Unipotent radicals of differential Galois group and integrals of solutions of inhomogeneous equations

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The reader is doubtless familiar with the idea of integration as the operation inverse to differentiation; and he is equally well aware that the integral (in this sense) is not always expressible in terms of elementary functions.
(Whittaker-Watson, A course of modern analysis, Chapter IV)

Résumé : Cet article fait suite à [2], où nous étudions le radical unipotent $R_u(G)$ du groupe de Galois différentiel d'un produit de deux opérateurs différentiels irréductibles. Nous passons ici au cas général, où nous donnons divers critères pour que ce radical unipotent soit 'aussi gros que possible'. À l'inverse, nous explicitons une situation fortement dégénérée, qui apparaît en présence d'opérateurs symétriquement autoduaux, dès que la filtration naturelle de $R_u(G)$ possède au moins deux crans; c'est l'analogue différentiel des dégénérescences mises en évidence par K. Ribet [12] dans l'étude des 1-motifs.

Let (K, ∂) be a differential field of characteristic 0 , with an algebraically closed constant field C , let $D_K = K[\partial]$ be the ring of differential operators with coefficients in K , and let N be an element of D_K . One of the motivations of the present paper is the following problem. Denote by V_N the C-vector-space of solutions of the equation $Ny = 0$ in a Picard-Vessiot extension K_N/K of N, and view the elements of K_N as 'known' (or, extending Whittaker and Watson's parlance, as 'elementary functions'). Given an element f in V_N , we then look for algebraic conditions on N and f deciding whether the integral $F = \partial^{-1} f := \int f$ of f , which is well defined up to addition of an element of C , still belongs to the field K_N, or whether it defines a 'new transcendent'. (It is easily seen that if F does not belong to K_N, then it is in fact truly transcendental over K_N.)

As recalled in the first part of the paper, the answer to this question follows from the study of the unipotent radical $R_u(G_M)$ of the differential Galois group $G_M = Gal_\partial(K_M/K)$

of the equation

$$My = 0, \text{where } M = N\partial.$$

This unipotent radical is easy to describe (cf. [1], [2], [3], and §1.2, Theorem 1.1 below) when one can write

$$M = L_2 L_1$$

as the product of two completely reducible operators L_2, L_1 (recall that an operator is *completely reducible* if it is the LCLM of irreducible elements of D_K). Criteria of a *linear* nature, involving the adjoint $\check{N}$ of the operator N, can then be derived to solve our problem. For instance (cf. [2]), when N itself is irreducible (in which case we can take $L_2 = N, L_1 = \partial$), the mere condition $1 \notin \check{N}(K)$ ensures that $\int f$ is transcendental over K_N for any non zero $f \in V_N$ (while these integrals are easily seen to belong to K_N if $1 \in \check{N}(K)$ [1]). This was generalized by Berman and Singer [1], whose work shows that if N is completely reducible, and $F = \int f$ is not transcendental over K_N, then N admits a right divisor N' such that $1 \in \check{N}'(K)$ and $N'f = 0$; cf. §1.3, Corollary 1.1.

In the second part of the paper, we turn to the study of a general differential operator M, given as a product

$$M = L_t L_{t-1} ... L_1$$

of $t \geq 3$ completely reducible operators. The unipotent radical of G_M , which identifies with the relative differential Galois group

$$Gal_\partial(K_M / K_{L_1} \ldots K_{L_t}) = R_u(G_M) \ ,$$

is then endowed with a natural filtration, and we give necessary and sufficient conditions for $R_u(G_M)$ to be 'as big as possible' with respect to this filtration (cf. §2.1, Theorem 2.1). This extends one aspect of the work of K. Boussel ([4], Théorème 2.3.1.a) on reducible hypergeometric equations, where $t = 3$. From then on, we assume that

$$M = L_3 L_2 L_1$$

has three completely reducible factors, and we concentrate on $Gal_\partial(K_M / K_{L_3 L_2} . K_{L_2 L_1})$, i.e. on the *second* step of the filtration (cf. §2.2, Theorem 2.2). In the same vein as above,

[1] The Picard-Vessiot extensions of adjoint operators coincide. Now, $K_N(\int f)$ lies in K_M, while the Picard-Vessiot extension of $\check{M} = -\partial \check{N}$ is differentially generated over $K_{\check{N}} = K_N$ by any solution of $\check{N} y = 1$; in particular, K_M / K_N depends only on the class of 1 in the cokernel $K / \check{N}(K)$ of $\check{N}$, and $K_M = K_N$ if this class vanishes.

we deduce a sufficient condition on $N = L_3 L_2$, which is still of a linear nature, for the integrals $\int f$ of the solutions of $Nf = 0, L_2 f \neq 0$ to define 'new transcendents' ; cf §2.2, Corollary 2.1.

However, as already noted in [4] in the hypergeometric case, degeneracies of an unexpected kind may occur on $R_u(G_M)$ as soon as $t = 3$, and in fact, the following elementary example makes it clear that such 'linear' criteria as those of §1 will not hold in general if N is not completely reducible. Take $K = C(z), \partial = d/dz, N = (\partial + \frac{2}{z})(\partial + \frac{1}{z})$ (so that $M = N\partial = (\partial + \frac{2}{z})(\partial + \frac{1}{z})\partial$ admits no decomposition in less than three completely reducible factors), and consider the solution $f = \frac{1}{z}\ell nz \in V_N$; then, we do have $\int f = \frac{1}{2}(\ell nz)^2 \in K_N$, although 1 does not lie in $(\partial - \frac{1}{z})(K)$ -and a fortiori not in $\check{N}(K)$.

In order to give a complete, but still manageable, description of what can happen, we restrict in the third part of the paper to the situation where L_2 is irreducible and L_3, L_1 are equivalent to ∂. For instance, going back to our problem on integrals $\int f$, we take

i) $L_1 = \partial$, $L_2 :=$ any irreducible operator, say L, and $L_3 = \partial_a := a\partial a^{-1} = \partial - \partial(a)/a$ for some non-zero element a of K.

Based on the results of §1 , and considering the adjoint of $M = \partial_a L \partial$ if necessary, we may then suppose that

ii) neither $N = \partial_a L$ nor $L\partial$ is completely reducible, i.e. that $a \notin L(K), 1 \notin \check{L}(K)$;

iii) restrict to solutions f of $Ny = 0$ such that $L(f) \neq 0$, and thus to the case $L(f) = a$ (whence the title of this paper).

From the exhaustive description of $R_u(G_M)$ given in §3, Theorems 3.1 and 3.2 (and summarized by Theorem 3 below), we derive a criterion of a *quadratic* nature, as follows.

Corollary 3 : *assume that $N = \partial_a L$ and $f \in V_N$ satisfy hypotheses i) , ii) , iii) above. Then either $F = \int f$ is transcendental over K_N, or it belongs to K_N. The latter case occurs if and only if there exists an element T in $K[\partial]$ such that $\check{L}\check{T} = -TL$ and $T(a) \equiv 1$ mod. $\check{N}(K)$; furthermore, if $T(a) = 1$, one has up to an additive constant: $F = \frac{1}{2}\{f, f\}_{TL}$.*

[Here, $\{,\}_{TL}$ denotes the bilinear concomitant attached to the element TL of $K[\partial]$; cf. [7], 5.3 . Its expression makes it clear that $\frac{1}{2}\{f, f\}_{TL} \in K_N$ - so that F does belong to K_N as soon as 1 is congruent to $T(a)$ mod. $\check{N}(K)$, in view of footnote (1) of the previous page.]

The first condition on T in the conclusion of Corollary 3 means that L is 'symmetrically autodual', while the second one implies that $M = \partial_a L \partial$ itself is autodual. The situation can thus be compared with the one encountered by K. Ribet ([12]; see also [8]) in the study of the Galois representations attached to 'deficient' 1-motives, which has indeed been the inspiration for the present work.

Our general description of $R_u(G_M)$, which pursues this analogy further, requires a case by case analysis. Denoting by $\sim$ the standard equivalence of operators in D_K, we shall prove in §3 :

Theorem 3 : *Let L be an irreducible element of order n of D_K, and let a, b be non-zero elements of K. Assume that $a \notin L(K), 1/b \notin \check{L}(K)$, and let $R_u(G_M)$ be the unipotent radical of the differential Galois group of $M = \partial_a L \partial_b$. Then :*

i) in the generic case where L is not autodual, $R_u(G_M)$ fills up the unipotent radical of the parabolic subgroup of $Aut(V_M)$ attached to the flag $0 \subset V_{\partial_b} \subset V_{L\partial_b} \subset V_M$; in particular, $dim(R_u(G_M)) = 2n + 1$;

ii) the same conclusion holds if L is autodual but $\partial_a L \not\sim \partial_{1/b}\check{L}$;

iii) if $\partial_a L \sim \partial_{1/b}\check{L}$ and L is antisymmetrically autodual, $R_u(G_M)$ is isomorphic to the simply connected form of the Heisenberg group attached to a pair of lagrangians of V_L; in particular, $dim(R_u(G_M)) = n + 1$;

iv.α) if $\partial_a L \sim \partial_{1/b}\check{L}$, L is symmetrically autodual, and M is not *autodual, $R_u(G_M)$ is isomorphic to the abelian group $V_L \oplus Hom(V_{\partial_b}, V_{\partial_a}) \simeq V_L \oplus C$; in particular, $dim(R_u(G_M))$ $= n + 1$;*

iv.β) if $\partial_a L \sim \partial_{1/b}\check{L}$, L is symmetrically autodual, and M is *autodual, $R_u(G_M)$ is isomorphic to the abelian group V_L; in particular, $dim(R_u(G_M)) = n$.*

Corollary 3 above corresponds to Case *iv.β* of this theorem. But note that Theorem 3 also contains a criterion of abelianity for the relative Galois group $R_u(G_M)$. It would be interesting to relate this aspect of our result to the recent theorem of Moralez-Ruiz and Ramis [11] on completely integrable Hamiltonian systems.

Finally, we like to point out that the computation of the unipotent radical of the Galois group of general products $M = L_t L_{t-1} \ldots L_1$ with which Section 2 is concerned may be viewed as a natural generalization of the study of the monodromy attached to Chen's iterated integrals. For instance, the k-th polylogarithm function $Li_k(z) = \int_{0 \leq t_1 \leq t_2 \ldots \leq t_k \leq z} \frac{dt_k}{t_k} \cdots \frac{dt_2}{t_2} \frac{dt_1}{(1-t_1)}$ satisfies $My = 0$, where

$$M = (\theta - \frac{1}{1-z})\theta^k \ , \ \theta = zd/dz,$$

and the methods of the present paper provide an easy way to compute the Zariski closure of its (well-known) monodromy group (cf. §2.2, Remark 2.3). The iterated integrals of Drinfeld type occurring in the study of Zagier's multi-zeta values would give further applications of this point of view.

Remark. In a sense, the main tools in the present study are Lemma 2.1 on subalgebras of nilpotent Lie algebras, Lemma 2.3 on subrepresentations in an extension, and Lemma 3.3 on the signs expressing the type of autoduality of differential operators. Although easy, they may present an independent interest, and have therefore been stated in a self-contained way.

1. Review of the completely reducible case.

In this section, we recall the cohomological method introduced in [2] to study the product of two differential operators, and rephrase in a more compact way the main results of [1], [2], [3].

1.1.The general setting.

To smooth up the translation between D_K-modules and differential equations, we shall use the following 'contragredient' notations.

(**N.1**) Boldface letters stand for (left) D_K-modules (of finite dimension over K). Thus, $\mathbf{1} = \mathbf{1}_K$ denotes K , with its standard D_K-action. For any $N \in D_K$, we set $\mathbf{V}_N = Hom_K(D_K/D_K N, \mathbf{1})$ endowed with its natural action of D_K . There then exist canonical C-isomorphisms between the space V_N of solutions of $Ny = O$ in K_N and the space $\mathbf{V}_N^\partial$ of horizontal vectors of $\mathbf{V}_N \otimes K_N$ on the one hand, and between the cokernels $K/N(K)$ and $\mathbf{V}_N(K)/\partial\mathbf{V}_N(K)$ on the other hand (cf. [2], p. 128) . The choice of the generator ∂ of D_K being fixed, there also exists a natural D_K-isomophism between $\mathbf{V}_N$ and $D_K/D_K\check{N}$, where $\check{N} = \sum_{i=0,\dots,n}(-1)^i\partial^i a_i$ denotes the adjoint of $N = \sum_{i=0,\dots,n} a_i\partial^i$ (cf. [9], 1.5.3).

(**N.2**) *Extensions.* For $P, Q \in D_K$, the D_K-module $Hom(\mathbf{V}_P, \mathbf{V}_Q)$ is isomorphic to $\mathbf{V}_Q \otimes \mathbf{V}_P^*$, where * denotes K- duals; its space of horizontal vectors identifies with $Hom(V_P, V_Q)$. We attach two extensions of D_K-modules to the decomposition $N = PQ$ in D_K. The 'standard' one, an element of $Ext(\mathbf{V}_P, \mathbf{V}_Q)$, is $\mathbf{V}_N$ itself, viewed (with a slight abuse of notations) as the exact sequence :

$$\mathbf{V}_{P\circ Q} \quad : \quad 0 \to \mathbf{V}_Q \to \mathbf{V}_N \to \mathbf{V}_P \to 0$$

deduced by transposition from multiplication to the right by $Q\ :\ D_K/D_K P \to D_K/D_K N$. Restricted to horizontal vectors, this induces on spaces of solutions the standard exact sequence

$$O \to V_Q \to V_N \to V_P \to O,$$

where the third arrow is given by the natural action of $Q \in D_K$ on K_N :

$$m_Q : f \to m_Q(f) = Q(f).$$

The second extension, denoted by $\mathbf{E}_N$ (or more properly $\mathbf{E}_{PoQ}$), lives in $Ext(\mathbf{1}, \mathbf{V}_Q \otimes \mathbf{V}_P^*)$: it is the pull-back, by the canonical map from $\mathbf{1}$ to $\mathbf{V}_P \otimes \mathbf{V}_P^* \simeq End(\mathbf{V}_P)$: $1 \mapsto id_{\mathbf{V}_P}$, of the extension deduced from $\mathbf{V}_N$ by tensorization by $\mathbf{V}_P^*$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{V}_Q \otimes \mathbf{V}_P^* & \longrightarrow & \mathbf{E}_N & \longrightarrow & \mathbf{1} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{V}_Q \otimes \mathbf{V}_P^* & \longrightarrow & \mathbf{V}_N \otimes \mathbf{V}_P^* & \longrightarrow & \mathbf{V}_P \otimes \mathbf{V}_P^* & \longrightarrow & 0
\end{array} \quad .$$

See [2] for a description of $\mathbf{E}_N$ as an element of the cokernel of ∂ in $Hom(\mathbf{V}_P, \mathbf{V}_Q)(K)$, and [1] for its matrix translation.

(**N.3**) *Splittings* : Let K'/K be a differential field extension. We say that a D_K-module $\mathbf{V}$ is *solvable* over K' if the $D_{K'}$-module $\mathbf{V} \otimes K' := \mathbf{V}_{K'}$ is isomorphic to a direct sum of copies of $\mathbf{1}_{K'}$. We say that a D_K-module extension $\mathbf{E}$ of $\mathbf{V}$ by some $\mathbf{W}$ *splits* over K' if the $D_{K'}$-module extension $\mathbf{E}_{K'}$ is isomorphic to the direct sum $\mathbf{V}_{K'} \oplus \mathbf{W}_{K'}$. We usually drop the reference to K' when $K' = K$. For instance, if $P = \partial_a := a\partial a^{-1}$ for some $a \in K$ and $L \in D_K$, the extension $\mathbf{V}_{\partial_a oL}$ of $\mathbf{1}_K$ by $\mathbf{V}_L$ splits if and only if a lies in $L(K)$ [2].

The following less standard notions will also be useful. We say that a decomposition $N = PQ$ is *totally unsplit* if no non-trivial pushout of the extension $\mathbf{E}_{PoQ}$ splits. We say that $N = PQ$ is *right unsplit* if no non-trivial pushout of the extension $\mathbf{V}_{PoQ}$ splits, or equivalently, if whenever a decomposition $Q = Q''Q'$ is given with $ord(Q'') \geq 1$, the extension $\mathbf{V}_{PoQ''}$ does not split. Similarly, we say that $N = PQ$ is *left unsplit* if no pullback of the extension $\mathbf{V}_{PoQ}$ splits, or equivalently, if whenever a decomposition $P = P''P'$ is given with $ord(P') \geq 1$, the extension $\mathbf{V}_{P'oQ}$ does not split. Note that PQ may be unsplit both to the right and to the left without being totally unsplit.

Let $N = PQ$ be the product of two arbitrary operators in $D_K = K[\partial]$. The Picard-Vessiot extension K_N contains the compositum $K_{P,Q} = K_P.K_Q$ of the Picard-Vessiot extensions of K_P and K_Q, and the map ξ_N (or, more properly ξ_{PoQ}, to indicate its dependence on the decomposition we start with):

$$\sigma \mapsto \xi_N(\sigma), \text{where } \xi_N(\sigma) : \{(f \text{ mod. } V_Q) \in V_N/V_Q \simeq V_P\} \mapsto \{\sigma f - f \in V_Q\}$$

[2] More generally, the association $a \mapsto \mathbf{V}_{\partial_a oL}$ induces a canonical isomorphism $K/L(K) \simeq Ext(\mathbf{1}_K, \mathbf{V}_L)$, cf [2], [5].

defines a natural embedding of the relative differential Galois group

$$U = Gal_\partial(K_N/K_{P,Q})$$

into the vectorial group $Hom(V_P, V_Q)$. Furthermore, a cohomological argument as in [2] (or a Kummer-type computation, cf. [3]) shows that it is $G_{P,Q}$-invariant, where

$$G_{P,Q} := Gal_\partial(K_{P,Q}/K)$$

acts on the abelian normal subgroup U of $G_N = Gal_\partial(K_N/K)$ by conjugation , and naturally on $Hom(V_P, V_Q)$. In particular, U identifies through ξ_N with a $G_{P,Q}$-invariant subspace of $Hom(V_P, V_Q)$, and hence with the full space $\mathbf{W}''^\partial$ of horizontal vectors of a D_K-submodule $\mathbf{W}''$ of

$$\mathbf{W} = Hom(\mathbf{V}_P, \mathbf{V}_Q) \simeq \mathbf{V}_Q \otimes \mathbf{V}_P^*$$

(for which we recall that $\mathbf{W}^\partial = Hom(V_P, V_Q)$).

Obvious constraints restrict the size of $\mathbf{W}''$, which we now describe in the setting of (**N.2**); see below for a concrete description in two useful cases. First notice that since it contains K_P and K_Q, the Picard-Vessiot extension K_N may alternatively be described as the smallest extension of $K_{P,Q}$ over which the D_K-module extension $\mathbf{V}_N$ splits. It therefore contains the smallest extension of $K_{P,Q}$ over which the D_K-module extension $\mathbf{E}_N$ splits, and in fact, these fields coincides, since already over K_P, $\mathbf{V}_N \otimes K_P$ occurs as a quotient of $\mathbf{E}_N \otimes K_P$. Consider now the (non-empty) set $\mathcal{E}$ of D_K-submodules $\mathbf{W}_1$ of $\mathbf{W}$ such that the extension $\mathbf{E}_N/\mathbf{W}_1$, viewed as the push-out of $\mathbf{E}_N$ by the natural projection from $\mathbf{W}$ to $\mathbf{W}/\mathbf{W_1}$, splits over K . Then, the image of the relative Galois group U under the corresponding projection $\mathbf{W}^\partial \to (\mathbf{W}/\mathbf{W}_1)^\partial \simeq \mathbf{W}^\partial/\mathbf{W}_1^\partial$ vanishes. In conclusion, $U = \mathbf{W}''^\partial$, hence $\mathbf{W}''$ itself, lies in the D_K-module $\mathbf{W}' = \bigcap_{\mathbf{W}_1 \in \mathcal{E}} \mathbf{W}_1$. (Beware, however, that $\mathbf{W}'$ will usually not belong to $\mathcal{E}$: one *cannot* in general speak of the 'smallest' D_K-submodule $\mathbf{W}_1$ of $\mathbf{W}$ such that the extension $\mathbf{E}_N/\mathbf{W}_1$ splits). On the other hand, the push-out $\mathbf{E}'' = \mathbf{E}_N/\mathbf{W}''$ of $\mathbf{E}_N$ by the projection $\mathbf{W} \to \mathbf{W}/\mathbf{W}''$ has trivial relative Galois group, so that the corresponding D_K-module is solvable over $K_{P,Q}$ and we may view the exact sequence of C-vector-spaces

$$0 \to \mathbf{W}^\partial/\mathbf{W}''^\partial \to \mathbf{E}''^\partial \to \mathbf{1}^\partial \to 0$$

as an extension of representations of the algebraic group $G_{P,Q}/C$.

1.2 The completely reducible case.

Assume now that P and Q are completely reducible operators. Then, $G_{P,Q}$ is a reductive group, any of its representations W satisfy

$$H^1(G_{P,Q}, W) = 0 \ ,$$

and in particular, the extension (of $G_{P,Q}$-representations) $\mathbf{E}''^{\partial}$ splits (cf. [2]). Therefore, the (D_K-module) extension $\mathbf{E}''$ splits over K, and $\mathbf{W}''$ belongs to the set $\mathcal{E}$. Hence, $\mathbf{W}'$ lies inside $\mathbf{W}''$, and we finally derive the equality $\mathbf{W}' = \mathbf{W}''$. Furthermore (cf. [1]), the complete reducibility of $\mathbf{W}$ now allows us to speak of the smallest element of the set $\mathcal{E}$. We can therefore state :

Theorem 1.1 : *let P, Q be completely reducible operators in D_K, and let $\mathbf{E}_N$ be the extension of $\mathbf{1}$ by $\mathbf{W} = Hom(\mathbf{V}_P, \mathbf{V}_Q)$ defined by the product $N = PQ$. Then, $U = Gal_\partial(K_N/K_P K_Q)$ is isomorphic to $\mathbf{W}'^{\partial}$, where $\mathbf{W}'$ is the smallest D_K-submodule of $\mathbf{W}$ such that the extension $\mathbf{E}_N/\mathbf{W}'$ of $\mathbf{1}$ by $\mathbf{W}/\mathbf{W}'$ splits over K . In particular, the relative Galois group $Gal_\partial(K_N/K_P K_Q)$ fills up $Hom(V_P, V_Q)$ if and only if $N = PQ$ is totally unsplit.*

Remark 1.1. When P and Q are completely reducible, the quotient $G_{P,Q}$ of G_N by its (vectorial) normal subgroup U is reductive, so that as pointed out in [1], U is the unipotent radical $R_u(G_N)$ of G_N and in particular, $R_u(G_N)$ is abelian.

Theorem 1.1 generalizes Proposition 2.1 of [1] and Corollaire 1 of [3]. As we shall need them later in the paper, we now show how to recover these statements from Theorem 1.1.

i) (cf. [1], §1) : Assume that P is completely reducible and that $Q = \partial_b = \partial - \partial(b)/b$ for some non-zero b in K, i.e. that $\mathbf{V}_Q$ is isomorphic to $\mathbf{1}$. Then $\mathbf{W}^{\partial} = Hom(V_P, V_Q) = Hom(V_P, C)$ can be identified with the vector-space $V_{\check{P}}$ of solutions of the adjoint of P, $\mathbf{W}'$ corresponds to a right factor $\check{P}''$ of $\check{P}$, i.e. to a decomposition $P = P''P'$, the extension $\mathbf{E}_N$ of $\mathbf{1} \simeq \mathbf{V}_{\check{Q}}$ by $\mathbf{W} \simeq \mathbf{V}_{\check{P}}$ is isomorphic to the standard extension $\mathbf{V}_{\check{N}} = \mathbf{V}_{\check{Q}\text{o}\check{P}}$ (cf. **N.2**) attached to the decomposition $\check{N} = \check{Q}\check{P}$, and under this isomorphism, its push-out to $\mathbf{W}/\mathbf{W}'$ becomes $\mathbf{V}_{\check{Q}\check{P}'\check{P}''}/\mathbf{V}_{\check{P}''}$, which is the standard extension $\mathbf{V}_{\check{Q}\check{P}'}$ attached to the product $\check{Q}\check{P}'$. Thus, the condition defining $\mathbf{W}'$ in the set $\mathcal{E}$ is equivalent to requiring that P'' be the left factor of minimal order of P such that if $P = P''P'$, then $P'\partial_b$ is completely reducible. We shall call this decomposition (which is the dual of one given by Berman-Singer in [1]) the *maximal decomposition* of P with respect to right multiplication by ∂_b. The relative Galois group $U = Gal_\partial(K_{P\partial_b}/K_P)$ is then isomorphic to the subspace $Hom(V_{P''}, C)$ of $Hom(V_P, C)$.

ii) (cf. [3], §2) Assume that P is irreducible, and that Q is a completely reducible operator of order $q \geq 1$, with trivial factors, i.e. such that $\mathbf{V}_Q$ is isomorphic to the direct sum of q copies of $\mathbf{1}$. The push-outs of the standard extension $\mathbf{V}_N \in Ext(\mathbf{V}_P, \mathbf{V}_Q)$ by each of the corresponding projections from $\mathbf{V}_Q$ to $\mathbf{1}$ are the standard extensions of $\mathbf{V}_P$ by $\mathbf{1}$ associated to products $P\partial_{b_1}, \ldots, P\partial_{b_q}$, for some non-zero elements $b_1, \ldots, b_q$ of K, and the relative differential Galois group $U = Gal(K_{PQ}/K_P)$ may be viewed as a subspace of $Hom(V_P, C^q)$. View similarly $\mathbf{E}_N$ as an extension of $\mathbf{1}$ by $\mathbf{W} = (\mathbf{V}_{\check{P}})^q$. By Schur's lemma (see [3], proof of Corollaire 1), the smallest D_K-submodule $\mathbf{W}'$ of $\mathbf{W}$ such that the extension $\mathbf{E}_N/\mathbf{W}'$ splits over K is the set of vectors $(v_1, \ldots, v_q) \in (\mathbf{V}_{\check{P}})^q$ satisfying : $c_1 v_1 + \ldots + c_q v_q = 0$ for all $(c_1, \ldots, c_q) \in C^q$ such that the equation $\check{P}(y) = c_1/b_1 + \ldots + c_q/b_q$ has a solution in K , and $U = Gal_\partial(K_{PQ}/K_P)$ is defined by the 'same' equations in $(V_{\check{P}})^q$.

1.3 Integrals of solutions of homogeneous equations.

Let us finally turn to the problem raised in the introduction to the paper. Let thus $N \in D_K$, and let F be an integral of a solution f of the equation $Ny = 0$, i.e. a solution of $MF = 0$, with $M = N\partial$, and $\partial F = f$. For any solution Φ of $My = 0$, the transcendence degree of the differential field $K_N < \Phi >= K_N(\Phi)$ over K_N is equal to the dimension of the orbit of Φ under the action of the relative Galois group $U = Gal_\partial(K_N(\Phi)/K_N)$, which, via the canonical identification $\xi_{N \circ \partial}$ of U with a subspace of $Hom(V_N, V_\partial) = Hom(V_N, C)$, is the same as the orbit of $\partial\Phi \in V_N$ under $\xi_{N\circ\partial}(U)$. Since this is a vector space in C , this orbit cannot be finite, so that as stated in the introduction to this paper, F either belongs to K_N or is transcendental over K_N .

Assume now that as in the first illustration of Theorem 1.1 above (cf. §1.2(i)), N is completely reducible, and let $N = N''N'$ be the maximal decomposition of N with respect to right multiplication by ∂ . Since $N'\partial$ is completely reducible, all solutions of $N'\partial y = 0$ lie in $K_{N'}$, and in particular, $\int f \in K_N$ as soon as $f \in V_{N'} \subset V_N$. On the other hand, if f is an element of V_N such that $N'f \neq 0$, i.e. such that the projection of f on $V_N/V_{N'} \simeq V_{N''}$ is not zero, then $Gal(K_{N\partial}/K_N) \simeq Hom(V_{N''}, C)$ acts non trivially on it and $\int f$ cannot belong to K_N . More generally (cf. [2], proof of Théorème 2), we can state :

Corollary 1.1 : *let $N = N''N'$ be the maximal decomposition of N w.r.t. right multiplication by ∂ of a completely reducible operator N , and let f be a solution of $Ny = 0$. Then $F = \int f$ belongs to K_N if $N'f = 0$, and is otherwise transcendental over K_N. More generally, given $f_1, \ldots, f_r$ in V_N, the functions $\int f_1, \ldots, \int f_r$ are algebraically dependent over K_N if and only if $N'f_1, \ldots, N'f_r$ are linearly dependent over C.*

As for the second illustration of Theorem 1.1, we encounter it (with $q = 2$) when simultaneously studying the integrals F, Φ of solutions f, ϕ of $Ly = 0, \Lambda y = 0$, where L and Λ are equivalent operators. Write S, T for elements of D_K providing such an equivalence, i.e. such that $\Lambda S = TL$, where S and L have no common right divisors. For any differential field extension K'/K , we then have a commutative diagram :

$$\begin{array}{ccccccc} V_L \cap K' & \longrightarrow & K' & \xrightarrow{L} & K' & \longrightarrow & K'/L(K') \\ \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T} \\ V_\Lambda \cap K' & \longrightarrow & K' & \xrightarrow{\Lambda} & K' & \longrightarrow & K'/\Lambda(K') \end{array}$$

where the first vertical arrow is the isomorphism induced by m_S on spaces of solutions in K', and the last one is the isomorphism on cokernels induced by m_T (cf. [1], 2.4). Let now $0 \neq b' \in K$ such that $0 \neq \check{S}(\frac{1}{b'}) := \frac{1}{b}$. Looking at adjoints, we find an operator S', not divisible to the right by ∂_b, such that $S\partial_b = \partial_{b'} S'$. Hence, $TL\partial_b = \Lambda\partial_{b'} S'$; moreover, S' and $L\partial_b$ have no common divisor to the right, since a solution y of both would satisfy $\partial_b y = 0$ (by the coprimality of S and L), contradicting that of S' and ∂_b. Thus, left multiplication by S' gives Galois equivariant isomorphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_{\partial_b} \simeq Cb & \longrightarrow & V_{L\partial_b} & \longrightarrow & V_L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle S'} & & \downarrow{\scriptstyle S'} & & \downarrow{\scriptstyle S} & & \\ 0 & \longrightarrow & V_{\partial_{b'}} \simeq Cb' & \longrightarrow & V_{\Lambda\partial_{b'}} & \longrightarrow & V_\Lambda & \longrightarrow & 0. \end{array}$$

The hypothesis $\check{S}(\frac{1}{b'}) \neq 0$ holds whenever $\frac{1}{b'} \notin \check{\Lambda}(K)$. Indeed, $\check{S}$ induces an isomorphism from the cokernel of $\check{\Lambda}$ in K to that of $\check{L}$. In view of the last statement of §1.2, case (ii), we therefore have:

Corollary 1.2 : *let L, Λ be two equivalent irreducible operators, with $TL = \Lambda S$ as above, let b', β be non zero elements of K, such that the classes of $\frac{1}{b'}$ and $\frac{1}{\beta}$ in $K/\check{\Lambda}(K)$ are linearly independent over C , and set $\frac{1}{b} = \check{S}(\frac{1}{b'})$. For any non zero solutions f, ϕ of $Ly = 0, \Lambda y = 0$, the functions $\int(\frac{1}{b}f)$ and $\int(\frac{1}{\beta}\phi)$ are algebraically independent over $K_L = K_\Lambda$.*

§2. Analysis of the general case.

2.1. The canonical filtrations.

Given a differential operator M *and a decomposition* of M as a product

$$M = L_t \dots L_1$$

of completely reducible operators, the dévissage of the differential Galois group G_M given in §1.1 generalizes as follows. The decomposition of M provides a G_M-equivariant filtration

$$Fil(V_M) \qquad 0 := V_0 \subset V_{L_1} := V_1 \subset V_{L_2 L_1} := V_2 \subset \ldots \subset V_{L_t \ldots L_1} := V_t = V_M$$

of its space of solutions, with $V_{L_i} \simeq V_i/V_{i-1}$ as semi-simple i-th quotient $(i = 1, \ldots, t)$, and $V_i/V_{i-k} \simeq V_{L_i \ldots L_{i-k+1}}$ for $k = 1, \ldots, i$. This induces a natural filtration $G_M^\bullet$ of normal subgroups $G_M^0 = G_M \supset G_M^1 \supset \ldots \supset G_M^{t-1} \supset G_M^t = 1$, with k-th step $(k = 0, \ldots, t)$:

$$\begin{aligned} G_M^k &= \{\sigma \in G_M, (\sigma - 1)V_i \subset V_{i-k} \ , i = k, \ldots, t\} \\ &= \{\sigma \in G_M, \sigma \mid_{V_i/V_{i-k}} = id_{V_i/V_{i-k}}, i = k, \ldots, t\} \\ &= Gal_\partial(K_{L_t \ldots L_1}/K_{L_k \ldots L_1} K_{L_{k+1} \ldots L_2} \ldots K_{L_t \ldots L_{t-k+1}}). \end{aligned}$$

Note that $G_M^\bullet$ is the restriction to G_M of the standard filtration on the parabolic subgroup

$$Aut(Fil(V_M)) = \{g \in Aut(V_M), gV_i = V_i, i = 1, \ldots, t\}$$

of $Aut(V_M)$ attached to the above flag. In particular, for each $k = 1, ..., t-1$, we get a natural injection of abelian groups

$$\begin{aligned} G_M^k/G_M^{k+1} &\simeq Gal_\partial(K_{L_{k+1} \ldots L_1} \ldots K_{L_t \ldots L_{t-k}}/K_{L_k \ldots L_1} \ldots K_{L_t \ldots L_{t-k+1}}) \\ &\hookrightarrow Aut^k(Fil(V_M))/Aut^{k+1}(Fil(V_M)) \simeq \oplus_{i=k+1,\ldots,t} Hom(V_{L_i}, V_{L_{i-k}}), \end{aligned}$$

which generalizes the map ξ_N of §1 , and shows that the above quotient lies in the center of G_M^1/G_M^{1+k}. The descending central series of the group G_M^1 is therefore finer than its G_M^k-filtration, and G_M^1 is unipotent. On the other hand, $G_M/G_M^1 = Gal_\partial(K_{L_1} \ldots K_{L_t}/K)$ admits $V_{L_1} \oplus \ldots \oplus V_{L_t}$ as a completely reducible faithful representation, hence is reductive. Thus,

$$G_M^1 = Gal_\partial(K_{L_t \ldots L_1}/K_{L_t} \ldots K_{L_1}) = R_u(G_M)$$

is the unipotent radical of G_M.

We begin this section with a necessary and sufficient set of conditions for $R_u(G_M)$ to coincide with the full unipotent radical $Aut^1(Fil(V_M))$ of $Aut(Fil(V_M))$ (i.e. to be truly 'as big as possible'). Recall (cf.(**N.3**)) that a product $N = PQ$ is totally unsplit if no non trivial pushout of the extension $\mathbf{E}_N$ splits.

Theorem 2.1 : *Let $M = L_1 \ldots L_t$ be a product of completely reducible elements of D_K. Then, $R_u(G_M) = Aut^1(Fil(V_M))$ if and only if the following two conditions hold.*

(C1) *for each $i = 1, \ldots, t-1$, the product $L_{i+1}L_i$ is totally unsplit;*

(C2) *the extensions $K_{L_{i+1}L_i}/K (i = 1, \ldots, t-1)$ are linearly disjoint over $K_{L_1} \ldots K_{L_t}$.*

Proof : by Theorem 1.1, Condition (C1) amounts to requiring that for each $i = 1, \ldots, t-1$, $Gal_\partial(K_{L_{i+1}L_i}/K_{L_{i+1}}K_{L_i})$ is isomorphic to $Hom(V_{i+1}/V_i, V_i/V_{i-1})$. Now, the extensions $K_{L_{i+1}L_i}/K_{L_{i+1}}K_{L_i}$ and $K_{L_1} \ldots K_{L_t}/K_{L_{i+1}}K_{L_i}$ are linearly disjoint, since their Galois groups, being unipotent for the former and reductive for the latter, cannot share a non-trivial quotient (cf. [1], 3.2 for another use of this argument). Thus, (C1) holds if and only if $Gal_\partial(K_{L_t} \ldots K_{L_{i+2}}K_{L_{i+1}L_i}K_{L_{i-1} \ldots K_{L_1}} / K_{L_t} \ldots K_{L_1}) \simeq Hom(V_{i+1}/V_i, V_i/V_{i-1})$ for all $i = 1, \ldots, t-1$. The disjointness assumption (C2) now means that the Galois group of the compositum $Gal_\partial(K_{L_2L_1} \ldots K_{L_tL_{t-1}}/K_{L_1} \ldots K_{L_t})$ is isomorphic to the direct sum of the groups $Hom(V_{i+1}/V_i, V_i/V_{i-1})(i = 1, \ldots, t-1)$. In other words, (C1) and (C2) hold if and only if

$$G_M^1/G_M^2 = Aut^1(Fil(V_M))/Aut^2(Fil(V_M)) \ ,$$

and the required equality on $R_u(G_M)$ follows from the following standard lemma on nilpotent Lie algebras.

Lemma 2.1 : *for $k \geq 1$, let n^k be the Lie algebra of $Aut^k(Fil(V_M))$, and let g be a subalgebra of n^1. Then, $g/g \cap n^2 = n^1/n^2$ if and only if $g = n^1$.*

Proof : n^1 is a nilpotent Lie algebra, whose derived algebra is n^2. We must thus show that a Lie subalgebra g of a nilpotent algebra n fills up n as soon as $g \to n/Dn$ is surjective. By recurrence, one checks that the descending central series of g satisfies $C^kg + C^{k+1}n = C^kn$, so that $C^kg \to C^kn/C^{k+1}n$ is surjective for all k's. But $C^kn = 0$ for $k >> 0$, so that $C^kg = C^kn$ for all k's.

Remark 2.1. There is a certain similarity between Theorem 2.1 and Kolchin's theorem on subgroups of products [10], in which (C2) would correspond to Kolchin's hypothesis in the case of products of abelian groups, and (C1) to its pair-wise assumption in the case of products of simple groups.

We now state a sufficient condition ensuring Condition (C2) .

Lemma 2.2 : *Let $M = L_1 \ldots L_t$ be a product of completely reducible elements of D_K. Then Condition* (C2) *holds true as soon as*
(C'2) *no pair amongst the D_K-modules $Hom(\mathbf{V}_{L_i}, \mathbf{V}_{L_{i+1}})$* $(i = 1, \ldots, t-1)$ *admit non-zero isomorphic submodules.*
Proof : denote by H' the compositum of any Picard-Vessiot extension H of K with $K' = K_{L_t} \ldots K_{L_1}$, and suppose (C2) doesn't hold. There then exists an index $i \geq 2$ such that $K'_{L_{i+1}L_i} \cap K'_{L_2L_1} \ldots K'_{L_iL_{i-1}} = K''$ is a strict extension of K'. Since K''/K' is a Picard-Vessiot extension, its Galois group is a quotient of both $Gal_\partial(K'_{L_{i+1}L_i}/K')$

and of $Gal_\partial(K'_{L_2L_1}\dots K'_{L_iL_{i-1}}/K')$. Moreover, since K'' is a Picard-Vessiot extension of K itself and since K''/K' is abelian, the corresponding subgroups are stable under the action of $Gal(K'/K)$ by conjugation. In other words, $Gal(K''/K')$ is a quotient of the two $Gal(K'/K)$-representations $Gal_\partial(K'_{L_{i+1}L_i}/K')$ and $Gal_\partial(K'_{L_2L_1}\dots K'_{L_iL_{i-1}}/K')$. Now, the former (resp. the latter) is a subrepresentation of $Hom(V_{L_{i+1}}, V_{L_i})$ (resp. $Hom(V_{L_2}, V_{L_1}) \oplus \dots \oplus Hom(V_{L_i}, V_{L_{i-1}})$). Complete reducibility then implies that one of the irreducible $Gal(K'/K)$-representations occurring in $Hom(V_{L_{i+1}}, V_{L_i})$ also occurs in $Hom(V_{L_{j+1}}, V_{L_j})$ for some $j < i$, and this contradicts (C'2).

In §3, we shall explain, in a special, but typical, situation, how to check the disjointness hypothesis (C2) when (C'2) is not satisfied, and more generally, how to compute G^1_M/G^2_M when (C2) is not satisfied. In particular, we shall see that (C2) is not necessary to ensure that G^2_M fills up $Aut^2(Fil(V_M))$. In the same vein, we now show how to relax Condition (C1) into a sufficient condition ensuring that G^2_M fills up $Aut^2(Fil(V_M))$. For simplicity, we henceforth restrict to the case of $t = 3$ factors. (Note that $Aut^2(Fil(V_M))$ then coincides with the center of $Aut^1(Fil(V_M))$, but this will have no bearing on our discussion, and in fact, G^2_M *need not* coincide with the center of G^1_M .)

2.2. Filling up $Aut^2(Fil(V_M))$ when $t = 3$.

Let thus $M = L_3L_2L_1$ be the product of three completely reducible operators in D_K, and set :

$$N_2 = L_3L_2, N_1 = L_2L_1.$$

We then have the towers of differential fields

$$\begin{tikzcd}
 & K_M & \\
 & K_{N_2}.K_{N_1} \arrow[u, "G^2_M \subset Hom(V_{L_3},V_{L_1})"'] & \\
K_{N_2}.K_{L_1} \arrow[ur, "Hom(V_{L_2},V_{L_1}) \supset U^1_1"] & & K_{N_1}.K_{L_3} \arrow[ul, "U^1_2 \subset Hom(V_{L_3},V_{L_2})"'] \\
 & K_{L_3}.K_{L_2}.K_{L_1} \arrow[ul, "U^1_2"] \arrow[ur, "U^1_1"'] & \\
 & K \arrow[u] &
\end{tikzcd}$$

where $Gal(K_M/K_{L_3}.K_{L_2}.K_{L_1}) = G^1_M$, and U^1_1, U^1_2 denote the two natural quotients of $G^1_M/G^2_M \subset Hom(V_{L_2}, V_{L_1}) \oplus Hom(V_{L_3}, V_{L_2})$ (the figure is drawn under the linear disjointness condition (C2), so that they also appear as subgroups), while the right lower corner stands for

$$
\begin{array}{ccccc}
 & & & & K_{N_1}.K_{L_3} \\
 & U^1_1 \nearrow & & & \uparrow \\
K_{L_3}.K_{L_2}K_{L_1} & & & & K_{N_1} \\
\uparrow & & & \nearrow U^1_1 & \\
K_{L_2}.K_{L_1} & & & & \\
\uparrow & & & & \\
K & & & &
\end{array}
$$

Contrary to that of Theorem 2.1, the next statement is asymmetric, and indeed, its proof is based on the study of the upper left side of the top tower. And in contrast with Theorem 1.1, it now relies on the action of the *non* reductive group $Gal_\partial(K_{N_2}.K_{L_1}/K)$. To ease notations, we write $K_{P,Q,\ldots,R}$ for a compositum $K_P.K_Q \ldots K_R$, and recall the definitions of (**N.3**).

Theorem 2.2 : $(t = 3.)$ *Assume that*

(C'1) $N_1 = L_2L_1$ *is totally unsplit, but* $N_2 = L_3L_2$ *is merely left unsplit;*

(C2) *the extensions* K_{N_1}/K *and* K_{N_2}/K *are linearly disjoint over* K_{L_1,L_2,L_3}.

Then, $Gal_\partial(K_M/K_{N_2}.K_{L_1}) \simeq Hom(V_{N_2}, V_{L_1})$, *and in particular,*

$$Gal_\partial(K_M/K_{N_2}.K_{N_1}) = G^2_M \simeq Aut^2(Fil(V_M)) = Hom(V_{L_3}, V_{L_1}).$$

[Note that (C'1) is weaker than (C1) : as soon as Q has order > 1, it may well happen that a product PQ is left unsplit, but that $Gal(K_{PQ}/K_{P,Q})$ does not fill $Hom(V_P, V_Q)$.]

Proof : the hypothesis on N_1 says that $Gal_\partial(K_{N_1}/K_{L_2,L_1}) \simeq Hom(V_{L_2}, V_{L_1})$, and the unipotent/reductive dichotomy implies that this group is equal to $Gal_\partial(K_{N_1,L_3}/K_{L_3,L_2,L_1})$ $:= U^1_1$. By (C2), we may extend the scalars to K_{N_2}, and obtain that $Gal_\partial(K_{N_2,N_1}/K_{N_2,L_1})$ is still isomorphic to $Hom(V_{L_2}, V_{L_1})$. Now, applying the general description given in §1.1 of the relative Galois group attached to the decomposition $M = N_2L_1$, we get an embedding $\xi_{N_2 o L_1}$ of $Gal_\partial(K_M/K_{N_2,L_1})$ into a $J = Gal_\partial(K_{N_2,L_1}/K_{L_1})$-submodule X of $Hom(V_{N_2}, V_{L_1})$. The latter Galois group J will in general not be reductive anymore, and we can't go on with the discussion of §1.2. Instead, we notice that in the natural projection π from $Hom(V_{L_3L_2}, V_{L_1})$ to $Hom(V_{L_2}, V_{L_1})$,

$$\pi X = \pi o \xi_{N_2 o L_1}(Gal_\partial(K_M/K_{N_2,L_1})) = \xi_{L_2 o L_1}(Gal(K_{N_2,N_1}/K_{N_2,L_1})).$$

Thus, X maps onto $Hom(V_{L_2}, V_{L_1})$. Now, with K_{L_1} as base field, the Galois representation V_{L_1} is a direct sum C^r of copies of the trivial representation C. The following lemma, with $K' = K_{L_1}, P = \check{L}_2, Q = \check{L}_3$, therefore concludes the proof of Theorem 2.2 (since by (**N.3**), a product PQ is right unsplit if no proper left divisor Q'' of Q yields a split extension $\mathbf{V}_{P\text{o}Q''}$, its statement is more or less a tautology).

Lemma 2.3 : *let $N = PQ$ be a right unsplit product of completely reducible operators in D_K, let $r \geq 1$, and let K'/K be a Picard-Vessiot extension with reductive Galois group. View V_N as a representation of $Gal_\partial(K'_N/K')$, where $K'_N = K_N.K'$. Then, any subrepresentation $X \subset (V_N)^r$ that projects onto $(V_P)^r$ fills up $(V_N)^r$.*

Proof : we first prove that $N = PQ$ remains a right unsplit product in the ring $D_{K'}$. Otherwise, there exists a decomposition $Q = Q''Q'$ in $D_{K'}$, with $ord(Q'') \geq 1$, such that $\mathbf{V}_{P\text{o}Q''}$ splits over K'. By Galois descent and the semisimplicity of V_Q, we may assume that Q'' belongs to D_K, so that $\mathbf{V}_{P\text{o}Q''}$ defines an element of $Ext_{D_K}(\mathbf{V}_P, \mathbf{V}_{Q''})$ dying in $Ext_{D_{K'}}(\mathbf{V}_P, \mathbf{V}_{Q''})$. Now, the kernel of the restriction map $Ext_{D_K}(\mathbf{V}_P, \mathbf{V}_{Q''}) \to Ext_{D_{K'}}(\mathbf{V}_P, \mathbf{V}_{Q''})$ is the cohomology group

$$H^1(Gal_\partial(K'/K), (Hom(V_P, V_{Q''}))^{Gal_\partial(K'/K)}) \ ,$$

which vanishes, since $Gal_\partial(K'/K)$ is here reductive. Thus, $\mathbf{V}_{P\text{o}Q''}$ splits over K, and this contradicts the hypothesis made on N. We are thus reduced to check the lemma when $K' = K$.

Suppose first that $r = 1$. Then, $X \cap V_Q$ is a Galois invariant subspace of V_Q, hence of the form $V_{Q'}$ for some decomposition $Q = Q''Q'$ of $Q \in D_K$. Since multiplication to the left by Q maps $X \subset V_{PQ}$ surjectively to V_P, we may view V_{PQ} as the pushout of $X \in Ext(V_P, V_{Q'})$ under the injection ι from $V_{Q'}$ to V_Q. Let $\pi = m_{Q'}$ be the natural projection from V_Q to $V_{Q''}$. Then $V_{PQ''} = \pi_* V_{PQ} = \pi_* \text{o} \iota_* X = (\pi \text{o} \iota)_*(X) = 0$ in $Ext(V_P, V_{Q''})$, so that the pushout $\mathbf{V}_{P\text{o}Q''}$ of $\mathbf{V}_{P\text{o}Q}$ splits. Since PQ is right unsplit, this forces $Q' = Q$, and $X = V_{PQ}$ fills up V_N. The general case $r > 1$ follows by intersecting X with the r factors V_N of $(V_N)^r$: each of these intersections projects onto V_P, hence fills up V_N, and X fills up $(V_N)^r$.

Going back to our initial problem on integrals, i.e. to the case $L_1 = \partial$, we deduce the following consequence of Theorem 2.2, in the style of Corollary 1.1.

Corollary 2.1 : *let $N = L_3 L_2$ be the product of two completely reducible operators. Assume that $L_2\partial$ and $L_3 L_2$ are left unsplit, and that no factor of $\mathbf{V}^*_{L_2}$ occurs as a factor of $Hom(\mathbf{V}_{L_3}, \mathbf{V}_{L_2})$. Then, for any C-linearly independent solutions $f_1, \ldots, f_r$ of $Ny = 0$, the functions $\int f_1, \ldots, \int f_r$ are algebraically independent over K_N .*

Remark 2.2. In the hypothesis of this corollary, we could have used (C2) instead of the stronger condition (C'2) from Lemma 2.2. But conditions of this type cannot be omitted: we shall see in §3 (Theorem 3.2, $(C3.\beta)$) a situation where although (C'1) (and in fact (C1)) is still satisfied, G_M^2 is trivial, and the analogue of Corollary 2.1 fails.

Remark 2.3 (Polylogarithms) . Let $k \geq 2$ be a rational integer, and let

$$M = R\theta^k \ , \text{ with } R = \theta - \frac{1}{1-z} \ , \ \theta = zd/dz.$$

A basis of solutions of $My = 0$ is given by $\{1, \ell n\ z, \frac{1}{2!}(\ell n\ z)^2, \dots, \frac{1}{(k-1)!}(\ell n\ z)^{k-1}, Li_k(z)\}$, where

$$Li_k(z) = \Sigma_{n \geq 1} \frac{z^n}{n^k}$$

is the k-th polylogarithm function. Its well-known monodromy group (see, for instance, [13], IV. prop. 3.4) is the semi-direct product $\mathbf{Z}^k \times_u \mathbf{Z}$ for the unipotent action u of $\mathbf{Z}$ on $Sym^{k-1}\mathbf{Z}^2 \simeq \mathbf{Z}^k$, and we get $\mathbf{G}_a^k \times_u \mathbf{G}_a$ for its Zariski closure, which is the differential Galois group G_M of the (Fuchsian) operator M over $K = C(z)$. The principle underlying Lemma 2.3 enables us to recover the latter from the mere knowledge of $Gal_\partial(K_{R\theta}.K_{\theta^k}/K) = Gal_\partial(K(\ell n(1-z), \ell n\ z)/K) \simeq \mathbf{G}_a \times \mathbf{G}_a$, as follows. First, note that $G_M = G_M^1$ is unipotent, since M is a successive extension of trivial D_K-modules (the rational function $f = \frac{z}{(1-z)}$ is a solution of $Ry = 0$). Now, write $M = RN$ as the product of R by the non completely reducible operator $N = \theta^k$, and accordingly, G_M as an extension of $G_N = Gal_\partial(K(\ell n(z)/K) \simeq \mathbf{G}_a$ by the relative differential Galois group

$$U = Gal_\partial(K_{RN}/K_N) \ ,$$

which the map $\xi_{R \circ N}$ of §1.1 still enables us to view as a G_N-subspace of V_N. Let us prove that U *fills up* V_N :

- firstly, U projects onto V_θ under the natural surjection $\pi := m_{\theta^{(k-1)}} : V_N \to V_N/V_{\theta^{(k-1)}} \simeq V_\theta$; indeed, this quotient $\pi(U)$ identifies with the differential Galois group of $R\theta$ over K_N, which, as stated above, is isomorphic to $Gal_\partial(K_{R\theta}/K) \simeq \mathbf{G}_a$.

- secondly, by Jordan theory, any subspace of $V_N = Sym^{k-1}V_{\theta^2}$ which is stable under $G_N = G_{\theta^2}$ is of the form $Sym^j V_{\theta^2} = V_{\theta^{j+1}}$ for some $j = 0, \dots, k-1$.

As in Lemma 2.3, we now deduce that $U = V_N \simeq \mathbf{G}_a^k$ from the fact that among these subrepresentations, only V_{θ^k} itself has a non trivial projection under π.

3. Integrals of solutions of inhomogeneous equations.

This last section is devoted to a precise description of $R_u(G_M)$ when the condition of linear independence (C2) of §2 fails to hold. To allow for a complete study, we restrict to the case where $M = L_3L_2L_1$ is the product of three operators such that

i) L_3 and L_1 are equivalent to ∂, i.e. $L_3 = \partial_a$ ($:= a\partial a^{-1} = \partial - \partial(a)/a$), $L_1 = \partial_b$ for some non zero elements $a, b \in K$;

ii) L_2 (denoted by L from now on) is irreducible; put $V := V_L, V^* := Hom(V_L, C)$ (the latter space can be identified with the solution space of the adjoint $\check{L}$ of L).

We first explain what the various conditions (C1),..., (C'2) of §2 concretely mean in the present situation, where we shall denote them by italic letters $(C1)$,..., $(C'2)$. We let n be the order of L.

Since L_3 and L_1 here have order one and L is irreducible, (C1) and (C'1) coincide, and are equivalent to requiring that $\partial_a L$ and $L\partial_b$ (or equivalently, its adjoint $\partial_{1/b}\check{L}$) are not completely reducible, i.e. that

$$(C1) \sim (C'1) \qquad a \notin L(K) \quad \text{and} \quad \frac{1}{b} \notin \check{L}(K)).$$

These conditions will be assumed throughout the rest of the paper.

Since L is irreducible and $V_{\check{L}}, Hom(V_L, C)$ are isomorphic Galois representations, denying (C'2) is equivalent to requiring that L and $\check{L}$ are equivalent operators (or, as we shall say, that L is *autodual*), i.e. that there exists elements $T, S \in D_K$ of order $< n$ such that S and L have no non trivial common divisor to the right and $TL = \check{L}S$. Then (cf. the diagram in §1.2), $\check{L}$ and T have no common divisors to the left, so that $\check{T}$ and L have no RCD either, and we deduce from the adjoint relation $\check{L}\check{T} = \check{S}L$ that the maps m_S and $m_{\check{T}}$ on the solution space V both define Galois equivariant isomorphisms from V to V^*. But V is an irreducible representation. By Schur's lemma, these isomorphisms must therefore be proportional, and there exists a non zero scalar $\lambda \in C$ such that $S = \lambda\check{T}$ (since both $\check{T}$ and S have order $< n$). Thus $TL = \check{L}S = \lambda\check{L}\check{T} = \lambda(\check{TL})$. Looking at initial coefficients, we get $\lambda = (-1)^{t+n}$. Finally, since L is irreducible, the coprimality condition is automatic and we may state the converse of Condition $(C'2)$ as :

Lemma 3.1 : *let $L \in D_K$ be irreducible of order n. Then, L and $\check{L}$ are equivalent operators if and only if :*
$(\neg C'2)$ there exists $T \neq 0 \in D_K$, of order $t < n$ such that $TL = (-1)^{t+n}(\check{TL})$.
[By Schur's lemma, T is then, unique up to multiplication by a non zero constant.]

Since Picard-Vessiot extensions are unchanged when taking adjoints, $K_L = K_{\check{L}}$, $K_{\partial_a L}$ $= K_{\check{L}\partial_{1/a}}$, and Condition (C2) of §2.1 here becomes : the extensions $K_{\check{L}\partial_{1/a}}$ and $K_{L\partial_b}$

are linearly disjoint over K_L. As we already saw in §2, Lemma 2.2, this certainly holds if L and $\check{L}$ are not equivalent. Consider now the converse case $(\neg C'2)$. Repeating the argument from §2, Lemma 2.2, and using the irreducibility of the Galois representation V^*, we find that if (C2) doesn't hold, then $K_{\check{L}\partial_{1/a}}$ and $K_{L\partial_b}$ actually coincide. In view of §1, corollary 1.2, this in turn is equivalent to requiring that a and the inverse image of $1/b$ under multiplication by $T : K/L(K) \to K/\check{L}(K)$ be C-linearly dependent modulo $L(K)$, i.e. that $T(a)$ and $1/b$ be linearly dependent over C mod. $\check{L}(K)$. Thus the converse of Condition $(C2)$ reads :

Lemma 3.2 : *Let $L \in D_K$ be irreducible of order n, and let a, b be non zero elements in K satisfying $(C1)$. The following conditions are equivalent.*

i) $(\neg C2)$ $K_{\partial_a L}$ and $K_{L\partial_b}$ are not linearly disjoint over K_L;

ii) there exists $T \neq 0 \in D_K$, of order $t < n$ such that $TL = (-1)^{t+n}(\check{TL})$ (i.e. $(\neg C'2)$ holds) and moreover, the images of $T(a)$ and $1/b$ in $K/\check{L}(K)$ are linearly dependent over C .

iii) $\partial_a L$ and the adjoint of $L\partial_b$ are equivalent operators.

Remark 3.1 : the D_K-modules $\mathbf{V}_{\partial_a L}$ and $\mathbf{V}_{\partial_{1/b}\check{L}}$ are then isomorphic, but even if $L = \check{L}$, $\mathbf{V}_{\partial_a \circ L}$ and $\mathbf{V}_{\partial_{1/b} 0 \check{L}}$ need not be isomorphic as extensions of $\mathbf{V}_\partial$ by $\mathbf{V}_L$: in this case, the second condition exactly means that one is the push-out of the other by a non-zero homothety.

As promised in the introduction, we now embark on a case-by case description of the unipotent radical $R_u(G_M)$ of the differential Galois group $G_M = Gal_\partial(K_M/K)$ of $M = \partial_a L \partial_b$. As in §2, we write $Fil(V_M)$ for the filtration of V_M attached to this decomposition of M, and $G^\bullet$ for the induced filtration of G_M. Bringing Theorem 2.1 and the previous discussion together, we first obtain:

Theorem 3.1: *Let $M = \partial_a L \partial_b$ with L irreducible satisfying $(C1) : a \notin L(K), \frac{1}{b} \notin \check{L}(K))$. Then $R_u(G_M) = Aut^1(Fil(V_M))$ if and only if either*

i) $(C'2) : L \not\sim \check{L}$; or

ii) $(\neg C'2) : L \sim \check{L}$, but $(C2) : \partial_a L \not\sim (\check{L\partial_b})$.

From now on, we assume that $(\neg C2)$ holds (cf. Lemma 3.2). We denote by T the element of D_K defined in this lemma (up to multiplication by a constant), and by $\phi = m_T$ the corresponding G_L-equivariant isomorphism

$$\phi : V \to V^*.$$

Because of Condition $(C1)$, the projections U_2^1 and U_1^1 of G_M^1/G_M^2 on each of the factors V, V^* of $Aut^1(Fil(V_M)/Aut^2(Fil(V_M))$ are surjective. By Schur's lemma, we deduce that the image of G_M^1/G_M^2 in $V \oplus V^*$ is the graph of a constant multiple of ϕ :

$$G_M^1/G_M^2 \simeq Graph(\phi) \subset V \oplus V^*.$$

[NB : As promised in §2, this gives an example where G_M^2/G_M^1 can be computed even though the assumption (C2) of linear disjointness does not hold : the above constant is given by the linear combination of $T(a)$ and $1/b$ appearing in Lemma 3.2.ii; cf. §1.2.ii).]

In order to complete the description of $R_u(G_M)$, we are thus left with the task of determining the subgroup

$$G_M^2 \subset Aut^2(Fil(V_M)) \simeq Hom(V_{\partial_a}, V_{\partial_b}) \simeq C$$

of its center (the group structure of $R_u(G_M) = G_M^1$ will automaticaly follow).

Identifying V with its bidual, we let

$$\tilde{\phi} : V \to V^*$$

denote the transpose of ϕ. Since V is irreducible, either

$(\neg C3)$ ϕ is antisymmetric, i.e. $\tilde{\phi} = -\phi$, or equivalently : G_L is contained in a symplectic subgroup of $Aut(V)$; we shall then say that L is *antisymmetrically autodual*; or

$(C3)$ ϕ is symmetric, i.e. $\tilde{\phi} = \phi$, or equivalently : G_L is contained in an orthogonal subgroup of $Aut(V)$; we shall then say that L is *symmetrically autodual*.

This dichotomy has a crucial effect on the answer to our problem, as follows.

Theorem 3.2 : *Assume M satisfies $(C1)$ and $(\neg C2)$, so that $T(a)$ and $1/b$ are linearly dependent over C modulo $\check{L}(K)$.*
$(\neg C3)$ *if L is antisymmetrically autodual, then $G_M^2 = Aut^2(Fil(V_M))$ and $R_u(G_M)$ is* not *abelian;*
$(C3)$ *if L is symmetrically autodual, then $R_u(G_M)$ is abelian and either*

$(C3.\alpha)$ *$T(a)$ and $1/b$ are C-linearly independent modulo $\check{L}\partial_{1/a}(K)$, in which case we again have: $G_M^2 = Aut^2(Fil(V_M))$; or*

$(C3.\beta)$ *$T(a)$ and $1/b$ are C-linearly dependent modulo $\check{L}\partial_{1/a}(K)$, in which case*

$$G_M^2 = 0.$$

Proof in the antisymmetric case $(\neg C3)$: the proof below follows the argument used in that of Theorem 2.1. Another proof, based on the computation of the cohomology group $H^1(G_M/G_M^2, V_{L\partial_b} \oplus V_{\partial_a L})$ as in [8] , would also be possible. Recalling the notations of Lemma 2.1 and identifying the images in $g^1/g^2 \simeq Lie(Graph(\phi))$ of the elements $X \in g^1$ to couples $(v, \phi(v))$ with $v = v_X \in V$, we deduce from an elementary matrix computation, and from the antisymmetry of ϕ that for all $X, X' \in g^1$,

$$[X, X'] = \phi(v_X)(v_{X'}) - \phi(v_{X'})(v_X) = \phi(v_X)(v_{X'}) - \tilde{\phi}(v_X)(v_{X'}) = 2\phi(v_X)(v_{X'}),$$

where we identify elements of $[g^1, g^1] \subset g^2 \subset n^2 \simeq C$ with numbers in C. Since ϕ is an isomorphism, not all such products can vanish, and $0 \neq [g^1, g^1]$. Thus, g^1, hence $R_u(G_M)$, is not abelian, and $G_M^2 \simeq C \simeq Aut^2(Fil(V_M)$. This concludes the proof of the case $(\neg C3)$.

Proof in the symmetric case $(C3)$: in the converse case $(C3)$ where $\phi = \tilde{\phi}$ is symmetric, the formula above gives $[X, X'] = 0$, and $R_u(G_M)$ is indeed abelian. We shall complete the proof of this case (hence of Theorem 3.2) as follows. Assuming that the conclusion of Case $(C3.\beta)$ has been established, we first derive that of $(C3.\alpha)$. We then (but independently !) treat $(C3.\beta)$. In the course of this proof, it is useful to keep in mind that $G_M^2 = 0$ if and only if there exists a solution F of $My = O$ which belongs to $K_{L\partial_b} = K_{\partial_a L}$ and such that $L\partial_b(F) \neq O$. Indeed, $G_M^2 = Gal_\partial(K_M/K_{L\partial_b}.K_{\partial_a L})$, while K_M is differentially generated over $K_{\partial_a L}$ by any element F of V_M not in $V_{L\partial_b}$.

Case $(C3.\alpha)$. Let $b \in K$ satisfy $(C3.\alpha)$. In view of $(\neg C2)$ (cf. Lemma 3.2.ii), we may assume without loss of generality that there exists an element u of K, not lying in $\partial_{1/a}(K)$, such that $1/b = T(a) + \check{L}(u)$. The differential extension $K_M = K_{\check{M}}$ of $K_{\check{L}\partial_{1/a}}$ is then generated by the solutions of the inhomogeneous equation :

$$\check{L}\partial_{1/a} y = T(a) + \check{L}(u).$$

Now, by the (assumed) conclusion of $(C3.\beta)$ and the reminder above, the solutions of

$$\check{L}\partial_{1/a} y = T(a)$$

all belong to $K_{\check{L}\partial_{1/a}}$. A standard result on inhomogeneous equations then implies that K_M is differentially generated over that field by the solutions of the equation

$$\check{L}\partial_{1/a} y = \check{L}(u),$$

and in particular, contains the solutions of the equation

$$\partial_{1/a} y = u.$$

Since $u \notin \partial_{1/a}(K)$, the Picard-Vessiot extension of K attached to $\partial_u \partial_{1/a} y = 0$ is not trivial, and we are reduced to proving that $K_{\check{L}\partial_{1/a}}$ and $K_{\partial_u \partial_{1/a}}$ are linearly disjoint over K. When $\check{L} \not\sim \partial$, this easily follows from Theorem 1.1, as shown in [3], Corollaire 2. And similarly (say now by Corollary 1.2 of §1 with $L = \Lambda = \partial_a$) in case $\check{L} \sim \partial$, i.e. $\check{L} = \partial_v$ for some non zero $v \in K$, if u and v are C-linearly independent modulo $\partial_{1/a}(K)$. But they must be so: since $1/b = T(a) + \partial_v(u)$ while $v \notin \partial_{1/a}(K)$ by $(C.1)$, $1/b$ and $T(a)$ would otherwise be linearly dependent modulo $\partial_v \partial_{1/a}(K) = \check{L}\partial_{1/a}(K)$.

[NB : in more intrisic terms, we have here used the structure of torsor under $Ext(\mathbf{V}_{\partial_a}, \mathbf{V}_{\partial_b})$ of the set of *extensions panachées* (cf. [6]) of $\mathbf{V}_{\partial_a L}$ by $\mathbf{V}_{L\partial_b}$.]

Case $(C3.\beta)$. We finally consider the case $(C3.\beta)$, where a proof can be given along the lines of [12], exhibiting $\mathbf{V}_M$ as a quotient of the symmetric square of $\mathbf{V}_{\partial_a L}$. But the following 'more differential' argument seems better fit to the present paper.

Lemma 3.3 : *let L be an irreducible operator, of order n. Assume that L is equivalent to its adjoint, and let T be the (unique up to a constant factor) operator of order $t < n$ such that $TL = (-1)^{t+n}\check{T}\check{L}$. Then, L is symmetrically autodual if and only if $t \equiv n-1 (\mathrm{mod}.2)$, i.e. $TL = -\check{T}\check{L}$.*

[If n is odd, the autoduality is necessarily symmetric, and this means that t is even. Note that in the classical case of 'truly' selfadjoint operators of order 2, where $L = \check{L}$ and $T = 1$ has order of the same parity as L, the wronskian is trivial, hence $G_L \subset SL_2(C) = Sp_2(C)$, and L is indeed antisymmetrically autodual.]

Proof: we first show that if Q is a (not necessarily irreducible) operator of order q such that $\check{Q} = (-1)^q Q$, then the isomorphism from V_Q to $V_{\check{Q}}$ given by the identity on K_Q induces a non-degenerate bilinear form β_Q on V_Q which is symmetric if q is odd and antisymmetric if q is even. Indeed, this pairing is the restriction to $V_Q \times V_{\check{Q}}$ of Lagrange's bilinear concomitant $\{,\}_Q$ (cf [7], 5.3, 9.31), a differential expression in two differential variables (u, v) with coefficients in the differential field generated by the coefficients of Q, which satisfies

$$uQ(v) - v\check{Q}(u) = \partial(\{u, v\}_Q).$$

In particular, $\{,\}_Q$ is C-linear in Q, and $\{u, v\}_Q = -\{v, u\}_{\check{Q}}$, so that in our situation, $\beta_Q(u, v) = (-1)^{q-1}\beta_Q(v, u)$ for u, v in V_Q.

Let now L and T satisfy the hypothesis of Lemma 3.3. Applying the previous remark to $Q = TL$, we obtain a Galois equivariant non degenerate bilinear form β_{TL} on V_{TL}, which is $(-1)^{t+n-1}$-symmetric. This induces on the subspace V_L of V_{TL} a Galois equivariant bilinear form β which certainly has the same type of symmetry. Moreover, β is still *non degenerate* on V_L: indeed, in view of the irreducibility of L, β would otherwise vanish identically; but V_L cannot be an isotropic subspace of β_{TL}, since $dim(V_{TL}) = t + n < 2n = 2dim(V_L)$. Thus, β induces a $(-1)^{t+n-1}$-symmetric equivariant isomorphism ϕ' between V_L and $V_{\check{L}}$, as was to be shewn (by Schur's lemma, ϕ' must be a multiple of $\phi = m_T : V_L \to V_{\check{L}}$).

Assume $(C3.\beta)$ holds. Since the extension $K_{\partial_{1/b}\check{L}\partial_{1/a}}/K_{\check{L}\partial_{1/a}}$ depends only on the class of the non zero constant multiples of $1/b$ modulo $\check{L}\partial_{1/a}(K)$, we may assume without loss of generality that $1/b = T(a)$. We are finally reduced to proving that if f satisfies the inhomogeneous equation $Lf = a$, and if $TL + \check{TL} = 0$, then the equation $\partial_{1/T(a)}y = f$ admits a solution F in $K_{\check{L}\partial_{1/a}} = K_{\partial_a L} = K_L < f >$ (i.e. that f is 'exact' with respect to the derivation $\partial_{1/T(a)}$ on $K_{\partial_a L}$).

To check this, we apply the definition of the bilinear concomitant to $Q = TL = -\check{Q}$ and $u = v = f$, and obtain:

$$\partial\{f, f\}_{TL} = 2fT(L(f)) = 2T(a)f$$

so that $F := \frac{1}{2T(a)}\{f, f\}_{TL}$ satisfies :

$$\partial_{1/T(a)}F = (1/T(a))\partial(T(a)F) = (1/2T(a))\partial(\{f, f\}_{TL}) = f.$$

Since F is a K-rational differential expression in an element of $K_{\partial_a L}$, it does belong this field. Thus, $Gal_\partial(K_M/K_{L\partial_a}) = G_M^2 = 0$, and the proof of Theorem 3.2 , hence the complete determination of the unipotent radical of G_M, is at last completed.

The proof of Corollary 3 , as stated in the introduction, easily follows from this computation of $R_u(G_M) = Gal(K_M/K_L)$, which implies that F belongs to $K_{\partial_a L}$ if and only if Condition $(C3.\beta)$ (with $b = 1$) holds. As for the proof of Theorem 3 itself, note that

• Cases (i) and (ii) of Theorem 3 correspond to Case $(C2)$, i.e. to Theorem 3.1. A concrete illustration of Case (ii) can be built up from the Picard-Fuchs equation L of a non isoconstant elliptic curve over K admitting two linearly independent K-rational sections.

• Case (iii) of Theorem 3 corresponds to Case $(\neg C.3)$ of Theorem 3.2. To recover the description of $R_u(G_M)$ as a Heisenberg group in this case, it suffices to choose a basis of V_L where the symplectic form is given by $\Sigma_{i=1,\ldots,n/2}\ (x_i\ y_{i+n/2} - x_{i+n/2}\ y_i)$.

• Cases (iv.α - β) of Theorem 3 are respectively Cases ($C3.\alpha - \beta$) of Theorem 3.2. Indeed:

Lemma 3.4 : *Assume $M = \partial_a L \partial_b$ satisfies ($C1$), ($\neg C2$) and ($C3$). Then, M is autodual if and only if ($C3.\beta$) holds.*

Proof : assume ($C1$) and ($\neg C2$). If ($C3.\alpha$) hold, $G_M^2 = C$ and one checks by a matrix computation that no non-degenerate bilinear form on V_M, invariant under a Levi subalgebra of $Lie(G_M)$, is invariant under $Lie(G_M^1)$ [3]. The converse statement can be proved in a similar way, but here is a direct argument. Since the operator T from Lemma 3.1 induces an isomorphism between the cokernels of L and $\check{L}$ in K, Condition ($C1$) implies that $T(a) \not\equiv 0$ mod $\check{L}(K)$. Let then T' be the unique operator such that $\partial_{T(a)} T = T' \partial_a$, hence $\check{T}\partial_{1/T(a)} = \partial_{1/a}\check{T}'$. In view of ($\neg C2$), we have

$$\partial_{T(a)}\check{L}\partial_{1/a}\check{T}' = \partial_{T(a)}\check{L}\check{T}\partial_{1/T(a)} = \pm\partial_{T(a)} T L \partial_{1/T(a)} = \pm T'\partial_a L \partial_{1/T(a)}.$$

Moreover, the same argument as in Corollary 1.2 implies that $\check{T}'$ and $\partial_a L \partial_{1/T(a)}$ have no common right divisor, and we conclude that $\partial_{T(a)}\check{L}\partial_{1/a}$ and its adjoint $\partial_a L \partial_{1/T(a)}$ are equivalent operators. Suppose now that as in Case ($C3.\beta$), the images of $T(a)$ and $1/b$ in $K/\check{L}\partial_{1/a}(K)$ are C-linearly dependent. Then, $\check{M} = \partial_{1/b}\check{L}\partial_{(1/a)}$ is equivalent to $\partial_{T(a)}\check{L}\partial_{1/a}$, and is therefore equivalent to its own adjoint M.

[3] This is in marked contrast with the Heisenberg group, which *does* stabilize such a non-degenerate bilinear form, so that in Case (iii) of Theorem 3, M is always autodual .

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