example is given in 5.3). Nevertheless one can always construct a central extension $H_1$ of $H$, such that

$$1 \to Z_1(F) \to H_1(F) \to H(F) \to 1$$

is an exact sequence, and an injective L-homomorphism

$$\mathcal{H} \to {}^LH_1$$

instead of an isomorphism. We refer the reader to [KS] for further details.

Consider now a Langlands parameter

$$\varphi \ : \ \mathcal{L}_F \ \to \ {}^LG$$

Let $S_\varphi$ denote the centralizer in $\check{G}$ of $\varphi(\mathbb{W}_{\mathbb{R}})$. We observe that any $s \in S_\varphi$ defines an endoscopic group $H$. As above, the group $\mathcal{H}$ generated in ${}^LG$ by the connected centralizer $\check{H}$ of $s$ in $\check{G}$ and the image of $\varphi$, is a split extension:

$$1 \to \check{H} \to \mathcal{H} \to W_F \to 1$$

and by construction $\varphi$ factors through $\mathcal{H}$. If there exists an admissible embedding $\eta : {}^LH \to {}^LG$ then one may factor $\varphi$ through it and one thus defines a Langlands parameter for $H$. In general one only obtains a parameter for $H_1$.

Now, denote by $\mathbf{S}_\varphi$ the quotient of $S_\varphi$ by its connected component $S_\varphi^0$ times $Z(\check{G})^\Gamma$ the center of ${}^LG$. In [She4] Diana Shelstad has established the

**Proposition 6.6.1** *For discrete series parameters there is an isomorphism between* $\mathbf{S}_\varphi$ *and* $\mathfrak{K}(\mathbb{R}, T\backslash G)$.

We give the proof when $G$ is semisimple and simply connected. Consider an elliptic torus $T$ in $G$. Let $\check{T}$ be the goup of complex characters of the lattice $\check{X}(T)$. Now consider a discrete series parameter $\varphi$. We may and will assume that $\check{T}$ contains $\varphi(\mathbb{C}^\times)$ the image of the subgoup $\mathbb{C}^\times \subset \mathbb{W}_\mathbb{R}$ but then $S_\varphi$ is the subgroup of invariant elements in $\check{T}$ under the Galois action from ${}^LT$ and hence $S_\varphi$ is the set of elements of order 2 in $\check{T}$. We observe that an element in $\mathfrak{K}(\mathbb{R}, T\backslash G)$ defines a complex character of $\check{X}(T)$. Hence to $\kappa \in \mathfrak{K}(\mathbb{R}, T\backslash G)$ corresponds an element $s \in \check{T}$ that moreover commutes with the Galois action. We thus get a bijective homomorphism

$$\mathfrak{K}(\mathbb{R}, T\backslash G) \to \mathbf{S}_\varphi$$

For $\mathfrak{p}$-adic fields the Langlands parametrization and the structure of $L$-packets of discrete series are not known in general but it can be checked in examples that the group $\mathbf{S}_\varphi$ is often bigger than $\mathfrak{K}$ and can even be non abelian.

## 6.7 Endoscopic transfer

Consider $T$ an elliptic torus and $\kappa$ an endoscopic character. Let $H$ be the endoscopic group defined by $(T, \kappa)$. We need some notation. We let $B_G$ be a Borel subgroup of $G$ containing $T$.

$$\Delta_B(\gamma) = \prod_{\alpha>0}(1 - \gamma^{-\alpha})$$

where the product is over the positive roots for for the order defined by $B$. There is only one choice of a Borel subgroup $B_H$ in $H$, containing $T_H$ compatible with the isomorphism $j : T_H \simeq T$.

**Proposition 6.7.1** *Assume there is an admissible embedding*

$$\eta : {}^L H \to {}^L G$$

*One can attach to the triple* $(G, H, \eta)$ *a character* $\chi_{G,H}$ *of* $T(\mathbb{R})$ *with the following property. Given* $f$ *a pseudo-coefficient for a discrete series on* $G$*, there is a function* $f^H$ *which is a linear combination of pseudo-coefficients for discrete series on* $H$ *such that for a* $\gamma = j(\gamma_H)$ *regular in* $T(\mathbb{R})$

$$S\mathcal{O}_{\gamma_H}(f^H) = \Delta_H^G(\gamma_H, \gamma)\mathcal{O}_\gamma^\kappa(f)$$

*where* $\Delta_H^G(\gamma_H, \gamma)$ *the "transfer factor" has the following expression:*

$$(-1)^{q(G)+q(H)}\chi_{G,H}(\gamma)\Delta_B(\gamma^{-1})\,.\,\Delta_{B_H}(\gamma_H^{-1})^{-1}$$

When $G(\mathbb{R}) = U(2,1)$ a proof has been given in 5.2, and a proof is given in 7.2 in general. The expression for the transfer factor is borrowed from Kottwitz [Ko2].

The character $\chi_{G,H}$ depends on the choice of the admissible embedding $\eta$. To obtain an unconditional and more canonical transfer we may change a little the requirements: using the "canonical" transfer factor one gets, instead of a transfer to $H(\mathbb{R})$, a transfer to $H_1(\mathbb{R})$ some covering group of $H(\mathbb{R})$ and thus one gets functions $f^{H_1}$ that transform according to some character of the kernel of

$$H_1(\mathbb{R}) \to H(\mathbb{R})$$

If an admissible embedding exists the character on the kernel is the restriction of some character of $H_1(\mathbb{R})$ which allows to twist the transfer so that it descends to $H(\mathbb{R})$. This has been explained in the case $G = U(2,1)$.

The transfer $f \mapsto f^H$ of pseudo-coefficients can be extended to all functions in $\mathcal{C}_c^\infty(G(\mathbb{R}))$; to define it one has to extend the correspondence $\gamma \mapsto \gamma_H$, called the norm, to all semisimple regular elements and the definition of the transfer factors to all tori. This is established by Shelstad in the series of four papers [She1], [She2], [She3] and [She4].

**Theorem 6.7.2** *Assume there is an admissible embedding*

$$\eta : {}^L H \to {}^L G$$

*One can define transfer factors* $\Delta_H^G(\gamma_H, \gamma)$ *such that for any* $f \in \mathcal{C}_c^\infty(G(\mathbb{R})$ *there exists a function* $f^H \in \mathcal{C}_c^\infty(H(\mathbb{R})$ *with*

$$S\mathcal{O}_{\gamma_H}(f^H) = \Delta_H^G(\gamma_H, \gamma)\mathcal{O}_\gamma^\kappa(f)$$

*whenever $\gamma_H$ is a norm of $\gamma$ semisimple regular and*

$$SO_{\gamma_H}(f^H) = 0$$

*if $\gamma_H$ is not a norm.*

It should be observed that the correpondence

$$f \mapsto f^H$$

is not a map since $f^H$ is not uniquely defined. In fact $f^H$ is only prescribed through its orbital integrals. But as regards invariant harmonic analysis this is a well defined object.

The geometric transfer

$$f \mapsto f^H$$

is dual of a transfer for representations. To any admissible irreducible representation $\sigma$ of $H(\mathbb{R})$ it corresponds an element $\sigma_G$ in the Grothendieck group of virtual representations of $G(\mathbb{R})$ as follows. Given $\varphi$ a Langlands parameter for $H$ then $\eta \circ \varphi$ is a Langlands parameter for $G^*$ where $\eta$ is the admissible embedding

$$\eta : {}^L H \to {}^L G$$

as above. Now consider $\Sigma$ the $L$-packet of admissible irreducible representation of $H(\mathbb{R})$ corresponding to $\varphi$ and $\Pi$ the $L$-packet of representations of $G(\mathbb{R})$, corresponding to $\eta \circ \varphi$ (that can be the empty set if this parameter is not relevant for $G$).

**Theorem 6.7.3** *There is a function*

$$\varepsilon : \Pi \to \pm 1$$

*such that, if we consider $\sigma_G$ in the Grothendieck group defined by*

$$\sigma_G = \sum_{\pi \in \Pi} \varepsilon(\pi)\, \pi$$

*then $\sigma \mapsto \sigma_G$ is the dual of the geometric transfer:*

$$\text{trace } \sigma_G(f) = \text{ trace } \sigma(f^H)$$

This is the Theorem 4.1.1 of [She4]. We shall prove it for discrete series of $U(2,1)$ below. We shall now give an expression for $\varepsilon(\pi)$ following section 5 of [She4].

Suppose we are given a complete set of inequivalent endoscopic groups $H$ and for each $H$ an admissible embedding

$$\eta : {}^L H \to {}^L G$$

Now consider a parameter

$$\varphi : \mathbb{W}_{\mathbb{R}} \to {}^L G$$

The connected centralizer of $s \in \mathbf{S}_\varphi$ is a group $\check{H}_s$ conjugate to $\check{H}$ for some $H$ and hence a conjugate of $\varphi$ factors through $\eta({}^L H)$ and defines an $L$-paquet $\Sigma_s$ of representations of $H(\mathbb{R})$. (The $L$-packet is not unique in general: it depends on the choice of the conjugate which may not be unique). On the other hand, using proposition 6.6.1 above Shelstad defines a pairing $< s, \pi >$ between $\mathbf{S}_\varphi$ and $\Pi(\varphi)$ and shows that

$$\varepsilon(\pi) = c(s) < s, \pi >$$

and hence the above identity

$$\sum_{\sigma \in \Sigma} \text{ trace } \sigma(f^H) = \sum_{\pi \in \Pi} \varepsilon(\pi) \text{ trace } \pi(f)$$

reads

$$\widetilde{\Sigma}_s(f^H) = \sum_{\pi \in \Pi} < s, \pi > \text{ trace } \pi(f)$$

where

$$\widetilde{\Sigma}_s(f^H) = c(s)^{-1} \sum_{\sigma \in \Sigma_s} \text{ trace } \sigma(f^H)$$

This can be inverted to give the theorem 5.2.9 of [She4]:

**Theorem 6.7.4**

$$\text{trace } \pi(f) = \frac{1}{\# \mathbf{S}_\varphi} \sum_{s \in \mathbf{S}_\varphi} < s, \pi > \widetilde{\Sigma}_s(f^H)$$

# 7 Discrete series and endoscopy

## 7.1 $L$-packets of discrete series over $\mathbb{R}$

We recall that, according to Harish-Chandra, the group $G(\mathbb{R})$ has a discrete series representation if and only if there exist an $\mathbb{R}$-elliptic torus $T$ in $G$. Assume from now on this is the case. They are parametrized as follows.

Given $T$, an $\mathbb{R}$-elliptic torus $T$, we consider $K$ a maximal compact subgroup of $G(\mathbb{R})$ containing $T(\mathbb{R})$. Consider a Borel subgroup $B_K$ in $K_{\mathbb{C}}$ containing $T$ and a Borel subgroup $B$ in $G$ containing $B_K$. Now consider a character of $T(\mathbb{R})$ defined by a

$$\lambda \in X^*(T) \otimes \mathbb{C}$$

which is $B$-dominant. There is a discrete series representation $\pi_{\lambda+\rho}$, where $\rho$ is half the sum of positive roots of $T$ in $B$, characterized by its character given on $T(\mathbb{R})$ by

$$\Theta_{\lambda+\rho} = (-1)^q \, \frac{\sum_{w \in \Omega_K} \varepsilon(w) e^{w(\lambda+\rho)}}{\sum_{w \in \Omega_G} \varepsilon(w) e^{w\rho}}$$

where

$$q = \frac{1}{2}(\dim G(\mathbb{R}) - \dim K)$$

An $L$-packet of discrete series is a set of representations $\pi_\mu$ where

$$\mu = w(\lambda + \rho) \quad \text{with} \quad w \in \Omega_{\mathbb{C}}$$

such that $\mu$ belongs to the Weyl chamber defined by $B_K$. We shall denote by $\Omega(B_K)$ the subset of such $w$. Clearly, $\Omega(B_K)$ is a set of representatives in $\Omega_{\mathbb{C}}(G,T)$ of the quotient $\Omega_{\mathbb{R}}(G,T)\backslash\Omega_{\mathbb{C}}(G,T)$.

**Proposition 7.1.1** *The set of representations in an $L$-packet of discrete series is in bijection with $\mathfrak{D}(\mathbb{R}, T\backslash G)$; the bijection depends on the choice of $B$.*

*Proof*: This follows from 6.4.2 and the above remarks.

□

Recall that the set $\mathfrak{D}(\mathbb{R}, T\backslash G)$ embeds in the abelian group $\mathfrak{E}(\mathbb{R}, T\backslash G)$. Thus we get a pairing between the $L$-packet and $\mathfrak{K}(\mathbb{R}, T\backslash G)$ the Pontryagin dual of $\mathfrak{E}(\mathbb{R}, T\backslash G)$. This pairing depends on the choice of $B$. We shall see that this pairing has another formulation using $L$-groups.

Some choices of $B$ seem to be better than others at least when $G$ is a quasi-split group: they are those that correspond to generic representations i.e. with a Whittaker model (with respect to some further choice). For example, for $G = U(2,1)$ and $K = U(2) \times U(1)$ once $B_K$ is chosen there are three Weyl chambers for $G$ that are contained in the Weyl chamber defined by $B_K$ in the Lie algebra of $T$. A Weyl chamber for $G$ that contains $\rho_K$ is a "good" choice. There is only one for $G = U(2,1)$. But for $G = SL(2)$, $\rho_K = 0$ and any choice is good.

## 7.2 General Discrete transfer

Consider a reductive Lie group $G(\mathbb{R})$ with compact maximal torus $T(\mathbb{R})$. In particular, $G(\mathbb{R})$ has discrete series. We choose an endoscopic character $\kappa$, this defines an endoscopic group $H$. We choose a Borel subgroup $B$ in $G$ containing $T$ and a compatible Borel subgroup $B_H$ in $H$. To simplify the discussion we assume that $\rho - \rho_H$ the difference of half sums of positive roots for $G$ and $H$ respectively defines a character of $T(\mathbb{R})$ and hence the canonical transfer factor:

$$\Delta(\gamma^{-1}) = (-1)^{q(G)-q(H)} \frac{\sum_{w \in \Omega_G} \varepsilon(w)\gamma^{w\rho}}{\sum_{w \in \Omega_H} \varepsilon(w)\gamma^{w\rho_H}}$$

is a well defined function.

Let $\mu$ be parameter for a discrete series and let $f_\mu$ be a pseudo-coefficient for $\pi_\mu$. Let $w_0\rho$ be the half sum of positive roots for the Weyl chamber defined by $\mu$ i.e. $\mu = w_0\mu_0$ where $\mu_0$ is $B$-dominant and regular. We have

$$(-1)^{q(G)}\mathcal{O}^\kappa_\gamma(\check{f}_\mu) = (-1)^{q(G)} \sum_{w \in \Omega_G/\Omega_K} \kappa(w)\mathcal{O}_{\gamma_w}(\check{f}_\mu)$$

$$= \sum_{w \in \Omega_G/\Omega_K} \kappa(w) \frac{\sum_{w' \in \Omega_K} \varepsilon(w')\gamma^{ww'\mu}}{\sum_{w_1 \in \Omega_G} \varepsilon(w_1)\gamma^{ww_1w_0\rho}}$$

so that

$$(-1)^{q(G)}\mathcal{O}^\kappa_\gamma(\check{f}_\mu) = \varepsilon(w_0) \frac{\sum_{w \in \Omega_G} \kappa(w)\varepsilon(w)\gamma^{w\mu}}{\sum_{w \in \Omega_G} \varepsilon(w)\gamma^{w\rho}}$$

Hence, we have

$$(-1)^{q(H)}\Delta(\gamma^{-1})\mathcal{O}^\kappa_\gamma(\check{f}_\mu) = \varepsilon(w_0)\frac{\sum_{w\in\Omega_G}\kappa(w)\varepsilon(w)\gamma^{w\mu}}{\sum_{w\in\Omega_H}\varepsilon(w)\gamma^{w\rho_H}}$$

**Lemma 7.2.1** *The function $\kappa$ is left invariant under $\Omega_H$:*

*Proof*: Let $w \in \Omega_G$ and $w_0 \in \Omega_H$ then the multiplicativity property of $\kappa$ shows that

$$\kappa(w_0 w) = \kappa^{w_0}(w)\kappa(w_0)$$

But, by definition of $H$, for any reflexion $s_\alpha \in \Omega_H$ one has

$$\kappa(s_\alpha) = \overline{\kappa}(\check{\alpha}) = 1 \quad \text{and} \quad \kappa^{s_\alpha} = \kappa$$

and hence, by induction on the length of $w_0$, one has $\kappa(w_0) = 1$ and $\kappa^{w_0}(w) = \kappa(w)$ so that

$$\kappa(w_0 w) = \kappa(w)$$

$\square$

Now introduce $\Omega_*(\mu)$ the set of representatives of the quotient $\Omega_H\backslash\Omega_G$ such that $w_*\mu$ is $B_H$ dominant. The above lemma shows that

$$(-1)^{q(H)}\Delta(\gamma^{-1})\mathcal{O}^\kappa_\gamma(\check{f}_\mu)$$

$$= \sum_{w_*\in\Omega_*(\mu)} \kappa(w_*)\varepsilon(w_* w_0)\frac{\sum_{w\in\Omega_H}\varepsilon(w)\gamma^{ww_*\mu}}{\sum_{w\in\Omega_H}\varepsilon(w)\gamma^{w\rho_H}}$$

Altogether, if we denote by $g_\nu$ a pseudo-coefficient for the discrete series of $H(\mathbb{R})$ with parameter $\nu$ we have

$$\Delta(\gamma)\mathcal{O}^\kappa_\gamma(f_\mu) = \sum_{w_*\in\Omega_*(\mu)} \kappa(w_*)\varepsilon(w_* w_0)S\mathcal{O}_\gamma(g_{w_*\mu})$$

and hence

$$f^H_\mu = \sum_{w_*\in\Omega_*(\mu)} a(w,\mu)\, g_{w\mu}$$

with

$$a(w,\mu) = \kappa(w_*)\varepsilon(w_* w_0)$$

and where $w$ is any element in the $\Omega_H$-class defined by $w_* \in \Omega_*(\mu)$.

Consider $\nu = w_*\mu = w_*w_0\mu_0$ and let $\Sigma$ be the $L$-packet of representations of $H(\mathbb{R})$ with parameter $\nu$. We have $w_\Sigma = w_*w_0 \in \Omega_*(\mu_0)$ and hence

$$a(w_\Sigma, \mu_0) = \kappa(w_\Sigma)\varepsilon(w_\Sigma) = \kappa^{w_\Sigma}(w_0^{-1})^{-1}\kappa(w_*)\varepsilon(w_\Sigma)$$

so that

$$a(w, \mu) = \kappa^{w_\Sigma}(w_0^{-1})\, a(w_\Sigma, \mu_0)$$

Let

$$< \kappa_\Sigma, \pi >= \kappa^{w_\Sigma}(w_0^{-1})$$

where $\pi$ is the discrete series for $G$ with parameter $\mu = w_0\mu_0$; then

$$a(w, \mu) =< \kappa_\Sigma, \pi > c(\Sigma, \pi_0)$$

with

$$c(\Sigma, \pi_0) = a(w_\Sigma, \mu_0)$$

and $\pi_0$ defined by $\mu_0$; altogether we get

$$f_\pi^H = \sum_\Sigma < \kappa_\Sigma, \pi > c(\Sigma, \pi_0) g_\sigma$$

the sum being over $L$-packets $\Sigma$ of discrete series for $H(\mathbb{R})$ with parameters in the orbit of $\mu_0$ and we have chosen some $\sigma \in \Sigma$. This is equivalent to

$$c(\Sigma, \pi_0)^{-1} \text{ trace } \Sigma(f^H) = \sum_\pi < \kappa_\Sigma, \pi > \text{ trace } \pi(f)$$

the sum being over discrete series for $G(\mathbb{R})$ with parameters in the orbit of $\mu_0$.

# 8 Further developments

## 8.1 K-groups

To get a more uniform treatment of endoscopy when $F$ is any local field it is better to introduce, following Adams-Barbasch-Vogan Kottwitz and Arthur, the K-group $\widetilde{G}$ associated to $G$. A K-group is a disjoint union of groups

$$\widetilde{G} = \coprod_{\alpha \in A} G_\alpha \ .$$

indexed by

$$A = \text{ Im } [H^1(F, G_{SC}) \to H^1(F, G)]$$

Each group $G_\alpha$ is the inner form of $G$ defined by $\alpha$ (or rather its image in the adjoint group) and we are given maps between $G_\beta$ and $G_\alpha$:

$$\psi_{\alpha\beta} : G_\beta \to G_\alpha$$

satisfying the obvious composition rules and such that $\psi_{\alpha\beta}$ defines $G_\beta$ as an inner form of $G_\alpha$: we are also given 1-cocycles $u_{\alpha\beta;\sigma} \in G_\alpha$ with

$$\psi_{\alpha\beta}\sigma(\psi_{\alpha\beta})^{-1} = \text{ Int } u_{\alpha\beta;\sigma}$$

and whose cohomology class belong to the image of the $H^1$ of the simply connected group $G_{\alpha,SC}$. When $F$ is non archimedean a theorem of Kneser shows that $A$ is trivial and hence one has always $\widetilde{G} = G$ but for $F = \mathbb{R}$ this is not so in general.

A stable conjugacy class of $\gamma \in G_\alpha(F)$ in the K-group $\widetilde{G}$ associated to $G$ is the set of $\gamma' \in G_\beta(F)$ for some $\beta$ and such that there is $x \in G_\alpha$ with

$$\psi_{\alpha\beta}(\gamma') = x^{-1}\gamma x \quad \text{and} \quad a_\sigma = x u_{\alpha\beta;\sigma}\sigma(x)^{-1} \in I_\gamma$$

The set of classes of such 1-cocycles $a$ build a set we denote by

$$\mathfrak{D}(F, I_\gamma\backslash\widetilde{G})$$

Assume that $\gamma$ is regular semisimple, then $I_\gamma$ is a torus $T$.

**Proposition 8.1.1** *Let $F$ be a local field and let $T$ be a torus. There is a bijective map*

$$\mathfrak{D}(F, T\backslash\widetilde{G}) \to \mathfrak{E}(F, T\backslash G)$$

*Proof*: Recall that

$$\mathfrak{E}(F, T\backslash G) = \ker[H^1(F, T) \to H^1_{ab}(F, G)]$$

By definition of the abelianized cohomology the class of a 1-cocycle $a$ with values in $T$ has a trivial image in $H^1_{ab}(F, G)$ if and only if

$$a_\sigma = x u_\sigma \sigma(x)^{-1} \in T$$

for some $x \in G$ and some 1-cocycle $u$ image of a 1-cocycle with values in $G_{SC}$. Hence the map is well defined and is obviously bijective.

$\square$

## 8.2 The twisted case

Another genralization is important for applications: the twisted case. We consider a group $G$ acting on the left on a space $L$

$$(x,\delta)\mapsto x\,\delta \quad , \quad x\in G \qquad \delta\in L$$

which is a principal homogeneous space under this action. We say that $L$ is a twisted $G$-space if we are given a $G$-equivariant map

$$\mathrm{Ad}_L : L\to \text{ Aut } G$$

This allows to define a right action of $G$ on $L$ such that

$$x\,\delta\,y = x\,(\mathrm{Ad}_L(\delta)\,y)\,\delta$$

By choosing a point $\delta_0\in L$ we get an isomorphism

$$L\to G\rtimes\theta$$

where

$$\theta=\mathrm{Ad}_L(\delta_0)$$

but this isomorphism is not canonical when $G$ has a non trivial center. This is why it is better to work with $L$ rather than with $G\rtimes\theta$. One can develop as above the notion of stable conjugacy inside $L(F)$ once stable centralizers are defined: given $\delta\in L(F)$ we say that $\delta'\in L(F)$ is stably conjugate if there is $x\in G(\overline{F})$ such that

$$x^{-1}\delta x=\delta' \quad \text{and} \quad x\sigma(x)^{-1}\in I_\delta$$

for all $\sigma\in \mathrm{Gal}(\overline{F}/F)$ and where $I_\delta$ is the group generated by $Z_L := Z(G)^\theta$ and the image of $G_{SC}^\delta$ the fixed points in $G_{SC}$ under the adjoint action of $\delta$. The stable centralizer $I_\delta$ may not be connected even if $\delta$ is regular semisimple i.e. when $G_{SC}^\delta$ is a torus.

We refer to [KS], [Lab3] and [Lab4] for a study of the twisted case in the trace formula context. As regards the real case Shelstad's results have been extended to the twisted case by Renard [Ren].

**Remark** – We warn the reader that the definition given for the stable centralizer in [Lab3] differs slightly from the one given above and used in [Lab4].

## 8.3 Trace formula stabilization

Last but not least the main motivation for studying endoscopy is the adelic trace formula. In the old paper [LL] the trace formula for $SL(2)$ was expressed as a sum of stable trace formulas for its endoscopic groups. The stabilization gives a better understanding of the space of automorphic forms and allows to establish cases of Langlands functoriality. This was the beginning of all this game.

Later on various other instances of endoscopy and of twisted endoscopy were treated like the Base Change for $GL(2)$ by Saito-Shintani and Langlands [Lan1] and for unitary groups in three variables by Rogawski [Rog]. Endoscopy also played a role at the place where it was first discovered: for computing the Zeta function of Shimura varieties.

At the present time the problem of stabilizing the trace formula which is to express the trace formula for $G$ (or more generally the trace formula for a twisted space) as a linear combination of stable trace formulas for its endoscopic groups is not completely solved.

Wonderful progress have been made by Jim Arthur in first establishing the trace formula for arbitrary reductive groups and then by developping techniques that have allowed him to get a conditional stabilization of the (non twisted) trace formula. The condition is about the endoscopic transfer for non archimedean fields. Observe that transfer factors have been defined for all local fields by Langland and Shelstad [LS]. One needs the existence of the transfer: given $f$ there exists $f^H$ satisfying the transfer identity recalled below and one needs also that the so-called fundamental lemma holds.

**The fundamental lemma**: *Assume that $f$ is the characteristic function of an hyperspecial maximal compact subgroup in $G(F)$ (when this makes sense) and let $f^H$ be the similar function for $H$. Then the transfer identity holds:*

$$S\mathcal{O}_{\gamma_H}(f^H) = \Delta_H^G(\gamma_H, \gamma)\mathcal{O}_\gamma^\kappa(f)$$

*whenever $\gamma_H$ is a norm of $\gamma$ and it vanishes if $\gamma_H$ is not a norm.*

In fact one also needs a weighted variant of it. These questions have resisted the attacks until recently although many important advances have been made by many people among which I should mention Waldspurger in particular: he has proved that the non archimedean transfer holds if the fundamental lemma is true.

A proof of the fundamental lemma for unitary groups has been obtained last year by Laumon and Ngô Bao Châu. But the weighted version of it is still lacking to finish the stabilization even for unitary groups and of course one needs the twisted analogues if one wants to deal with the twisted case.

Things are now progressing faster. For example recent progress have been made by Waldspurger to essentially reduce the twisted case to the ordinary one and more recently Ngô has announced (during the winter 2007) results that imply the fundamental lemma in full generality, even in the twisted case; only the weighted case is still pending.

As a last word I would like to say that endoscopy is not the final target and some are looking

**beyond endoscopy**

but this is another story.

# 9 Bibliography


[Bo] **M. V. Borovoi**, *Abelian Galois cohomology of Reductive groups*, Mem. Amer. Math. Soc. **132** (1998), No. 626.

[A] **J. Arthur**, *A Paley-Wiener theorem for real reductive groups*, Acta Math. **150** (1983), 1-89.

[CD] **L. Clozel P. Delorme**, *Pseudo-coefficients et cohomologie des groupes réductifs*, C.R. Acad. Sci. Paris **300** No. 11 (1985), 385-387.

[D] **P. Delorme**, *Multipliers for the convolution algebra of left and right $K$-finite compactly supported smooth functions on a semi-simple Lie group*, Invent. Math. **75** (1984), 9-23.

[Ka] **D. Kazhdan**, *Cuspidal geometry of $p$-adic groups*, J. Analyse Math **47** (1986), 1–36.

[Ko1] **R. Kottwitz**, *Sign changes in harmonic analysis on reductive groups*, Trans. Amer. Math. Soc. **278** (1983), p. 289-297.

[Ko2] **R. Kottwitz**, *Shimura Varieties and $\lambda$-adic Representations*, Perspectives in Math. **10** Academic Press (1988), pp. 161-210.

[KS] **R. Kottwitz D. Shelstad**, *Foundations of Twisted Endoscopy*, Astérisque **255** (1999).

[Lab1] **J.-P. Labesse**, *Cohomologie, L-groupes et fonctorialité*, Compositio **55** (1985), p. 163-184.

[Lab2] **J.-P. Labesse**, *Pseudo-coefficients très cuspidaux et $K$-théorie*, Math. Ann. **291** (1991), p. 607-616.

[Lab3] **J.-P. Labesse**, *Cohomologie, Stabilisation et changement de base* , Astérisque **257** (1999).

[Lab4] **J.-P. Labesse**, *Stable twisted trace formula: elliptic terms*, Journ. Inst. Math. Jussieu **3** (2004), p. 473-530.

[LL] **J.-P. Labesse R.P. Langlands**, *L-indistinguishability for $SL_2$*, Can. J. Math. **31** (1979), p. 726-785.

[Lan1] **R.P. Langlands**, *Base Change for $GL(2)$*, Annals of Math. Studies **96** Princeton Univ. Press (1980).

[Lan2] **R.P. Langlands**, *On the classification of irreducible representations of real algebraic groups*, Math. Surveys and Monographs **31** AMS (1989). p 101-170

[LS] **R.P. Langlands D. Shelstad**, *On the Definition of Transfer Factors*, Math. Annalen **278** (1987). p 219-271

[Ren] **D. Renard**, *Twisted endoscopy for real groups*, Journ. Inst. Math. Jussieu **2** (2003), p. 529-566.

[Rog] **J. Rogawski**, *Automorphic Representations of Unitary Groups in Three Variables*, Annals of Math. Studies **123** (1990).

[She1] **D. Shelstad**, *Characters and inner forms of a quasi-split group over* $\mathbb{R}$, Compositio Math. **39** (1979), p. 11-45.

[She2] **D. Shelstad**, *Orbital integrals and a family of groups attached to a real reductive group*, Ann. Sci. E.N.S. **12** (1979), p. 1-31.

[She3] **D. Shelstad**, *Embeddings of L-groups*, Can. J. Math. **33** (1981), p. 513-558.

[She4] **D. Shelstad**, *L-Indistinguishability for Real Groups*, Math. Annalen **259** (1982), p. 385-430.