# DECOMPOSITION THEOREM AND ABELIAN FIBRATION

NGÔ BAO CHÂU

The text is the account of a talk given in Bonn in a conference in honour of M. Rapoport. We will review the proof of the support theorem which is the main geometric ingredient in the proof of the fundamental lemma. The original source is the chapter 7 of [15].

## 1. Support in the decomposition of a pure direct image

Let $X$ be a smooth algebraic variety over a field $k$ and $f : X \to S$ be a proper morphism. By Deligne's theorem [5], the direct image $f_*\mathbb{Q}_\ell$ is a pure complex i.e. for the perverse $t$-structure, the cohomology ${}^p\mathrm{H}^j(f_*\mathbb{Q}_\ell)$ are pure perverse sheaves. After a base change to $S \otimes_k \bar{k}$, ${}^p\mathrm{H}^j(f_*\mathbb{Q}_\ell)$ is direct sum of simple perverse sheaves according to Beilinson, Bernstein and Deligne [1].

Let us recall the description of simple perverse sheaves over $S \otimes_k \bar{k}$. Let $Z$ be a closed irreducible subscheme of $S \otimes_k \bar{k}$, $i : Z \hookrightarrow S \otimes_k \bar{k}$ denoting the closed immersion. Let $U$ be a smooth open dense subscheme of $Z$, $j : U \hookrightarrow Z$ denoting the open immersion. Let $\mathcal{K}$ be an irreducible local system on $U$. Then

$$K = i_* j_{!*}\mathcal{K}[\dim(Z)]$$

is a simple perverse sheaf on $S \otimes_k \bar{k}$, $j_{!*}$ being Goresky-MacPherson's intermediate extension. Moreover, it is known that all simple perverse sheaves over $S \otimes_k \bar{k}$ are of this form, see [1]. In particular the closed reduced subscheme $Z$ of $S \otimes_k \bar{k}$ is completely determined by $K$ and is called the support of $K$.

One of the basic invariant one can attach to the proper morphism $f$ with smooth source is set of supports of the simple perverse sheaves occurring in the perverse cohomology of $f_*\mathbb{Q}_\ell$, and in particular, in which circumstances the only present support is $S \otimes_k \bar{k}$.

The knowledge of the support is useful in proving certain equalities of number of points. Let $k$ be a finite field. Let $f_1 : X_1 \to S$ and $f_2 : X_2 \to S$ be morphisms as above. Assume that the only support in $f_{i,*}\mathbb{Q}_\ell$ is $S \otimes_k \bar{k}$ for $i \in \{1, 2\}$. Assume there exists a dense open subset $U \subset S$ such that for all extension $k'/k$, for all $s \in U(k')$ we have the equality

$$\sharp X_{1,s}(k') = \sharp X_{2,s}(k').$$

Then the above equality is also true for all $s \in S(k)$. This is a simple application of Deligne's purity theorem, Grothendieck-Lefschetz fixed points formula and Chebotarev's theorem. Since the only present support is $S \otimes_k \bar{k}$, the class of ${}^p\mathrm{H}^j(f_*\mathbb{Q}_\ell)$ in the appropriate Grothendieck group is determined by its restriction to $U$ which is in turn determined by the traces of the Frobenius operator $\sigma_s$ attached to closed points $s$ of $U'$. In [12], Laumon explained an induction process for the above strategy in a more general case when we know the set of support which might contains

*Date*: 2009.

more than $S \otimes_k \bar{k}$. In any case, the control on the set of supports allows a principle of continuation of equality of number of points in the fibers of $f_1$ and $f_2$.

In general however, it is difficult to determine the support of the simple perverse constituents of the direct image. Here is nevertheless a few examples where such a determination is possible. First assume $f : X \to S$ is a proper and smooth morphism. Since the source $X$ is already supposed to be smooth, so is the target $S$. In this case, the ordinary cohomology $\mathrm{H}^j(f_*\mathbb{Q}_\ell)$ are local system on $S$ and they coincide with perverse cohomology. Thus the only present support is $S \otimes_k \bar{k}$.

Less obvious examples consist in small and semi-small morphisms in Goresky-MacPherson's sense. A proper morphism $f : X \to S$ is called semi-small if $\dim(X \times_S X) \leq \dim(X)$ and small if $\dim(X \times_S X - \Delta X) < \dim(X)$. Here $\Delta X$ denote the diagonal of $X$ in $X \times_S X$. Assume as always $X$ smooth. A simple dimension computation shows that if $f$ is small then $f_*\mathbb{Q}_\ell[\dim X]$ is a perverse sheaf and the only present spport is $S \otimes_k \bar{k}$. If $f$ is semi-small then $f_*\mathbb{Q}_\ell[\dim X]$ is a perverse sheaf and the support of its constituent are the images of maximal dimension components of $X \times_S X$ in $S$. Springer resolution of the nilpotent cone in a semi-simple Lie algebra is a semi-small morphism and Grothendieck-Springer simultaneous resolution is a small morphism. These examples had quite an impact in representation theory, see [13] for instant.

Let us consider another simple but important example which was pointed out to me by Goresky. Let $f : X \to S$ be a relative proper curve. Assume that $X/k$ is smooth of dimension 2, $S/k$ smooth of dimension 1 and the fibers $X_s$ are irreducible curves. Then one can prove that the only support in $f_*\mathbb{Q}_\ell$ is $S$. The proof is an application of Poincaré duality. We are going to explain it in some details since this argument will be important in the sequel. Consider the decomposition

$$f_*\mathbb{Q}_\ell[2] = \bigoplus K[n]$$

where $K$ are simple perverse sheaves on $S \otimes_k \bar{k}$ and $n$ are integers. By Poincaré duality, we know if $K[n]$ occurs in the above direct sum then so does $K^\vee[-n]$. Let $Z$ be a support of certain simple summand $K[n]$. Assume $\dim(Z) = 0$. Then $K[n]$ contribute in ordinary cohomological degree $-n$ and $K^\vee[-n]$ contribute in degree $n$. But $f_*\mathbb{Q}_\ell[2]$ only have ordinary cohomological degree $0, -1, -2$. It follows that $n = 0$. But $\mathrm{H}^0(f_*\mathbb{Q}_\ell[2]) = \mathrm{H}^2(f_*\mathbb{Q}_\ell)$ is a constant sheaf of rank one in which a punctual sheaf can't contribute as a direct factor. This yields a contradiction.

This argument provides in fact a general upper bound for the codimension of the support.

**Proposition 1** (Goresky-MacPherson). *Assume $X/k$ smooth. Let $f : X \to S$ a proper morphism of relative dimension $d$. Let $Z$ be the support of a simple perverse sheaf occurring in $f_*\mathbb{Q}_\ell$. Then we have the inequality*

$$\mathrm{codim}(Z) \leq d.$$

In this degree of generality, the above inequality seems to be of little use. In concrete situations, we have to do a little bit more to extract precise informations of the set of supports. The good point is Poincaré duality argument is very malleable to be adapted to different specific situation.

## 2. Algeraic abelian fibration

Algebraic abelian fibration is a somewhat vague terminology for a denegerating family of abelian varieties. It is however difficult to coin exactly what an abelian fibration is. We are going to introduce instead a loose notion of weak abelian fibration by keeping the properties that are conserved by arbitrary base change and a more restrictive notion of $\delta$-regular abelian fibration. A good notion of algebraic abelian fibration must be somewhere in between.

A *weak abelian fibration* will consist in a proper morphism $f : M \to S$ equipped with an action of a smooth commutative group scheme $g : P \to S$ i.e. we have an action of $P_s$ on $M_s$ depending algebraically on $s \in S$. In this section, it is convenient to assume $P$ have connected fibers. In general, we can replace $P$ by the open sub-group schemes of neutral components. We will require the following three properties to be satisfied.

(1) The morphism $f$ and $g$ have the same relative dimension $d$.
(2) The action has *affine stabilizers* : for all geometric points $s \in s$, $m \in M_s$, the stabilizer $P_{s,m}$ of $m$ is an affine subgroup of $P_s$. We cas rephrase this property as follows. According to Chevalley, for all geometric point $s \in S$, there exists an exact sequence

$$1 \to R_s \to P_s \to A_s \to 1$$

where $A_s$ is an abelian variety and $R_s$ is a connected affine commutative group. Then for all geometric points $s \in s$, $m \in M_s$, the stabilizer $P_{s,m}$ is a subgroup of $R_s$.
(3) The group scheme $P$ to have a *polarizable Tate module*. Let $\mathrm{H}_1(P/S) = \mathrm{H}^{2g-1}(g_! \mathbb{Q}_\ell)$ with fiber $\mathrm{H}_1(P/S)_s = T_{\mathbb{Q}_\ell}(P_s)$. This is a sheaf for the étale topology of $S$ whose the stalk over a geometric point $s \in S$ is the $\mathbb{Q}_\ell$-Tate module of $P_s$. The Chevalley exact sequence induces

$$0 \to T_{\mathbb{Q}_\ell}(R_s) \to T_{\mathbb{Q}_\ell}(P_s) \to T_{\mathbb{Q}_\ell}(A_s) \to 0.$$

We require that locally for the étale topology there exists an alternating form $\psi$ on $\mathrm{H}_1(P/S)$ such that over any geometric point $s \in S$, $\psi$ is null on $T_{\mathbb{Q}_\ell}(R_s)$ and induces a non-degenerating form on $T_{\mathbb{Q}_\ell}(A_s)$.

We observe that the notion of weak abelian fibration is conserved by arbitrary base change. In particular, the generic fiber of $P$ is not necessarily an abelian variety. We are going now to introduce a strong restriction called *$\delta$-regularity* which implies in particular that the generic $P$ is an abelian variety.

Let's stratify $S$ be the invariant by the dimension $\delta(s) = \dim(R_s)$ of the affine part of $P_s$. We know $\delta$ is a function upper semi-continuous. Let denote

$$S_\delta = \{s \in S | \delta(s) = \delta\}$$

which is a locally closed subset of $S$. The group scheme $g : P \to S$ is $\delta$-regular if

$$\mathrm{codim}(S_\delta) \geq \delta.$$

A *$\delta$-regular abelian fibration* is a weak abelian fibration $f : M \to S$ equipped with action of $\delta$-regular group scheme $g : P \to S$ .

We observe that $\delta$-regularity is conserved by flat base change.

For a $\delta$-regular abelian fibration, the open subset $S_0$ is a non empty open subset i.e. generically $P$ is an abelian variety. Combined with the affineness of stabilizer and with the assumption $f$ and $g$ having the same relative dimension, it follows that the generic fiber of $f$ is a finite union of abelian varieties.

Considering a typical example. Let $X \to S$ be a family curve with reduced irreducible fibers with plane singularities. Let $P = \mathrm{Jac}_{X/S}$ be the relative Jacobian. Let $M = \overline{\mathrm{Jac}}_{X/S}$ be the compactified relative Jacobian. For every $s \in S$, $P_s$ classifies invertible sheaves on $X_s$, $M_s$ classifies rank one torsion-free sheaves on $X_s$ and $P_s$ acts on $M_s$ by tensor product. The Weil pairing defines a polarization of the Tate module $\mathrm{H}_1(P/S)$. For every geometric point $s \in S$, we can check that

$$\delta(s) = \dim \mathrm{H}^0(X_s, c_* \mathcal{O}_{\tilde{X}_s} / \mathcal{O}_{X_s})$$

is Serre's $\delta$-invariant of $X_s$. Here $c : \tilde{X}_s \to X_s$ denote the normalization of $X_s$. It is well known that the $\delta$-regularity is true for a versal deformation of curve with plane singularities, see [6].

In general, it is not obvious to prove the $\delta$-regularity of a given weak abelian fibration.

One family of examples is concerned with algebraic integrable systems over the field of complex numbers. As we will see, in this case the existence of the symplectic form implies the $\delta$-regularity. Let $f : M \to S$ and $g : P \to S$ form a weak abelian fibration. Assume $M$ is a complex smooth algebraic variety of dimension $2d$ equipped with a symplectic form and $S$ is smooth of dimension $\dim(S) = \dim(M)/2$. Assume that for every $m \in M$ over $s \in S$, the tangent space $T_m M_s$ to the fiber is coisotropic i.e. its orthogonal $(T_m M_s)^{\perp}$ with respect to the symplectic form is contained itself. The tangent application $T_m M \to T_s S$ defines by duality a linear map

$$T_s^* S \to T_m M$$

by identifying $T_m^* M$ with $T_m M$ using the symplectic form. Assume that $\mathrm{Lie}(P_s) = T_s^* S$ and the infinitesimal action of $P_s$ on $M_s$ at the point $m \in M_s$ is given by the above linear map. Consider the Chevalley exact sequence

$$1 \to R_s \to P_s \to A_s \to 1$$

of $P_s$. The connected affine subgroup $R_s$ acting on the proper scheme $M_s$ must have a fixed point according to Borel. Denote $m$ a fixed point. The map $P_s \to M$ given by $p \mapsto pm$ factors through $A_s$ so that the infinitesimal map $\mathrm{Lie}(P_s) \to T_m M$ factors through $\mathrm{Lie}(A_s)$. By duality, for every point $m \in M_s$ fixed under the action of the affine part $R_s$, the image of the tangent application

$$T_m M \to T_s S$$

is contained in $\mathrm{Lie}(A_s)^*$ which is a subspace of codimension $\delta(s)$ independent of $m$. In characteristic zero, we can derive from this tangent vectors computation the $\delta$-regularity. Essentially. when $s$ moves in such a way $\delta(s)$ remains constant, the tangent direction of the motion of $s$ can't get away from a fixed subvector space $\mathrm{Lie}(A_s)^*$ of $T_s S$ of codimension $\delta(s)$.

Unfortunately, this argument does not work well in positive characteristic. In the case of Hitchin fibration, we can use a kind of global-local argument. One can define a local variant of the $\delta$-invariant. The similar computation of the codimension of $\delta$-constant strata can be derived from Goresky-Kottwitz-MacPherson's result [?]. One can use Riemann-Roch's type argumet to obtain the global estimate from the local estimates in certain circumstance as in [15]. In loc. cit, we proved a weaker form of $\delta$-regularity good enough to prove local statement as the fundamental lemma but unsatisfying from the point of view of Hitchin fibration. We hope to be able to remove this caveat in future works.

## 3. Support in the direct image by an abelian fibration

We are now summarizing chapter 7 of [15].

**Theorem 2** (Support)**.** *Let $f : M \to S$ and $g : P \to S$ be a $\delta$-regular abelian fibration of relative dimension $d$ with the total space $M$ smooth over $k$. Let $K$ be a simple perverse sheaf occuring in $f_*\mathbb{Q}_\ell$. Then there exists an open subset $U$ of $S \otimes_k \bar{k}$ such that $U \cap Z \neq \emptyset$ and a non trivial local system $L$ on $U \cap Z$ such that $L$ is a direct factor of $\mathrm{R}^{2d} f_* \mathbb{Q}_\ell|_U$. In particular, if the geometric fibers of $f$ are irreducible then $Z = S \otimes_k \bar{k}$.*

We prove in fact for any weak abelian fibration an estimate on the codimension o $Z$ which is an improvement of Goresky-MacPherson inequality.

**Proposition 3** ($\delta$-Inequality)**.** *Let $f : M \to S$ equipped with $g : P \to S$ be a weak abelian fibration of relative dimension $d$ with total space $M$ smooth over the base field $k$. Let $K$ be a simple perverse sheaf occurring in $f_*\mathbb{Q}_\ell$. Let $Z$ be the support of $K$. Let $\delta_Z$ be the minimal value of $\delta$ on $Z$. Then we have the inequality*

$$\mathrm{codim}(Z) \leq \delta_Z.$$

*If the equality occurs then there exists an open subset $U$ of $S \otimes_k \bar{k}$ such that $U \cap Z \neq \emptyset$ and a non trivial local system $L$ on $U \cap Z$ such that $L$ is a direct factor of $\mathrm{R}^{2d} f_* \mathbb{Q}_\ell|_U$. In particular, if the geometric fibers of $f$ are irreducible then $Z = S \otimes_k \bar{k}$.*

It is immediate that the $\delta$-inequality implies the support theorem.

What follows now is a false start which gives however an intuitive idea about the $\delta$-inequality. The $\delta$-inequality is an improvement of Goresky-MacPherson's inequality $\mathrm{codim}(Z) \leq d$ in the case of abelian fibration. It can be even reduced to this inequality if we make the following lifting assumptions on an neighborhood around a point $s \in Z$:

- there exists a lift $A_s$ to an abelian scheme $A_{S'}$ on an étale neighborhood $S'$ of $s$,
- there exists a homomorphism $A_{S'} \to P_{S'} = P \times_S S'$ such that over the point $s$, its composition with the projection $P_s \to A_s$ is an isogeny of the abelian variety $A_s$.

Under these assumptions, we have an action of abelian scheme $A_{S'}$ on $M_{S'} = M \times_S S'$ with finite stabilizers. Consider the quotient $[M_{S'}/A_{S'}]$ which is an algebraic stack proper and smooth over $S'$ of relative dimension $\delta_Z$. The $\delta$-inequality follows from the fact that the morphism $M_{S'} \to [M_{S'}/A_{S'}]$ is proper and smooth and Goresky-MacPherson's inequality for the morphism $[M_{S'}/A_{S'}] \to S'$.

In pratice, the above lifting assumption almost never happen because in most of the cases, the generic fiber of $P$ is an irreducible abelian variety. Our strategy is in fact to imitate the above proof in homology level instead of geometry.

Consider the complex of homology $\Lambda_P = g_! \mathbb{Q}_\ell[2d](d)$ of $P$ which is concentrated in the degrees $-2d, -2d+1, \ldots, 0$. For all $s \in S$, $\mathrm{H}^{-1}(\Lambda_P)_s$ is the $\mathbb{Q}_\ell$-Tate module $T_{\mathbb{Q}_\ell}(P_s)$. For all $i = 1, \ldots, d$, $\mathrm{H}^{-i}(\Lambda_P)_s = \wedge^i T_{\mathbb{Q}_\ell}(P_s)$.

Assume the $S$-group scheme $g : P \to S$ acts on the $S$-scheme $f : M \to S$. Then the trace map induces a canonical map $\Lambda_P \otimes f_! \mathbb{Q}_\ell \to f_! \mathbb{Q}_\ell$. When $f = g$,

$\Lambda_P \otimes \Lambda_P \to \Lambda_P$ is fiberwise the Pontryagin product. Thus $\Lambda_P$ has an algebra structure for which $f_!\mathbb{Q}_\ell$ is a $\Lambda_P$-module. In our case, $f$ is proper thus $f_*\mathbb{Q}_\ell$ is a $\Lambda_P$-module on the homology of a commutative algebraic group. We have to be aware however that $\Lambda_P$ and $f_*\mathbb{Q}_\ell$ are objects of a triangular category.

The hypothetical lifting assumption can be replaced by the following homological property : *$f_!\mathbb{Q}_\ell$ is free over the abelian part of $\Lambda_P$*. It is not immediately clear what we mean by the abelian part of $\Lambda_P$ and by the freeness for objects in derived category. In fact, it require quite a few preparations to formulate this freeness property in the right way.

Consider the decomposition of the $n$-th perverse cohomology ${}^p\mathrm{H}^n(f_*\mathbb{Q}_\ell)$ into a direct sum of simple perverse sheaves over $S\otimes_k \bar{k}$. We can put together those simple perverse sheaves having the same support so that we have a canonical decomposition

$$ {}^p\mathrm{H}^n(f_*\mathbb{Q}_\ell) = \bigoplus_{\alpha\in\mathfrak{A}} K_\alpha^n $$

where $\mathfrak{A}$ is a certain finite set of indices $\alpha$ with a distinct closed irreducible subscheme $Z_\alpha \subset S\otimes_k \bar{k}$ being attached to each $\alpha\in\mathfrak{A}$ and $K_\alpha^n$ being the sum of simple perverse sheaves of support $Z_\alpha$ occuring in ${}^p\mathrm{H}^n(f_*\mathbb{Q}_\ell)$. For all $\alpha\in\mathfrak{A}$, let denote

$$ K_\alpha = \bigoplus_n K_\alpha^n[-n]. $$

Let $x_\alpha$ be a geometric generic point of $Z_\alpha$ and let denote $S_\alpha$ the strict henselization of $S\otimes_k \bar{k}$ at $x_\alpha$. The restriction of $K_\alpha$ to $S_\alpha$ is just a graded vector space supported by the closed point $x_\alpha$ of $S_\alpha$. We have a canonical structure of graded $\Lambda_{P_{x_\alpha}}$ on $K_{\alpha,x_\alpha}$. Let

$$ 1\to R_\alpha \to P_{x_\alpha} \to A_\alpha \to 1 $$

be Chevalley canonical exact sequence attached to $P_{x_\alpha}$. For any homological lifting $T_{\mathbb{Q}_\ell}(A_\alpha)\to T_{\mathbb{Q}_\ell}(T_{P_\alpha})$, we have a homomorphism of graded algebras $\Lambda_{A_\alpha}\to\Lambda_{P_{x_\alpha}}$ and thus a structure of $\Lambda_{A_\alpha}$-module on $K_{\alpha,x_\alpha}$.

**Proposition 4** (Freeness). *For all $\alpha\in\mathfrak{A}$, for all homological lifting $T_{\mathbb{Q}_\ell}(A_\alpha)\to T_{\mathbb{Q}_\ell}(T_{P_\alpha})$, $K_{\alpha,x_\alpha}$ is a free $\Lambda_{A_\alpha}$-module.*

We can derive the $\delta$-inequality from this freeness property as follows. It amounts to prove that for all $\alpha\in\mathfrak{A}$, $\mathrm{codim}(Z_\alpha)\le\delta_{x_\alpha}$ and if equality occurs then $K^n_{\alpha,x_\alpha}[-n]\neq 0$ in highest degree occurs as a direct factor of $\mathrm{R}^{2d}f_*\mathbb{Q}_\ell[-2d]|_{S_\alpha}$. The freeness property implies that the set

$$ \{n\in\mathbb{Z}\mid K^n_{\alpha,x_\alpha}\neq 0\} $$

is an union of integral intervals of length $2(d-\delta_{x_\alpha})$. By Poincaré duality, this set is symmetric with respect to the integer $\dim(M)$. Thus there exist $n\ge\dim(M)+d-\delta_\alpha$ such that $K^n_{\alpha,x_\alpha}\neq 0$. But $K^n_{\alpha,x_\alpha}[-n]$ is a vector space supported by $x_\alpha$ and placed on the degree $n-\dim(Z_\alpha)$ so that

$$ n-\dim(Z_\alpha)\le 2d $$

which implies eventually $\mathrm{codim}(Z_\alpha)\le\delta_\alpha$. If equality occurs then the punctual sheaf $K^n_{\alpha,x_\alpha}[n-\dim(Z_\alpha)]$ is a direct summand of $\mathrm{R}^{2d}f_*\mathbb{Q}_\ell|_{S_\alpha}$. In particular, if $f$ has only irreducible fibers around $x_\alpha$, $\mathrm{R}^{2d}f_*\mathbb{Q}_\ell|_{S_\alpha}$ is the constant sheaf which can not admit the punctual sheaf as a direct summand unless $S_\alpha=\{x_\alpha\}$. In this case, we have $Z_\alpha = S\otimes_k\bar{k}$.

Before sketching the proof of the freeness property, let us have a look at the freeness property fiberwise. Let $P$ be a smooth connected commutative group over an algebraically closed field $k$ which acts on a proper scheme $M/k$ with affine stabililizer. Then $\Lambda_P$ acts on $\mathrm{H}^*(M)$. Let $1 \to R \to P \to A \to 1$ be the canonical Chevalley exact sequence. Any homological lifting $T_{\mathbb{Q}_\ell}(A) \to T_{\mathbb{Q}_\ell}(P)$ induces a homomorphism of algebras $\Lambda_A \to \Lambda_P$. Via any of such homomorphisms, $\mathrm{H}^*(M)$ is a free module over $\Lambda_A$.

If $P = A$ is an abelian variety, then $A$ acts on $M$ with finite stabilizer. The quotient $[M/A]$ is an algebraic stack which is proper over $k$. Using the degeration of Leray spectral sequence, one can prove that there exists an isomorphism

$$\mathrm{H}^*(M) = \mathrm{H}^*([M/A]) \otimes \Lambda_A$$

that makes $\mathrm{H}^*(M)$ a free $\Lambda_A$-module.

If $P$ is a smooth connected commutative group defined over a finite field, there exists a quasi-lifting $A \to P$ i.e. such that the composition $A \to P \to A$ is the multiplication by a certain non-zero integer $N$. Indeed, the group of extension $\mathrm{Ext}^1(A, R)$ is a finite group in the case where $k$ is a finite field and thus must be annihilated by a non-zero integer divisible enough. The quasi-lifting induces a canonical homological lifting $T_{\mathbb{Q}_\ell}(A) \to T_{\mathbb{Q}_\ell}(P)$ with respect to which the freeness $\mathrm{H}^*_c(M)$ on $\Lambda_A$ can be proved by reducing to the case where $P$ is an abelian variety. For an arbitrary homological lifting $T_{\mathbb{Q}_\ell}(A) \to T_{\mathbb{Q}_\ell}(P)$, the freeness can be established by the following homogeneity argument due to Deligne. Since the quasi-lifting $A \to P$ induces a canonical linear lifting $T_{\mathbb{Q}_\ell}(A) \to T_{\mathbb{Q}_\ell}(P)$, the space of all linear liftings can be identified with the $\mathbb{Q}_\ell$-vector space $\mathrm{Hom}(T_{\mathbb{Q}_\ell}(A), T_{\mathbb{Q}_\ell}(R))$. The space of all linear liftings $T_{\mathbb{Q}_\ell}(A) \to T_{\mathbb{Q}_\ell}(P)$ for which $\mathrm{H}^*(M)$ is a free module over $\Lambda_A$ is an open subset of this vector space which is stable under Frobenius and which contains 0. This open subset is then necessarily the whole vector space $\mathrm{Hom}(T_{\mathbb{Q}_\ell}(A), T_{\mathbb{Q}_\ell}(R))$. The case of an arbitrary field $k$ can be reduced to finite fields by the standard spreading out argument.

We come now to the proof of the freeness property over a strict henselian base. Consider the restriction of the summands $K_{\alpha'}$ to $S_\alpha$. Only those $K_{\alpha'}$ with $x_\alpha \in Z_{\alpha'}$ survive to this restriction. In that case, we have $\dim(Z_{\alpha'}) > \dim(Z_\alpha)$. Let us prove the proposition by a downward induction with respect to the integer $\dim(Z_\alpha)$. By induction hypothesis, we know the freeness property for $\alpha' \neq \alpha$ such that $x_\alpha \in Z_{\alpha'}$. The induction hypothesis is however concerned with a freeness property over $S_{\alpha'}$ instead of $S_\alpha$ so that we will need to migrate it to $S_\alpha$. This follows essentially from the fact that the agebra $\Lambda_{A_{\alpha'}}$ is a free module over $\Lambda_{A_\alpha}$. This in turns follows from the assumption $T_{\mathbb{Q}_\ell}(P)$ polarizable.

Except $K_\alpha|_{S_\alpha}$, all other perverse summands $K_{\alpha'}|_{S_\alpha}$ of $F_*\mathbb{Q}_\ell|_{S_\alpha}$ are now known to be free $\Lambda_{A_\alpha}$-modules. We want to prove $K_\alpha|_{S_\alpha}$ is also a free $\Lambda_{A_\alpha}$-module. For this, we look at the spectral sequence associated with the perverse filtration of $f_*\mathbb{Q}_\ell$ with respect to the fiber functor at $x_\alpha$. The spectral sequence degenerates at $E_2$ by an application of the decomposition theorem. The abutment is the cohomology of the special fiber $M_{x_\alpha}$ which is known to be a free $\Lambda_{A_\alpha}$-module. Now, except the summand of simple perverse sheaves of supported $s$, all other direct summand contribute a free $\Lambda_{A_s}$-module to the abutment. We can prove that the remaining summand also contributes a free $\Lambda_{A_s}$-module to the abutment by using a peculiar property of the ring $\Lambda_{A_\alpha}$ : its injective modules are also projective. We also have to use the fact that the contribution $K_\alpha|_{S_\alpha}$ occurs as one subquotient in a filtration of the abutment with only three layers.

## 4. Application to Hitchin fibration

Fix a proper smooth irreducible curve $X$ defined over a field $k$ and a line bundle $D$ of large degree on $X$. Let $G$ be a semi-simple group. We put semi-simple instead of reductive only for simplify the exposition. We can construct from this datum a Hitchin fibration $f : \mathcal{M} \to \mathcal{A}$ and the anisotropic part of it $f^{\rm ani} : \mathcal{M}^{\rm ani} \to \mathcal{A}^{\rm ani}$ which is a proper morphism. This construction has been recalled in [4]. In the sequel, we will write $f : \mathcal{M} \to \mathcal{A}$ instead of $f^{\rm ani} : \mathcal{M}^{\rm ani} \to \mathcal{A}^{\rm ani}$. As recalled in loc. cit. there exists a group scheme $g : \mathcal{P} \to \mathcal{A}$ which acts on $f : \mathcal{M} \to \mathcal{A}$ which satisfies the axioms of a weak abelian fibration.

When $k$ is the field of complex numbers and $D$ is the canonical sheaf of $X$, $\mathcal{M}$ is equipped with a sympectic form and $f : \mathcal{M} \to \mathcal{A}$ is an algebraic completely integrable system. In that case our weak abelian fibration is $\delta$-regular. We can in this case apply theorem 2. In order to get the most satisfying result, we need to take care of the action of sheaf $\pi_0(\mathcal{P})$ of connected components of fibers of $\mathcal{P}$ with the interpolating property $\pi_0(\mathcal{P})_a = \pi_0(\mathcal{P}_a)$ for all $a \in \mathcal{A}$. Over our anisotropic base, $\pi_0(\mathcal{P})$ is a sheaf of finite abelian groups. It is known that for every $a \in \mathcal{A}$, $\pi_0(\mathcal{P}_a)$ acts simply transitively on the set of irreducible components of $\mathcal{M}_a$. The action of $\mathcal{P}$ in the perverse sheaves ${}^p\mathrm{H}^n(f_*\mathbb{Q}_\ell)$ factors through the sheaf of connected components $\pi_0(\mathcal{P})$. Let's denote ${}^p\mathrm{H}^j(f_*\mathbb{Q}_\ell)_{st}$ the largest direct factor on which $\pi_0(\mathcal{P})$ acts trivially.

**Corollary 5.** *Under the above assumptions, if a simple perverse sheaf $K$ is a direct factor of ${}^p\mathrm{H}^j(f_*\mathbb{Q}_\ell)_{st}$ then the support of $K$ is $\mathcal{A} \otimes_k \bar{k}$.*

By theorem 2, it suffices to prove that the stable part $\mathrm{H}^{2d}(f_*\mathbb{Q}_\ell)_{st}$ of the top ordinary cohomology is the constant sheaf $\mathbb{Q}_\ell$. But this follows from the fact that for all geometric point $a \in \mathcal{A}$, $\pi_0(\mathcal{P}_a)$ acts simply transitively on the set of irreducible components of $\mathcal{M}_a$.

If $G_1$ and $G_2$ are dual semi-simple groups in Langlands' sense, or more generally having isogenous root data as in [15], generic fibers of $f_1 : \mathcal{M}_a \to \mathcal{A}_1$ and $f_2 : \mathcal{M}_2 \to \mathcal{A}_2$ are isogenous abelian schemes if we ignore the finite groups of connected components and of isomorphisms. This implies that ${}^p\mathrm{H}^j(f_{1,*}\mathbb{Q}_\ell)_{st}$ and ${}^p\mathrm{H}^j(f_{2,*}\mathbb{Q}_\ell)_{st}$ are isomorphic local system over an open part of $\mathcal{A}$. It follows from support theorem that there exists an isomorphism between those two perverse sheaves.

If $k$ is a finite fields, by counting points in corresponding fibers of $f_1$ and $f_2$ we obtain Waldspurger's non-standard variant of the fundamental lemma. However, over a finite field, we have not yet proved the $\delta$-regularity of the fibration. We can prove that after cutting off a closed subset of $\mathcal{A}^{\rm bad}$, over the remaining part $\mathcal{A}^{\rm good} = \mathcal{A} - \mathcal{A}^{\rm bad}$, $g : \mathcal{P} \to \mathcal{A}$ becomes a $\delta$-regular and moreover $\mathcal{A}^{\rm good}$ is still big enough to contain all informations about local orbital integrals.

As in [14], endoscopy theory is closely related to the sheaf $\pi_0(\mathcal{P})$ interpolating the groups of connected components of the fibers of $\mathcal{P}$. It can be useful to understand first the case of SL(2) exposed in the last section of [14]. In that case, by direct computation one sees that $\pi_0(\mathcal{P})$ is a quotient of the constant sheaf $\mathbb{Z}/2\mathbb{Z}$ which is non trivial over the locus consisting of the parameter $a \in \mathcal{A}$ coming from a parameter $a_H \in \mathcal{A}_H$ for certain one-dimensional unramified twisted torus $H$. This locus is a disjoint union of smooth irreducible subscheme of $\mathcal{B}_H$ indexed by those one-dimensional unramified twisted torus $H$. The action of $\mathbb{Z}/2\mathbb{Z}$ on ${}^p\mathrm{H}^j(f_*\mathbb{Q}_\ell)$ induces a decomposition

$$ {}^p\mathrm{H}^j(f_*\mathbb{Q}_\ell) ={}^p \mathrm{H}^j(f_*\mathbb{Q}_\ell)_+ \oplus^p \mathrm{H}^j(f_*\mathbb{Q}_\ell)_- $$

where ${}^p\mathrm{H}^j(f_*\mathbb{Q}_\ell)_+$ will be called the stable part and ${}^p\mathrm{H}^j(f_*\mathbb{Q}_\ell)_-$ is will be called the endoscopic part. It is obvious that the restriction of ${}^p\mathrm{H}^j(f_*\mathbb{Q}_\ell)_-$ on the open complement of $\bigcup \mathcal{B}_H$ is trivial. It can be proved that up to twisting ${}^p\mathrm{H}^j(f_*\mathbb{Q}_\ell)_-$ are local system supported on the union of $\mathcal{B}_H$ which are comparable with the the cohomology of Hitchin fibration for those tori.

In the general case, it is agreeable to base change to certain étale covering $\tilde{\mathcal{A}} \to \mathcal{A}$, see [4, 5.1] so that over $\tilde{\mathcal{A}}$, we have a canonical surjective homomorphism $\mathbf{X}_* \to \pi_0(\mathcal{P})$ from the constant sheaf of values the group $\mathbf{X}_*$ of cocharacters of a fixed maximal torus $T$. For every character of finite order $\kappa : \mathbf{X}_* \to {\mathbb{Q}_\ell}^\times$, the locus $\tilde{a} \in \tilde{\mathcal{A}}$ such that $\kappa$ factor through $\mathbf{X}_* \to \pi_0(\mathcal{P}_a)$ can be identified canonically with disjoint union of $\tilde{\mathcal{A}}_H$ for a certain unramified endoscopic groups $H$ attached to $\kappa$. We have a finite action of $\mathbf{X}_*$ on ${}^p\mathrm{H}^j(f_*\mathbb{Q}_\ell)$ which induces a direct decomposition following characters of finite order. The $\kappa$-part ${}^p\mathrm{H}^j(f_*\mathbb{Q}_\ell)_\kappa$ have a trivial restriction to the open complement of the disjoint union $\bigcup_H \tilde{\mathcal{A}}_H$ over the finite set of unramified endoscopic groups attached to $\kappa$.

Over a base field $k$ characteristic zero, since $\delta$-regularity is available, the support theorem 2 implies :

**Corollary 6.** *Any simple perverse constituent of* ${}^p\mathrm{H}^j(f_*\mathbb{Q}_\ell)_\kappa$ *of* $f_*\mathbb{Q}_\ell|_{\tilde{\mathcal{A}}}$ *must have support* $\tilde{\mathcal{A}}_H$ *where* $H$ *is one of the unramified endoscopic groups attached to* $\kappa$ *.*

This makes possible a comparaison between the restriction of ${}^p\mathrm{H}^j(f_*\mathbb{Q}_\ell)_\kappa$ to $\tilde{\mathcal{A}}_H$ and the stable part of the cohomology of Hitchin fibration for $H$.

Over a base field $k$ of positive characteristic, we will have to throw away a closed subset of $\tilde{\mathcal{A}}_H$ which might violate the $\delta$-regularity before applying the $\delta$-inequality 3. We obtain in that way [4, 5.5]. As in the non-standard case, the remaining part $\tilde{\mathcal{A}}_H^{\mathrm{good}}$ of $\tilde{\mathcal{A}}_H$ still contain all informations about local orbital integrals. Thus after completing the exercices of counting points on Hitchin fibers and affine Springer fibers, as well as the approximation of local conjugacy class by global one, we obtain a proof of the fundamental lemma for Lie algebra which was conjectures by Langlands and Shelstad in positive characteristic case. In vertu of theorem of Waldspurger [16], and of Cluckers-Hales-Loeser [2], the fundamental lemma for $p$-adic fields is also proved at least large prime $p$. By Hales, we know also the fundamental lemma for $p$-adic fields for all $p$.

## References


[1] Beilinson A., Bernstein J., Deligne P.: Faisceaux pervers. *Astérisque* 100 (1982).
[2] Cluckers R., Hales T., Loeser F.. Chapter in this volume.
[3] Dat, J.-F., Lemme fondamental et endoscopie, une approche géométrique, *Séminaire Bourbaki* 940 novembre 2004.
[4] Dat, J.F., Ngo Dac Tuan, Chapter in this volume.
[5] Deligne P. La conjecture de Weil II. *Publ. Math. de l'I.H.E.S.* 52 (1980) 137–252.
[6] Fantechi B., Göttsche L., Van Straten D. : Euler number of the compactified Jacobian and multiplicity of rational curves. *J. Alg. Geom.* 8 (1999), 115-133.

[7] Goresky M., Kottwitz R., MacPherson R.: Homology of affine Springer fiber in the unramified case. *Duke Math. J.* 121 (2004) 509–561.

[8] Goresky M., Kottwitz R., MacPherson R.: Purity of equivalued affine Springer fibers. *Represent. Theory* 10 (2006), 130–146.

[9] Hales T. On the fundamental lemma for standard endoscopy: reduction to unit elements. *Canad. J. Math.* 47 (1995), no. 5, 974–994.

[10] Hitchin N.: Stable bundles and integrable connections. *Duke Math. J.* 54 (1987) 91–114.

[11] Langlands R. *Les débuts d'une formule des traces stables.* Publications de l'Université Paris 7, 13 (1983).

[12] Laumon, G. Transformation de Fourier, constantes d'quations fonctionnelles et conjecture de Weil. *Inst. Hautes tudes Sci. Publ. Math.* No. 65 (1987), 131–210.

[13] Lusztig, G. *Intersection cohomology methods in representation theory.* Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 155–174

[14] Ngô B.C. Fibration de Hitchin et endoscopie. *Inv. Math.* 164 (2006) 399–453.

[15] Ngô B.C. Le lemme fondamental pour les algèbres de Lie. Preprint arXiv.

[16] Waldspurger J.-L. Endoscopie et changement de caractéristique, prépublication.