UNRAMIFIED REPRESENTATIONS OF UNITARY GROUPS

by

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Introduction

Let F be a non-Archimedean locally compact field, of residual characteristic p and let E/F be an unramified quadratic extension. Denote by $U(n)$ an unramified unitary group attached to the extension E/F.

The goal of this chapter is to make explicit the transfer of *unramified* representations in the following cases:

(1) Quadratic base change from $U(n)$ to $\mathrm{GL}(n)_{\mathrm{E}}$.
(2) Endoscopic transfer from $U(a) \times U(n-a)$ to $U(n)$.

These results are used in chapter 4A of this volume and in Book 2 to construct automorphic representations. We recall the classification of spherical representations in terms of Satake parameters for any reductive group and then make it explicit for linear and unitary groups (not necessarily unramified).

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1. Notation

1.1. Let F be a non-Archimedean locally compact field, of residual characteristic p. For every finite extension L over F we will denote by $\mathscr{O}_{\mathrm{L}}$ its ring of integers, $\mathfrak{p}_{\mathrm{L}}$ its maximal ideal, ϖ_{L} a uniformizing parameter and k_{L} its residue field. We denote by q_{L} the cardinality of k_{L}.

Let W_{F} be the absolute Weil group of F and, if L is a finite extension over F, let $W_{\mathrm{L/F}}$ be the Weil group of L/F. If L/F is a Galois extension, let $\Gamma(\mathrm{L/F})$ be the Galois group. The symbol Γ will denote $\Gamma(\overline{\mathrm{F}}/\mathrm{F})$, where $\overline{\mathrm{F}}$ is an algebraic closure of F. The norm and the trace maps will be denoted by $\mathrm{N_{L/F}}$ and $\mathrm{tr_{L/F}}$ respectively.

1.2. Let G be a connected reductive group over F. As usual, if there is no confusion, we will not distinguish between G and the group G(F) of F-points of G. Fix a minimal F-parabolic subgroup P_0 of G and let M_0 be a Levi factor of P_0. We denote by $\mathscr{W}_{\mathrm{G}}$ the *spherical* Weyl group, defined to be

$$\mathscr{W}_{\mathrm{G}} = \mathscr{N}_{\mathrm{G}}(\mathrm{M}_0)/\mathrm{M}_0,$$

where $\mathscr{N}_{\mathrm{G}}(\mathrm{M}_0)$ is the normalizer of M_0 in G. A parabolic subgroup P of G will be called standard if it contains P_0. In this case we will denote by M_P the unique Levi factor of P containing M_0, by $\mathrm{N_P}$ the unipotent radical of P, P^- the opposite of P (such that $\mathrm{P} \cap \mathrm{P}^- = \mathrm{M_P}$) and $\mathrm{N_P^-} = \mathrm{N_{P^-}}$. Set $\mathrm{N}_0 = \mathrm{N}_{\mathrm{P}_0}$.

1.3. In this chapter, all representations are assumed to be smooth and admissible – that is, by a representation of G we understand a pair (π, V) where V is a vector space over $\mathbb{C}$ and π is a group homomorphism from G into GL(V) such that the stabilizer of every vector in V is an open subgroup of G and, for every compact open subgroup K of G, the space $\mathrm{V^K}$ of K-invariant vectors in V is finite dimensional.

We denote by Irr(G) the set of equivalence classes of irreducible representations of G. Given π a representation of G we will denote by $\widetilde{\pi}$ the contragredient representation of π and let $\mathrm{JH}(\pi)$ be the set of irreducible constituents of π.

1.4. Let P be a standard parabolic subgroup of G and let (τ, V) be a representation of the Levi factor M_P of P, regarded as a representation of P on which N_P acts trivially. We denote by $i_\mathrm{P}^\mathrm{G}(\tau)$, the representation of G unitarily induced from τ; this is the representation by right translation in the space of functions $\phi : \mathrm{G} \to \mathrm{V}$ satisfying the following conditions:

(a) One has $\phi(pg) = \delta_\mathrm{P}^{1/2}(p)\tau(p)\phi(g)$, for $p \in \mathrm{P}$, $g \in \mathrm{G}$ and where δ_P is the modulus function.

(b) There exists a compact open subgroup K of G such that $\phi(gk) = \phi(g)$ for $g \in \mathrm{G}$ and $k \in \mathrm{K}$.

The factor $\delta_\mathrm{P}^{1/2}$ is there to ensure that $i_\mathrm{P}^\mathrm{G}(\tau)$ is unitary if τ is (hence the term unitary induction). The functor i_P^G preserves finite length, *i.e.* if τ is a finite length representation of M_P, then $i_\mathrm{P}^\mathrm{G}(\tau)$ is also a representation of finite length.

1.5. A representation π of G is called cuspidal if it is not a composition factor of any representation of the form $i_\mathrm{P}^\mathrm{G}(\tau)$ with P a proper parabolic subgroup of G and τ a representation of M_P. This is equivalent to every coefficient of π being compactly supported modulo the center of G.

A cuspidal datum is a pair (M, ρ) where M is a Levi subgroup of G and ρ is a cuspidal representation of G. Two cuspidal data, (M, ρ), (M', ρ') are conjugate if there exists $g \in \mathrm{G}$ such that

$$\mathrm{Ad}\, g : M \xrightarrow{\sim} M',$$
$$\mathrm{Ad}\, g : \rho \xrightarrow{\sim} \rho'.$$

If π is an irreducible representation of G, there exists, up to conjugacy, a unique cuspidal datum (M, ρ) such that π is a composition factor of $i_\mathrm{P}^\mathrm{G}(\rho)$. We call it the cuspidal support of π and write it $\mathrm{supp}(\pi)$.

1.6. Let K be a compact open subgroup of G. The Hecke algebra of G relative to K, denoted by $\mathscr{H}(\mathrm{G}, \mathrm{K})$, is the space of compactly supported functions $f : \mathrm{G} \to \mathbb{C}$ which are left and right K-invariant. This is an algebra under the operation of convolution, relative to some choice of Haar measure μ_G on G. It has a unit element, denoted by e_K, which is $\mu_\mathrm{G}^{-1}(\mathrm{K})$ times the characteristic function of K.

The Hecke algebra $\mathscr{H}(\mathrm{G})$ is then defined as

$$\mathscr{H}(\mathrm{G}) := \bigcup_{\mathrm{K}} \mathscr{H}(\mathrm{G}, \mathrm{K}),$$

where K runs through a basis of neighborhood of 1 consisting of compact open subgroups.

Representations of G on a complex vector space V correspond bijectively to the admissible representations of $\mathscr{H}(\mathrm{G})$ on V, where the action of $\mathscr{H}(\mathrm{G})$ on V is defined as usual by

$$\pi(f)v = \int_{\mathrm{G}} f(g)\pi(g)v\,dg. \tag{1.1}$$

1.7. Let (B,T) be a Borel pair, *i.e.*, a pair consisting of a maximal split torus T of G and a Borel subgroup B containing T. It gives rise to a reduced based root datum

$$\psi(\mathrm{G},\mathrm{B}) = (\mathrm{X}^*(\mathrm{T}), \Delta^*, \mathrm{X}_*(\mathrm{T}), \Delta_*)$$

where $\mathrm{X}^*(\mathrm{T})$ (resp. $\mathrm{X}_*(\mathrm{T})$) is the character (resp. co-character) group, $\Delta^* \subset \mathrm{X}^*(\mathrm{T})$ is the subset of simple roots of T which are positive with respect to B and Δ_* is the set of co-roots associated to the roots in Δ^*. Up to canonical isomorphism, $\psi(\mathrm{G},\mathrm{B})$ is independent of the choice of (B,T).

Let ${}^L\mathrm{G}$ be the *L-group* of G, that is ${}^L\mathrm{G}$ is the semi-direct product of $\widehat{\mathrm{G}}$ – the complex connected reductive group whose reduced based root datum is dual to that of G – and the Weil group W_{F}. See [**Bo1**] or the introduction of this volume [**Har**] for more information on the L-group.

Let H and G be two connected reductive groups. A homomorphism $\xi : {}^L\mathrm{H} \to {}^{\mathrm{L}}\mathrm{G}$ is called an L-homomorphism if

(i) it is a homomorphism over W_{F}, that is, the following diagram commutes

$$\begin{array}{ccc} {}^L\mathrm{H} & \xrightarrow{\xi} & {}^L\mathrm{G} \\ \downarrow & & \downarrow \\ W_F & = & W_{\mathrm{F}} \end{array}$$

where the vertical arrows are the projection onto the factor W_{F} of the L-group,

(ii) ξ is continuous, and

(iii) the restriction of ξ to $\widehat{\mathrm{H}}$ is a complex analytic homomorphism $\xi : \widehat{\mathrm{H}} \to \widehat{\mathrm{G}}$.

The principle of functoriality predicts that, at least when G is quasi-split, to each such L-homomorphism there is associated a correspondence (not a function!) from irreducible representations of H to irreducible representations of G. The original idea comes from the fact studied in this chapter: we will see the lifting of unramified representations of H to unramified representations of G in the case where H and G are unramified groups.

2. Spherical representations

In this section, given a maximal compact subgroup K of G, we recall the classification of K-*spherical* representations of G – that is, irreducible representations of G having a non-zero K-fixed vector – in terms of Satake parameters. All the proofs can be found in the articles of Cartier [**Car**] and Borel [**Bo1**] at Corvallis. One can also consult the original paper of Satake [**Sat**].

2.1. Let K be a good, special, maximal compact subgroup of G. The original definition of such a compact subgroup appears in [**BT**] in terms of the apartment of G. There always exists such a group. The relevant properties of such a group that we need in the sequel are:

Iwasawa decomposition. $\mathrm{G} = \mathrm{P}_0\mathrm{K}$.

Cartan decomposition. $\mathrm{G} = \mathrm{K}\Lambda^+\mathrm{K}$, where Λ^+ is defined as in [**Be2**, §2.2].

Iwahori decomposition. There exists an Iwahori subgroup $\mathrm{I} \subset \mathrm{K}$ such that $\mathrm{I} = (I \cap \mathrm{N}_0^-)(I \cap \mathrm{M}_0)(I \cap \mathrm{N}_0)$ (unique factorization).

The subgroup $\mathrm{M}_0 \cap \mathrm{K}$ is the unique maximal compact subgroup of M_0; it is normal in M_0 and we have $\mathrm{M}_0/\mathrm{M}_0 \cap \mathrm{K} \simeq \mathbb{Z}^d$ where d is the rank of G.

Remark 2.1. — In the case where M_0 is a split torus defined over F, then $\mathrm{M}_0(\mathrm{F})$ is isomorphic to $\mathrm{F}^{\times d}$ and the map taking $\alpha^\vee$ to $\alpha^\vee(\varpi_\mathrm{F})$ induces an isomorphism between the group of co-characters $X_*(\mathrm{M}_0)$ and $\mathrm{M}_0/\mathrm{M}_0 \cap \mathrm{K}$.

Denote by $X^{un}(\mathrm{M}_0)$ the set of unramified characters of M_0, that is, the characters of M_0 which are trivial on $\mathrm{M}_0 \cap \mathrm{K}$. The Weyl group $\mathscr{W} = \mathscr{W}_\mathrm{G}$ acts on $X^{un}(\mathrm{M}_0)$ by

$$(w\chi)(m) = \chi(w^{-1}mw),$$

for $\chi \in X^{un}(\mathrm{M}_0)$, $m \in \mathrm{M}_0$ and $w \in \mathscr{W}$. If we fix $m_1, \dots, m_d$ some generators of $\mathrm{M}_0/\mathrm{M}_0 \cap \mathrm{K}$, we get an isomorphism:

$$\begin{array}{rcl} \mathbb{C}^{\times d} & \xrightarrow{\sim} & X^{un}(\mathrm{M}_0) \\ z & \mapsto & \chi_z, \end{array} \tag{2.1}$$

given by $\chi_z(m_i) = z_i$, for $i = 1, \dots, d$ and $z = (z_1, \dots, z_d)$.

Remark 2.2. — The unramified characters of M_0 are hence in bijection with the $\mathbb{C}$-rational points of a split torus defined over $\mathbb{C}$. This particular fact is not an unimportant accident, as we will see.

Let $\chi \in X^{un}(\mathrm{M}_0)$. By the Iwasawa decomposition we easily check that the space of K-fixed vectors $i_{\mathrm{P}_0}^{\mathrm{G}}(\chi)^{\mathrm{K}}$ is one-dimensional generated by the *canonical spherical vector*

$$\psi_{\chi,\mathrm{K}} : \mathrm{G} \to \mathbb{C}, \qquad mnk \mapsto \delta_{\mathrm{P}}^{1/2}(m)\chi(m),$$

for $m \in \mathrm{M}_0$, $n \in \mathrm{N}_0$ and $k \in \mathrm{K}$.

As the functor of K-invariants is exact, there is one unique irreducible composition factor of $i_{\mathrm{P}_0}^{\mathrm{G}}(\chi)$ which is K-spherical. Thus we get a map

$$X^{un}(\mathrm{M}_0) \to \{\text{K-spherical representations of G}\}. \tag{2.2}$$

Theorem 2.3. — *The map defined by equation* (2.2) *induces a bijection*

$$X^{un}(\mathrm{M}_0)/\mathscr{W} \xrightarrow{\sim} \{\text{K-}\textit{spherical representations of } \mathrm{G}\}.$$

We prove injectivity with a light version of Bernstein-Zelevinski geometric lemma. Using the Iwahori decomposition and the Borel-Matsumoto theorem [**Bo2**], we get that the map is surjective. Remark that the inverse map is given by $\pi \mapsto \mathrm{supp}(\pi)$.

2.2. Another way of defining an inverse map is by using the Satake isomorphism [**Sat**]. Consider the spherical Hecke algebra $\mathscr{H}(\mathrm{M}_0, \mathrm{M}_0 \cap \mathrm{K})$. For elementary reasons it is a polynomial algebra

$$\mathscr{H}(\mathrm{M}_0, \mathrm{M}_0 \cap \mathrm{K}) \simeq \mathbb{C}\left[x_1^{\pm 1}, \ldots, x_d^{\pm 1}\right]$$

where x_i corresponds to the characteristic function of $m_i\,(\mathrm{M}_0 \cap \mathrm{K})$, and we recall that $m_1, \ldots, m_d$ are some fixed generators of $\mathrm{M}_0/\mathrm{M}_0 \cap \mathrm{K}$.

The Weyl group $\mathscr{W}$ acts on $\mathscr{H}(\mathrm{M}_0, \mathrm{M}_0 \cap \mathrm{K})$ in the obvious manner: for $f \in \mathscr{H}(\mathrm{M}_0, \mathrm{M}_0 \cap \mathrm{K})$, $w \in \mathscr{W}$, $m \in \mathrm{M}_0$,

$${}^{w}f(m) = f(w^{-1}mw).$$

Theorem 2.4. — *The $\mathbb{C}$-algebra homomorphism (the Satake transform):*

$$\mathscr{S} : \mathscr{H}(\mathrm{G}, \mathrm{K}) \longrightarrow \mathscr{H}(\mathrm{M}_0, \mathrm{M}_0 \cap \mathrm{K})$$

given by

$$(\mathscr{S}f)(m) = \delta_{\mathrm{P}_0}^{1/2}(m) \int_{\mathrm{N}_0} f(mn)dn = \delta_{\mathrm{P}_0}^{-1/2}(m) \int_{\mathrm{N}_0} f(nm)dn,$$

where dn denotes a Haar measure on N_0*, is injective with image* $\mathscr{H}(\mathrm{M}_0, \mathrm{M}_0 \cap \mathrm{K})^{\mathscr{W}}$*, the algebra of* $\mathscr{W}$*-invariant elements of* $\mathscr{H}(\mathrm{M}_0, \mathrm{M}_0 \cap \mathrm{K})$*.*

Let χ be any unramified character of M_0 and fix a Haar measure on M_0 such that $\int_{\mathrm{M}_0 \cap \mathrm{K}} dm = 1$. The map (Fourier transform) $f \mapsto \int_{\mathrm{M}_0} f(m)\chi(m)dm$ is an algebra homomorphism from $\mathscr{H}(\mathrm{M}_0, \mathrm{M}_0 \cap \mathrm{K})$ to $\mathbb{C}$, and, by varying χ, we get in this way all such homomorphisms.

Define a linear map $\omega_\chi : \mathscr{H}(\mathrm{G}, \mathrm{K}) \to \mathbb{C}$ by

$$\omega_\chi(f) = \int_{\mathrm{M}_0} \mathscr{S} f(m)\chi(m)dm.$$

Proposition 2.5. — *Any algebra homomorphism from* $\mathscr{H}(\mathrm{G}, \mathrm{K})$ *into* $\mathbb{C}$ *is of the form* ω_χ *for some unramified character* χ *of* M_0*. Moreover, one has* $\omega_\chi = \omega_{\chi'}$ *if, and only if, there exists an element* $w \in \mathscr{W}$ *such that* $\chi' = w\chi$*.*

2.3. Now let π be a K-spherical representation of G. As we have supposed that the compact subgroup K is special, the Hecke algebra $\mathscr{H}(\mathrm{G}, \mathrm{K})$ is commutative. The subspace of all K-fixed vectors in π, being irreducible, is then one-dimensional.

Thus, we obtain an algebra homomorphism:

$$\lambda_\pi : \mathscr{H}(\mathrm{G}, \mathrm{K}) \to \mathbb{C},$$

defined by $\pi(f)v_0 = \lambda_\pi(f)v_0$, with v_0 any K-fixed vector, and where the action of $\mathscr{H}(\mathrm{G}, \mathrm{K})$ on the representation space of π is defined by (1.1). One can check, using for example [**Car**, §1.5], that we also have

$$\lambda_\pi(f) = \mathrm{tr}\,(\pi(f))\,. \tag{2.3}$$

So, by Proposition 2.5, there exists an unramified character χ_π, unique up to conjugacy by $\mathscr{W}$ such that

$$\lambda_\pi = \omega_{\chi_\pi}$$

. The next proposition is proved in [**Gar**].

Proposition 2.6. — *The map* $\pi \mapsto \chi_\pi$ *is the inverse of the homomorphism defined in Theorem 2.3.*

Proof. — Let π be a K-spherical representation and let $\chi \in \mathrm{X}^{un}(\mathrm{M}_0)$ be such that π is a subrepresentation of $i_{\mathrm{P}_0}^{\mathrm{G}}(\chi)$. Let ψ be the canonical spherical vector. Then for any $f \in \mathscr{H}(\mathrm{G},\mathrm{K})$, and any $g \in \mathrm{G}$ we have

$$\begin{aligned}\lambda_\pi(f)\psi(g) &= i_{\mathrm{P}_0}^{\mathrm{G}}(\chi)(f)\psi(g)\\ &= \int_{\mathrm{G}} f(h)\psi(gh)dh.\end{aligned}$$

Normalizing the Haar measure on K to be 1, we deduce that:

$$\begin{aligned}\lambda_\pi(f) &= \lambda_\pi(f)\psi(1)\\ &= \int_{\mathrm{G}} f(h)\psi(h)dh\\ &= \int_{\mathrm{P}_0}\int_{\mathrm{K}} f(pk)\psi(pk)\delta^{-1}(p)dpdk\\ &= \int_{\mathrm{P}_0} f(p)\psi(p)\delta^{-1}(p)dp\\ &= \int_{\mathrm{P}_0} f(p)\chi(p)\delta^{-1/2}(p)dp\\ &= \int_{\mathrm{N}_0}\int_{\mathrm{M}_0} f(nm)\chi(nm)\delta^{-1/2}(nm)dndm\\ &= \int_{\mathrm{M}_0} \mathscr{S}f(m)\chi(m)dm\\ &= \omega_\chi.\end{aligned}$$

□

Depending upon the choice $m_1, \dots, m_d$ of some generators for the quotient $\mathrm{M}_0/\mathrm{M}_0 \cap \mathrm{K}$, the *Satake parameters* attached to π are the images

$$\chi_\pi(m_1), \dots, \chi_\pi(m_d).$$

2.4. A connected reductive group is said to be *unramified* over F if it is quasi-split and splits over an unramified extension of F. One can consult [**Cas**] where these groups and their spherical representations are treated in detail. The special properties of these groups are:

(1) A connected reductive group G is unramified if, and only if, it has *hyperspecial* maximal compact subgroups. The original definition of these compact subgroups appears

in [**Tit**] in terms of hyperspecial points in the Bruhat-Tits building of G, generalizing the concept of special compact subgroups.

(2) There exists a group scheme X over $\mathscr{O}_{\mathrm{F}}$ such that $\mathrm{G} = X_{\mathrm{F}}$ and $X_{k_{\mathrm{F}}}$ is a connected reductive group. The compact subgroup $X(\mathscr{O}_{\mathrm{F}})$ is hyperspecial. See, for example [**Mil**].

(3) The action of W_{F} on $\widehat{\mathrm{G}}$ factors through the projection of W_{F} onto $\Gamma(\mathrm{F}^{un}/\mathrm{F})$, where F^{un} is the maximal unramified extension of F. It is hence determined by the action of the Frobenius element.

(4) Denote by $\mathrm{T} = \mathrm{M}_0$ a maximal torus contained in a Borel subgroup $\mathrm{B} = \mathrm{P}_0$ of G. Let A be a maximal split torus and K a hyperspecial maximal compact subgroup. Then the embedding of A into T induces an isomorphism of the lattices

$$\mathrm{A}/\mathrm{A} \cap \mathrm{K} \xrightarrow{\sim} \mathrm{T}/\mathrm{T} \cap \mathrm{K}. \tag{2.4}$$

This last property gives us the idea for using the L-group for classifying spherical representations. The dual group of A is a complex torus $\widehat{\mathrm{A}}$. So we have a set of canonical isomorphisms:

$$\begin{aligned} \mathrm{Hom}\left(\mathrm{T}/\mathrm{T}\cap\mathrm{K}, \mathbb{C}^\times\right) &= \mathrm{Hom}\left(\mathrm{A}/\mathrm{A}\cap\mathrm{K}, \mathbb{C}^\times\right) \\ &= \mathrm{Hom}\left(\mathrm{X}_*(\mathrm{A}), \mathbb{C}^\times\right) \\ &= \mathrm{Hom}\left(\mathrm{X}^*(\widehat{\mathrm{A}}), \mathbb{C}^\times\right), \end{aligned} \tag{2.5}$$

which is, by definition, the group of points of $\widehat{\mathrm{A}}$. Here, the first equality comes from property (4), the second one from the fact that A is split and Remark 2.1, and the third equality by definition of the dual group.

The embedding of A into T gives rise to a surjection from $\widehat{\mathrm{T}}$ to $\widehat{\mathrm{A}}$ which, by (2.5) associates to each element in $\widehat{\mathrm{T}}$ an unramified character of T. We have an action coming from the Weyl group on $X^{un}(\mathrm{T})$ and action of the Frobinius element on $\widehat{\mathrm{T}}$. Let's see that these actions are compatible.

2.5. Suppose until the end of this section that G is an unramified group. We say that a representation of G is K-*unramified* if it is K-spherical where K is a hyperspecial compact subgroup of G.

Remark 2.7. — The notion of unramified representation depends on the choice of the hyperspecial maximal compact subgroup K. However, the hyperspecial maximal compact subgroups form a single orbit under the action of the adjoint group G_{ad} of G (*cf.* [**Tit**]).

Fix an element $\mathfrak{F} \in W_{\mathrm{F}}$ whose projection to $\Gamma(\mathrm{F}^{un}/\mathrm{F})$ is the Frobenius element. We say that two elements g_1' and g_2' of $\widehat{\mathrm{G}}$ are $\mathfrak{F}$-*conjugate* if there exists $h \in \widehat{\mathrm{G}}$ such that $g_2' = h^{-1} g_1' h^{\mathfrak{F}}$.

The following theorem was first stated and proved in [**La1**]. See also [**La2**] and [**Bo1**].

Theorem 2.8. — (1) *Every semisimple* $\widehat{\mathrm{G}}$*-conjugacy class in* $\widehat{\mathrm{G}} \rtimes \mathfrak{F}$ *contains an element of the form* $t' \rtimes \mathfrak{F}$ *with* $t' \in \widehat{\mathrm{T}}$.

(2) *The surjection* $\widehat{\mathrm{T}} \to \widehat{\mathrm{A}}$ *which associates to each element* $t' \in \widehat{\mathrm{T}}$*, by* (2.5)*, an unramified character* $\chi_{t'}$ *of* T *is such that two elements* t_1' *and* t_2' *of* $\widehat{\mathrm{T}}$ *are* $\mathfrak{F}$*-conjugate if, and only if, the unramified characters* $\chi_{t_1'}$ *and* $\chi_{t_2'}$ *of* T *are conjugate under the action of the Weyl group* $\mathscr{W}$.

2.6. Combining Theorems 2.3 and 2.8, we deduce that K-unramified representations are in bijective correspondence with the $\mathfrak{F}$-conjugacy classes of semisimple elements in $\widehat{\mathrm{G}} \rtimes \mathfrak{F}$. Furthermore, each such class can be represented by an element of the form $(t, \mathfrak{F})$, with $t \in \widehat{\mathrm{T}}$ fixed under $\mathfrak{F}$. To sum up, when G is an unramified group the set $\mathrm{Irr}^{\mathrm{K}\text{-}un}(\mathrm{G})$ of K-unramified representations of G is in canonical bijection with:

(1) $X^{un}(\mathrm{M}_0)/\mathscr{W}$.
(2) $\mathrm{Hom}\,(\mathscr{H}(\mathrm{G},\mathrm{K}), \mathbb{C})$.
(3) Semi-simple $\widehat{\mathrm{G}}$-conjugacy classes in the coset $\widehat{\mathrm{G}} \rtimes \mathfrak{F}$.
(4) Equivalence classes of unramified L-parameters of G, that is, commuting diagrams

$$\begin{array}{ccc} \Gamma(\mathrm{F}^{un}/\mathrm{F}) & \xrightarrow{\ \phi\ } & {}^{L}\mathrm{G} \\ \downarrow & \swarrow & \\ \Gamma(\mathrm{F}^{un}/\mathrm{F}) & & \end{array}$$

where the vertical arrow is the identity map from $\Gamma(\mathrm{F}^{un}/\mathrm{F})$ to itself and $\phi(w)$ is semisimple, for all $w \in \Gamma(\mathrm{F}^{un}/\mathrm{F})$.

For (4), remark that an unramified parameter ϕ is determined by the semisimple element $\phi(\mathfrak{F}) = g_\phi \rtimes \mathfrak{F}$.

2.7. Let H and G be two unramified connected reductive groups over F and let

$$\xi : {}^{L}\mathrm{H} \to {}^{\mathrm{L}}\mathrm{G}$$

be an L-homomorphism. We deduce a map from semi-simple conjugacy classes in $\widehat{\mathrm{H}} \rtimes \mathfrak{F}$ to semi-simple conjugacy classes in $\widehat{\mathrm{G}} \rtimes \mathfrak{F}$. Thus if we fix some hyperspecial maximal compact subgroups $\mathrm{K_H}$ and $\mathrm{K_G}$ in H and G respectively, using the preceding results, we deduce a lift, called the *natural unramified lift*:

$$\widetilde{\xi} : \mathrm{Irr}^{\mathrm{K_H}\text{-}un}(\mathrm{H}) \to \mathrm{Irr}^{\mathrm{K_G}\text{-un}}(\mathrm{G}). \tag{2.6}$$

We will make this lift explicit in some special cases in section 4.

We get also a natural map from $\mathrm{Hom}\,(\mathscr{H}(\mathrm{H}, \mathrm{K_H}), \mathbb{C})$ to $\mathrm{Hom}\,(\mathscr{H}(\mathrm{G}, \mathrm{K_G}), \mathbb{C})$ and hence we deduce a lift from the relative Hecke algebras:

$$b(\xi) : \mathscr{H}(\mathrm{G}, \mathrm{K_G}) \to \mathscr{H}(\mathrm{H}, \mathrm{K_H}). \tag{2.7}$$

By equation (2.3), this map is characterized by the property $\mathrm{tr}\,(\pi\,(b(f))) = \mathrm{tr}\left(\widetilde{\xi}(\pi)(f)\right)$, for $\pi \in \mathrm{Irr}^{\mathrm{K_H}\text{-}un}(\mathrm{H})$ and $f \in \mathscr{H}(\mathrm{G}, \mathrm{K_G})$.

3. Basic structure of linear and unitary groups

3.1. Linear groups. — $\mathrm{GL}_n(\mathrm{F})$ is by definition the multiplicative group of invertible matrices in $End_{\mathrm{F}}(\mathrm{F}^n)$. It is endowed with the inherited topology. The identity has a countable basis of neighborhoods that are compact open subgroups; in particular, $\mathrm{GL}_n(\mathrm{F})$ is a locally compact topological group. As such, its Haar measure is left and right invariant (the group is unimodular).

3.1.1. Denote by $\mathrm{GL}_n(\mathscr{O}_\mathrm{F})$ the subgroup of $\mathrm{GL}_n(\mathrm{F})$ of elements $g \in End_{\mathscr{O}_\mathrm{F}}(\mathscr{O}_\mathrm{F}^n)$ such that $\det g$ is a unit in $\mathrm{F}^\times$.

Theorem 3.1. — $\mathrm{GL}_n(\mathrm{F})$ *contains a unique conjugacy class of maximal compact subgroups, and each such subgroup is open. One element in this class is* $\mathrm{GL}_n(\mathscr{O}_\mathrm{F})$.

3.1.2. Let $\alpha = (n_1, \ldots, n_r)$ be a partition of the integer n. We denote by M_α the subgroup of $\mathrm{GL}_n(\mathrm{F})$ of invertible matrices which are diagonal by blocks of size n_i and P_α the subgroup of upper triangular matrices by blocks of size n_i. A standard parabolic subgroup of $\mathrm{GL}_n(\mathrm{F})$ is a subgroup of the form P_α and its Levi factor is M_α.

Thus a minimal Levi factor M_0 is the subgroup of diagonal matrices $\mathrm{M}_{(1,\ldots,1)}$, which is isomorphic to n copies of $\mathrm{F}^\times$. For commodity, we will write elements in M_0 by

$diag(\lambda_1, \ldots, \lambda_n)$, with $\lambda_i \in \mathrm{F}^\times$ for $1 \le i \le n$. The spherical Weyl group is isomorphic to the group of permutations $\mathscr{S}_n$. Here $\mathscr{S}_n$ acts on M_0 by permutations on the matrix entries λ_i, $1 \le i \le n$.

3.1.3. The dual group of $\mathrm{GL}_n(\mathrm{F})$ is $\mathrm{GL}_n(\mathbb{C})$. To verify this we identify X^* and X_* with $\mathbb{Z}^n$ under the standard pairing $\langle e_i, e_j\rangle = \delta_{ij}$, and let

$$\Delta^* = \Delta_* = \{e_i - e_{i+1} : 1 \le i \le n-1\}\,.$$

The Galois action on it is trivial since $\mathrm{GL}_n(\mathrm{F})$ is a split group, thus ${}^L\mathrm{GL}_n(\mathrm{F}) = \mathrm{GL}_n(\mathbb{C}) \times W_\mathrm{F}$ (direct product).

3.1.4. The set of unramified representations is, by Theorem 2.3, in bijection with the n-tuples $(\chi_1, \ldots, \chi_n)$ of unramified characters of $\mathrm{F}^\times$ up to permutation. To such an n-tuple we associate the conjugacy class in $\mathrm{GL}_n(\mathbb{C})$ of the diagonal element $diag(\chi_1(\varpi), \ldots, \chi_n(\varpi))$, where ϖ is a uniformizing parameter of F.

3.2. Unitary groups. — Let E/F be a quadratic extension and let σ be the conjugation of E with respect to F. We also write $\sigma(x) = \overline{x}$. The character of order two of $\mathrm{F}^\times$ associated to E/F by local class field theory will be denoted by $\omega_{\mathrm{E/F}}$. If χ is a character of $\mathrm{E}^\times$, $\overline{\chi}$ will denote the character $\overline{\chi}(x) = \chi(\overline{x})$.

Let V be an n-dimensional vector space over E. A *hermitian form* on V is a pairing

$$h : \mathrm{V} \times \mathrm{V} \to \mathrm{E}$$

that is σ-linear in the first variable, linear in the second variable:

$$h(\alpha v, \beta w) = \overline{\alpha}\beta h(v, w)$$

and satisfies $h(w, v) = \sigma\left(h(v, w)\right)$ for $v, w \in \mathrm{V}$ and $\alpha, \beta \in \mathrm{E}$. We always assume h to be non-degenerate, *i.e.* for $v \in \mathrm{V}$, $v \neq 0$, there exists $w \in V$ such that $h(v, w) \neq 0$. We say that two hermitian vector spaces are *isometric* if there is an E-linear isomorphism between them that identifies the hermitian forms. Such a map is called an *isometry.* The group of isometries of a hermitian space into itself is called a unitary group, *i.e.* $U(V)$ is the subgroup of $g \in \mathrm{GL(V)}$ that preserve h:

$$h(g(v), g(w)) = h(v, w), \qquad \text{for } v, w \in \mathrm{V}.$$

This relation defines an algebraic group over F. We say that $n = \dim_\mathrm{E} \mathrm{V}$ is the degree the unitary group $U(\mathrm{V})$. A hermitian space is called anisotropic if, for all $v \in \mathrm{V}$, $v \neq 0$, we have $h(v, v) \neq 0$. A subspace W of V is called totally isotropic if $h(w, w) = 0$ for all

$w \in \mathrm{W}$. If V and V$'$ are two hermitian vector spaces, one constructs the hermitian vector space (orthogonal sum) $\mathrm{V} \perp \mathrm{V}'$ in the obvious way.

Example 3.2. — (1) For $n = 1$, let $a \in \mathrm{F}$. We define the hermitian space $\mathrm{E}(a)$ as the E-vector space of dimension 1 where the hermitian form h is defined by:

$$h(e, e') = \overline{e} a e'.$$

(2) For $n = 2$, the hyperbolic plane H over E is the E-vector space of dimension 2 with the product:

$$h\left((e_1, e_2), (e_1', e_2')\right) = \overline{e}_1 e_2' + \overline{e}_2 e_1'.$$

(3) For $n = 2$, the anisotropic hermitian space $\mathrm{W}_2(a_1, a_2)$ is the E-vector space of dimension 2, $\mathrm{W}_2(a_1, a_2) = \mathrm{E}(a_1) \perp \mathrm{E}(a_2)$ with a_1, a_2 not equal to zero and $-a_1/a_2 \notin \mathrm{N}_{\mathrm{E}/\mathrm{F}}(\mathrm{E}^\times)$. All anisotropic hermitian spaces of dimension 2 are isometric.

In general, by a theorem of Landherr [**L**], for each n there are exactly two different classes of isomorphism of n-dimensional hermitian spaces over E:

(1) For $n = 2m + 1$ odd, let $\mathrm{V}^\pm \simeq mH \perp \mathrm{W}^\pm$, where $\mathrm{W}^\pm \simeq \mathrm{E}(a)$ (see example 3.2) depending on whether $a \in \mathrm{N}_{\mathrm{E}/\mathrm{F}}(\mathrm{E}^\times)$ or not.

(2) For $n = 2m$ even, let $\mathrm{V}^+ \simeq nH$ and $\mathrm{V}^- \simeq (n-1)H \perp \mathrm{W}_2^-$ where W_2^- is an anisotropic space of dimension 2.

This decomposition is called the Witt decomposition. The number of hyperbolic planes appearing in this decomposition is called the Witt index $w(\mathrm{V})$ of V.

3.2.1. If n is odd, $U(\mathrm{V}^+)$ is isomorphic to $U(\mathrm{V}^-)$ and it is always a quasi-split group. We will denote it by $U(n)$. If n is even, then $U(\mathrm{V}^+)$ is not isomorphic to $U(\mathrm{V}^-)$. We usually write $U(\mathrm{V}^+) = U(m, m)$; it is a quasi-split group while $U(\mathrm{V}^-)$ is not. A unitary group U is thus unramified if, and only if, E/F is an unramified extension and U is isomorphic to $U(\mathrm{V}^+)$ (see Paragraph 2.4).

3.2.2. The number of conjugacy classes of maximal compact subgroups of a unitary group $U(\mathrm{V})$ is equal to $w(\mathrm{V}) + 1$. By [**Tit**], two of them consist of special compact subgroups and, when $U(\mathrm{V})$ is unramified, they are also hyperspecial if the degree of $U(\mathrm{V})$ is even while, if the degree is odd, just one conjugacy class of maximal compact subgroups consists of hyperspecial compact subgroups.

3.2.3. Let (V, h) be a hermitian space. A self dual flag Φ is a decreasing sequence of spaces

$$\mathrm{V} = \mathrm{V}_{-d} \supsetneq \mathrm{V}_{1-d} \supsetneq \cdots \supsetneq \mathrm{V}_{-1} \supsetneq \mathrm{V}_0 \supseteq \mathrm{V}_0^{\perp} \supsetneq \mathrm{V}_1 \supsetneq \cdots \supsetneq \mathrm{V}_d = \{0\},$$

where, for $i = -d, \dots, d$, $i \neq 0$, we have

$$\mathrm{V}_i^{\perp} := \{v \in \mathrm{V} : h(v, v_i) = 0, \forall v_i \in \mathrm{V}_i\} = \mathrm{V}_{-i}.$$

For such a flag Φ and $i = 1, \dots, d$, we can always choose a totally isotropic hermitian subspace $\mathrm{W}_i \subset \mathrm{V}_{-i}$ such that $\mathrm{V}_{-i} = \mathrm{V}_{-i+1} \oplus \mathrm{W}_i$, and a hermitian space W_0 such that $\mathrm{V}_0 = \mathrm{V}_0^{\perp} \oplus \mathrm{W}_0$.

Parabolic subgroups of $U(\mathrm{V})$ are stabilizers of self-dual flags. For a given flag Φ, denote by P_Φ its associated standard parabolic subgroup. Then, the Levi factor of P_Φ is isomorphic to

$$\mathrm{M}_\Phi \simeq \prod_{i=1}^{d} \mathrm{Aut}_{\mathrm{E}}(\mathrm{W}_i) \times U(\mathrm{W}_0).$$

In particular, for any hermitian space V of Witt index $w(\mathrm{V}) = m$, the minimal Levi subgroup M_0 of the unitary group $U(V)$ is isomorphic to m copies of $\mathrm{E}^\times$ times $U(\mathrm{V}_{an})$, where

$$U(\mathrm{V}_{an}) = \begin{cases} 1, & \text{if } \dim \mathrm{V} = 2m \ (\mathrm{V} \simeq \mathrm{V}^+, \\ U(1), & \text{if } \dim \mathrm{V} = 2m+1 \\ U(\mathrm{W}_2^-), & \text{if } \dim \mathrm{V} = 2m+2 \ (\mathrm{V} \simeq \mathrm{V}^-, \end{cases}$$

For commodity, we will write elements in M_0 by $diag(\lambda_1, \dots, \lambda_m, u_o)$, with $\lambda_i \in \mathrm{E}^\times$ and $u_0 \in U(\mathrm{V}_{an})$. The spherical Weyl group $\mathscr{W}$ is isomorphic to $\mathscr{S}_m \rtimes \mathbb{Z}_2^m$. Here $\mathscr{S}_m$ acts on M_0 by permutations on the matrix entries λ_i, $1 \leq i \leq m$. If c_i is the non-trivial element of the i-th copy of $\mathbb{Z}_2$, then c_i changes λ_i into $\overline{\lambda_i}^{-1}$.

3.2.4. Let K be a good special maximal compact subgroup of $U(\mathrm{V})$. The set of K-spherical representations of $U(\mathrm{V})$ is, by Theorem 2.3, in canonical bijection with the m-tuples $(\chi_1, \dots, \chi_m)$ of unramified characters of $\mathrm{E}^\times$ where m is the Witt index $w(\mathrm{V})$ of V and two m-tuples $(\chi_1, \dots, \chi_m)$ and $(\chi'_1, \dots, \chi'_m)$ correspond to the same K-spherical representation if there exists a permutation $s \in \mathscr{S}_m$ such that, for $1 \leq i \leq m$, $\chi'_{s(i)}$ equals χ_i or $\overline{\chi_i}^{-1}$.

3.2.5. Let $U(n)$ be a unitary group of degree n (quasi-split or not). Over E, $U(n)$ is isomorphic to $\mathrm{GL}_n|_{\mathrm{E}}$, and $U(n)$ is an outer form of $\mathrm{GL}_n|_{\mathrm{E}}$. It follows that $\widehat{U(n)} = \mathrm{GL}_n(\mathbb{C})$ and the Galois action factors through $\Gamma(\mathrm{E}/\mathrm{F})$.

Let Φ_n be the $n \times n$ matrix whose ij entry is $(-1)^{i+1}\delta_{i,n-j+1}$;

$$\Phi_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & -1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & (-1)^n & \dots & 0 & 0 \\ (-1)^{n+1} & 0 & \dots & 0 & 0 \end{pmatrix} \tag{3.1}$$

Then $g \mapsto \Phi_n {}^t g^{-1} \Phi_n^{-1}$ is the unique outer automorphism of $\mathrm{GL}_n(\mathbb{C})$ which preserves the standard splitting (defined with respect to the upper-triangular Borel subgroup and the standard basis for the simple root spaces). Hence the non-trivial element σ of $\Gamma(\mathrm{E}/\mathrm{F})$ acts on $\widehat{U(n)} = \mathrm{GL}_n(\mathbb{C})$ by this automorphism

$$\sigma(g) = \Phi_n {}^t g^{-1} \Phi_n^{-1}.$$

An action of W_{F} on $\widehat{U(n)}$ is defined by projection onto $\Gamma(\mathrm{E}/\mathrm{F})$. The L-group of $U(n)$ is the semi-direct product of $\widehat{U(n)}$ with W_{F} with respect to this action.

3.2.6. Suppose $U(n)$ is an unramified group of degree n – that is E/F is an unramified extension and there exists a positive integer m (the Witt index) such that $U(n)$ is isomorphic to $U(2m+1)$ or to $U(m,m)$. Fix a hyperspecial maximal compact subgroup K of $U(n)$ and denote by w_σ a fixed element of $W_{\mathrm{E}/\mathrm{F}}$ whose projection to $\Gamma(\mathrm{E}/\mathrm{F})$ is σ. Then $W_{\mathrm{F}} = W_{\mathrm{E}} \cup w_\sigma W_{\mathrm{E}}$.

As we have seen in Paragraph 3.2.4, K-unramified representations are classified by m-tuples $(\chi_1, \dots, \chi_m)$ of unramified characters of $\mathrm{E}^\times$, where two m-tuples $(\chi_1, \dots, \chi_m)$ and $(\chi'_1, \dots, \chi'_m)$ correspond to the same K-unramified representation if, and only if, there exists a permutation $s \in \mathscr{S}_m$ such that, for $1 \le i \le m$, $\chi'_{s(i)}$ equals χ_i or $\overline{\chi_i}^{-1} = \chi_i^{-1}$.

Fix a maximal torus T in $U(n)$ and a maximal split torus A in T. Then T is isomorphic to $\mathrm{Res}_{\mathrm{E}/\mathrm{F}}\mathrm{A}$ so that $\widehat{\mathrm{A}} \simeq \mathbb{C}^{\times m}$ and $\widehat{\mathrm{T}} \simeq \mathbb{C}^{\times 2m}$. The diagonal embedding A into T gives rise to a natural projection form $\widehat{\mathrm{T}}$ onto $\widehat{\mathrm{A}}$ given by $(t_1, t_2) \mapsto t_1 t_2^{-1}$ where $t_i \in \mathbb{C}^{\times m}$ for $i = 1, 2$.

So, by Paragraph 2.4, to an m-tuple $(\chi_1, \dots, \chi_m)$ of unramified characters of $\mathrm{E}^\times$ we can associate the conjugacy class in ${}^L U(n)$ of the diagonal element

$$diag\left(\chi_1^{1/2}(\varpi), \dots, \chi_m^{1/2}(\varpi), \chi_m^{-1/2}(\varpi), \dots, \chi_1^{-1/2}(\varpi)\right) \rtimes w_\sigma, \quad \text{if } n = 2m \text{ is even,}$$
$$diag\left(\chi_1^{1/2}(\varpi), \dots, \chi_m^{1/2}(\varpi), 1, \chi_m^{-1/2}(\varpi), \dots, \chi_1^{-1/2}(\varpi)\right) \rtimes w_\sigma, \quad \text{if } n = 2m+1 \text{ is odd.}$$

Of course the w_σ-conjugacy class does not depend on the choice of the square root of the $\chi_i(\varpi)$ for $1 \le i \le m$.

4. Some lifting problems

In this section, we make explicit, in terms of Satake parameters, *base change* lifting and *endoscopic transfer*. We will deal only with the unramified case, that is we will suppose that E/F is a quadratic unramified extension and, for n a positive integer, we will write $U = U(n)$ the quasi-split unitary group of degree n. So, if we denote by θ the automorphism $g \mapsto \Phi_n {}^t\overline{g}^{-1}\Phi_n^{-1}$ of $\mathrm{GL}_n(\mathrm{E})$ with Φ_n defined as in (3.1), then U is isomorphic to the subgroup of fixed points by θ in $\mathrm{GL}_n(\mathrm{E})$. We still fix w_σ an element of $W_{\mathrm{E/F}}$ whose projection to $\Gamma(\mathrm{E/F})$ is σ.

Denote by μ the unique unramified character of $\mathrm{E}^\times$ of order 2.

4.1. Quadratic base change. —

4.1.1. Set $\mathrm{G} = \mathrm{Res}_{\mathrm{E/F}}(U)$. The dual group of G is $\widehat{\mathrm{G}} = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ [**Bo1**, §I.5], where the Weil group acts on $\widehat{\mathrm{G}}$ through its projection onto $\Gamma(\mathrm{E/F})$ and σ acts by $\sigma(x,y) = (\Phi_n {}^t y^{-1}\Phi_n^{-1}, \Phi_n {}^t x^{-1}\Phi_n^{-1})$ with Φ_n as in (3.1).

There is a natural bijection between w_σ-conjugacy classes of $\widehat{\mathrm{G}}$ and semi-simple conjugacy classes of $\mathrm{GL}_n(\mathbb{C})$, sending the conjugacy class of $(g_1, g_2) \rtimes w_\sigma$ in ${}^L\mathrm{G}$ to the conjugacy class of $(g_1 \Phi_n {}^t g_2^{-1}\Phi_n^{-1})$ in $\mathrm{GL}_n(\mathbb{C})$. This bijection reflects the fact that G(F) is isomorphic to $U(\mathrm{E})$.

4.1.2. There are two natural L-homomorphisms (the *base change lifts*):

$$BC : \ {}^L U \to {}^L\mathrm{G} \tag{4.1}$$
$$BC' : \ {}^L U \to {}^L\mathrm{G}$$

defined by $BC(g,w) = (g,g,w)$ and $BC'(g,w) = \alpha(w)BC(g,w)$ where $\alpha(w)$ is the 1-cocycle defined by

$$\alpha(w) = \begin{cases} (\mu(w), \mu(w)), & \text{if } w \in W_{\mathrm{E}}, \\ (\mu(w_0), -\mu(w_0)), & \text{if } w = w_0 w_\sigma, w_0 \in W_{\mathrm{E}}. \end{cases}$$

where we regard here μ as a character of W_{E} via local class field theory.

4.1.3. We now make base change explicit for unramified representations. We fix a hyperspecial maximal compact subgroup K in U . We deduce, by (2.6), two morphisms:

$$\begin{aligned} \widetilde{BC} &: \ \mathrm{Irr}^{\mathrm{K}\text{-}un}(U) \to \mathrm{Irr}^{un}(\mathrm{G}) \\ \widetilde{BC'} &: \ \mathrm{Irr}^{\mathrm{K}\text{-}un}(U) \to \mathrm{Irr}^{un}(\mathrm{G}). \end{aligned} \tag{4.2}$$

Theorem 4.1. — (1) *Let π be a* K*-unramified representation of $U(n)$ and denote by $(\chi_1(\varpi), \dots, \chi_m(\varpi))$ its Satake parameters. Then the Satake parameters of $\widetilde{BC}(\pi)$ are*

$$\begin{aligned} (\chi_1(\varpi), \dots, \chi_m(\varpi), \chi_m^{-1}(\varpi), \dots, \chi_1^{-1}(\varpi)), &\quad \textit{if } n = 2m \textit{ is even}, \\ (\chi_1(\varpi), \dots, \chi_m(\varpi), 1, \chi_m^{-1}(\varpi), \dots, \chi_1^{-1}(\varpi)), &\quad \textit{if } n = 2m+1 \textit{ is odd}. \end{aligned}$$

(2) $\widetilde{BC'}(\pi) = \mu(\det) \otimes \widetilde{BC}(\pi)$.

Proof. — Let π be a K-unramified representation of $U(n)$ and $(\chi_1, \dots, \chi_m)$ its Satake parameters. By 3.2.6, π is also parametrized by the conjugacy class in ${}^L U$ of the diagonal element

$$(t, w_\sigma) = \begin{cases} diag\left(\chi_1^{1/2}(\varpi), \dots, \chi_m^{1/2}(\varpi), \chi_m^{-1/2}(\varpi), \dots, \chi_1^{-1/2}(\varpi)\right) \rtimes w_\sigma, & \text{if } n = 2m, \\ diag\left(\chi_1^{1/2}(\varpi), \dots, \chi_m^{1/2}(\varpi), 1, \chi_m^{-1/2}(\varpi), \dots, \chi_1^{-1/2}(\varpi)\right) \rtimes w_\sigma, & \text{if } n = 2m+1. \end{cases}$$

The conjugacy class $BC(t, w_\sigma) = (t, t, w_\sigma)$ in ${}^L\mathrm{G}$ can be regarded, by 4.1.1, as the conjugacy class of $(t\Phi_n t^{-1}\Phi_n^{-1})$ in $\mathrm{GL}_n(\mathbb{C})$, that is, by the conjugacy class of the element

$$\begin{aligned} diag\left(\chi_1(\varpi), \dots, \chi_m(\varpi), \chi_m^{-1}(\varpi), \dots, \chi_1^{-1}(\varpi)\right), &\quad \text{if } n = 2m \text{ is even}, \\ diag\left(\chi_1(\varpi), \dots, \chi_m(\varpi), 1, \chi_m^{-1}(\varpi), \dots, \chi_1^{-1}(\varpi)\right), &\quad \text{if } n = 2m+1 \text{ is odd}. \end{aligned}$$

We deduce the first part of the theorem with 3.2.6. To prove (2) recall that, by the Langlands correspondence, the cocycle α corresponds to the character of G defined by $g \mapsto \mu \circ \det g$. □

Corollary 4.2. — *$\widetilde{BC}$ is an injective map.*

Proof. — Let π and π' be two K-unramified representations of U and $(\chi_1(\varpi), \dots, \chi_m(\varpi))$ and $(\chi'_1(\varpi), \dots, \chi'_m(\varpi))$ respectively its Satake parameters. Suppose $\widetilde{BC}(\pi) \simeq \widetilde{BC}(\pi')$. Then, up to a permutation in $\mathscr{S}_{2m}$, we have that the sets $(\chi_1, \dots, \chi_m, \chi_m^{-1}, \dots, \chi_1^{-1})$ and $(\chi'_1, \dots, \chi'_m, \chi'^{-1}_m, \dots, \chi'^{-1}_1)$ are equal. Hence there exists a permutation $s \in \mathscr{S}_m$ such that, for $1 \leq i \leq m$, $\chi'_{s(i)}$ equals χ_i or χ_i^{-1}. Thus, $\pi \simeq \pi'$.

□

Remark 4.3. — For any representation π of G denote by π^θ the representation $g \mapsto \pi(\theta(g))$. We say that π is θ-invariant if $\pi \simeq \pi^\theta$. One could naively think that the image of $\widetilde{BC}$ is the set of θ-invariant representations of G. But this set is bigger as it also contains some representations coming from endoscopic groups (see next chapter). For example, the unramified representation of $GL_2(\mathrm{E})$ with Satake parameters $(1, \mu)$ is θ-invariant and comes (see Paragraph 4.2.3 for more details) from the endoscopic group $U(1) \times U(1)$ where in the first factor we take the standard base change $\widetilde{BC}$ and in the second we use the twisted base change $\widetilde{BC}'$.

4.2. Endoscopic transfer. —

4.2.1. Recall from [**Ro1**] that, for a unitary group U of degree n, the elliptic endoscopic groups are the quasi-split unitary groups $\mathrm{H} = \mathrm{U(a)} \times \mathrm{U(b)}$ where a and b are positive integers with $a + b = n$. The embedding ${}^L\mathrm{H} \to {}^\mathrm{L}\mathrm{U}$ depends on the choice of characters μ_a and μ_b of $\mathrm{E}^\times$ extending respectively the characters $w^a_{\mathrm{E/F}}$ and $w^b_{\mathrm{E/F}}$.

Then the embedding $\xi_{\mu_a,\mu_b} : {}^L\mathrm{H} \to {}^\mathrm{L}\mathrm{U}$ is defined by (*cf.* [**Ro2**]):

$$\begin{array}{rcll}
(g_1, g_2) \rtimes 1 & \mapsto & \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} \rtimes 1 & \\
1 \rtimes w & \mapsto & \begin{pmatrix} \mu_b(w) 1_a & \\ & \mu_a(w) 1_b \end{pmatrix} \rtimes w & \text{for } w \in W_\mathrm{E} \\
1 \rtimes w_\sigma & \mapsto & \begin{pmatrix} \Phi_a & \\ & \Phi_b \end{pmatrix} \Phi_n^{-1} \rtimes w_\sigma, &
\end{array}$$

where Φ_m, $m = a, b, n$ is defined as in (3.1) and we regard μ_a and μ_b as characters of W_E via local class field theory.

4.2.2. Fix some hyperspecial maximal compact subgroups K_a, K_b and K in $U(a)$, $U(b)$ and $U(n)$ respectively. We suppose here that μ_a and μ_b are unramified characters, that is, for $i = a, b$:

$$\mu_i = \begin{cases} \mu, & \text{if } i \equiv 1 \mod 2 \\ 1, & \text{otherwise.} \end{cases}$$

The following theorem is now straightforward:

Theorem 4.4. — *Let π and π' be a K_a-unramified representation of $U(a)$ and π' be a K_b-unramified representation of $U(b)$. Let a', b' be the Witt indexes of $U(a)$ and $U(b)$ respectively. Denote by $(\chi_1(\varpi), \dots, \chi_{a'}(\varpi))$ and $(\chi'_1(\varpi), \dots, \chi'_{b'}(\varpi))$ respectively its Satake parameters. Then the Satake parameters of $\widetilde{\xi}_{\mu_a,\mu_b}(\pi, \pi')$ (see (2.6)) are:*

$$(\mu_b\chi_1(\varpi), \dots, \mu_b\chi_{a'}(\varpi), \mu_a\chi'_1(\varpi), \dots, \mu_a\chi_{b'}(\varpi)),$$
$$\textit{if } a \textit{ and } b \textit{ are not both odd integers,}$$
$$(\mu\chi_1(\varpi), \dots, \mu\chi_{a'}(\varpi), \mu, \mu\chi'_1(\varpi), \dots, \mu\chi_{b'}(\varpi)),$$
$$\textit{if } a \textit{ and } b \textit{ are both odd integers.}$$

Remark 4.5. — Notice that the rank of $H = U(a) \times U(b)$ is the same as that of U unless $n = a + b$ is even and a, b are both odd. In the second (exceptional) case, the rank of H is one less than that of U

4.2.3. For global purposes, we study now the split case. Using local Langlands correspondence [**HT**], [**Hen**], this can be done in a much greater generality, but we shall restrict ourselves just to an easy example. Let $\mathrm{H} = \mathrm{GL_a(E)} \times \mathrm{GL_b(E)}$ and set $\mathrm{G} = \mathrm{GL}_n(\mathrm{E})$ with $n = a+b$ and let μ_a and μ_b be unramified characters of $\mathrm{E}^\times$. We define an L-homomorphism $\xi_{\mu_a,\mu_b} : {}^L\mathrm{H} \to {}^{\mathrm{L}}\mathrm{G}$ by

$$(g_1, g_2) \quad \mapsto \quad \begin{pmatrix} \mu_b(\varpi)g_1 & \\ & \mu_a(\varpi)g_2 \end{pmatrix}.$$

It is now clear that if π and π' are unramified representations of $\mathrm{GL}_a(\mathrm{E})$ and $\mathrm{GL}_b(\mathrm{E})$ respectively and $(\chi_1(\varpi), \dots, \chi_a(\varpi))$ and $(\chi'_1(\varpi), \dots, \chi'_b(\varpi))$ are its respective Satake parameters, then the Satake parameters of $\widetilde{\xi}_{\mu_a,\mu_b}(\pi, \pi')$ are:

$$(\mu_b \circ \chi_1(\varpi), \dots, \mu_b \circ \chi_a(\varpi), \mu_a \circ \chi'_1(\varpi), \dots, \mu_a \circ \chi_b(\varpi)).$$

Hence $\widetilde{\xi}_{\mu_a,\mu_b}(\pi, \pi')$ is a composition factor of the parabolically induced representation

$$i^{\mathrm{G}}_{\mathrm{P}_{(a,b)}} \left((\pi\, \mu_b \circ \det) \otimes (\pi'\, \mu_b \circ \det)\right) \tag{4.3}$$

where we see $(\pi\,\mu_b\circ\det)\otimes(\pi'\,\mu_b\circ\det)$ as a representation of the Levi subgroup $\mathrm{H}\simeq\mathrm{M}_{(\mathrm{a,b})}$ of $\mathrm{P}_{(a,b)}$. In particular, if π and π' are unitary representations, by Theorem A.2, the representation (4.3) is irreducible and hence isomorphic to $\widetilde{\xi}_{\mu_a,\mu_b}(\pi,\pi')$.

A
On the classification of irreducible representations of linear groups

By the work of Silberger [**Sil**] and Borel-Wallach [**BW**], extending the results of Langlands to the p-adic case, the problem of classifying irreducible representations of a connected reductive group G over F is reduced to the study of *tempered* representations. These representations appear as composition factors of parabolically induced representations of discrete series representations of Levi subgroups of G. So identifying the tempered dual of G consists of two problems:

(1) Determine the discrete series representations of the Levi subgroups (in terms of cuspidal representations or by the theory of types).

(2) Decompose the resulting parabolically induced representations.

Neither problem is resolved in any generality. The general theory of irreducible representations of GL_n over a non-Archimedean local field, however, is well understood. In this appendix, we recall the construction of irreducible representations in terms of cuspidal data. For a more detailed exposition and historical notes one can consult [**Mo1**], see also [**Rod**], [**BZ1**] and [**Ze1**].

A.1. Discrete series. — For a general reductive group, one does not know how to classify the discrete series in terms of cuspidal representations. But for $\mathrm{GL}_n(\mathrm{F})$ this has been done by Bernstein and Zelevinsky [**Ze1**]. An understanding of cuspidal representations in terms of types is due to Bushnell and Kutzko [**BK**].

Let ρ be a cuspidal representation of $\mathrm{GL}_r(\mathrm{F})$ and let a be a positive integer. We denote by $\delta(a,\rho)$ the unique irreducible quotient of the parabolically induced representation:

$$i_{\mathrm{P}}^{\mathrm{GL}_n(\mathrm{F})}\left(\rho|\det|^{-\frac{a-1}{2}}\otimes\rho|\det|^{-\frac{a-3}{2}}\otimes\cdots\otimes\rho|\det|^{\frac{a-1}{2}}\right),$$

where P denotes the standard parabolic subgroup associated to the partition $(r,r,\ldots,r)$.

Theorem A.1. — (1) *All representations of the form $\delta(a,\rho)$ with a a positive integer and ρ a unitary cuspidal representation of $\mathrm{GL}_r(\mathrm{F})$ are irreducible discrete series representations.*

(2) *Conversely, let* π *be an irreducible discrete series representation of* $\mathrm{GL}_n(\mathrm{F})$*. There exist a unique divisor* r *of* n *and a unique irreducible unitary cuspidal representation* ρ *of* $\mathrm{GL}_r(\mathrm{F})$ *such that, if we set* $a = \frac{n}{r}$*, then* π *is isomorphic to* $\delta(a, \rho)$*.*

A.2. Tempered representations. — The parabolically induced representation of a discrete series representation, in the case of $\mathrm{GL}_n(\mathrm{F})$, is irreducible (see, for example [**Jac**]). We have a deeper result, proved by Bernstein [**Be1**]:

Theorem A.2. — *Let* P *be a standard parabolic subgroup of* $\mathrm{GL}_n(\mathrm{F})$ *and let* $\mathrm{M_P}$ *be its Levi factor. Let* ρ *be an irreducible unitary representation of* $\mathrm{M_P}$*. Then* $i_{\mathrm{P}}^{\mathrm{GL}_n(\mathrm{F})}(\rho)$ *is irreducible.*

A similar theorem for the inner forms of $\mathrm{GL}_n(\mathrm{F})$ has been proved by V. Sécherre [**Sec**].

A.3. Irreducible representations. — The understanding of the unitary dual of $\mathrm{GL}_n(\mathrm{F})$ is due to Tadić [**Tad**]. The Langlands correspondence in this case is due to Harris-Taylor [**HT**] and Henniart [**Hen**]. The Langlands quotient theorem, in this case, reads:

Theorem A.3. — *Let* π *be an irreducible representation of* $\mathrm{GL}_n(\mathrm{F})$*. There exist a partition* $\alpha = (n_1, \ldots, n_r)$ *of* n *and, for* $1 \leq i \leq r$*, a unique (up to isomorphism) tempered representation* τ_i *of* $\mathrm{GL}_{n_i}(\mathrm{F})$ *and a unique real number* t_i *with*

$$t_1 > t_2 > \cdots > t_r,$$

such that π *is the unique irreducible quotient of*

$$i_{\mathrm{P}_\alpha}^{\mathrm{GL}_n(\mathrm{F})}\left(\tau_1|\det|^{t_1} \otimes \tau_2|\det|^{t_2} \otimes \cdots \otimes \tau_r|\det|^{t_r}\right).$$

We note $\pi = L\left(\tau_1|\det|^{t_1}, \ldots, \tau_r|\det|^{t_r}\right)$ *and, if* $\pi' = L\left(\tau_1'|\det|^{t_1'}, \ldots, \tau_{r'}'|\det|^{t_{r'}'}\right)$ *is another representation of* $\mathrm{GL}_n(\mathrm{F})$*, then* π *is isomorphic to* π' *if, and only if,* $r = r'$ *and for all* $1 \leq i \leq r$*,* $\tau_i|\det|^{t_i} \simeq \tau_i'|\det|^{t_i'}$*.*

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