

# MOTIVIC ZETA FUNCTIONS OF INFINITE DIMENSIONAL LIE ALGEBRAS

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## 1. INTRODUCTION

1.1. In the present paper we associate motivic zeta functions to certain classes of infinite dimensional Lie algebra over a field  $k$  of characteristic zero. Included in these classes are the important cases of loop algebras, affine Kac-Moody algebras, the Virasoro algebra and Lie algebras of Cartan type. These zeta functions take their values in the Grothendieck ring of algebraic varieties over  $k$  and are built by encoding in some manner the  $k$ -subalgebras of a given codimension. This construction is done by an adaptation of the idea of the motivic Igusa zeta function recently introduced in [6].

Let us recall the definition of the Grothendieck ring  $\mathcal{M}$  of algebraic varieties over  $k$ . This is the ring generated by symbols  $[S]$ , for each  $S$  an algebraic variety over  $k$ , that is, a reduced separated scheme of finite type over  $k$ , with the relations

- (1)  $[S] = [S']$  if  $S$  is isomorphic to  $S'$ ;
- (2)  $[S] = [S \setminus S'] + [S']$  if  $S'$  is closed in  $S$ ;
- (3)  $[S \times S'] = [S][S']$ .

The idea leading to this construction comes from the analogy with the zeta functions capturing the subalgebra lattice of  $p$ -adic Lie algebras. More precisely, in [17] the zeta function of a finite dimensional Lie algebra  $L_p$  over  $\mathbb{Z}_p$  was defined as

$$\begin{aligned} \zeta_{L_p}(s) &:= \sum_{H \leq L_p} |L_p : H|^{-s} \\ &= \sum_{n=0}^{\infty} a_{p^n}(L_p) p^{-ns}, \end{aligned}$$

where the first sum is taken over finite index  $\mathbb{Z}_p$ -subalgebras, that is  $\mathbb{Z}_p$ -submodules which define subalgebras in  $L_p$  and  $a_{p^n}(L_p)$  is the number of these of index  $p^n$ . This is a well-defined series since there are only finitely many of each for a given finite index. Note that since  $\mathbb{Z}$  is dense in  $\mathbb{Z}_p$ , finite index  $\mathbb{Z}_p$ -subalgebras are the same as finite index  $\mathbb{Z}$ -subalgebras. The zeta function captures in some analytic manner the lattice of subalgebras. It was proved in the same paper that, when  $L_p$  is additively isomorphic to  $\mathbb{Z}_p^d$ , the zeta function is a rational function in  $p^{-s}$ . The proof depends on Denef's results on  $p$ -adic definable integrals. Explicit formulae in terms of an associated resolution of singularities are given in [13] for  $\zeta_{L_p}(s)$  in the case that  $L_p = L \otimes \mathbb{Z}_p$  for some Lie algebra  $L$  defined over  $\mathbb{Z}$ .

In general an infinite dimensional Lie algebra over a field  $k$  of characteristic zero contains an infinite number of subalgebras of each given finite codimension. So it doesn't make sense to count them in a conventional way and try to encode this counting function in a Dirichlet or Poincaré series with integer coefficients. Instead, in the cases we consider in the present paper, the set of subalgebras of a fixed finite codimension form a constructible subset of some Grassmannian, so we may associate to it an element of the Grothendieck ring  $\mathcal{M}$ , and we can construct the zeta function as a Poincaré series with coefficients in  $\mathcal{M}$ .

**1.2. An example : counting  $k[[t]]$ -submodules of  $k[[t]]^2$ .** To illustrate the thinking behind this, it is instructive to see how we would count  $k[[t]]$ -submodules of finite codimension in the free  $k[[t]]$ -module  $L = k[[t]]^2$ . Let  $e_1, e_2$  be a basis for  $L$ , then each subalgebra  $H$  of codimension 1 can be represented by coordinates in a  $k[[t]]$ -basis with respect to this standard basis. We can record these coordinates in a matrix whose rows are the coordinates for the basis of the subalgebra  $H$ . By normalizing the matrix into upper triangular form we can choose a unique representative

$$\begin{pmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{pmatrix}$$

for each  $H$ . We have two cases:

- (1)  $m_{22} \in t(k[[t]])^*$ , and  $m_{11} \in (k[[t]])^*$ . To get a unique representative we can assume that  $m_{11} = 1$  and  $m_{22} = t$ . By subtracting  $k[[t]]$ -combinations of the second row from the first we can choose  $m_{12} \in k$ . So the subalgebras of this sort give a space of subalgebras which looks like  $k$ .
- (2) If  $m_{22} \in (k[[t]])^*$ , and  $m_{11} \in t(k[[t]])^*$  then we get a unique representative of the form

$$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}.$$

This gives just one subalgebra of codimension 1.

In the case of  $(\mathbb{Z}_p^2, +)$  this analysis showed us that there were  $p + 1$  subalgebras of index  $p$ . Here we would like to say that there are  $\mathbb{A}_k^1 + 1$  subalgebras, with  $\mathbb{A}_k^1$  the affine line over  $k$ , since, for any field  $K$  containing  $k$ , the  $K[[t]]$ -submodules of  $K[[t]]^2$  are parametrized by  $K + 1 = \mathbb{A}_k^1(K) + 1$ . For that to make sense one is naturally led to work in the Grothendieck ring  $\mathcal{M}$ . So if we set  $\mathbf{L} := [\mathbb{A}_k^1]$  in  $\mathcal{M}$ , we can interpret our analysis of the subalgebras in  $((k[[t]])^2, +)$  as saying there are  $\mathbf{L} + 1$  subalgebras of codimension 1.

On the other hand if we consider counting all vector subspaces of codimension 1, *i.e.* ignoring the action of  $t$ , then there are really too many subspaces to be able to assign an element of the Grothendieck ring  $\mathcal{M}$ . In fact, this is similar to the situation when  $k = \mathbb{F}_p$ . There are  $1 + p + \dots + p^n$   $\mathbb{F}_p[[t]]$ -submodules of index  $p^{n+1}$  in  $(\mathbb{F}_p[[t]])^2$  but an infinite number of additive  $\mathbb{F}_p$ -subspaces of index  $p^{n+1}$ .

**1.3. Stability and the motivic zeta function.** The key to being able to assign an element of the Grothendieck ring to subalgebras of a fixed codimension is a notion we call *stability*. The context in which we shall work is that of  $\mathbb{N}$ -filtered or  $\mathbb{Z}$ -filtered

Lie algebras over a field  $k$  of characteristic zero. These are infinite dimensional Lie algebras  $L$  with a filtration  $L = \bigcup L_i$  indexed by  $\mathbb{N}$  or  $\mathbb{Z}$  consisting of subalgebras  $L_i$  of finite codimension in  $L_0$ . By a class  $\mathcal{X}$  of subalgebras of  $L$ , we shall mean the data, for every field  $K$  which is a finite extension of  $k$ , consisting of a family  $\mathcal{X}(K)$  of subalgebras of  $L \otimes K$ . We will sometimes, with a slight abuse, identify  $\mathcal{X}$  with the union of the various sets  $\mathcal{X}(K)$ . Let  $A_n(\mathcal{X})(K)$  denote the set of subalgebras in  $\mathcal{X}(K)$  of codimension  $n$  in  $L \otimes K$ . We say that a class of subalgebras  $\mathcal{X}$  of  $L$  is *stable* if, for every  $n$ , there exists some  $f(n)$  such that, for every  $K$ , if  $H$  belongs to  $A_n(\mathcal{X})(K)$ , then  $H$  contains  $L_{f(n)} \otimes K$ .

In this case it is possible to show for certain classes of subalgebras  $\mathcal{X}$  that  $A_n(\mathcal{X})$  is a constructible subset of the Grassmannian  $\text{Gr}(L/L_{f(n)})$ . We can then define the concept of a *motivic zeta function*:

$$P_{L,\mathcal{X}}(T) := \sum_{n=0}^{\infty} [A_n(\mathcal{X})] T^n$$

where the coefficients  $[A_n(\mathcal{X})]$  belong to  $\mathcal{M}$ . This zeta function will be our interpretation of how to add the series

$$\sum_{H \in \mathcal{X}} T^{\text{codim } H},$$

which is then an element of  $\mathcal{M}[[T]]$ .

We also introduce a variant of this function, which has not been considered previously in the  $p$ -adic case, which counts subalgebras commensurable with the subalgebra  $L_0$  of infinite codimension in a  $\mathbb{Z}$ -filtered Lie algebra  $L$ . A subalgebra  $H$  is commensurable with  $L_0$  if  $H \cap L_0$  has finite codimension in  $H$  and  $L_0$ . We then define the codimension of  $H$  in  $L_0$  to be the sum of the codimension of  $H \cap L_0$  in  $H$  and  $L_0$ .

Our approach to proving the stability of subalgebras in a filtered Lie algebra is a new concept we call *well-covered*. This means essentially that elements in the  $n$ -th term of the grading can be realised in many ways as commutators of elements further up the grading. We prove that affine Kac-Moody Lie algebras (which are built from loop algebras  $L \otimes k[[t]]$ , with  $L$  a finite dimensional Lie algebra over  $k$ ), the Virasoro algebra and Lie algebras of Cartan type are well-covered and hence have an associated motivic zeta function.

On the other hand we show that a finitely generated free Lie algebra is not well-covered since an Engel element like  $(a, b, b, \dots, b)$  of length  $n$  cannot be realised as a commutator in more than one way. Hence we can show that such Lie algebras don't have an associated motivic zeta function.

**1.4. Rationality of the motivic zeta function.** Once the motivic zeta function is defined, it is natural to consider the question of its rationality. By using results from [7] and [6], we are able to prove the rationality of motivic zeta functions counting  $k[[t]]$ -subalgebras in  $k[[t]]$ -Lie algebras of the form  $L_k \otimes k[[t]]$  where  $L_k$  is a finite dimensional Lie algebra over  $k$ , a field of characteristic zero.

Denote by  $\mathcal{M}_{\text{loc}}$  the ring  $\mathcal{M}[\mathbf{L}^{-1}]$  obtained by localization and define by  $\mathcal{M}[T]_{\text{loc}}$  the subring of  $\mathcal{M}_{\text{loc}}[[T]]$  generated by  $\mathcal{M}_{\text{loc}}[T]$  and the series  $(1 - \mathbf{L}^a T^b)^{-1}$  with  $a \in \mathbb{Z}$  and  $b \in \mathbb{N}$ .

**1.5. Theorem.** *Let  $k$  be a field of characteristic zero.*

- (1) *If  $L$  is a finite dimensional free  $k[[t]]$ -Lie algebra of the form  $L = L_k \otimes_k k[[t]]$ , with  $L_k$  a Lie algebra over  $k$ , and  $\mathcal{X}(K)$  is the set of all  $K[[t]]$ -subalgebras of  $L \otimes K$ , then  $P_{L,\mathcal{X}}(T)$  is well-defined and is rational, belonging to  $\mathcal{M}[T]_{\text{loc}}$ .*
- (2) *If  $L$  is a finite dimensional  $k((t))$ -Lie algebra and  $L_0$  is a choice of some  $k[[t]]$ -Lie subalgebra of the form  $L_0 = L_k \otimes_k k[[t]]$ , with  $L_k$  a Lie algebra over  $k$ , and  $\mathcal{X}(K)$  is the set of all  $K[[t]]$ -subalgebras of  $L \otimes K$  commensurable with  $L_0 \otimes K$ , then  $P_{L_0,\mathcal{X}}(T)$  is well-defined and is rational, belonging to  $\mathcal{M}[T]_{\text{loc}}$ .*

The key to the proof of this Theorem is to express the motivic zeta function as a motivic integral as developed by Denef and the second author [7] and [6]. In fact we are naturally led to introduce the concept of a motivic measure on the infinite Grassmannian.

**1.6. Implications for  $p$ -adic zeta functions.** If  $L$  is a Lie algebra over  $\mathbb{Z}$ , then the motivic zeta function of  $L \otimes \mathbb{C}[[t]]$  has implications for the zeta functions of  $L \otimes \mathbb{Z}_p$  as we range over primes  $p$ . In particular we can show, by “taking the trace of Frobenius” of the motivic zeta function that the explicit formulae calculated in [13] and alluded to in 1.1 for  $\zeta_{L \otimes \mathbb{Z}_p}(s)$  are canonical and hence independent of a choice of resolution and simplicial decomposition of a cone which are involved in the calculation.

We also define the concept of the topological zeta function of  $L$  which is the analogue of the topological Igusa zeta function defined in [5]. This can be thought of as the limit as  $p \mapsto 1$  of the explicit formulae expressing  $\zeta_{L \otimes \mathbb{Z}_p}(s)$ . To show that this is well-defined we show that this is the same as taking the Euler characteristic of the associated motivic zeta function, similarly as in [6].

## CONTENTS

1.	Introduction	1
2.	Definition of the motivic zeta function	5
3.	Motivic integration and rationality of Poincaré series	20
4.	Motivic integration on the infinite Grassmannian	24
5.	Rationality of motivic zeta functions of infinite dimensional Lie algebras	27
6.	$k[[t]]$ -powered nilpotent groups	32
7.	Motivic and $p$ -adic cone integrals	34
8.	Topological zeta functions	40
9.	Examples	42

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**Notations and terminology.** We shall use the notation  $\mathbb{N} = \{0, 1, 2, \dots\}$ . For  $k$  a field, we denote by  $\mathbb{A}_k^n$  the affine space  $\text{Spec } k[x_1, \dots, x_n]$ . By an algebraic variety over  $k$ , we shall always mean a scheme of finite type over  $k$ , which is separated and reduced. Except explicitly stated otherwise, all Lie algebras we consider will be assumed to be Lie algebras over  $k$ . We shall consider a measure  $\tilde{\mu}$  taking values in  $\mathcal{M}_{\text{loc}}$  and a measure  $\mu$  taking values in the completion  $\widehat{\mathcal{M}}$  of  $\mathcal{M}_{\text{loc}}$ .

## 2. DEFINITION OF THE MOTIVIC ZETA FUNCTION

We fix a field  $k$ . In this section we explore some infinite dimensional Lie algebras for which one can define a motivic zeta function.

**2.1. Definition.** (1) An  $\mathbb{N}$ -filtered Lie algebra  $L$  is a Lie algebra equipped with a filtration by subspaces  $L_j$ ,  $j \in \mathbb{N}$ , of finite codimension with  $L_{j+1} \subset L_j$  satisfying  $(L_j, L_k) \subset L_{j+k}$  and  $\bigcap_{i \in \mathbb{N}} L_i = 0$ . We shall assume for convenience that  $L_0 = L$ . (2) Similarly a  $\mathbb{Z}$ -filtered Lie algebra comes with a filtration by subspaces  $L_j$ ,  $j \in \mathbb{Z}$ , with  $L_{j+1} \subset L_j$  satisfying  $(L_j, L_k) \subset L_{j+k}$  where we assume that  $L_j$  is commensurable with  $L_0$  and  $L = \bigcup_{j \in \mathbb{Z}} L_j$  and  $\bigcap_{i \in \mathbb{Z}} L_i = 0$ .

The filtration defines a (formal) topology on  $L$  defined by the neighbourhood base of zero consisting of subspaces  $L_j$ . A subspace  $H$  is commensurable with  $L_0$  if the intersection  $H \cap L_0$  has finite codimension in both  $H$  and  $L_0$ . We define the codimension of  $H$  in  $L_0$  in this case as the sum of the codimension of  $H \cap L_0$  in  $H$  and  $L_0$ . Note that in an  $\mathbb{N}$ -filtered Lie algebra the  $L_j$  are ideals but this is not necessarily the case in a  $\mathbb{Z}$ -filtered Lie algebra since  $(L_{-j}, L_k) \subset L_{k-j}$ .

**2.2. Remark.** We can carry out the analysis of this section just under the assumption that the Lie algebra  $L$  comes only with a formal topology of subspaces  $L_j$  of finite codimension without insisting that  $(L_j, L_k) \subset L_{j+k}$ . However essentially all the cases where we can get the theory up and running apply to the examples where the chain of subspaces is actually a filtration. However in any particular case it may not be necessary to assume anything this strong. It may be possible to show that subalgebras are a constructible subset of the finite Grassmannian  $\text{Gr}(L/L_j)$ .

**2.3. Examples of filtered Lie algebras.** (1) A good example of an  $\mathbb{N}$ -filtration on  $L$  is the lower central series. In this case  $L_0 = L_1$  and  $L_i = \gamma_i(L) = (\gamma_{i-1}(L), L)$ . We are insisting that the filtration be of subalgebras of finite codimension. If the first layer  $L_1/L_2$  is finite dimensional then this implies the same for the other layers  $L_i/L_{i+1}$ . If  $(L, L)$  has infinite codimension then the theory won't get up and running,

since there will be too many subalgebras of codimension 1. Note that, by definition of an  $\mathbb{N}$ -filtration,  $L_i$  contains  $\gamma_i(L_1)$ .

(2) Another example arises if  $L$  has some  $\mathbb{N}$ -grading, so that  $L = \bigoplus_{j \geq 0} L(j)$  and suppose that  $L(j)$  is finite dimensional then put  $L_k = \bigoplus_{j \geq k} L_j$ .

(3) If  $L$  has the structure of a  $k[[t]]$ -module then put  $L_k = t^k L$ .

(4) If  $L$  comes with a  $\mathbb{Z}$ -grading  $L = \bigoplus_{j \in \mathbb{Z}} L(j)$  then we can define  $L_k = \bigoplus_{j \geq k} L(j)$  for  $k \in \mathbb{Z}$  and consider subalgebras commensurable with  $L_0$ .

(5) Finally if  $L$  is a  $k((t))$ -Lie algebra with a choice of  $k[[t]]$ -subalgebra  $L_0$  we take  $L_k = t^k L_0$  for  $k \in \mathbb{Z}$ .

**2.4. Constructible sets of subalgebras in Grassmannians.** It is the subalgebras of finite codimension that are closed with respect to the formal topology that we shall seek to count. Such subalgebras contain then some term  $L_j$  of the neighbourhood base and can therefore be viewed as a point of the finite Grassmannian  $\text{Gr}(L/L_j)$ . This follows because if  $H$  is closed,  $H = \bigcap_{i \in \mathbb{N}} (H + L_i)$  and since  $H$  has finite codimension, the chain  $H + L_i$  must stabilize at some point.

In the case of a closed commensurable subalgebra  $H$  in a  $\mathbb{Z}$ -filtered Lie algebra, there exists some  $j$  with  $L_{-j} \supseteq H \supseteq L_j$ . Hence these will be points of the finite Grassmannian  $\text{Gr}(L_{-j}/L_j)$ . For convenience we set  $X_j = \text{Gr}(L/L_j)$  or  $\text{Gr}(L_{-j}/L_j)$  according to which situation we find ourselves.

*2.5. Remark.* In section 4 we shall introduce the concept of infinite Grassmannians which can be thought of as the direct limit of these finite Grassmannians. These are considered for example in Section 7.2 (i) of [23]. The set of subalgebras  $\mathcal{X}$  can then be thought of as subsets of these infinite Grassmannians. We shall see later that in some cases it is possible to put a motivic measure directly on an infinite Grassmannian and express the motivic zeta functions of this section as integrals over  $\mathcal{X}$  with respect to this measure.

*2.6. Remark.* Note that not all subalgebras of finite codimension will be closed with respect to a chosen formal topology. For example in  $\bigoplus_{i \in \mathbb{N}} ke_i$  with respect to the formal topology  $L_j = \bigoplus_{i \geq j} ke_i$ , the subspace

$$\langle e_0 - e_1, e_1 - e_2, \dots, e_k - e_{k+1}, \dots \rangle$$

has codimension 1 but contains no term of the filtration  $L_j$ .

*2.7. Remark.* There is a one-to-one correspondence preserving the codimension between closed subalgebras of finite codimension of  $L$  and closed subalgebras of finite codimension of the completion of  $L$  with respect to the formal topology.

**2.8.** Let  $Y$  be an algebraic variety of finite type over  $k$ . By the set underlying  $Y$  we shall always mean the set of closed points of  $Y$ . These points correspond to rational points of  $Y$  over fields  $K$  which are finite extensions of  $k$ . The boolean algebra of constructible subsets of  $Y$  is the smallest family of subsets of  $Y$  containing all Zariski closed subsets of  $Y$  and stable by taking finite unions and complements. It follows from the definitions that to any constructible subset  $A$  of  $Y$  one can associate a canonical element  $[A]$  of the Grothendieck ring  $\mathcal{M}$ .

We shall now view the finite Grassmannian  $X_j = \text{Gr}(L/L_j)$  or  $\text{Gr}(L_{-j}/L_j)$  as an algebraic variety of finite type over  $k$ . Hence points of  $X_j$  will be linear spaces defined over some field  $K$  which is a finite extension of  $k$ .

By a class  $\mathcal{X}$  of subalgebras of  $L$ , we shall mean the data, for every field  $K$  which is a finite extension of  $k$ , consisting of a family  $\mathcal{X}(K)$  of subalgebras of  $L \otimes K$ . For every  $l, n \in \mathbb{N}$ , and every field  $K$  which is a finite extension of  $k$ , we define the subsets:

$$\begin{aligned} A_n(\mathcal{X})(K) &:= \{H \in \mathcal{X}(K) : \text{codim}_{L \otimes K} H = n\} \\ A_{l,n}(\mathcal{X})(K) &:= \{H \in A_n(\mathcal{X})(K) : H \supseteq L_l \otimes K\}. \end{aligned}$$

Hence  $A_{l,n}(\mathcal{X})(K)$  can be thought of then as a subset of  $X_l(K)$ , and the union of these subsets when varying  $K$  is a subset  $A_{l,n}(\mathcal{X})$  of the finite Grassmannian  $X_l$ .

**2.9. Definition.** We call  $\mathcal{X}$  a *constructible class of subalgebras* if  $A_{l,n}(\mathcal{X})$  is a constructible subset of  $X_l$ , for every  $l, n \in \mathbb{N}$ .

**2.10. Examples of constructible classes of subalgebras.** For  $L$  an  $\mathbb{N}$ -filtered Lie algebra, we define  $\mathcal{X}^{\leq}$  as the class of closed subalgebras, *i.e.*  $\mathcal{X}^{\leq}(K)$  is the set of all closed subalgebras of  $L \otimes K$ , and  $\mathcal{X}^{\triangleleft}$  as the class of closed subalgebras which are ideals, *i.e.*  $\mathcal{X}^{\triangleleft}(K)$  is the set of all closed subalgebras of  $L \otimes K$  which are ideals of  $L \otimes K$ .

If  $L$  is a  $\mathbb{Z}$ -filtered Lie algebra, the corresponding classes of commensurable subalgebras are defined as follows: for  $*$   $\in \{\leq, \triangleleft\}$ ,  $\mathcal{X}_0^*(K)$  is the set of subalgebras  $H$  in  $\mathcal{X}^*(K)$  which are commensurable with  $L_0 \otimes K$ .

**2.11. Lemma.** *Let  $L$  be an  $\mathbb{N}$ -filtered Lie algebra then for  $*$   $\in \{\leq, \triangleleft\}$ ,  $\mathcal{X}^*$  is a constructible class of subalgebras.*

*Proof.* Choose a basis  $e_1, \dots, e_r$  for  $L/L_l$ . Let  $\beta : \mathbb{A}_k^{2r} \rightarrow \mathbb{A}_k^r$  denote the bilinear form defining the Lie bracket in  $L/L_l$  with respect to this basis. Let  $\text{Tr}_{r,k}$  denote the  $k$ -variety of upper triangular matrices of order  $r$ . Hence the  $K$ -points of  $\text{Tr}_{r,k}$  are just the upper triangular matrices of order  $r$  with coefficients in  $K$ . We have a morphism

$$h : \text{Tr}_{r,k} \longrightarrow \text{Gr}(L/L_l)$$

which takes the matrix  $(m_{ij})$  to the subspace spanned by

$$\{m_{11}e_1 + \dots + m_{1r}e_r, \dots, m_{rr}e_r\}.$$

Let  $\mathcal{M}_{l,n}^*$  denote the inverse image of  $A_{l,n}(\mathcal{X}^*)$  under this map. Since there is a finite partition of  $\text{Gr}(L/L_l)$  into locally closed subvarieties over which  $h$  induces morphisms which are locally trivial fibrations for the Zariski topology, it suffices to show that  $\mathcal{M}_{l,n}^*$  is a constructible subset of  $\text{Tr}_{r,k}$ .

- (1) a matrix  $(m_{ij})$  in  $\text{Tr}_{r,k}(K)$  defines a subspace of codimension  $n$  in  $L \otimes K$  if and only if the diagonal  $(m_{11}, \dots, m_{rr})$  contains exactly  $n$  zero entries.

(2<sup>≤</sup>) a matrix  $(m_{ij})$  in  $\mathrm{Tr}_{r,k}(K)$  defines a subalgebra of  $L \otimes K$  if and only if for each  $1 \leq i < j \leq r$  there exist  $Y_{ij}^1, \dots, Y_{ij}^r \in K$  such that

$$\beta(\mathbf{m}_i, \mathbf{m}_j) = \sum_{k=1}^r Y_{ij}^k \mathbf{m}_k$$

where  $\mathbf{m}_i$  denotes the  $i$ th row of  $(m_{ij})$ ;

(2<sup>∠</sup>) a matrix  $(m_{ij})$  in  $\mathrm{Tr}_{r,k}(K)$  defines an ideal of  $L \otimes K$  if and only if for each  $1 \leq i, j \leq r$  there exist  $Y_{ij}^1, \dots, Y_{ij}^r \in K$  such that

$$\beta(\mathbf{e}_i, \mathbf{m}_j) = \sum_{k=1}^r Y_{ij}^k \mathbf{m}_k$$

where  $\mathbf{e}_i$  denotes the  $r$ -tuple with 1 in the  $i$ -th entry and zeros elsewhere.

The two conditions on a matrix  $(m_{ij})$  in (1) and (2\*) define  $\mathcal{M}_{l,n}^*$  as a constructible subset of  $\mathrm{Tr}_{r,k}$ .  $\square$

If  $L$  is a finite dimensional  $k[[t]]$ -Lie algebra then we can also define the classes of subalgebras  $\mathcal{X}_t^{\leq}$  with  $\mathcal{X}_t^{\leq}(K)$  the set of closed subalgebras of  $L \otimes K$  which are  $K[[t]]$ -submodules of  $L \otimes K$  and  $\mathcal{X}_t^{\leq}$  defined by  $\mathcal{X}_t^{\leq}(K) = \mathcal{X}_t^{\leq}(K) \cap \mathcal{X}^{\leq}(K)$ .

**2.12. Lemma.** *Let  $L$  be a finite dimensional  $k[[t]]$ -Lie algebra whose filtration consists of the ideals  $L_j = t^j L$ . Then for  $*$   $\in \{\leq, \angle\}$ ,  $\mathcal{X}_t^*$  are constructible classes of subalgebras.*

*Proof.* We just have to add the constructible condition:

(3) a matrix  $(m_{ij})$  in  $\mathrm{Tr}_{r,k}(K)$  defines a  $K[[t]]$ -submodule of  $L \otimes K$  if and only if for each  $1 \leq i \leq r$  there exist  $Y_i^1, \dots, Y_i^r \in K$  such that

$$t\mathbf{m}_i = \sum_{k=1}^r Y_i^k \mathbf{m}_k.$$

$\square$

We shall see later a better reason for these subalgebras being constructible when we realise  $\mathcal{X}_t^*$  as a semi-algebraic subset in a suitable arc space.

**2.13. Lemma.** *Suppose that  $L$  is a  $\mathbb{Z}$ -filtered Lie algebra. Then  $\mathcal{X}_0^{\leq}$  is a constructible set of subalgebras.*

*Proof.* We want to put a condition on a subspace of  $L_{-j}/L_j$  that it is a subalgebra of codimension  $n$  in  $L_0$ . The trouble is now that  $L_j$  is not necessarily an ideal in  $L_{-j}$ . However we use the fact that  $(L_{-j}, L_{2j}) \subset L_j$ . Let  $e_0, e_{-1}, \dots, e_{-s_1}$  be a basis for  $L_{-2j}/L_0$  with  $e_0, e_{-1}, \dots, e_{-s}$  a basis for  $L_{-j}/L_0$ , and  $e_1, \dots, e_{r_1}$  be a basis for  $L_0/L_{2j}$  with  $e_1, \dots, e_r$  a basis for  $L_0/L_j$ . Let  $\beta : \mathbb{A}_k^{s+r_1+1} \times \mathbb{A}_k^{s+r_1+1} \rightarrow \mathbb{A}_k^{s_1+r+1}$  be the bilinear map defining the Lie bracket from  $L_{-j}/L_{2j} \rightarrow L_{-2j}/L_j$  with respect to these choices of basis.

There is still a morphism

$$\mathrm{Tr}_{s+r+1,k} \rightarrow \mathrm{Gr}(L_{-j}/L_j)$$

which takes a matrix  $(m_{ij})$ ,  $-s \leq i \leq j \leq r$ , to the subspace spanned by

$$\{m_{-s,-s}e_{-s} + \cdots + m_{-s,r}e_r, \dots, m_{rr}e_r\}.$$

(1) a matrix  $(m_{ij})$  in  $\text{Tr}_{s+r+1,k}(K)$  defines a subspace of codimension  $n$  in  $L_0 \otimes K$  if and only if the diagonal  $(m_{11}, \dots, m_{rr})$  contains exactly  $n_1$  zero entries and the diagonal  $(m_{-s,-s}, \dots, m_{00})$  contains  $n_2$  non zero entries and  $n_1 + n_2 = n$ .

(2 $\leq$ ) the subspace  $H = \langle m_{-s,-s}e_{-s} + \cdots + m_{-s,r}e_r, \dots, m_{rr}e_r \rangle + L_j \otimes K$  defines a subalgebra of  $L \otimes K$  if and only if, for  $-s \leq i < j \leq r$  and for all  $\lambda = (\lambda_{r+1}, \dots, \lambda_{r_1})$ ,  $\mu = (\mu_{r+1}, \dots, \mu_{r_1}) \in K^{r_1-r}$  there exist  $Y_{ij}^{-s}, \dots, Y_{ij}^r \in K$  such that

$$\beta(\mathbf{m}_i, \lambda, \mathbf{m}_j, \mu) = \left( 0, \dots, 0, \sum_{k=-s}^r Y_{ij}^k \mathbf{m}_k \right)$$

where there are  $s_1 - s$  zeros. The point is that we are going to be guaranteed  $(H, L_{2j}) \subset H$  so we just need to check that  $H$  is a subalgebra modulo  $L_{2j}$  which is finite dimensional and therefore leads to a constructible subset.  $\square$

Proving that ideals in a  $\mathbb{Z}$ -filtered Lie algebra define a constructible set is slightly more problematic since we need to check that the action of all the  $L_{-j}$  stabilise the candidate ideal  $H$ . However, if  $L$  is finitely generated we need only check these finite number of generators:

**2.14. Lemma.** *Suppose that  $L$  is a finitely generated  $\mathbb{Z}$ -filtered Lie algebra. Then  $\mathcal{X}_0^{\leq}$  is a constructible set of subalgebras.*

*Proof.* Let  $f_1, \dots, f_d$  be a finite set of generators. Then, for any fixed  $j$ , there exists some  $N(j)$  and  $M(j)$  such that  $(f_i, L_{N(j)}) \subset L_j$  and  $(f_i, L_{-j}) \subset L_{-M(j)}$ . So we can carry out the same analysis essentially as the previous lemma. Let  $e_0, e_{-1}, \dots, e_{-s_1}$  be a basis for  $L_{-M(j)}/L_0$  with  $e_0, e_{-1}, \dots, e_{-s}$  a basis for  $L_{-j}/L_0$ , and  $e_1, \dots, e_{r_1}$  be a basis for  $L_0/L_{N(j)}$  with  $e_1, \dots, e_r$  a basis for  $L_0/L_j$ . Let  $\varphi_i : \mathbb{A}_k^{s+r_1+1} \rightarrow \mathbb{A}_k^{s_1+r+1}$  define the action of  $f_i$  taking  $L_{-j}/L_{N(j)} \rightarrow L_{-M(j)}/L_j$  with respect to these choices of basis. Then

(2 $\leq$ ) the subspace  $H = \langle m_{-s,-s}e_{-s} + \cdots + m_{-s,r}e_r, \dots, m_{rr}e_r \rangle + L_j \otimes K$  defines an ideal of  $L \otimes K$  if and only if it is a subalgebra of  $L \otimes K$  and for  $-s \leq j \leq r$  and  $i = 1, \dots, d$  for all  $\lambda = (\lambda_{r+1}, \dots, \lambda_{r_1}) \in K^{r_1-r}$  there exist  $Y_{ij}^{-s}, \dots, Y_{ij}^r \in K$  such that

$$\varphi_i(\mathbf{m}_i, \lambda) = \left( 0, \dots, 0, \sum_{k=-s}^r Y_{ij}^k \mathbf{m}_k \right)$$

where there are  $s_1 - s$  zeros.  $\square$

Let  $L$  be a finite dimensional  $k((t))$ -Lie algebra and  $L_0$  be some choice of a  $k[[t]]$ -submodule of the same dimension. Set  $L_j = t^{-j}L_0$  for  $j \in \mathbb{Z}$ . Note that in this case there are no commensurable ideals unless  $L$  is abelian. The reason for this is as follows. Let  $e_1, \dots, e_d$  be a basis for the  $k[[t]]$ -submodule  $L_0$ . Then commensurable ideals must be contained in the centre of  $L$  since if  $H \leq L_{-j}$  is an ideal and  $(e_i, h) =$

$x \in L_k \setminus L_{k+1}$  then  $(t^{-k-j-1}e_i, h) = t^{-k-j-1}x \notin H$ . But the centre  $Z(L)$  is a  $k((t))$ -submodule. So if there exists a commensurable ideal  $H$  then there exists some  $L_j \leq H \leq Z(L)$ . Hence  $Z(L) = L$ . So the case of commensurable ideals in finite dimensional  $k((t))$ -Lie algebras is not an interesting one.

In the same manner as above we can prove:

**2.15. Lemma.** *Suppose that  $L$  is a finite dimensional  $k((t))$ -Lie algebra. Then  $\mathcal{X}_{t,0}^{\leq}$ , defined by*

$$\mathcal{X}_{t,0}^{\leq}(K) = \{H \in \mathcal{X}_0^{\leq}(K) : H \text{ is a } K[[t]]\text{-submodule of } L \otimes K\},$$

*is a constructible class of subalgebras.*

*Proof.* Again we just have to add the constructible condition:

(3) a matrix  $(m_{ij})$  in  $\text{Tr}_{s+r+1,k}(K)$  defines a  $K[[t]]$ -submodule if and only if, for each  $-s \leq i \leq r$ , there exist  $Y_i^{-s}, \dots, Y_i^r \in K$  such that

$$tm_i = \sum_{k=-s}^r Y_i^k \mathbf{m}_k.$$

□

**2.16. Stable classes of subalgebras and motivic zeta functions.** If  $\mathcal{X}$  is a constructible class of subalgebras then  $A_{l,n}(\mathcal{X})$  defines an element  $[A_{l,n}(\mathcal{X})]$  of the Grothendieck ring  $\mathcal{M}$ . The only way that this can have a limit as  $l \rightarrow \infty$  is that the series stabilize at some point.

**2.17. Definition.** We shall call a constructible class of subalgebras  $\mathcal{X}$  a *stable class of subalgebras* if for every  $n$  there exists some  $l$  such that  $A_{l,n} = A_{k,n}$  for all  $k \geq l$ . When this is the case we set  $[A_n] = [A_{l,n}]$  and we define the *motivic zeta function* of  $L$  and  $\mathcal{X}$  as

$$P_{L,\mathcal{X}}(T) = \sum_{n=0}^{\infty} [A_n] T^n.$$

Notice we can now say why it isn't sensible to count  $k$ -subalgebras of finite codimension in  $k[[t]]^2$  with trivial Lie structure since they are not a stable system of subalgebras. As we said above, there are too many such subalgebras. In fact  $\mathcal{X}^{\leq}$  (respectively  $\mathcal{X}^{\triangleleft}$ ) will not be stable for all  $\mathbb{N}$ -filtered or  $\mathbb{Z}$ -filtered Lie algebras  $L$  for which  $\overline{(H, H)}$  (respectively  $\overline{(L_0, H)}$ ) has infinite codimension for any closed subalgebra  $H$  (respectively ideal  $H$ ) of finite codimension.

**2.18. Stability of ideals.** If we only want to count ideals then we can prove stability for all  $\mathbb{N}$ -filtered Lie algebras, for which the closure of terms in the lower central series of  $L_1$  have finite codimension:

**2.19. Theorem.** *Suppose  $L = \bigcup_{i \in \mathbb{N}} L_i$  is an  $\mathbb{N}$ -filtered Lie algebra and  $\overline{\gamma_i(L_1)}$  has finite codimension for all  $i$  (where  $\overline{\gamma_i(L)}$  is the closure of the  $i$ th term of the lower central series with respect to the formal topology defined by  $L_i$ ). Then  $\mathcal{X}^{\triangleleft}(L)$  is a stable class of subalgebras.*

*Proof.* Suppose  $H$  is a closed ideal of codimension  $n$  in  $L$ . Let us show that each ideal  $H \cap \bigcup_{i \geq 1} L_i$  of codimension  $n$  in  $\bigcup_{i \geq 1} L_i$  contains some fixed term of the filtration. We can assume  $L_0 = L_1$ . Let us consider  $H_i = (H \cap \gamma_i(L)) \gamma_{i+1}(L)$  where  $\gamma_i(L)$  is the lower central series of  $L$  (which might be distinct from the closure of the lower central series with respect to the filtration  $L_i$ ). The ideal  $H$  is closed and hence contains some term of the filtration  $L_m \supset \gamma_m(L)$ . There must be some  $1 \leq i \leq n+1$  such that  $H_i = \gamma_i(L)$  else the ideal would have codimension greater than  $n$ . Hence  $H + \gamma_{i+1}(L) \supset \gamma_i(L)$ . So  $H \cap \gamma_i(L) \equiv \gamma_i(L) \pmod{\gamma_{i+1}(L)}$ . Now  $H \cap \gamma_{i+1}(L) \supset (L, H \cap \gamma_i(L))$  since  $H$  is an ideal. Hence  $H \cap \gamma_{i+1}(L) \equiv \gamma_{i+1}(L) \pmod{\gamma_{i+2}(L)}$  and therefore  $H + \gamma_{i+2}(L) \supset H + \gamma_{i+1}(L) \supset \gamma_i(L)$ . Continuing this analysis shows that  $H \cap \gamma_{i+e}(L) \equiv \gamma_{i+e}(L) \pmod{\gamma_{i+e+1}(L)}$  for all  $e$  hence  $H + \gamma_{i+e+1}(L) \supset H + \gamma_{i+e}(L) \supset \gamma_i(L)$ . But  $H + \gamma_{i+e+1}(L) = H$  for some  $e$  since  $H$  is a closed subalgebra of finite codimension. This implies that  $H \supset \gamma_{n+1}(L)$ . Since  $H$  is closed,  $H \supset \overline{\gamma_{n+1}(L)} \supset L_{f(n)}$ . Similarly, if  $K$  is a finite extension of  $k$ , every closed ideal of codimension  $n$  in  $L \otimes K$  contains  $L_{f(n)} \otimes K$ , whence  $\mathcal{X}^\triangleleft(L)$  is a stable class of subalgebras.  $\square$

**2.20. Corollary.** *Suppose  $L = \bigcup_{i \in \mathbb{N}} L_i$  is an  $\mathbb{N}$ -filtered Lie algebra and  $L_0 = L_1$ . Then  $\mathcal{X}^\triangleleft(L)$  is a stable class of subalgebras if and only if  $\overline{\gamma_i(L)}$  has finite codimension for all  $i$ .*

Note that  $\overline{\gamma_i(L)}$  has finite codimension for all  $i$  if and only if  $\overline{(L_1, L_1)}$  has finite codimension.

2.21. *Remark.* There is a case missing in the above analysis: when  $L_0 \neq L_1$ ,  $\overline{(L_0, H)}$  has finite codimension for all closed ideals  $H$  of finite codimension but  $\overline{(L_1, L_1)}$  has infinite codimension.

2.22. **Stability of subalgebras.** We now describe several important categories of Lie algebras with stable classes of subalgebras. Note that for two classes of constructible subalgebras  $\mathcal{X}^\natural(L) \subset \mathcal{X}^\flat(L)$  once we have proved that  $\mathcal{X}^\flat(L)$  is stable, then  $\mathcal{X}^\natural(L)$  will also be stable.

**2.23. Theorem.** (1) *Let  $L$  be a finitely dimensional Lie algebra over  $k[[t]]$  and define the formal topology on  $L$  via  $L_l = t^l L$ . Then  $\mathcal{X}_t^*$  is a stable constructible class of subalgebras for  $* \in \{\leq, \triangleleft\}$ .*

(2) *Suppose that  $L$  is a finite dimensional  $k((t))$ -Lie algebra. Then  $\mathcal{X}_{t,0}^*$  for  $* \in \{\leq, \triangleleft\}$  is a stable class of subalgebras.*

*Proof.* This follows from the fact that a  $K[[t]]$ -submodule  $H$  of codimension  $n$  in  $L \otimes K$  must contain  $L_n \otimes K$ . In (2) we apply this to  $H \cap (L_0 \otimes K)$  which must have codimension bounded by  $n$ . Recall that the case  $\mathcal{X}_{t,0}^\triangleleft$  either is empty or else  $L$  is abelian and  $\mathcal{X}_{t,0}^\triangleleft = \mathcal{X}_{t,0}^\leq$ .  $\square$

2.24. **Loop algebras.** Examples of Lie algebras that are finite-dimensional as  $\mathbb{C}[[t]]$ -Lie algebras arise as the positive parts of loop algebras. These are defined as the complexified algebra of smooth maps from the circle  $S^1$  (or more generally some

compact complex manifold) into a finite dimensional  $\mathbb{C}$ -Lie algebra  $\mathfrak{g}_{\mathbb{C}}$ , and denoted  $L\mathfrak{g}_{\mathbb{C}}$ . We can decompose  $L\mathfrak{g}_{\mathbb{C}}$  into its Fourier components:

$$L\mathfrak{g}_{\mathbb{C}} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\mathbb{C}} \cdot t^k.$$

(One uses the notation  $L\mathfrak{g}$  to denote the set of loops, *i.e.* the set of smooth maps before we have complexified.) This is the decomposition into eigenspaces of the action of the circle which rotates the loops bodily. We define the positive part of the loop algebra to be

$$L_0\mathfrak{g}_{\mathbb{C}} = \bigoplus_{k \geq 0} \mathfrak{g}_{\mathbb{C}} \cdot t^k.$$

This is then isomorphic to the  $\mathbb{C}[[t]]$ -Lie algebra  $\mathfrak{g}_{\mathbb{C}} \otimes \mathbb{C}[[t]]$ .

These examples are then graded Lie algebras, unlike a general  $\mathbb{C}[[t]]$ -Lie algebra arising from a bilinear form defined over  $\mathbb{C}[[t]]$  on a finite dimensional  $\mathbb{C}[[t]]$ -module, which are not in general graded but just  $\mathbb{N}$ -filtered.

Note that  $L_0\mathfrak{g}_{\mathbb{C}} \otimes \mathbb{C}((t))$  is the  $\mathbb{C}$ -subalgebra generated by the rational loops sometimes denoted  $L_{\text{rat}}\mathfrak{g}_{\mathbb{C}}$  (see section 3.5 of [23]). Also it should be pointed out that  $L_{\text{rat}}\mathfrak{g}_{\mathbb{C}}$  is  $\mathbb{Z}$ -filtered with  $L_{\text{rat}}\mathfrak{g}_{\mathbb{C}} = \bigcup_{i \in \mathbb{Z}} t^i L_0\mathfrak{g}_{\mathbb{C}}$  whilst the Hilbert space of all loops  $L\mathfrak{g}_{\mathbb{C}}$  is something bigger containing  $L_{\text{rat}}\mathfrak{g}_{\mathbb{C}}$  as a dense subspace.

There is another important  $\mathbb{C}$ -subalgebra  $L_{\text{pol}}\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}} \otimes \mathbb{C}[z, z^{-1}]$  generated by all the loops which are given by Laurent polynomials in  $z$  and  $z^{-1}$ . This is the same as the Lie algebra of polynomial maps from  $\mathbb{C}^*$  into  $\mathfrak{g}$  -  $\mathbb{C}^*$  is the complexification of the circle (see 2.2 of [23]). In the case that  $\mathfrak{g}_{\mathbb{C}}$  is a simple finite dimensional  $\mathbb{C}$ -Lie algebra, the subalgebras of polynomial loops are what the Kac-Moody algebras are built from. Kac-Moody Lie algebras arise out of the existence of interesting central extensions that these loop algebras have. We shall see that affine Kac-Moody algebras are examples of infinite dimensional Lie algebras for whom the class of all subalgebras is a stable class.

**2.25. Kac-Moody Lie algebras.** A Kac-Moody Lie algebra over a field  $k$  of characteristic zero is defined in the first instance via a presentation associated to a generalized Cartan matrix (see [21]). The affine Kac-Moody algebras are those whose Cartan matrices have corank 1. They can then be realised as algebras whose derived group is the universal central extension of the Lie algebra  $L_{\text{pol}}\mathfrak{g}_k = \mathfrak{g}_k \otimes k[t, t^{-1}]$  of polynomial loops from  $k^*$  into a simple finite dimensional Lie algebra  $\mathfrak{g}_k$ . More generally if  $\mathfrak{g}_k = \mathfrak{g}_{1,k} \oplus \cdots \oplus \mathfrak{g}_{q,k}$  is a decomposition of  $\mathfrak{g}_k$  into simple factors then the product of algebras  $k \cdot \tilde{\bigoplus} \tilde{L}\mathfrak{g}_{j,k}$  is an affine Kac-Moody Lie algebra where  $k \cdot d \tilde{\bigoplus} \tilde{L}\mathfrak{g}_{j,k}$  means the semi-direct product where the factor  $k \cdot d$  is generated by the derivation  $d = t \frac{d}{dt}$  of  $\tilde{L}\mathfrak{g}_{j,k}$  and  $\tilde{L}\mathfrak{g}_{j,k}$  is the universal central extension of the polynomial loop algebra  $L_{\text{pol}}\mathfrak{g}_{j,k}$  which looks additively like  $k \cdot c \oplus L_{\text{pol}}\mathfrak{g}_{j,k}$ . Not all are covered by this construction. The other affine Kac-Moody Lie algebras arise by taking the same construction associated to the *twisted loop algebras*.

These are defined for a choice of outer automorphism  $\alpha$  of  $\mathfrak{g}_k$  of finite order  $m$  and for fields  $k$  containing primitive  $m$ -th roots of unity. One replaces  $L_{\text{pol}}\mathfrak{g}_k$  by its

subalgebra  $L_{(\alpha)}\mathfrak{g}_k$  consisting of the loops  $f \in L_{\text{pol}}\mathfrak{g}_k$  which are equivariant:

$$f(\varepsilon^{-1}t) = \alpha(f(t))$$

where  $\varepsilon$  is a primitive  $m$ -th root of unity. The automorphism gives rise to a decomposition of the finite dimensional Lie algebra:

$$\mathfrak{g}_k = \bigoplus_{j \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_j,$$

where  $\mathfrak{g}_j$  is the eigenspace of  $\alpha$  for the eigenvalue  $\varepsilon^j$ . Conversely any  $\mathbb{Z}/m\mathbb{Z}$ -grading of  $\mathfrak{g}_k$  arises like this since the linear transformation of  $\mathfrak{g}_k$  given by multiplying the vectors of  $\mathfrak{g}_j$  by  $\varepsilon^j$  is an automorphism  $\alpha$  of  $\mathfrak{g}_k$  which has order  $m$ . The twisted loop algebra then has the following description as a  $\mathbb{Z}$ -graded Lie algebra:

$$L_{(\alpha)}\mathfrak{g}_k = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_{n \bmod m} \otimes t^n.$$

(This is sometimes referred to as a loop algebra by algebraists because of the cyclic filtration and  $L_{\text{pol}}\mathfrak{g}_k$  as the loop algebra with trivial  $C_m$ -grading. This seems to be a little confusing and perhaps arose out of a misunderstanding that the original name loop arose from the setting over  $\mathbb{C}$  of maps from the circle (hence the name loop) rather than a graded Lie algebra whose filtration somehow comes from ‘looping’ the finite graded Lie algebra  $\mathfrak{g}$ .)

For more details see section 5.3 of [23]. Note that in our context there is little difference between taking polynomial loops or the rational loop space because polynomial loops are dense in the rational loops and the lattices of closed subalgebras of finite codimension will be in one to one correspondence.

**2.26. Theorem.** *If  $L$  is an affine Kac-Moody Lie algebra with no infinite dimensional abelian quotients and  $L_0$  is its positive part, then for  $*$   $\in \{\leq, \triangleleft\}$ ,  $\mathcal{X}^*(L_0)$  and  $\mathcal{X}_0^*(L)$  are stable classes of subalgebras.*

*Proof.* Let  $H$  be a closed subalgebra of  $L_0$  of codimension  $n$ . Suppose  $\mathfrak{g}_k = \mathfrak{g}_{1,k} \oplus \cdots \oplus \mathfrak{g}_{q,k}$  is the decomposition of the underlying simple Lie algebra. Then

$$L_0 = \bigoplus_{i=1}^q L_{(\alpha_i),0}\mathfrak{g}_{i,k} \oplus k \cdot c_i \oplus k \cdot d_i$$

where  $c_i$  is central,  $d_i$  is a derivation and  $\alpha_i$  is an automorphism of  $\mathfrak{g}_k$  which has order  $m$  ( $k$  is assumed to have primitive  $m$ -th roots of unity) and

$$L_{(\alpha),r}\mathfrak{g}_k = \bigoplus_{n \geq r} \mathfrak{g}_{n \bmod m} \otimes t^n.$$

We can restrict our attention to showing that a subalgebra  $H_i$  of codimension  $n_i$  in  $L_{(\alpha_i),0}\mathfrak{g}_{i,k}$  must contain some fixed term  $L_{(\alpha_i),f(n_i)}\mathfrak{g}_{i,k}$  of the filtration on  $L_{(\alpha_i),0}\mathfrak{g}_{i,k}$  depending only on  $n_i$ . For convenience we drop the subscript  $i$  and since we are only considering one twisted algebra at a time we put  $L_r = L_{(\alpha),r}\mathfrak{g}_k$ .

The assumption that  $L$  has no infinite abelian sections means that  $\mathfrak{g}_k$  is not abelian and hence  $\mathfrak{g}_k = (\mathfrak{g}_k, \mathfrak{g}_k)$  is perfect. This implies in turn that  $(L_{im}, L_{jm}) = L_{(i+j)m}$ . Let

$$H_i = (H \cap L_{im}) + L_{(i+1)m}/L_{(i+1)m}.$$

The codimension of  $H$  is then equal to the sum of the codimensions of  $H_i$  in  $L_{im}/L_{(i+1)m}$ . Therefore  $H_i \neq L_{im}/L_{(i+1)m}$  for at most  $n$  values of  $i$ . Therefore consider  $N \geq 2n$ , there must be some value of  $i$  with  $1 \leq i \leq N/2$  such that both  $H_i = L_{im}/L_{(i+1)m}$  and  $H_{N-i} = L_{(N-i)m}/L_{(N-i+1)m}$ . Hence  $H_N \supset (H_i, H_{N-i}) = L_{Nm}/L_{(N+1)m}$  for all  $N$ . Thus  $H \supset L_{2nm}$ .  $\square$

Note that an affine Kac-Moody Lie algebra is finitely generated hence  $\mathcal{X}_0^{\leq}(L)$  is constructible.

The key to this proof is that in affine Kac-Moody Lie algebras, each element can be realised as a commutator in many different ways.

In fact we can see that the argument applies to the following filtered Lie algebras:

**2.27. Definition.** Call an  $\mathbb{N}$ -filtered Lie algebra *well-covered* if it has the property that for each  $n$  there exists some  $f(n)$  such that  $L_{f(n)} = (L_i, L_{f(n)-i})$  for  $n+1$  values of  $i < f(n)/2$ . Call a  $\mathbb{Z}$ -filtered Lie algebra well-covered if its positive part is well-covered.

The proof of the theorem above can then easily be adapted to prove:

**2.28. Theorem.** (1) If the  $\mathbb{N}$ -filtered Lie algebra  $L$  is well-covered then  $\mathcal{X}^*(L)$  for  $*$   $\in \{\leq, \triangleleft\}$  are stable classes of subalgebras.

(2) If the  $\mathbb{Z}$ -filtered Lie algebra  $L$  is well-covered then  $\mathcal{X}_0^{\leq}(L)$  is a stable class of subalgebras.

The affine Kac-Moody Lie algebras (or rather the underlying loop algebras) make up the bulk of the classification of the simple  $\mathbb{Z}$ -graded Lie algebras over an algebraically closed field of characteristic zero whose graded pieces have finite growth. According to the classification of such Lie algebras proved by Mathieu [22] the remaining Lie algebras are those of Cartan type and the Virasoro algebra.

**2.29. Virasoro algebra.** The Virasoro algebra is defined for any field  $k$  of characteristic zero as the unique central extension of the Witt algebra. The Witt algebra  $\mathfrak{d}$  is defined by:

$$\mathfrak{d} := \text{Der}k[t, t^{-1}] = \bigoplus_{j \in \mathbb{Z}} k \cdot d_j$$

where  $d_j = -t^{j+1} \frac{d}{dt}$  with the following commutation relations:

$$[d_i, d_j] = (i - j)d_{i+j}.$$

The Lie algebra  $\mathfrak{d}$  has a unique (up to isomorphism) non-trivial central extension by a 1-dimensional centre,  $k \cdot c$  say, called the Virasoro algebra  $\text{Vir}$ , which is defined by the following commutation relations:

$$[d_i, d_j] = (i - j)d_{i+j} + (i^3 - i)\delta_{i,-j}c \text{ where } i, j \in \mathbb{Z}.$$

The Virasoro algebra, denoted  $\text{Vir}$ , plays an important role in the representation theory of affine Kac-Moody Lie algebra and in quantum field theory. It is a  $\mathbb{Z}$ -graded subalgebra in  $\text{Der} \tilde{\mathbf{L}}\mathfrak{g}_k$ , the group of derivations of the algebra  $\tilde{\mathbf{L}}\mathfrak{g}_k$ . When  $k = \mathbb{C}$ , it is the complexified Lie algebra of smooth vector fields on the circle. The positive part of this Lie algebra  $\mathfrak{d}^+ := \bigoplus_{j \geq 1} k \cdot d_j$  also arises as the graded Lie algebra of the so-called Nottingham group over  $k$ . This is defined as the group whose underlying set is the set of formal power series  $tk[[t]]$  and whose group operation is substitution of power series. It is a certain subgroup of the automorphism group of  $k[[t]]$ , called the wild automorphisms.

**2.30. Cartan subalgebras.** The Cartan algebras are defined as follows. Let  $n \geq 1$  and let  $\mathbf{W}_n$  be the algebra of derivations of the polynomial ring  $R = k[X_1, \dots, X_n]$ . So  $\mathbf{W}_n = RD_1 + \dots + RD_n$ , where  $D_i = \frac{d}{dX_i}$ . Thus  $\mathbf{W}_n$  acts on the Grassmann algebra of Kahler differential forms on  $R$ . We define three subalgebras of  $\mathbf{W}_n$ .

(1) The subalgebra  $\mathbf{S}_n$ , called the *special algebra*, consists of those derivations annihilating the differential form  $\nu = dX_1 \wedge \dots \wedge dX_n$ . We can describe this set of derivations as follows:

$$\mathbf{S}_n = \left\{ \sum_{j=1}^n a_j D_j : \sum_{j=1}^n D_j(a_j) = 0 \right\}.$$

It is additively spanned by  $D_{ij}(f) = D_j(f)D_i - D_i(f)D_j$ , where  $1 \leq i, j \leq n$  and  $f$  ranges over  $R$ .

(2) Suppose now that  $n = 2m$  is even then  $\mathbf{H}_{2m}$ , called the *hamiltonian algebra*, consists of derivations annihilating  $\omega = \sum_{1 \leq i \leq m} dX_i \wedge dX_{m+i}$ . Define the following notation:

$$\begin{aligned} j' &= j + m \text{ and } \sigma(j) = 1 \text{ if } 1 \leq j \leq m \\ j' &= j - m \text{ and } \sigma(j) = -1 \text{ if } m + 1 \leq j \leq 2m. \end{aligned}$$

Then we have the following description of the hamiltonian algebra:

$$\mathbf{H}_{2m} = \left\{ \sum_{j=1}^{2m} a_j D_j : \sigma(j') D_i(a_{j'}) = \sigma(i') D_j(a_{i'}), 1 \leq i, j \leq 2m \right\}.$$

Every element of  $\mathbf{H}_{2m}$  can be represented as  $D_H(f)$  for some  $f \in R$  where

$$D_H(f) = \sum_{j=1}^{2m} \sigma(j) D_j(f) D_{j'}.$$

(3) Finally if  $n = 2m + 1$  is odd then  $\mathbf{K}_{2m+1}$ , called the *contact algebra*, consists of derivations  $\partial$  such that  $\partial \cdot \alpha = f \cdot \alpha$  for some polynomial  $f$  where

$$\alpha = dX_{2m+1} + \sum_{1 \leq i \leq m} X_i dX_{m+i} - X_{m+i} dX_i.$$

For every derivation  $D = \sum_{j=1}^n a_j \frac{d}{dX_j}$  define

$$u(D) = \sum_{j=1}^{2m} \sigma(j) X_j \frac{da_{j'}}{dX_n} - \frac{da_n}{dX_n}.$$

Then  $D$  is contained in  $\mathbf{K}_{2m+1}$  if and only if for each  $1 \leq i \leq 2m$

$$\sigma(i) a_{i'} - \sum_{j=1}^{2m} \sigma(j) X_j \frac{da_{j'}}{dX_i} - \frac{da_n}{dX_i} = \sigma(i) X_{i'} u(D).$$

Every element of  $\mathbf{K}_{2m+1}$  can be represented as  $D_K(f) = \sum_{j=1}^{2m+1} f_j D_j$  where

$$\begin{aligned} f_j &= X_j D_{2m+1}(f) + \sigma(j') D_{j'}(f) \text{ for } j \leq 2m \\ f_{2m+1} &= 2f - \sum_{j=1}^{2m} \sigma(j) X_j f_{j'}. \end{aligned}$$

The Lie algebras  $\mathbf{W}_n, \mathbf{S}_n, \mathbf{H}_{2m}$  and  $\mathbf{K}_{2m+1}$  are called *Lie algebras of Cartan type*. The usual grading of the polynomial ring  $k[X_1, \dots, X_n]$  induces a grading of  $\mathbf{W}_n, \mathbf{S}_n$  and  $\mathbf{H}_{2m}$ . If  $n = 2m+1$  and  $m \geq 1$  then there exists a unique grading of  $k[X_1, \dots, X_{2m+1}]$  such that  $X_1, \dots, X_{2m}$  are homogeneous of degree 1 and  $X_{2m+1}$  is homogeneous of degree 2. The induced grading for  $\mathbf{K}_{2m+1}$  is called the natural grading of  $\mathbf{K}_{2m+1}$ .

**2.31. Theorem.** *The Virasoro algebra Vir and the Lie algebras of Cartan type  $\mathbf{W}_n, \mathbf{S}_n, \mathbf{H}_{2m}$  and  $\mathbf{K}_{2m+1}$  are well-covered.*

*Proof.* This depends on the fact that any derivation

$$X_1^{m_1} \dots X_i^{m_i+1} \dots X_n^{m_n} \frac{d}{dX_i}$$

with  $m_1 + \dots + m_n = N$  can be realised as a linear multiple of the Lie product

$$\left( X_1^{r_1} \dots X_i^{r_i+1} \dots X_n^{r_n} \frac{d}{dX_i}, X_1^{s_1} \dots X_i^{s_i+1} \dots X_n^{s_n} \frac{d}{dX_i} \right)$$

where  $r_j + s_j = m_j$  and  $r_i \neq s_i$ . For the Lie algebras Vir and  $\mathbf{W}_n$  the result follows, since we can then realise the derivation in the grading of weight  $N$  as a Lie product of elements of weight  $i$  and  $N - i$  for all values of  $i < N/2$ .

We shall have to work harder to get the corresponding result for the other Cartan algebras  $\mathbf{S}_n, \mathbf{H}_{2m}$  or  $\mathbf{K}_{2m+1}$ , since we need to realise elements of weight  $N$  as sums of commutators of elements of weight  $i$  and  $N - i$  which still stabilize the form defining the corresponding Cartan algebra.

For  $\mathbf{S}_n$  let us show why, for  $\mathbf{X}^{\mathbf{c}} = X_1^{c_1} \dots X_n^{c_n}$  where  $\langle \mathbf{c} \rangle = c_1 + \dots + c_n = nN$ , that  $D_{ij}(\mathbf{X}^{\mathbf{c}})$  can be realised as a linear combination of commutators of elements of weight  $nN - t$  and  $t$  for  $N$  values of  $t$ .

We are going to use the following identity which can easily be checked. Suppose that  $b_j = 0$  then

$$(2.31.1) \quad [D_{ki}(\mathbf{X}^{\mathbf{a}}), D_{kj}(\mathbf{X}^{\mathbf{b}})] = a_i b_k D_{kj}(\mathbf{X}^{\mathbf{a}+\mathbf{b}-\mathbf{e}_i-\mathbf{e}_k}) + a_k b_k D_{ij}(\mathbf{X}^{\mathbf{a}+\mathbf{b}-2\mathbf{e}_k})$$

Since  $\langle \mathbf{c} \rangle = nN$ , we know that there is some  $c_k \geq N$ .

Case 1. Suppose firstly that  $k \neq i, j$ . Then choose  $\mathbf{a}$  and  $\mathbf{b}$  with  $\mathbf{c} = \mathbf{a} + \mathbf{b} - 2\mathbf{e}_k$ ,  $b_j = 0$ ,  $a_i = 0$  and  $a_k = c_k + 1 - l$  and  $b_k = l + 1$ . For  $l = 0, \dots, c_k$ , we get

$$[D_{ki}(\mathbf{X}^{\mathbf{a}}), D_{kj}(\mathbf{X}^{\mathbf{b}})] = a_k b_k D_{ij}(\mathbf{X}^{\mathbf{a}+\mathbf{b}-2\mathbf{e}_k}) = \lambda D_{ij}(\mathbf{X}^{\mathbf{c}})$$

with  $\lambda \neq 0$  and  $D_{ki}(\mathbf{X}^{\mathbf{a}})$  of weight  $a_1 + \dots + a_{k-1} + a_{k+1} + \dots + a_n + c_k + 1 - l$  and  $D_{kj}(\mathbf{X}^{\mathbf{b}})$  of weight  $b_1 + \dots + b_{k-1} + b_{k+1} + \dots + b_n + l + 1$ .

Case 2. If  $k = i$ , then choose some  $s \in \{1, \dots, n\} \setminus \{i, j\}$  and define  $\mathbf{a}$  and  $\mathbf{b}$  with  $\mathbf{c} = \mathbf{a} + \mathbf{b} - 2\mathbf{e}_s$ ,  $b_j = 0$ ,  $a_s = 1$  and  $a_i = c_i + 1 - l$  and  $b_i = l + 1$ . Form

$$[D_{si}(\mathbf{X}^{\mathbf{a}}), D_{sj}(\mathbf{X}^{\mathbf{b}})] = a_i b_s D_{sj}(\mathbf{X}^{\mathbf{a}+\mathbf{b}-\mathbf{e}_i-\mathbf{e}_s}) + a_s b_s D_{ij}(\mathbf{X}^{\mathbf{a}+\mathbf{b}-2\mathbf{e}_s})$$

where note that  $a_s \neq 0 \neq b_s$ . To deal with  $a_i b_s D_{sj}(\mathbf{X}^{\mathbf{a}+\mathbf{b}-\mathbf{e}_i-\mathbf{e}_s})$  we can use the analysis of case 1 since the weight is concentrated still at  $c_i$  but now we are considering a derivation  $D_{sj}$  with  $i \neq s, j$ . Hence choose  $\mathbf{a}' = \mathbf{a} + \mathbf{e}_i - \mathbf{e}_s$  then  $\mathbf{a} + \mathbf{b} - \mathbf{e}_i - \mathbf{e}_s = \mathbf{a}' + \mathbf{b} - 2\mathbf{e}_i$ , and  $a'_s = 0$ ,  $a'_i = c_i + 2 - l$  and  $b_i = l + 1$  where  $l = 0, \dots, c_i$ . (Note that the weight at  $X_i$  of  $D_{sj}(\mathbf{X}^{\mathbf{a}+\mathbf{b}-\mathbf{e}_i-\mathbf{e}_s})$  is now  $c_i + 1$ .) Hence we have

$$[D_{si}(\mathbf{X}^{\mathbf{a}}), D_{sj}(\mathbf{X}^{\mathbf{b}})] - (a_i b_s) (a'_i b'_i)^{-1} [D_{is}(\mathbf{X}^{\mathbf{a}'}) , D_{ij}(\mathbf{X}^{\mathbf{b}})] = \lambda D_{ij}(\mathbf{X}^{\mathbf{c}})$$

with  $\lambda \neq 0$ ,  $D_{si}(\mathbf{X}^{\mathbf{a}})$  and  $D_{is}(\mathbf{X}^{\mathbf{a}'})$  have weight  $a_1 + \dots + a_{i-1} + a_{i+1} + \dots + a_n + c_i + 1 - l$  and  $D_{sj}(\mathbf{X}^{\mathbf{b}})$  and  $D_{ij}(\mathbf{X}^{\mathbf{b}})$  have weight  $b_1 + \dots + b_{i-1} + b_{i+1} + \dots + b_n + l + 1$ . This proves then that  $\mathbf{S}_n$  is well-covered.

For the Hamiltonian algebra, we can use the following identity (see Chapter 4 Lemma 4.3 (1) of [29])

$$\begin{aligned} [D_H(\mathbf{X}^{\mathbf{a}}), D_H(X_i^{b_i})] &= D_H(D_H(\mathbf{X}^{\mathbf{a}})(X_i^{b_i})) \\ &= D_H\left(\sum_{j=1}^{2m} \sigma(j') D_{j'}(f) D_j(X_i^{b_i})\right) \\ &= D_H(\sigma(i') a_{i'} \mathbf{X}^{\mathbf{a}-\mathbf{e}_{i'}} b_i X_i^{b_i-1}). \end{aligned}$$

Consider now realising  $D_H(\mathbf{X}^{\mathbf{c}})$  of weight  $Nn$ . So there exists some  $i$  such that  $c_i \geq N$ . We then put  $b_i = l + 1$  and  $\mathbf{a} = \mathbf{c} + \mathbf{e}_{i'} - b_i + 1$  with  $l = 0, \dots, c_i$  and hence can express  $D_H(\mathbf{X}^{\mathbf{c}})$  as a commutator of elements of the Hamiltonian algebra of weight  $Nn - l$  and  $l$  for  $l = 0, \dots, c_i$ . Hence  $\mathbf{H}_{2m}$  is well-covered.

For the contact algebra  $\mathbf{K}_n = \mathbf{K}_{2m+1}$ , we shall use the following identity (see Chapter 4 Proposition 5.2 of [29]):

$$[D_K(f), D_K(g)] = D_K(D_K(f)(g) - 2gD_{2m+1}(f)).$$

This algebra is slightly trickier than the previous algebras. We shall need to consider realising  $D_K(\mathbf{X}^{\mathbf{c}})$  of weight  $4Nn(n-1)$  as a linear combination of commutators of weight  $4Nn(n-1) - l$  and  $l$  for  $N$  different values of  $l$ . Recall that in this algebra  $X_n$  has weight two whilst the other variables have weight one. There exists some  $i$  with  $c_i \geq 2N(n-1)$ .

Case 1. Suppose that  $i \neq n$ . We prove by induction on  $c_n$  that we can express  $D_K(\mathbf{X}^{\mathbf{c}})$  as a linear combination of commutators of weight  $4Nn(n-1) - l$  and  $l$  for

$l = 0, \dots, N$ . Put  $f = \mathbf{X}^{\mathbf{a}}$  and  $g = X_i^{b_i}$  where  $\mathbf{a} = \mathbf{c} + \mathbf{e}_{i'} - l\mathbf{e}_i$  and  $b_i = l + 1$ . Then

$$(2.31.2) \quad [D_K(\mathbf{X}^{\mathbf{a}}), D_K(X_i^{b_i})] = D_K(X_i a_n \mathbf{X}^{\mathbf{a} - \mathbf{e}_n} b_i X_i^{b_i - 1}) + D_K(\sigma(i') a_{i'} \mathbf{X}^{\mathbf{a} - \mathbf{e}_{i'}} b_i X_i^{b_i - 1})$$

Suppose that  $c_n = a_n = 0$ . Then we are done. We then suppose by induction that we have proved the claim for  $c_n - 1$ . Then the identity (2.31.2) shows how to express  $D_K(\mathbf{X}^{\mathbf{c}})$  as a linear combination of commutators as desired since the first expression on the right hand side of (2.31.2) can be dealt with by the inductive hypothesis.

Case 2. Suppose now that  $i = n$ . We show how to shift the weight from  $X_n$  to the other variables such that eventually we are in a position to invoke case 1.

We use the following identity

$$[D_K(X_n^{b_n}), D_K(\mathbf{X}^{\mathbf{a}})] = D_K(\langle X_n^{b_n}, \mathbf{X}^{\mathbf{a}} \rangle)$$

where

$$\begin{aligned} \langle X_n^{b_n}, \mathbf{X}^{\mathbf{a}} \rangle &= D_K(X_n^{b_n})(\mathbf{X}^{\mathbf{a}}) - 2\mathbf{X}^{\mathbf{a}} D_n(X_n^{b_n}) \\ &= \left( \left( \left( \sum_{j=1}^{2m} a_j \right) - 2 \right) b_n + 2a_n \right) \mathbf{X}^{\mathbf{a}} X_n^{b_n - 1} \\ &\quad - \sum_{j=1}^{2m} \sigma(j) b_n a_n \mathbf{X}^{\mathbf{a} + (b_n - 2)\mathbf{e}_n + \mathbf{e}_j + \mathbf{e}_{j'}}. \end{aligned}$$

Note that we shall let  $b_n = l + 1$  for  $l = 0, \dots, N$  and  $a_n = c_n - l$ . Since  $c_n \geq 2N(n - 1)$  this will mean that  $\left( \left( \sum_{j=1}^{2m} a_j \right) - 2 \right) b_n + 2a_n$  is always non-zero. The choice of  $c_n \geq 2N(n - 1)$  means that we can keep on applying the above identity to realise the expressions  $\sigma(j) b_n a_n \mathbf{X}^{\mathbf{a} + (b_n - 2)\mathbf{e}_n + \mathbf{e}_j + \mathbf{e}_{j'}}$  as commutators  $[D_K(X_n^{b'_n}), D_K(\mathbf{X}^{\mathbf{a}'})]$  with an error term which has the values of  $c_i$  increasing for  $i \neq n$  and  $c_n$  decreasing whilst still ensuring that  $c_n > N$  so that we can choose  $b'_n = l + 1$  for  $l = 0, \dots, N$ . Eventually the error terms will have some  $c_i$  with  $c_i > N$  and we can apply case 1 to finish the realisation.  $\square$

**2.32. Corollary.** *If  $L$  is a simple  $\mathbb{Z}$ -graded Lie algebra over an algebraically closed field of characteristic zero of finite growth then  $\mathcal{X}_0^*(L)$  and  $\mathcal{X}^*(L_0)$  for  $* \in \{\leq, \triangleleft\}$  are stable classes of subalgebras.*

The proof above depends on the classification proved by Mathieu [22] and a case by case analysis of each class of Lie algebra in the classification. It may be possible that there is a more direct argument based on the simplicity of the  $\mathbb{Z}$ -graded Lie algebra. Note that since  $L$  is simple  $\mathcal{X}_0^{\triangleleft}(L)$  is actually empty or consists of  $L$  if  $L_0$  has finite codimension in  $L$ .

As we have pointed out, an  $\mathbb{N}$ -filtered Lie algebra has many ideals so we cannot expect it to be simple. However in this context, the concept of being simple is replaced by that of being *just infinite* - that is an infinite dimensional Lie algebra all of whose proper quotients are finite dimensional. It is conjectured by Shalev and Zelmanov (see [28] section 6.5) that the just infinite  $\mathbb{N}$ -filtered Lie algebras over an

algebraically closed field of characteristic zero for which  $L_0 = L_1$  and  $\dim(L_i/L_{i+1})$  are uniformly bounded are in the following list:

- (1)  $L$  is soluble;
- (2) The completion of  $L$  is commensurable with the positive part of a loop algebra;
- (3) The completion of  $L$  is commensurable to the completion of  $\partial^+$ .

Note that only (2) and (3) will have stable subalgebras.

**2.33. Non-stability of free graded Lie algebras.** Stability for infinite Lie algebras is the analogue of determining when a group has only a finite number of subgroups of each given index which can then be counted using a standard Dirichlet series. For groups the condition that the group be finitely generated (either as an abstract group or a topological group) was sufficient to ensure this condition. Note that being finitely generated will not suffice in the context of infinite dimensional Lie algebras as the following Theorem indicates:

**2.34. Theorem.** *Let  $L = \bigoplus_{i \geq 1} \gamma_i(L)/\gamma_{i+1}(L)$  be the free two generated infinite dimensional graded Lie algebra where  $\gamma_i(L)$  is the lower central series. Then for each  $i$  there exist closed subalgebras of codimension 2 not containing  $\gamma_i(L)$ .*

*Proof.* Let  $x$  and  $y$  be the free generators. If  $w$  is any Lie word in  $x$  and  $y$  we define the length  $l(w)$  to be the number of terms in  $w$ . We take a Hall set  $H = \{w_i : i \geq 1\}$  as a basis for  $L$  (see [2] II.2.10) which is defined as follows:

- (1) if  $w_i \in H$  and  $w_j \in H$  and  $l(w_i) < l(w_j)$  then  $i < j$ ;
- (2)  $w_1 = x, w_2 = y$  and  $w_3 = (xy)$ ;
- (3) an element  $w$  of length  $\geq 3$  belongs to  $H$  if and only if it is of the form  $(a(bc))$  with  $a, b, c$  in  $H$ ,  $(bc) \in H$  and  $b \leq a < bc$  and  $b < c$  where the ordering is defined by the ordering on the index set.

For example the construction provided by Proposition 11 of [2] II.2.10 starts with the following basis for each layer  $\gamma_i(L)/\gamma_{i+1}(L)$ :

$$\begin{array}{lll}
 \gamma_1(L)/\gamma_2(L) & w_1 = x & w_2 = y \\
 \gamma_2(L)/\gamma_3(L) & w_3 = (xy) & \\
 \gamma_3(L)/\gamma_4(L) & w_4 = (x(xy)) & w_5 = (y(xy)) \\
 \gamma_4(L)/\gamma_5(L) & w_6 = (x(x(xy))) & w_7 = (y(x(xy))) & w_8 = (y(y(xy))) \\
 \gamma_5(L)/\gamma_6(L) & w_9 = (x(x(x(xy)))) & w_{10} = (y(x(x(xy)))) & w_{11} = (y(y(x(xy)))) \\
 & w_{12} = (y(y(y(xy)))) & w_{13} = ((xy)(x(xy))) & w_{14} = ((xy)(y(xy)))
 \end{array}$$

Let  $n_i - 1$  be the dimension of  $\gamma_1(L)/\gamma_i(L)$ . Then we claim that the additive subspace of  $L$  generated by the following basis elements is actually a subalgebra:

$$H_i = \bigoplus_{l \neq 1, n_i} kw_l.$$

So this is a vector subspace of codimension 2 which skips the basis elements  $w_1 = x$  and  $w_{n_i} = (x(x \dots (xy) \dots))$  where  $x$  appears  $i - 1$  times.

The Hall basis has the property that each word  $w_l$  has a unique decreasing factorization  $w_l = (w_j w_k)$  with  $j < k < l$  (see Corollary 4.7 of [24]).

We prove by induction on the length of  $w_j$  for  $1 < j < n_i$  and  $k \neq 1, n_i$  that  $(w_j w_k) = \sum_{l \neq 1, n_i} a_l w_l$ . If  $w_j$  has length 1 then  $w_j = y$  and by condition (3)  $(yw_k)$  is in

*H.* The property of unique factorization implies it is not the element  $w_{n_i} = (xw_{n_i-1})$ . Suppose we have proved that  $(w_j w_k) = \sum_{l \neq 1, n_i} a_l w_l$  for all words  $w_j$  of length less than  $m$  and take  $w_j$  of length  $m > 1$ . There is a unique decreasing factorization  $w_j = (w_{j_1} w_{j_2})$  with  $l(w_{j_i}) < l(w_j)$ . Using the Jacobi identity we can rewrite

$$\begin{aligned} (w_j w_k) &= ((w_{j_1} w_{j_2}) w_k) \\ &= (w_{j_1} (w_{j_2} w_k)) - (w_{j_2} (w_{j_1} w_k)). \end{aligned}$$

Now we can use our induction hypothesis applied to  $w_{j_1}$  and  $w_{j_2}$ .  $\square$

Obviously the same argument applies to the  $d$ -generated free Lie algebra. So the existence of a motivic zeta function for an infinite dimensional Lie algebra does not appear to be so straightforward. The condition of being well-covered will suffice, but this is a special sort of property which depends on the internal workings of a Lie algebra. It would be nice to have a more transparent Lie theoretic condition which will ensure the stability of the class of subalgebras. We therefore raise the following:

**2.35. Problem.** Determine a criterion for the class of all subalgebras of finite codimension to be stable in an  $\mathbb{N}$ -filtered or  $\mathbb{Z}$ -filtered Lie algebra.

From now on we shall focus on the situation of a finite dimensional  $k[[t]]$ -Lie algebra and  $\mathcal{X}$  a class of  $k[[t]]$ -subalgebras.

### 3. MOTIVIC INTEGRATION AND RATIONALITY OF POINCARÉ SERIES

In this section, we review material from [7] and [8] which will be used in the present paper.

**3.1. Scheme of arcs.** For  $X$  a variety over  $k$ , we will denote by  $\mathcal{L}(X)$  the *scheme of germs of arcs on  $X$* . It is a scheme over  $k$  and for any field extension  $k \subset K$  there is a natural bijection

$$\mathcal{L}(X)(K) \simeq \text{Mor}_{k\text{-schemes}}(\text{Spec } K[[t]], X)$$

between the set of  $K$ -rational points of  $\mathcal{L}(X)$  and the set of germs of arcs with coefficients in  $K$  on  $X$ . We will call  $K$ -rational points of  $\mathcal{L}(X)$ , for  $K$  a field extension of  $k$ , arcs on  $X$ , and  $\varphi(0)$  will be called the origin of the arc  $\varphi$ . More precisely the scheme  $\mathcal{L}(X)$  is defined as the projective limit

$$\mathcal{L}(X) := \varprojlim \mathcal{L}_n(X)$$

in the category of  $k$ -schemes of the schemes  $\mathcal{L}_n(X)$  representing the functor

$$R \mapsto \text{Mor}_{k\text{-schemes}}(\text{Spec } R[t]/t^{n+1}R[t], X)$$

defined on the category of  $k$ -algebras. The existence of  $\mathcal{L}_n(X)$  is well known (cf. [7]) and the projective limit exists since the transition morphisms are affine. We shall denote by  $\pi_n$  the canonical morphism, corresponding to truncation of arcs,

$$\pi_n : \mathcal{L}(X) \longrightarrow \mathcal{L}_n(X).$$

The schemes  $\mathcal{L}(X)$  and  $\mathcal{L}_n(X)$  will always be considered with their reduced structure.

**3.2. Semi-algebraic geometry.** From now on we will denote by  $\bar{k}$  a fixed algebraic closure of  $k$ , and by  $\bar{k}((t))$  the fraction field of  $\bar{k}[[t]]$ , where  $t$  is one variable. Let  $x_1, \dots, x_m$  be variables running over  $\bar{k}((t))$  and let  $\ell_1, \dots, \ell_r$  be variables running over  $\mathbb{Z}$ . A *semi-algebraic* (resp.  *$k[[t]]$ -semi-algebraic*) condition  $\theta(x_1, \dots, x_m; \ell_1, \dots, \ell_r)$  is a finite boolean combination of conditions of the form

- (1)  $\text{ord}_t f_1(x_1, \dots, x_m) \geq \text{ord}_t f_2(x_1, \dots, x_m) + L(\ell_1, \dots, \ell_r)$
- (2)  $\text{ord}_t f_1(x_1, \dots, x_m) \equiv L(\ell_1, \dots, \ell_r) \pmod{d}$
- (3)  $h(\overline{ac}(f_1(x_1, \dots, x_m)), \dots, \overline{ac}(f_{m'}(x_1, \dots, x_m))) = 0$ ,

where  $f_i$  are polynomials with coefficients in  $k$  (resp.  $f_i$  are polynomials with coefficients in  $k[[t]]$ ),  $h$  is a polynomial with coefficients in  $k$ ,  $L$  is a polynomial of degree  $\leq 1$  over  $\mathbb{Z}$ ,  $d \in \mathbb{N}$ , and  $\overline{ac}(x)$  is the coefficient of lowest degree of  $x$  in  $\bar{k}((t))$  if  $x \neq 0$ , and is equal to 0 otherwise. Here we use the convention that  $\infty + \ell = \infty$  and  $\infty \equiv \ell \pmod{d}$ , for all  $\ell \in \mathbb{Z}$ . In particular the condition  $f(x_1, \dots, x_m) = 0$  is a semi-algebraic condition (resp. a  $k[[t]]$ -semi-algebraic condition), for  $f$  a polynomial over  $k$  (resp. over  $k[[t]]$ ). A subset of  $\bar{k}((t))^m \times \mathbb{Z}^r$  defined by a semi-algebraic (resp.  $k[[t]]$ -semi-algebraic) condition is called *semi-algebraic* (resp.  *$k[[t]]$ -semi-algebraic*). One defines similarly semi-algebraic and  $k[[t]]$ -semi-algebraic subsets of  $K((t))^m \times \mathbb{Z}^r$  for  $K$  an algebraically closed field containing  $\bar{k}$ .

A function  $\alpha : \bar{k}((t))^m \times \mathbb{Z}^n \rightarrow \mathbb{Z}$  is called *simple* (resp.  *$k[[t]]$ -simple*) if its graph is semi-algebraic (resp.  $k[[t]]$ -semi-algebraic).

Let  $X$  be an algebraic variety over  $k$ . For  $x \in \mathcal{L}(X)$ , we denote by  $k_x$  the residue field of  $x$  on  $\mathcal{L}(X)$ , and by  $\tilde{x}$  the corresponding rational point  $\tilde{x} \in \mathcal{L}(X)(k_x) = X(k_x[[t]])$ . When there is no danger of confusion we will often write  $x$  instead of  $\tilde{x}$ . A *semi-algebraic family of semi-algebraic subsets* (resp.  *$k[[t]]$ -semi-algebraic family of  $k[[t]]$ -semi-algebraic subsets*) (for  $n = 0$  a semi-algebraic subset (resp.  $k[[t]]$ -semi-algebraic subset))  $A_\ell$ ,  $\ell \in \mathbb{N}^n$ , of  $\mathcal{L}(X)$  is a family of subsets  $A_\ell$  of  $\mathcal{L}(X)$  such that there exists a covering of  $X$  by affine Zariski open sets  $U$  with

$$A_\ell \cap \mathcal{L}(U) = \left\{ x \in \mathcal{L}(U) : \theta(h_1(\tilde{x}), \dots, h_m(\tilde{x}); \ell) \right\},$$

where  $h_1, \dots, h_m$  are regular functions on  $U$  and  $\theta$  is a semi-algebraic condition (resp.  $k[[t]]$ -semi-algebraic condition). Here the  $h_i$ 's and  $\theta$  may depend on  $U$  and  $h_i(\tilde{x})$  belongs to  $k_x[[t]]$ .

Let  $A$  be a semi-algebraic subset (resp.  $k[[t]]$ -semi-algebraic subset) of  $\mathcal{L}(X)$ . A function  $\alpha : A \times \mathbb{Z}^n \rightarrow \mathbb{Z} \cup \{\infty\}$  is called *simple* (resp.  *$k[[t]]$ -simple*) if the family of subsets  $\{x \in \mathcal{L}(X) : \alpha(x, \ell_1, \dots, \ell_n) = \ell_{n+1}\}$ ,  $(\ell_1, \dots, \ell_{n+1}) \in \mathbb{N}^{n+1}$ , is a semi-algebraic family of semi-algebraic subsets (resp. a  *$k[[t]]$ -semi-algebraic family of  $k[[t]]$ -semi-algebraic subsets*) of  $\mathcal{L}(X)$ .

An important fact is that if  $A$  is a  $k[[t]]$ -semi-algebraic subset of  $\mathcal{L}(X)$ , then  $\pi_n(A)$  is a constructible subset of  $\mathcal{L}_n(X)$  (cf. Proposition 1.7 of [8]).

**3.3. Remark.** Motivic integration is developed in [7] for semi-algebraic subsets of  $\mathcal{L}(X)$ , when  $X$  is an algebraic variety over  $k$ , and simple functions. In the paper [8], this is extended to what is called there  *$t$ -semi-algebraic subsets of  $\mathcal{L}(X)$*  and  *$t$ -simple functions*, which are defined similarly as  $k[[t]]$ -semi-algebraic subsets of  $\mathcal{L}(X)$

and  $k[[t]]$ -simple functions, but replacing  $k[[t]]$  by  $k[t]$ . In fact, as mentioned in Remark 1.18 of [8], all results in §1 of [8] (before 1.17) may be extended, with similar proofs, to cover the case of  $k[[t]]$ -semi-algebraic subsets and  $k[[t]]$ -simple functions. Hence when we shall quote a result from [8], we shall use its extension to  $k[[t]]$ -semi-algebraic subsets and  $k[[t]]$ -simple functions without further comment.

**3.4. Motivic integration.** From now on we assume  $X$  is a smooth algebraic variety over  $k$  of pure dimension  $d$ . Let  $A$  be a  $k[[t]]$ -semi-algebraic subset  $\mathcal{L}(X)$ . We say  $A$  is stable at level  $n$  if  $A = \pi_n^{-1}\pi_n(A)$ . Remark that if  $A$  is stable at level  $n$ , then  $A$  is stable at level  $m$ , for any  $m \geq n$ . We say  $A$  is stable if it is stable at some level. Denote by  $\mathbf{B}^t$  the set of all  $k[[t]]$ -semi-algebraic subsets of  $\mathcal{L}(X)$ , and by  $\mathbf{B}_0^t$  the set of all  $A$  in  $\mathbf{B}^t$  which are stable. Clearly there is a unique additive measure

$$\tilde{\mu} : \mathbf{B}_0^t \longrightarrow \mathcal{M}_{\text{loc}}$$

satisfying

$$\tilde{\mu}(A) = [\pi_n(A)] \mathbf{L}^{-(n+1)d},$$

when  $A$  is stable at level  $n$ . In fact, the condition of being  $k[[t]]$ -semi-algebraic is superfluous here. By the same formula one may define  $\tilde{\mu}(A)$  for  $A$  cylindrical at level  $n$ , *i.e.* subsets of  $\mathcal{L}(X)$  of the form  $A = \pi_n^{-1}(C)$  with  $C$  constructible. One says  $A$  is cylindrical if it is cylindrical at some level.

Let  $A$  be in  $\mathbf{B}_0^t$  and let  $\alpha : A \rightarrow \mathbb{Z}$  be a  $k[[t]]$ -simple function (or, more generally, assume  $A$  and the fibers of  $\alpha$  are cylindrical). By Lemma 2.4 of [7] and Lemma A.3 of [8],  $|\alpha|$  is bounded, and we can define

$$(3.4.1) \quad \int_A \mathbf{L}^{-\alpha} d\tilde{\mu} := \sum \mathbf{L}^{-n} \tilde{\mu}(\alpha^{-1}(n)),$$

the sum on the right hand side being finite.

**3.5. Completion.** We now explain how one extends  $\tilde{\mu}$  to non stable  $k[[t]]$ -semi-algebraic subsets by using a completion of  $\mathcal{M}_{\text{loc}}$ . This is indeed similar to the use of real numbers for defining  $p$ -adic integrals. The material here will only be used in section 7. So we denote by  $\widehat{\mathcal{M}}$  the completion of  $\mathcal{M}_{\text{loc}}$  with respect to the filtration  $F^m \mathcal{M}_{\text{loc}}$  where  $F^m \mathcal{M}_{\text{loc}}$  is the subgroup generated by  $\{[S] \mathbf{L}^{-i} : i - \dim S \geq m\}$ . We will also denote by  $F^\cdot$  the filtration induced on  $\widehat{\mathcal{M}}$ .

In [7] and [8] the following is shown<sup>1</sup>: There exists a unique map  $\mu : \mathbf{B}^t \rightarrow \widehat{\mathcal{M}}$  satisfying the following three properties.

- (1) If  $A \in \mathbf{B}^t$  is stable at level  $n$ , then  $\mu(A) = [\pi_n(A)] \mathbf{L}^{-(n+1)d}$ .
- (2) If  $A \in \mathbf{B}^t$  is contained in  $\mathcal{L}(S)$  with  $S$  a reduced closed subscheme of  $X \otimes k[[t]]$  with  $\dim_{k[[t]]} S < \dim X$ , then  $\mu(A) = 0$ .
- (3) Let  $A_i$  be in  $\mathbf{B}^t$  for each  $i$  in  $\mathbb{N}$ . Assume that the  $A_i$ 's are mutually disjoint and that  $A := \bigcup_{i \in \mathbb{N}} A_i$  is  $k[[t]]$ -semi-algebraic. Then  $\sum_{i \in \mathbb{N}} \mu(A_i)$  converges in  $\widehat{\mathcal{M}}$  to  $\mu(A)$ .

Moreover we have:

<sup>1</sup>in fact *loc. cit.* also covers the case of singular varieties

(4) If  $A$  and  $B$  are in  $\mathbf{B}^t$ ,  $A \subset B$  and if  $\mu(B) \in F^m \widehat{\mathcal{M}}$ , then  $\mu(A) \in F^m \widehat{\mathcal{M}}$ .

This unique map  $\mu$  is called the *motivic volume* on  $\mathcal{L}(X)$  and is denoted by  $\mu_{\mathcal{L}(X)}$  or  $\mu$ . For  $A$  in  $\mathbf{B}^t$  and  $\alpha : A \rightarrow \mathbb{Z} \cup \{\infty\}$  a  $k[[t]]$ -simple function, one defines the motivic integral

$$(3.5.1) \quad \int_A \mathbf{L}^{-\alpha} d\mu := \sum_{n \in \mathbb{Z}} \mu(A \cap \alpha^{-1}(n)) \mathbf{L}^{-n}$$

in  $\widehat{\mathcal{M}}$ , whenever the right hand side converges in  $\widehat{\mathcal{M}}$ , in which case we say that  $\mathbf{L}^{-\alpha}$  is integrable on  $A$ . If the function  $\alpha$  is bounded from below, then  $\mathbf{L}^{-\alpha}$  is integrable on  $A$ , because of (4).

**3.6. Rationality.** The following results are proved in section 5 of [7] for semi-algebraic families of semi-algebraic subsets and simple functions. For simplicity results are stated here only for smooth varieties.

**3.7. Theorem.** *Let  $X$  be a smooth algebraic variety over  $k$  of pure dimension  $d$ . Let  $A_n$ ,  $n \in \mathbb{Z}^r$ , be a semi-algebraic family of semi-algebraic subsets of  $\mathcal{L}(X)$  and let  $\alpha : \mathcal{L}(X) \times \mathbb{Z}^r \rightarrow \mathbb{N}$  be a simple function. Assume that  $A_n$  and the fibers of  $\alpha(-, n) : A_n \rightarrow \mathbb{N}$  are stable, for every  $n \in \mathbb{N}^r$ . Then the power series*

$$\sum_{n \in \mathbb{N}^r} T^n \int_{A_n} \mathbf{L}^{-\alpha(-, n)} d\tilde{\mu}$$

in the variable  $T = (T_1, \dots, T_r)$  belongs to the subring of  $\mathcal{M}_{\text{loc}}[[T]]$  generated by  $\mathcal{M}_{\text{loc}}[T]$  and the series  $(1 - \mathbf{L}^{-a} T^b)^{-1}$ , with  $a \in \mathbb{N}$  and  $b \in \mathbb{N}^r \setminus \{0\}$ .

**3.8. Theorem.** *Let  $X$  be a smooth algebraic variety over  $k$  of pure dimension  $d$ . Let  $A_n$ ,  $n \in \mathbb{Z}^r$ , be a semi-algebraic family of semi-algebraic subsets of  $\mathcal{L}(X)$  and let  $\alpha : \mathcal{L}(X) \times \mathbb{Z}^r \rightarrow \mathbb{N}$  be a simple function. Then the power series*

$$\sum_{n \in \mathbb{N}^r} T^n \int_{A_n} \mathbf{L}^{-\alpha(-, n)} d\mu$$

in the variable  $T = (T_1, \dots, T_r)$  belongs to the subring of  $\widehat{\mathcal{M}}[[T]]$  generated by the image in  $\widehat{\mathcal{M}}[[T]]$  of  $\mathcal{M}_{\text{loc}}[T]$ ,  $(\mathbf{L}^i - 1)^{-1}$  and  $(1 - \mathbf{L}^{-a} T^b)^{-1}$ , with  $i \in \mathbb{N} \setminus \{0\}$ ,  $a \in \mathbb{N}$ ,  $b \in \mathbb{N}^r \setminus \{0\}$ .

**3.9. Corollary.** *For any semi-algebraic subset  $A$  of  $\mathcal{L}(X)$ , the measure  $\mu(A)$  is in  $\overline{\mathcal{M}}_{\text{loc}}[((\mathbf{L}^i - 1)^{-1})_{i \geq 1}]$ , where  $\overline{\mathcal{M}}_{\text{loc}}$  is the image of  $\mathcal{M}_{\text{loc}}$  in  $\widehat{\mathcal{M}}$   $\square$*

**3.10. Remark.** By replacing  $T_i$  by  $T_i \mathbf{L}^{m_i}$  in Theorems 3.7 and 3.8, one sees that the condition “ $\alpha$  takes values in  $\mathbb{N}$ ” may be replaced by the condition “ $\alpha$  is bounded from below by a linear function of the  $\mathbb{Z}^r$ -variable”.

**3.11. Remark.** It seems quite likely that Theorems 3.7 and 3.8 remain true if semi-algebraic is replaced everywhere by  $k[[t]]$ -semi-algebraic and simple by  $k[[t]]$ -simple. However it does not seem that the proofs may be adapted directly to that more general situation, since there might be some “bad reduction at  $t = 0$ ”.

## 4. MOTIVIC INTEGRATION ON THE INFINITE GRASSMANNIAN

All the material in 4.1 and 4.2 is contained in [1], or is directly adapted, replacing SL by GL, from statements in [1].

4.1. We work over a field  $k$  of characteristic 0. We shall consider the infinite Grassmannian as a functor on the category of  $k$ -algebras. (A  $k$ -algebra will always be assumed to be associative, commutative and unitary.) A natural framework is that of  $k$ -spaces and  $k$ -groups in the sense of [1]. By definition, a  $k$ -space (resp. a  $k$ -group) is a functor from the category of  $k$ -algebras to the category of sets (resp. of groups) which is a sheaf for the faithfully flat topology (see [1] for a definition). The category of schemes over  $k$  can be viewed as a full subcategory of the category of  $k$ -spaces. Schemes will always be assumed to be quasi-compact and quasi-separated. An important feature is that direct limits exist in the category of  $k$ -spaces, so we can say a  $k$ -space (resp. a  $k$ -group) is an *ind-scheme* (resp. an *ind-group*) if it is the direct limit of a directed system of schemes (resp. of group schemes).

We fix a positive integer  $d$ . Let  $t$  be an indeterminate. We consider the  $k$ -groups  $\mathbf{GL}_d(k[[t]])$  and  $\mathbf{GL}_d(k((t)))$  respectively defined by  $R \mapsto \mathrm{GL}_d(R[[t]])$  and  $R \mapsto \mathrm{GL}_d(R((t)))$ .

For  $n \geq 0$ , we denote by  $G_{(n)}(R)$  the set of matrices  $A(t)$  in  $\mathrm{GL}_d(R((t)))$  such that both  $A(t)$  and  $A(t)^{-1}$  have a pole of order  $\leq n$ . This defines a subfunctor  $G_{(n)}$  of  $\mathbf{GL}_d(k((t)))$ . One can show (cf. [1]), that the  $k$ -group  $\mathbf{GL}_d(k[[t]])$  is an affine group scheme and that the  $k$ -group  $\mathbf{GL}_d(k((t)))$  is an ind-group, being the direct limit of the sequence of schemes  $(G_{(n)})$ ,  $n \geq 0$ .

4.2. The infinite Grassmannian  $\mathcal{G}r$  may be defined as the quotient

$$\mathbf{GL}_d(k((t)))/\mathbf{GL}_d(k[[t]])$$

in the category of  $k$ -spaces. It is the sheaf, for the faithfully flat topology, associated to the presheaf  $R \mapsto \mathrm{GL}_d(R((t)))/\mathrm{GL}_d(R[[t]])$ . Now we have to explain why  $\mathcal{G}r$  is indeed the infinite Grassmannian. For any  $k$ -algebra  $R$ , consider the set  $W(R)$  of  $R[[t]]$ -submodules  $L$  of  $R((t))^d$  such that, for some  $n \geq 0$ ,  $t^n R[[t]]^d \subset L \subset t^{-n} R[[t]]^d$  and  $L/t^n R[[t]]^d$  is a projective  $R$ -module. By Proposition 2.3 of [1], the  $k$ -space  $\mathcal{G}r$  is isomorphic to the functor  $R \mapsto W(R)$ . Under this isomorphism, the group scheme action of  $\mathbf{GL}_d(k((t)))$  on  $\mathcal{G}r$  corresponds to the natural action of  $\mathrm{GL}_d(R((t)))$  on  $R[[t]]$ -submodules of  $R((t))^d$ . Denote by  $\mathcal{G}r_{(n)}$  the image of  $G_{(n)}$  in  $\mathcal{G}r$ . Under the preceding isomorphism  $\mathcal{G}r_{(n)}$  may be identified with the Grassmannian  $\mathrm{Gr}_t(t^{-n} k[[t]]^d / t^n k[[t]]^d)$ , whose  $K$ -rational points, for  $K$  a field containing  $k$ , are the  $K$ -linear subspaces of the finite dimensional  $K$ -vector space  $t^{-n} K[[t]]^d / t^n K[[t]]^d$  which are stable by multiplication by  $t$ ; in particular  $\mathcal{G}r_{(n)}$  is a projective variety. Furthermore,  $\mathcal{G}r$  as a  $k$ -space is naturally isomorphic to the direct limit of the system of projective varieties  $\mathcal{G}r_{(n)}$ , hence  $\mathcal{G}r$  is an ind-scheme. The canonical morphism of  $k$ -spaces  $\Theta : \mathbf{GL}_d(k((t))) \rightarrow \mathcal{G}r$  is a locally trivial fibration for the Zariski topology, *i.e.*  $\mathcal{G}r$  is covered by open subsets over which  $\Theta$  is a product.

4.3. Denote by  $M_d$  the affine  $k$ -space of  $d$  by  $d$  matrices. There is canonical immersion of  $k$ -schemes  $\iota : \mathbf{GL}_d(k[[t]]) \hookrightarrow \mathcal{L}(M_d)$ , which identifies  $\mathbf{GL}_d(k[[t]])$  with the open subscheme defined by  $\det \neq 0$ . In particular we can consider  $\mathbf{GL}_d(k[[t]])$  as a semi-algebraic subset of  $\mathcal{L}(M_d)$ . A subset of  $\mathbf{GL}_d(k[[t]])$  will be called semi-algebraic (resp.  $k[[t]]$ -semi-algebraic) if it is semi-algebraic (resp.  $k[[t]]$ -semi-algebraic) as a subset of  $\mathcal{L}(M_d)$ . The space  $\mathbf{GL}_d(k((t)))$  being an ind-scheme, one can associate to it an underlying set, which by abuse we shall denote by the same letter. We shall say a subset  $A$  of  $\mathbf{GL}_d(k((t)))$  is bounded  $k[[t]]$ -semi-algebraic (resp. stable bounded  $k[[t]]$ -semi-algebraic, resp. cylindrical), if, for some integer  $n$  in  $\mathbb{N}$ ,  $t^n A$  is a  $k[[t]]$ -semi-algebraic (resp. stable  $k[[t]]$ -semi-algebraic, resp. cylindrical) subset of  $\mathbf{GL}_d(k[[t]])$ .

We may extend the measures  $\mu$  and  $\tilde{\mu}$  to bounded  $k[[t]]$ -semi-algebraic, stable bounded  $k[[t]]$ -semi-algebraic and cylindrical subsets respectively, by defining

$$\mu(t^{-n}A) = \mathbf{L}^{d^2 n} \mu(A) \quad \text{and} \quad \tilde{\mu}(t^{-n}A) = \mathbf{L}^{d^2 n} \tilde{\mu}(A),$$

for  $A$  a  $k[[t]]$ -semi-algebraic (resp. stable  $k[[t]]$ -semi-algebraic or cylindrical) subset of  $\mathbf{GL}_d(k[[t]])$ , which is independent from the choice of the integer  $n$ . One should also remark that the measures  $\mu$  and  $\tilde{\mu}$  are invariant under  $\mathbf{GL}_d(k[[t]])$ -action.

4.4. For any integer  $m$  in  $\mathbb{Z}$ , the functor

$$R \longmapsto \left\{ M \in \mathbf{GL}_d(R((t))) : \text{ord}_t \det M = m \right\}$$

defines a subspace of  $\mathbf{GL}_d(k((t)))$ , which we shall denote by  $\mathbf{GL}_d(k((t)))[m]$ . Clearly  $\mathbf{GL}_d(k((t)))[m]$  is an ind-scheme.

If  $H$  is a linear subspace of  $t^{-n}K[[t]]^d/t^nK[[t]]^d$ , for  $K$  a field containing  $k$ , we set

$$(4.4.1) \quad \text{index}(H) := \dim t^{-n}K[[t]]^d/H - nd.$$

We define  $\mathcal{G}r_{(n)}[m]$  as the projective variety parametrizing subspaces in the Grassmannian  $\text{Gr}_t(t^{-n}k[[t]]^d/t^nk[[t]]^d)$  which are of index  $m$ . For fixed  $m$ , the varieties  $\mathcal{G}r_{(n)}[m]$  form an inductive system and we denote by  $\mathcal{G}r[m]$  the corresponding ind-scheme. For every  $m$ , the fibration  $\Theta : \mathbf{GL}_d(k((t))) \rightarrow \mathcal{G}r$  restricts to a fibration  $\Theta : \mathbf{GL}_d(k((t)))[m] \rightarrow \mathcal{G}r[m]$ .

4.5. Let  $A$  be a subset of  $\mathcal{G}r$ . We say  $A$  is gr-stable at level  $n$  if  $A$  is a constructible subset of  $\mathcal{G}r_{(n)}$ . Note that in this case  $A \cap \mathcal{G}r[m]$  is constructible in  $\mathcal{G}r_{(n)}[m]$  for every  $m$ . Hence we can define the motivic measure  $\tilde{\mu}_{\mathcal{G}r}(A)$  of  $A$  as the element

$$\tilde{\mu}_{\mathcal{G}r}(A) := \sum_{m \in \mathbb{Z}} \frac{|A \cap \mathcal{G}r[m]|}{\mathbf{L}^{md}}$$

in  $\mathcal{M}_{\text{loc}}$ , which makes sense, the sum on the right hand side being finite.

Clearly if  $A$  is gr-stable at level  $n$  then  $\Theta^{-1}(t^n A)$  is cylindrical at level  $2n$ . The relation with the previously defined measure  $\tilde{\mu}$  is given by the following Proposition.

**4.6. Proposition.** *Let  $A$  be a gr-stable subset of  $\mathcal{G}r$ . Then*

$$\tilde{\mu}_{\mathcal{G}r}(A) = (1 - \mathbf{L}^{-1})^{-1} \dots (1 - \mathbf{L}^{-d})^{-1} \tilde{\mu}(\Theta^{-1}(A)).$$

*Proof.* We may assume  $A$  is contained in  $\mathcal{G}r_{(n)}[m]$ . By Lemma 4.7 applied to  $r = m + nd$  and  $p = 2n$ ,

$$[\pi_{2n}(\Theta^{-1}(t^n A))] = [t^n A](\mathbf{L}^d - 1) \dots (\mathbf{L}^d - \mathbf{L}^{d-1}) \mathbf{L}^{2nd^2 - d(m+nd)},$$

since  $[\mathrm{GL}_{d,k}] = (\mathbf{L}^d - 1) \dots (\mathbf{L}^d - \mathbf{L}^{d-1})$ . We deduce that

$$\tilde{\mu}(\Theta^{-1}(t^n A)) = [t^n A](1 - \mathbf{L}^{-1}) \dots (1 - \mathbf{L}^{-d}) \mathbf{L}^{-nd^2} \mathbf{L}^{-md}.$$

The result follows since

$$\tilde{\mu}(\Theta^{-1}(t^n A)) = \tilde{\mu}(t^n \Theta^{-1}(A)) = \mathbf{L}^{-nd^2} \tilde{\mu}(\Theta^{-1}(A))$$

and  $[t^n A] = [A]$ .  $\square$

**4.7. Lemma.** *For any integer  $p \geq 0$ , the morphism*

$$\pi_p(\Theta^{-1}(\mathrm{Gr}_t(k[[t]]^d/t^p k[[t]]^d)[r])) \longrightarrow \mathrm{Gr}_t(k[[t]]^d/t^p k[[t]]^d)[r]$$

*is a locally trivial fibration for the Zariski topology with fiber  $\mathrm{GL}_{d,k} \times \mathbb{A}_k^{pd^2 - dr}$ . Here  $\mathrm{GL}_{d,k}$  denotes the algebraic variety of invertible  $d$  by  $d$  matrices over  $k$ .*

*Proof.* By taking a cover of the Grassmannian  $\mathrm{Gr}_t(k[[t]]^d/t^p k[[t]]^d)[r]$  by open Schubert cells corresponding to different choices of bases of the lattice  $k[[t]]^d$ , one deduces the result from the following elementary Lemma 4.8.  $\square$

**4.8. Lemma.** *Let  $p \geq m_d \geq \dots \geq m_1 \geq 0$  be integers. Set  $r = \sum_{1 \leq i \leq d} m_i$ . Let  $U$  be the subscheme of  $\mathcal{L}(\mathbb{M}_d)$  consisting of triangular matrices  $(a_{ij})$  with  $a_{ii} = t^{m_i}$ ,  $a_{ij} = 0$  for  $j < i$ ,  $m_i \leq \mathrm{ord}_t a_{ij}$  and  $\mathrm{deg}_t a_{ij} < m_j$  for  $j > i$ . Consider the morphism  $\varphi : U \times \mathbf{GL}_d(k[[t]]) \rightarrow U \times \mathcal{L}_p(\mathbb{M}_d)$  sending  $(A, M)$  to  $(A, \pi_p(AM))$ . Then the image  $W$  of  $\varphi$  is an algebraic variety over  $k$  which is isomorphic to  $U \times \mathrm{GL}_{d,k} \times \mathbb{A}_k^{pd^2 - dr}$  by an isomorphism compatible with the projection  $W \rightarrow U$ .*

Now we can express our zeta functions as integrals, defined as in 3.4.1, with respect to the measure  $\tilde{\mu}_{\mathcal{G}r}$  on the Grassmannian:

**4.9. Theorem.** *Let  $\mathcal{X}$  be a class of constructible  $k[[t]]$ -subalgebras. Then*

$$P_{L,\mathcal{X}}(\mathbf{L}^{-s}) = \int_{\mathcal{X} \subset \mathrm{Gr}_0^d(L)} \mathbf{L}^{\mathrm{index}(H)d - (\mathrm{codim} H)s} d\tilde{\mu}_{\mathcal{G}r}(H).$$

*Proof.* Let  $\mathcal{X}_n$  be the subalgebras of codimension  $n$ . Then  $\mathcal{X}_n \subset \mathrm{Gr}^d(t^{-n}L/t^nL)$  and

$$\begin{aligned} [\mathcal{X}_n] &= \sum_{m \in \mathbb{Z}} [\mathcal{X}_n \cap \mathcal{G}r[m]] \\ &= \sum_{m \in \mathbb{Z}} \frac{[\mathcal{X}_n \cap \mathcal{G}r[m]]}{\mathbf{L}^{md}} \mathbf{L}^{md} \\ &= \sum_{m \in \mathbb{Z}} \int_{\mathcal{X}_n \cap \mathcal{G}r[m]} \mathbf{L}^{\mathrm{index}(H)d} d\tilde{\mu}_{\mathcal{G}r}(H) \\ &= \int_{\mathcal{X}_n} \mathbf{L}^{\mathrm{index}(H)d} d\tilde{\mu}_{\mathcal{G}r}(H). \end{aligned}$$

$\square$

## 5. RATIONALITY OF MOTIVIC ZETA FUNCTIONS OF INFINITE DIMENSIONAL LIE ALGEBRAS

In this section we shall prove the following rationality result:

**5.1. Theorem.** *Let  $k$  be a field of characteristic zero.*

- (1) *Let  $L$  be a finite dimensional free  $k[[t]]$ -Lie algebra of the form  $L = L_k \otimes_k k[[t]]$ , with  $L_k$  a Lie algebra over  $k$ . Let  $\mathcal{X}_t^{\leq}$  (resp.  $\mathcal{X}_t^{\triangleleft}$ ) be the class of subalgebras such that  $\mathcal{X}_t^{\leq}(K)$  (resp.  $\mathcal{X}_t^{\triangleleft}(K)$ ) is the set of all  $K[[t]]$ -subalgebras (resp.  $K[[t]]$ -ideals) of  $L \otimes K$ , for every field  $K$  which is a finite extension of  $k$ . Then  $P_{L, \mathcal{X}_t^*}(T)$  for  $* \in \{\leq, \triangleleft\}$  is rational, belonging to  $\mathcal{M}[T]_{\text{loc}}$ .*
- (2) *Let  $L$  be a finite dimensional  $k((t))$ -Lie algebra and  $L_0$  be a choice of some  $k[[t]]$ -Lie subalgebra of the form  $L_0 = L_k \otimes_k k[[t]]$ , with  $L_k$  a Lie algebra over  $k$ . Let  $\mathcal{X}_{t,0}^{\leq}$  be the class of subalgebras such that  $\mathcal{X}_{t,0}^{\leq}(K)$  is the set of all  $K[[t]]$ -subalgebras  $L \otimes K$  commensurable with  $L_0 \otimes K$ , for every field  $K$  which is a finite extension of  $k$ . Then  $P_{L, \mathcal{X}_{t,0}^{\leq}}(T)$  is rational, belonging to  $\mathcal{M}[T]_{\text{loc}}$ .*

Let  $L$  be a finite dimensional free  $k[[t]]$ -Lie algebra of dimension  $d$ . Choosing a basis allows us to identify  $L$  additively with  $k[[t]]^d$  and we may view the  $\mathcal{X}_t^{\leq}$  as a subset of the Grassmannian  $\mathcal{G}r$ .

**5.2. Theorem.** *With the preceding notations,  $\Theta^{-1}(\mathcal{X}_t^*)$  is a  $k[[t]]$ -semi-algebraic subset of  $\mathbf{GL}_d(k[[t]])$ , for  $* \in \{\leq, \triangleleft\}$ , and*

$$(5.2.1) \quad P_{L, \mathcal{X}_t^*}(\mathbf{L}^{-s}) = \int_{\mathcal{X}_t^*} \mathbf{L}^{(d-s)\text{codim } H} d\tilde{\mu}_{\mathcal{G}r}(H)$$

$$(5.2.2) \quad = (1 - \mathbf{L}^{-1})^{-1} \dots (1 - \mathbf{L}^{-d})^{-1} \int_{\Theta^{-1}(\mathcal{X}_t^*)} \mathbf{L}^{(d-s)\text{ord}_t M} d\tilde{\mu}.$$

Furthermore, if  $L = L_k \otimes_k k[[t]]$ , with  $L_k$  a Lie algebra over  $k$ , then  $\Theta^{-1}(\mathcal{X}_t^*)$  is a semi-algebraic subset of  $\mathbf{GL}_d(k[[t]])$ .

*Proof.* The first line (5.2.1) follows from Theorem 4.9 and the remark that  $\text{codim } H = \text{index}(H)$  if  $H$  is a subalgebra of  $L$ .

Let  $A^* = \Theta^{-1}(\mathcal{X}_t^*)$ . Let  $M = (m_{ij})$  be in  $\mathbf{GL}_d(k[[t]])$  and write  $\mathbf{m}_i = m_{i1}e_1 + \dots + m_{id}e_d$ , for  $i = 1, \dots, d$ . We have  $M \in A^{\leq}$  if and only if, for every  $1 \leq i, j \leq d$ , there exist  $Y_{ij}^1, \dots, Y_{ij}^d \in K[[t]]$  such that  $\beta(\mathbf{m}_i, \mathbf{m}_j) = \sum_{k=1}^d Y_{ij}^k \mathbf{m}_k$ , for some finite field extension  $K$  of  $k$ . Here  $\beta$  is the bilinear mapping  $L \times L \rightarrow L$  corresponding to the product in  $L$ . Let  $C_j$  denote the matrix whose rows are  $\mathbf{c}_i = \beta(e_i, e_j)$ . We then have

$$\beta(\mathbf{m}_i, \mathbf{m}_j) = \mathbf{m}_i \left( \sum_{l=j}^d m_{jl} C_l \right).$$

Then  $M \in A^{\leq}$  if and only if, for every  $1 \leq i, j \leq d$ , one can solve the matrix equation

$$(5.2.3) \quad \mathbf{m}_i \left( \sum_{l=j}^d m_{jl} C_l \right) = (Y_{ij}^1, \dots, Y_{ij}^d) M$$

for  $Y_{ij}^1, \dots, Y_{ij}^d \in K[[t]]$ , with  $K$  a finite field extension of  $k$ . Let  $M'$  denote the adjoint matrix, we can then rewrite (5.2.3) as

$$\mathbf{m}_i \left( \sum_{l=j}^d m_{jl} C_l \right) M' = (\det(M) Y_{ij}^1, \dots, \det(M) Y_{ij}^d).$$

Let  $g_{ijk}(m_{rs})$  denote the  $k$ -th entry of the  $d$ -tuple  $\mathbf{m}_i \left( \sum_{l=j}^d m_{jl} C_l \right) M'$ . Then the set  $A^\leq$  has the following description:

(5.2.4)

$$A^\leq = \{(m_{ij}) \in \mathcal{L}(M_d) : \text{ord}_t(\det(M)) \leq \text{ord}_t(g_{ijk}(m_{rs})) \text{ for } i, j, k \in \{1, \dots, d\}\}.$$

The set  $A^\leq$  is therefore  $k[[t]]$ -semi-algebraic. Let  $\Phi^\leq(M)$  denote the conjunction of conditions  $\text{ord}_t(\det(M)) \leq \text{ord}_t(g_{ijk}(m_{rs}))$ .

Since  $\text{ord}_t : \mathcal{L}(M_d) \rightarrow \mathbb{N} \cup \{\infty\}$  is a simple function this implies that  $A_n$  is a semi-algebraic set and in particular is constructible. As we promised in section 2 this provides an alternative way to show the constructibility of this class of subalgebras. We prove that  $A^\natural$  is  $k[[t]]$ -semi-algebraic. We have  $M \in A^\natural$  if and only if, for every  $1 \leq i, j \leq d$ , there exist  $Y_{ij}^1, \dots, Y_{ij}^d \in K[[t]]$ , with  $K$  some finite field extension of  $k$ , such that  $\beta(\mathbf{m}_i, \mathbf{e}_j) = \sum_{k=1}^d Y_{ij}^k \mathbf{m}_k$ . Let  $g_{ijk}^\natural(m_{rs})$  denote the  $k$ -th entry of the  $d$ -tuple  $\mathbf{m}_i C_j M'$ . Then an argument similar to the above implies that the set  $A^\natural$  has the following description:

$$A^\natural = \{(m_{ij}) \in \mathcal{L}(M_d) : \text{ord}_t(\det(M)) \leq \text{ord}_t(g_{ijk}^\natural(m_{rs})) \text{ for } i, j, k \in \{1, \dots, d\}\}.$$

Hence  $A^\natural$  is  $k[[t]]$ -semi-algebraic. Indeed it is defined by  $\Phi^\natural$ , the definable condition which is the conjunction of conditions  $\text{ord}_t(\det(M)) \leq \text{ord}_t(g_{ijk}^\natural(m_{rs}))$ . Equation (5.2.2) follows from Proposition 4.6 and the last statement is clear, since when  $L$  is of the form  $L_k \otimes_k k[[t]]$ , the above definable conditions are all semi-algebraic.  $\square$

5.3. *Remark.* The following provides an easier integral expression in general to calculate. It is similar to the integral expression that was used in [17]. Let  $Y$  denote the variety  $\text{Tr}_{d,k}$  of  $d \times d$  upper triangular matrices which is isomorphic to the affine space  $\mathbb{A}_k^{d(d+1)/2}$ . Let  $\tilde{\nu}$  denote the measure on stable  $k[[t]]$ -semi-algebraic subsets of  $\mathcal{L}(Y)$ . For  $i = 1, \dots, d$  define the simple function  $\alpha_i : \mathcal{L}(Y) \rightarrow \mathbb{Z}$  by  $\alpha_i(M) = \text{ord}_t(m_{ii})$ . Then

$$\begin{aligned} & (1 - \mathbf{L}^{-1})^{-1} \dots (1 - \mathbf{L}^{-d})^{-1} \int_{A^*} \mathbf{L}^{(d-s)\text{ord}_t(M)} d\tilde{\mu} \\ &= (1 - \mathbf{L}^{-1})^{-d} \int_{A^* \cap \mathcal{L}(Y)} \mathbf{L}^{-\text{ord}_t(M) s} \mathbf{L}^{\alpha_1(M) + \dots + i\alpha_i(M) + \dots + d\alpha_d(M)} d\tilde{\nu}. \end{aligned}$$

5.4. **Commensurable subalgebras.** We turn now to the issue of commensurable subalgebras. The following discussion will also apply to counting commensurable subalgebras in a  $\mathbb{Q}_p$ -Lie algebra, an issue which has previously not been discussed. We identify  $L$  with  $k((t))^d$  by a choice of a basis  $e_1, \dots, e_d$  for the  $k[[t]]$ -submodule  $L_0$  which is then identified with  $k[[t]]^d$ . The mapping  $\Theta : \mathbf{GL}_d(k((t))) \rightarrow \mathcal{G}r$  sends

a matrix  $M$  to the lattice  $\langle \mathbf{m}_1, \dots, \mathbf{m}_d \rangle$  spanned by

$$\mathbf{m}_i = m_{ii}e_i + \dots + m_{id}e_d,$$

for  $i = 1, \dots, d$ . The index then of  $H = \langle \mathbf{m}_1, \dots, \mathbf{m}_d \rangle$  (as defined in (4.4.1)) is just  $\text{ord}_t(\det M)$ .

5.5. *Remark.* For subalgebras of  $L_0 \otimes K$  the index of  $H$  is the same as the codimension. We might have considered encoding a zeta function of commensurable subalgebras according to index but note that there are an infinite number of commensurable submodules of index 0 whilst there is only one of codimension 0.

We now give a description of the codimension as a function of the entries of  $M$ . Recall that if the entries of  $M$  are in the field  $K$  containing  $k$ , the codimension of  $H = \langle \mathbf{m}_1, \dots, \mathbf{m}_d \rangle$  in  $L \otimes K$  is defined as the sum of the codimension of  $H \cap (L_0 \otimes K)$  in  $L_0 \otimes K$  and in  $H$ :

**5.6. Lemma.** *Let  $M$  be a  $K$ -rational point of  $\mathbf{GL}_d(k((t)))$  and define*

$$n_i = \min \{0, \text{ord}_t(m_{ij}) : j = 1, \dots, d\}.$$

*Then the codimension of  $H = \langle \mathbf{m}_1, \dots, \mathbf{m}_d \rangle$  in  $L \otimes K$  is the sum of  $\text{ord}_t(\det(m_{ij}t^{n_i}))$  and  $\sum_{i=1}^d n_i$ .*

*Proof.* This follows from the fact that  $H \cap (L_0 \otimes K) = \langle t^{n_1}\mathbf{m}_1, \dots, t^{n_d}\mathbf{m}_d \rangle$ . Hence the codimension of  $H \cap (L_0 \otimes K)$  is given as usual by  $\text{ord}_t(\det(m_{ij}t^{n_i}))$ . The codimension of  $H \cap (L_0 \otimes K)$  in  $H$  on the other hand is given by  $\sum_{i=1}^d n_i$ .  $\square$

We define then the function

$$\text{codim}(M) := \text{ord}_t(\det(m_{ij}t^{n_i})) + \sum_{i=1}^d n_i = \text{ord}_t(\det(m_{ij})) + 2 \sum_{i=1}^d n_i$$

which is a simple function.

5.7. We want to convert the Poincaré series into integrals over arc spaces. To do this we think of  $k((t))$  as consisting of two pieces:  $k[[t]]$  identified with  $k[[t]]$  via  $v \mapsto v$  and  $k((t)) \setminus k[[t]]$  identified with  $tk[[t]]$  via  $v \mapsto v^{-1}$ . Notice that the measure on  $k((t)) \setminus k[[t]]$  translates under this identification to the measure  $\mathbf{L}^{2\text{ord}_t(v)} \mu_k[[t]]$  on  $tk[[t]]$ . This follows because the set  $t^{-n}k[[t]]^*$  has measure  $\mathbf{L}^n$  whilst  $t^n k[[t]]^*$ , its image under  $v \mapsto v^{-1}$ , has measure  $\mathbf{L}^{-n}$ .

We can partition  $\mathbf{GL}_d(k((t)))$  into  $2^{d^2}$  subsets  $\mathbf{GL}_d(k((t)))_A$  which we can index with an element of  $A \in M_d(\{\pm 1\})$  so that

$$\mathbf{GL}_d(k((t)))_A = \{M \in \mathbf{GL}_d(k((t))) : \text{ord}_t(m_{ij}) \geq 0 \text{ if and only if } A_{ij} = 1\}.$$

We then identify each  $\mathbf{GL}_d(k((t)))_A$  with a subset of  $\mathcal{L}(M_d)$  by the morphism  $\Xi_A : M \mapsto \left(m_{ij}^{A_{ij}}\right)$ . Note that  $\Xi_A(\Xi_A(M)) = M$ .

Now remark, that with the notations used in the proof of Proposition 5.2, a matrix  $M$  is in  $\Theta^{-1}(\mathcal{X}_0^{\leq})$  if and only if  $\Phi^{\leq}(M)$  is true where  $\Phi^{\leq}$  was defined as the conjunction of statements  $\text{ord}_t(\det(M)) \leq \text{ord}_t(g_{ijk}(m_{rs}))$  appearing in (5.2.4). Hence the subset  $\Xi_A(\Theta^{-1}(\mathcal{X}_{t,0}^{\leq}) \cap \mathbf{GL}_d(k((t)))_A)$  is a  $k[[t]]$ -semi-algebraic subset of

$\mathcal{L}(M_d)$  defined by the condition  $\Phi^{\leq}(\Xi_A M)$  is true. Furthermore, when  $L_0$  is of the form  $L_k \otimes_k k[[t]]$ , the definable conditions considered are all semi-algebraic, hence  $\Xi_A(\Theta^{-1}(\mathcal{X}_{t,0}^{\leq}) \cap \mathbf{GL}_d(k((t)))_A)$  is semi-algebraic in this case. Hence we have the following result:

**5.8. Theorem.** *With the preceding notations,  $\Xi_A(\Theta^{-1}(\mathcal{X}_{t,0}^{\leq}) \cap \mathbf{GL}_d(k((t)))_A)$  is a  $k[[t]]$ -semi-algebraic subset of  $\mathcal{L}(M_d)$ , and*

$$\begin{aligned}
P_{L, \mathcal{X}_{t,0}^{\leq}}(\mathbf{L}^{-s}) &= \int_{\Theta^{-1}(\mathcal{X}_{t,0}^{\leq})} \mathbf{L}^{\text{ord}_t(\det M)d - (\text{codim } M)s} d\tilde{\mu} \\
(5.8.1) \quad &= \sum_{A \in M_d(\{\pm 1\})} \int_{\Xi_A(\Theta^{-1}(\mathcal{X}_{t,0}^{\leq}) \cap \mathbf{GL}_d(k((t)))_A)} \mathbf{L}^{\text{ord}_t(\det \Xi_A(M))d - (\text{codim } \Xi_A(M))s} \\
&\quad \times \mathbf{L}^{\sum_{A_{ij}=-1} 2\text{ord}_t m_{ij}^{-A_{ij}}} d\tilde{\mu}.
\end{aligned}$$

Furthermore, if  $L_0 = L_k \otimes_k k[[t]]$ , with  $L_k$  a Lie algebra over  $k$ , then

$$\Xi_A(\Theta^{-1}(\mathcal{X}_{t,0}^{\leq}) \cap \mathbf{GL}_d(k((t)))_A)$$

is a semi-algebraic subset of  $\mathcal{L}(M_d)$ .

*Proof.* Everything has already been proven, except for (5.8.1), which follows from Proposition 4.6.  $\square$

**5.9. Proof of Theorem 5.1.** Let us first prove (1). Since, by Theorem 5.2, we have (5.2.2) the result follows from Theorem 3.7, Remark 3.10 and the last statement in Theorem 5.2. Similarly to deduce (2) from Theorem 5.8, Theorem 3.7 and Remark 3.10, one needs to check that there exists an integer  $N$  such that

$$N \text{codim } \Xi_A(M) > \text{ord}_t(\det \Xi_A(M))d + \sum_{A_{ij}=-1} 2\text{ord}_t m_{ij}^{-A_{ij}}.$$

This follows from the fact that  $\text{codim } \Xi_A(M) > \text{ord}_t(\det \Xi_A(M))$  and if  $A_{ij} = -1$  then  $\text{ord}_t m_{ij}^{-1} < n_i$ . Hence if we take  $N = 3d$  we are done.  $\square$

**5.10. Remark.** If Theorem 3.7 remains true when semi-algebraic is replaced everywhere by  $k[[t]]$ -semi-algebraic and simple by  $k[[t]]$ -simple, as suggested in Remark 3.11, then Theorem 5.1 remains true with the same proof, without assuming that  $L$  is obtained by extension of scalars from a Lie algebra  $L_k$  defined over  $k$ .

**5.11. Commensurable subalgebras in  $p$ -adic Lie algebras.** Note that the analysis above can also be applied to any finite dimensional  $\mathbb{Q}_p$ -Lie algebra  $\mathcal{L}$  with a choice of  $\mathbb{Z}_p$ -lattice  $L$  where we define the commensurable zeta function as:

$$\zeta_{\mathcal{L}, L}(s) = \sum_{H \in \mathcal{X}(\mathcal{L}, L)} |L + H : L \cap H|^{-s}$$

where  $\mathcal{X}(\mathcal{L}, L)$  denotes the set of  $\mathbb{Z}_p$ -subalgebras commensurable with  $L$ . This is the analogue of what we did above where the index  $|L + H : L \cap H|$  corresponds to the codimension since  $\text{codim } H = \text{codim}(L_0 + H : L \cap H)$  in the  $k[[t]]$ -setting. We shall

consider later, see Corollary 7.8, the question of zeta functions of  $L \otimes \mathbb{Z}_p$ , where  $L$  is  $\mathbb{Z}$ -Lie algebra, but here the analysis applies to a Lie algebra with no underlying  $\mathbb{Z}$ -structure. So it is worth recording the following result for  $\zeta_{\mathcal{L},L}(s)$ , proved as we said by the same analysis as above replacing  $\mathbf{L}$  by  $p$ ,  $k[[t]]$  by  $\mathbb{Z}_p$ ,  $k((t))$  by  $\mathbb{Q}_p$ , and Theorem 3.7 by Theorem 3.2 of [4].

**5.12. Theorem.** *Let  $\mathcal{L}$  be a finite dimensional  $\mathbb{Q}_p$ -Lie algebra with a choice of  $\mathbb{Z}_p$ -lattice  $L$  such that  $L \otimes \mathbb{Q}_p = \mathcal{L}$ . Then  $\zeta_{\mathcal{L},L}(s)$  is a rational function in  $p^{-s}$ .*

**5.13.  $\mathbb{F}_p[[t]]$ -Lie algebras.** We mention that there is another open rationality question which fits into this context. This concerns Lie algebras in characteristic  $p$ . For example the following is conjectured in section 5 of [11]:

**5.14. Conjecture.** *Let  $L$  be a finite dimensional Lie algebra over  $\mathbb{F}_p[[t]]$  additively isomorphic to  $\mathbb{F}_p[[t]]^d$ . Then*

$$\zeta_L(s) = \sum_{H \leq L} |L : H|^{-s}$$

*is a rational function in  $p^{-s}$  where the sum is taken over  $\mathbb{F}_p[[t]]$ -subalgebras  $H$ .*

**5.15.** The techniques involved in proving such a result in characteristic zero (quantifier elimination and Hironaka’s resolution of singularities) do not exist at present in characteristic  $p$ . However, in [11] the rationality was proved for one class of  $\mathbb{F}_p[[t]]$ -algebras. Take  $L$  to be an algebra additively isomorphic to  $\mathbb{Z}^d$  and set  $L_p = L \otimes \mathbb{F}_p[[t]]$ , then one can prove that for almost all primes  $\zeta_{L_p}(s)$  is a rational function in  $p^{-s}$ . In fact one gets that  $\zeta_{L_p}(s) = \zeta_{L \otimes \mathbb{Z}_p}(s)$  for almost all primes. The proof actually follows the theme of this current paper, that there is an integral over the additive Haar measure on  $(\mathbb{F}_p[[t]])^d$  representing this zeta function  $\mathbb{F}_p[[t]]$  which formally looks the same as the integral representing  $\zeta_{L \otimes \mathbb{Z}_p}(s)$ . Macintyre then observed that the calculation of this latter integral will carry over formally to the former essentially for any primes for which one does not divide by in this calculation. See also Theorem 8.3.2 of [9] for much more general results along these lines.

**5.16.** If we are going to consider the possible rationality of some of the other motivic zeta functions considered in this paper then it is necessary to introduce the concept of *polynomial growth* for  $(L, \mathcal{X})$ . This is going to be a necessary requirement if we are to prove that the zeta function  $P_{L,\mathcal{X}}(T)$  can be expressed as an element of  $\mathcal{M}[T]_{\text{loc}}$ .

**5.17. Definition.** We say that  $(L, \mathcal{X})$  has *polynomial growth* if there exists some  $d \in \mathbb{N}$  such, that for all  $n$ ,  $A_n(\mathcal{X})$  is a constructible set and

$$\dim A_n(\mathcal{X}) \leq dn.$$

Recall a constructible set is of dimension  $\leq m$ , if it may be partitioned into finitely many locally closed pieces with dimension  $\leq m$ .

It is a necessary condition that  $(L, \mathcal{X})$  have polynomial growth for the zeta function to be rational, because the coefficients of a rational series satisfy a linear recurrences of finite order. For example, when  $L$  is a finite dimensional  $k[[t]]$ -algebra additively isomorphic to  $k[[t]]^d$  and  $\mathcal{X}$  the class of commensurable  $k[[t]]$ -subalgebras,

then  $A_n(\mathcal{X})$  is contained in  $\mathcal{G}r_{(n)}$  whose dimension is bounded by  $2nd^2$ , the dimension of  $M_d(t^{-n}k[[t]]/t^nk[[t]])$ .

**5.18. Problem.** Characterize the  $\mathbb{N}$ -filtered or  $\mathbb{Z}$ -filtered infinite dimensional Lie algebras for which  $\mathcal{X}^{\leq}$  has polynomial growth.

**5.19. Polynomial growth.** If we consider the algebra  $\mathfrak{d}^+ := \bigoplus_{j \geq 1} k \cdot d_j$ , where  $(d_i, d_j) = (i - j)d_{i+j}$  and  $\mathcal{X}$  is the class of all subalgebras, then this does not have polynomial growth. However, if we define  $\mathcal{X}^d$  to be the class of  $d$ -generated subalgebras, we find that  $(\mathfrak{d}^+, \mathcal{X}^d)$  does have polynomial growth.

**5.20. Problem.** Calculate  $P_{\mathfrak{d}^+, \mathcal{X}^d}(T)$  for  $d \geq 2$ . Is it rational?

More generally,  $(L, \mathcal{X}^d)$  has polynomial growth if  $L$  is a well-covered Lie algebra (where recall a subalgebra of codimension  $n$  in  $L \otimes K$  contains  $L_{f(n)} \otimes K$  for some function  $f(n)$ ) and  $\dim(L/L_{f(n)}) \leq n^c$  for some fixed  $c$ . This is true for example for all the simple graded Lie algebras of finite growth. However we are unable to prove that the corresponding zeta function encoding these  $d$ -generated subalgebras is rational.

## 6. $k[[t]]$ -POWERED NILPOTENT GROUPS

6.1. In [17] subgroups of finite index in a torsion-free finitely generated nilpotent group  $G$  were encoded in zeta functions. It was shown there how, for almost all primes  $p$ , there is a one-to-one correspondence between subgroups of finite index in the pro- $p$  completion  $\widehat{G}_p$  of  $G$  and subalgebras of finite index in  $L \otimes \mathbb{Z}_p$  for an associated Lie algebra  $L$  over  $\mathbb{Z}$ . The pro- $p$  completion has the structure of a  $\mathbb{Z}_p$ -powered nilpotent group.

We can use the same ideas to show that the motivic zeta functions counting subalgebras in a finite dimensional  $k[[t]]$ -Lie algebra are also encoding the subgroup structure of a  $k[[t]]$ -powered nilpotent group  $G$  where subgroups are taken to be  $k[[t]]$ -closed subgroups.

An  $R$ -powered nilpotent group is a nilpotent group with an action of the ring  $R$  satisfying various axioms which can be found in Chapter 6 of P. Hall's notes on nilpotent groups [19].

We shall be interested in a certain type of  $R$ -powered nilpotent group which arises when  $R$  is a binomial ring in the following way. Take a torsion-free finitely generated nilpotent group  $G$  with a choice of Malcev basis  $x_1, \dots, x_d$ . The group  $G$  can then be identified with the set of elements  $\mathbf{x}(\mathbf{a}) = x_1^{a_1} \dots x_d^{a_d}$  with  $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d$ . In [19] P. Hall proved that there exist polynomials over  $\mathbb{Q}$  in suitably many variables which define mappings

$$\begin{aligned} \lambda &: \mathbb{Z}^d \times \mathbb{Z} \rightarrow \mathbb{Z} \\ \mu &: \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathbb{Z}^d \end{aligned}$$

such that for  $\mathbf{a}_1$  and  $\mathbf{a}_2 \in \mathbb{Z}^d$  and  $k \in \mathbb{Z}$

$$\begin{aligned} \mathbf{x}(\mathbf{a})^k &= \mathbf{x}(\lambda(\mathbf{a}, k)) \\ \mathbf{x}(\mathbf{a}_1) \cdot \mathbf{x}(\mathbf{a}_2) &= \mathbf{x}(\mu(\mathbf{a}_1, \mathbf{a}_2)). \end{aligned}$$

Since these polynomials over  $\mathbb{Q}$  take integer values, they can be represented as  $\mathbb{Z}$ -linear combinations of binomial polynomials. Suppose  $R$  is a binomial ring, *i.e.*  $\binom{r}{n}$  is a well-defined element of  $R$  for  $r \in R$  and  $n \in \mathbb{Z}$ . Then we can use the polynomials  $\lambda$  and  $\mu$  to define the structure of an  $R$ -powered nilpotent group on the set of elements  $\mathbf{x}(\mathbf{a}) = x_1^{a_1} \dots x_d^{a_d}$  with  $\mathbf{a} = (a_1, \dots, a_d) \in R^d$ .

Therefore to every finitely generated torsion-free nilpotent group  $G$  we can define a  $k[[t]]$ -powered group which we shall denote  $G^{k[[t]]}$ . There exists a  $\mathbb{Q}$ -rational representation of  $G^{k[[t]]}$  embedding it as a subgroup of  $\mathrm{Tr}_n^1(k[[t]])$ , the upper triangular matrix group of dimension  $n$  with 1's on the diagonal. The image of  $G^{k[[t]]}$  under the map  $\log$  defines a nilpotent  $k[[t]]$ -subalgebra  $\mathcal{L} = \log G^{k[[t]]}$  of  $\mathrm{Tr}_n^0(k[[t]])$ , the upper triangular matrix group of dimension  $n$  with 0's on the diagonal. (Note that  $\log$  is a polynomial map since  $(U - 1)^n = 0$  if  $U \in \mathrm{Tr}_n^1(k[[t]])$ .) The Lie algebra  $\mathcal{L}$  is  $L_{\mathbb{Q}} \otimes k[[t]]$  where  $L_{\mathbb{Q}}$  is the Lie algebra corresponding to  $G^{\mathbb{Q}}$  under the Malcev correspondence. In general the image of  $G$  under  $\log$  is not a Lie algebra. However there is some integer  $f$  such that  $L = \log G^f$  is a  $\mathbb{Z}$ -Lie algebra. This is the Lie algebra mentioned in the first paragraph used by Grunewald, Segal and Smith to set up a one-to-one correspondence between subgroups in  $G^{\mathbb{Z}_p} = \widehat{G}_p$  and  $L \otimes \mathbb{Z}_p$  for all primes not dividing  $f$  (see Theorem 4.1 of [17]).

A subgroup  $H$  of  $G^{k[[t]]}$  is  $k[[t]]$ -closed if  $h^r \in H$  for all  $r \in k[[t]]$  and  $h \in H$ . We shall define the codimension of such a subgroup as the length  $n$  of the maximal chain of  $k[[t]]$ -closed subgroups  $H_i$   $0 \leq i \leq n$  with  $H_i$  contained in  $H_{i-1}$  with infinite index and  $H_0 = G^{k[[t]]}$  and  $H_n = H$ .

The claim is then that  $\log$  gives a way of getting from  $k[[t]]$ -closed subgroups to  $k[[t]]$ -subalgebras to prove the following:

**6.2. Theorem.** *There is a one-to-one correspondence between the set of  $k[[t]]$ -closed subgroups of  $G^{k[[t]]}$  of finite codimension and the set of  $k[[t]]$ -subalgebras of finite codimension in  $\mathcal{L}$  which preserves the codimension.*

*Proof.* We need to prove that the image of a  $k[[t]]$ -closed subgroup  $H$  under the map  $\log$  is a  $k[[t]]$ -subalgebra. The proof of Lemma 1 in Chapter 6 of [26] can be applied in this setting to prove that  $m\mathbb{Z} \log H \subset \log H$ . But since  $H$  is  $k[[t]]$ -closed and  $k[[t]] \supset \mathbb{Q}$  we have  $h^{1/m} \in H$ . Hence  $\mathbb{Z} \log H \subset \log H$ , *i.e.*  $\log H$  is additively closed. Once we have this then Corollary 3 of Chapter 6 of [26] implies that  $\log H$  is also closed under the Lie bracket by applying the inverse of the Campbell-Hausdorff series. We have implicitly used the identity  $r \log h = \log h^r$  above when  $r \in \mathbb{Q}$ . This is true in fact for all values of  $r \in k[[t]]$ . This follows from the fact that there exist polynomials  $f_1(\mathbf{a}), \dots, f_d(\mathbf{a})$  such that  $\log x_1^{a_1} \dots x_d^{a_d} = f_1(\mathbf{a}) \log x_1 + \dots + f_d(\mathbf{a}) \log x_d$ . Since  $r \log h = \log h^r$  is true for all  $h \in G$  and  $r \in \mathbb{Z}$  we get that the following is a polynomial identity for all  $\mathbf{a} \in k[[t]]^d$  and  $r \in \mathbb{Z}$

$$f_i(\lambda(\mathbf{a}, r)) = r f_i(\mathbf{a}).$$

Hence this must be a formal identity of polynomials which is then true for all values of  $r \in k[[t]]$ . Conversely, if  $M$  is a  $k[[t]]$ -subalgebra of  $\mathcal{L}$  then  $\exp M$  is well-defined on  $\mathrm{Tr}_n^0(k[[t]])$  being a finite polynomial map. By using the Campbell-Hausdorff series one obtains directly that  $\exp M$  is a subgroup and the identity  $r \log h =$

$\log h^r$  again implies that  $\exp M$  is a  $k[[t]]$ -closed subgroup taking  $h = \exp m$  and using the identity  $\log \exp(m) = m$ . Since  $\log$  and  $\exp$  are bijective maps, we have a one-to-one correspondence as detailed in the statement of the Theorem. The correspondence preserves codimension since this is defined by lengths of chains of  $k[[t]]$ -closed subgroups or  $k[[t]]$ -subalgebras.  $\square$

We can use this to deduce the existence and rationality of a motivic zeta function associated to  $G^{k[[t]]}$ , a certain type of loop group.

**6.3. Corollary.** *Let  $G$  be a finitely generated torsion-free nilpotent group and  $G^{k[[t]]}$  the associated  $k[[t]]$ -powered nilpotent group. For any field  $K$  which is a finite extension of  $k$ , let  $\mathcal{X}(K)$  denote the set of  $K[[t]]$ -closed subgroups of  $G^{K[[t]]}$  and let  $A_n(K)$  denote the set of  $K[[t]]$ -closed subgroups of  $G^{K[[t]]}$  of codimension  $n$ . There is a natural structure of constructible set on  $A_n = \cup_K A_n(K)$ , hence*

$$P_{G^{k[[t]]}, \mathcal{X}}(T) := \sum_{n \in \mathbb{N}} [A_n] T^n$$

is a well-defined element of  $\mathcal{M}[[T]]$ . We call it the motivic zeta function of  $G^{k[[t]]}$ . It is equal to the motivic zeta function  $P_{\mathcal{L}, \mathcal{X}_t^{\leq}}(T)$  of the associated  $k[[t]]$ -Lie algebra  $\mathcal{L} = \log G^{k[[t]]} = \log G^{\mathbb{Q}} \otimes_{\mathbb{Q}} k[[t]]$ . Hence  $P_{G^{k[[t]]}, \mathcal{X}}(T)$  is rational, belonging to  $\mathcal{M}[T]_{\text{loc}}$ .

We have a similar result for normal  $k[[t]]$ -closed subgroups in  $G^{k[[t]]}$  since the one-to-one correspondence in Theorem 6.2 sends normal subgroups to ideals.

## 7. MOTIVIC AND $p$ -ADIC CONE INTEGRALS

The motivic integrals we have defined to capture the subalgebras in a  $k[[t]]$ -algebra are examples of integrals that the first author and Grunewald studied in the  $p$ -adic setting in [13] called cone integrals. We make the same definition in the motivic setting:

**7.1. Definition.** We call an integral

$$Z_{\mathcal{D}, \text{geom}}(s) := \int_V \mathbf{L}^{-\text{ord}_t(f_0)s - \text{ord}_t(g_0)} d\mu$$

a *motivic cone integral* if  $f_0(\mathbf{x})$  and  $g_0(\mathbf{x})$  are polynomials in  $k[[t]][x_1, \dots, x_m]$  and there exist polynomials  $f_i(\mathbf{x}), g_i(\mathbf{x}), i = 1, \dots, l$ , in  $k[[t]][x_1, \dots, x_m]$  such that

$$V = \{\mathbf{x} \in \mathcal{L}(\mathbb{A}_k^m) : \text{ord}_t(f_i(\mathbf{x})) \leq \text{ord}_t(g_i(\mathbf{x})) \text{ for } i = 1, \dots, l\}.$$

The set  $\mathcal{D} = \{f_0, g_0, f_1, g_1, \dots, f_l, g_l\}$  is called the *set of cone integral data*.

**7.2. Definition.** Let  $L$  be a  $k[[t]]$ -algebra additively isomorphic to  $(k[[t]])^d$  (or a  $\mathbb{Z}$ -Lie algebra additively isomorphic to  $\mathbb{Z}^d$ ). Then we define  $Z_{L, \text{geom}}(s)$ , the *motivic cone integral associated to  $L$* , to be

$$\begin{aligned} Z_{L, \text{geom}}(s) &:= \int_A \mathbf{L}^{-\text{sord}_t(M)} d\mu \\ &= Z_{\mathcal{D}, \text{geom}}(s) \end{aligned}$$

where

$$A = \{(m_{ij}) \in \mathcal{L}(M_d) : \text{ord}_t(\det(M)) \leq \text{ord}_t(g_{ijk}(m_{rs})) \text{ for } i, j, k \in \{1, \dots, d\}\}$$

and  $\mathcal{D}$  is the corresponding cone data.

In section 5 we therefore established that

$$P_{L, \mathcal{X}_t^{\leq}}(\mathbf{L}^{-s}) = (1 - \mathbf{L}^{-1})^{-1} \dots (1 - \mathbf{L}^{-d})^{-1} Z_{L, \text{geom}}(s - d).$$

The set  $V$  in the definition of the motivic cone integral is  $k[[t]]$ -semi-algebraic and therefore measurable; the functions  $\text{ord}_t(f_0)$  and  $\text{ord}_t(g_0)$  are  $k[[t]]$ -simple functions. However, in this more general setting, we don't know whether the fibres of  $\text{ord}_t(f_0) : V \rightarrow \mathbb{N}$  and  $\text{ord}_t(g_0) : V \rightarrow \mathbb{N}$  are weakly stable. This is why we need to use the completion  $\widehat{\mathcal{M}}$  of  $\mathcal{M}_{\text{loc}}$  with respect to the filtration  $F^m \mathcal{M}_{\text{loc}}$ , introduced in 3.5, and to replace the measure  $\tilde{\mu}$  by the measure  $\mu$ . Hence, Theorem 3.8 gives us the following:

**7.3. Theorem.** *Assume the polynomials  $f_i(\mathbf{x}), g_i(\mathbf{x}), i = 0, \dots, l$ , have their coefficients in  $k$ . The motivic cone integral  $Z_{\mathcal{D}, \text{geom}}(s)$  belongs to the subring  $\overline{\mathcal{N}}$  of the ring  $\widehat{\mathcal{M}}[[T]]$  which is generated by the image in  $\widehat{\mathcal{M}}[[T]]$  of  $\mathcal{M}_{\text{loc}}[T], (\mathbf{L}^i - 1)^{-1}$  and  $(1 - \mathbf{L}^{-a}T^b)^{-1}$  with  $i \in \mathbb{N}, a \in \mathbb{N}, b \in \mathbb{N} \setminus \{0\}$ , where  $T = \mathbf{L}^{-s}$ .*

In the algebra setting we had that the cone conditions were in fact stable and hence, by Theorem 3.7, the associated zeta function  $Z_{L, \text{geom}}(s)$  is an element of the subring of  $\mathcal{M}_{\text{loc}}[[T]]$ , where  $T = \mathbf{L}^{-s}$ . However in the more general setting here, we are required to take infinite sums of elements of  $\mathcal{M}_{\text{loc}}$  which are then only defined in the associated complete ring  $\widehat{\mathcal{M}}$ . At the moment it is not known if the image of  $\mathcal{M}_{\text{loc}}$  is the same as  $\widehat{\mathcal{M}}$  since there might be some kernel. The infinite sums of elements of  $\mathcal{M}_{\text{loc}}$  and the terms like  $(\mathbf{L}^i - 1)^{-1}$  which don't occur in Theorem 3.7 appear for example when we consider the constant term of these motivic zeta functions which take the following form:

$$\int_{V \cap \{\mathbf{x} : \text{ord}_t(f_0(\mathbf{x})) = 0\}} \mathbf{L}^{-\text{ord}_t(g_0)} d\mu.$$

Even if  $g_0 = 0$  we get terms coming from the cone conditions like  $(\mathbf{L}^i - 1)^{-1}$  as we shall see when we derive an explicit expression for motivic cone integrals.

In [13] the first author and Grunewald produced an explicit formula for  $p$ -adic cone integrals extending Denef's explicit formula for the Igusa zeta function. We can do the same thing for the motivic cone integrals. As we shall explain, this will impact on how canonical the explicit formula for  $p$ -adic cone integrals is. It will also allow us to define a topological zeta function associated to  $p$ -adic cone integrals and ultimately to the zeta function of an algebra defined over  $\mathbb{Z}$ .

7.4. Assume the polynomials  $f_i(\mathbf{x}), g_i(\mathbf{x}), i = 0, \dots, l$ , have their coefficients in  $k$ . Let  $F = f_0 g_0 \dots f_l g_l$  and denote by  $D$  the divisor defined by  $F = 0$ . Let  $(Y, h)$  be a resolution of  $F$ . By this, we mean that  $Y$  is a smooth and connected algebraic variety,  $h : Y \rightarrow X = \mathbb{A}_k^m$  is proper, that the restriction  $h : Y \setminus h^{-1}(D) \rightarrow X \setminus D$  is an isomorphism, and that  $(h^{-1}(D))_{\text{red}}$  has only normal crossings as a subvariety of

$Y$ . Let  $E_i$ ,  $i \in T$  be the irreducible (smooth) components of  $(h^{-1}(D))_{\text{red}}$  where  $D$  is the divisor defined by  $F = 0$  in  $X = \mathbb{A}^m$ . For each  $i \in T$ , denote by  $N_i(f_j)$  and  $N_i(g_j)$  the multiplicity of  $E_i$  in the divisor of  $f_j \circ h$  and  $g_j \circ h$  on  $Y$  and by  $\nu_i - 1$  the multiplicity of  $E_i$  in the divisor of  $h^*dx$ , where  $dx$  is a local non vanishing volume form. For  $i \in T$  and  $I \subset T$ , we consider the schemes  $E_i^\circ = E_i \setminus \bigcup_{j \neq i} E_j$ ,  $E_I = \bigcap_{i \in I} E_i$  and  $E_I^\circ = E_I \setminus \bigcup_{j \in T \setminus I} E_j$ . When  $I = \emptyset$ , we have  $E_\emptyset = Y$ .

Define a closed cone  $\overline{D_T}$  as follows:

$$\overline{D_T} := \left\{ (x_1, \dots, x_t) \in \mathbb{R}_{\geq 0}^t : \sum_{j=1}^t N_j(f_i)x_j \leq \sum_{j=1}^t N_j(g_i)x_j \text{ for } i = 1, \dots, l \right\},$$

where  $\text{card}T = t$  and  $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$ . Denote the lattice points in  $\overline{D_T}$  by  $\overline{\Delta_T}$ , i.e.  $\overline{\Delta_T} = \overline{D_T} \cap \mathbb{N}^t$ . We can write this cone as a disjoint union of open simplicial pieces that we shall call  $R_k$ ,  $k = 0, 1, \dots, w$ . We shall assume that  $R_0 = (0, \dots, 0)$  and that the next  $q$  pieces are all the open one-dimensional edges in the cone  $\overline{D_T}$ : for  $k = 1, \dots, q$ ,

$$R_k = \{\alpha \mathbf{e}_k = \alpha(q_{k1}, \dots, q_{kt}) : \alpha > 0\}.$$

Since these are all the edges, for any  $k \in \{0, \dots, w\}$  there exists some subset  $M_k \subset \{1, \dots, q\}$  such that

$$R_k = \left\{ \sum_{j \in M_k} \alpha_j \mathbf{e}_j : \alpha_j > 0 \text{ for all } j \in M_k \right\}.$$

Note that  $m_k := \text{card}M_k \leq t$ . We can choose the decomposition of the cone  $\overline{D_T}$  so that for any  $k \in \{0, \dots, w\}$

$$R_k \cap \mathbb{N}^t = \left\{ \sum_{j \in M_k} l_j \mathbf{e}_j : l_j \in \mathbb{N}_{>0} \text{ for all } j \in M_k \right\},$$

as follows from [3], p.123-124.

Define for each  $k = 1, \dots, q$  the following non-negative constants:

$$(7.4.1) \quad \begin{aligned} A_k &= \sum_{j=1}^t q_{kj} N_j(f_0) \\ B_k &= \sum_{j=1}^t q_{kj} (N_j(g_0) + \nu_j). \end{aligned}$$

For each  $I \subset T$  define:

$$\begin{aligned} D_I &= \{(k_1, \dots, k_t) \in \overline{D_T} : k_i > 0 \text{ if } i \in I \text{ and } k_i = 0 \text{ if } i \in T \setminus I\} \\ \Delta_I &= D_I \cap \mathbb{N}^t. \end{aligned}$$

So  $\overline{D_T} = \bigcup_{I \subset T} D_I$ , a disjoint union.

For each  $I \subset T$  there is then a subset  $W_I \subset \{0, \dots, w\}$  so that

$$D_I = \bigcup_{k \in W_I} R_k.$$

If  $k \in W_I$ , we set  $I_k = I$ . We can now state the explicit formula for motivic cone integrals:

**7.5. Theorem.** *Assume the polynomials  $f_i(\mathbf{x}), g_i(\mathbf{x})$ ,  $i = 0, \dots, l$ , have their coefficients in  $k$ . The following equality holds in  $\mathcal{N}$ :*

$$(7.5.1) \quad Z_{\mathcal{D}, \text{geom}}(s) = \sum_{k=0}^w (\mathbf{L} - 1)^{|I_k|} \mathbf{L}^{-m} [E_{I_k}^{\circ}] \prod_{j \in M_k} \frac{\mathbf{L}^{-(A_j s + B_j)}}{(1 - \mathbf{L}^{-(A_j s + B_j)})}.$$

*Proof.* The proof works just the same as the one in [13] for the explicit formula for  $p$ -adic cone integrals, using Lemma 3.4 of [7] for performing the change of variable, cf. the proof of Proposition 2.2.2 of [6].  $\square$

7.6. Our goal now is to explain the relationship between motivic cone integrals and  $p$ -adic cone integrals. We begin with some preliminaries. We assume  $k = \mathbb{Q}$  (in fact we could assume as well that  $k$  is a number field or even only a field of finite type over  $\mathbb{Q}$ ). For any variety  $X$  over  $\mathbb{Q}$ , one can choose a model  $\mathcal{X}$  of  $X$  over  $\mathbb{Z}$ , and consider the number of points  $n_p(X)$  of the reduction of  $\mathcal{X}$  modulo  $p$ , for  $p$  a prime number. Of course, for some prime numbers  $p$ ,  $n_p(X)$  may depend of the model  $\mathcal{X}$ , but, for a given  $X$ , the numbers  $n_p(X)$  are well defined for almost all  $p$ . If we denote by  $\mathcal{P}$  the set of all primes, the sequence  $n_p(X)$  is well defined as an element of the ring  $\mathbb{Z}^{\mathcal{P}'}$ , where, for any ring  $R$ , we set  $R^{\mathcal{P}'} := \prod_{p \in \mathcal{P}} R / \bigoplus_{p \in \mathcal{P}} R$ . Moreover, counting points being additive for disjoint unions and multiplicative for products, the sequence  $n_p(X)$  in  $\mathbb{Z}^{\mathcal{P}'}$  only depends on the class of  $X$  in  $\mathcal{M}$  and may be extended to a ring morphism  $n : \mathcal{M} \rightarrow \mathbb{Z}^{\mathcal{P}'}$ . Setting  $n_p(\mathbf{L}^{-1}) = 1/p$ , one may extend uniquely  $n$  to a ring morphism  $n : \mathcal{M}_{\text{loc}} \rightarrow \mathbb{Q}^{\mathcal{P}'}$ . Furthermore, by Lemma 8.1.1 of [9], the morphism  $n$  factors through the image  $\overline{\mathcal{M}}_{\text{loc}}$  of  $\mathcal{M}_{\text{loc}}$  in  $\widehat{\mathcal{M}}$ . Recall that we denoted by  $\overline{\mathcal{N}}$  the subring of the ring  $\widehat{\mathcal{M}}[[T]]$  which is generated by the image in  $\widehat{\mathcal{M}}[[T]]$  of  $\mathcal{M}_{\text{loc}}[T]$ ,  $(\mathbf{L}^i - 1)^{-1}$  and  $(1 - \mathbf{L}^{-a} T^b)^{-1}$  with  $i \in \mathbb{N}$ ,  $a \in \mathbb{N}$ ,  $b \in \mathbb{N} \setminus \{0\}$ . Hence, sending  $(\mathbf{L}^i - 1)^{-1}$  and  $(1 - \mathbf{L}^{-a} T^b)^{-1}$  to  $(p^i - 1)^{-1}$  and  $(1 - p^{-a} T^b)^{-1}$ , respectively, one can uniquely extend  $n$  to a ring morphism  $n : \overline{\mathcal{N}} \rightarrow \mathbb{Q}[[T]]^{\mathcal{P}'}$ .

7.7. Assume  $k = \mathbb{Q}$  and that  $f_i$  and  $g_i$ , for  $i = 0, \dots, l$ , are polynomials with coefficients in  $\mathbb{Q}$ . Then, for every prime  $p$ , we can consider the  $p$ -adic cone integral

$$Z_{\mathcal{D}, p}(s) = \int_{V_p} p^{-\text{ord}_p(f_0)s - \text{ord}_p(g_0)} d\mu_p,$$

with

$$V_p = \{\mathbf{x} \in \mathbb{Q}_p^m : \text{ord}_p(f_i(\mathbf{x})) \leq \text{ord}_p(g_i(\mathbf{x})) \text{ for } i = 1, \dots, l\}.$$

Here  $\text{ord}_p$  stands for  $p$ -adic valuation and  $d\mu_p$  denotes the additive Haar measure on  $\mathbb{Q}_p^m$ , normalized so that  $\mathbb{Z}_p^m$  has volume 1.

By [13], for almost all  $p$ ,

$$Z_{\mathcal{D},p}(s) = \sum_{k=0}^w (p-1)^{|I_k|} p^{-m|E_{I_k}^\circ(\mathbb{F}_p)|} \prod_{j \in M_k} \frac{p^{-(A_j s + B_j)}}{(1 - p^{-(A_j s + B_j)})}.$$

Hence Theorem 7.5 has the following corollary:

**7.8. Corollary.** *For almost all primes  $p$ ,*

$$n_p(Z_{\mathcal{D},\text{geom}}(s)) = Z_{\mathcal{D},p}(s).$$

(In the left hand side  $T = \mathbf{L}^{-s}$ , while in the right hand side  $T = p^{-s}$ .)

In particular, we have the following relationship between the motivic zeta function of a Lie algebra and the zeta functions counting finite index  $\mathbb{Z}_p$ -subalgebras:

**7.9. Corollary.** *Let  $L$  be a Lie-algebra of finite rank defined over  $\mathbb{Z}$ . For almost all  $p$ ,*

$$n_p(P_{L \otimes_{\mathbb{Q}} \mathbb{Q}[[t]], \mathcal{A}_t^{\leq}}(\mathbf{L}^{-s})) = \zeta_{L \otimes_{\mathbb{Z}} \mathbb{Z}_p}(s).$$

7.10. In fact, we should mention that a result such as Corollary 7.8 holds in much wider generality. Let  $k$  be a finite extension of  $\mathbf{Q}$  with ring of integers  $\mathcal{O}$  and  $R = \mathcal{O}[\frac{1}{N}]$ , with  $N$  a non zero multiple of the discriminant. For  $x$  a closed point of  $\text{Spec } R$ , we denote by  $K_x$  the completion of the localization of  $R$  at  $x$ , by  $\mathcal{O}_x$  its ring of integer, and by  $\mathbf{F}_x$  the residue field at  $x$ , a finite field of cardinality  $q_x$ . Let  $f(x)$  be a polynomial in  $k[x_1, \dots, x_m]$  (or more generally a definable function in the first order language of valued fields with variables and values taking place in the valued field and with coefficients in  $k$ ) and let  $\varphi$  be a formula in the language of valued fields with coefficients in  $k$ , free variables  $x_1, \dots, x_m$  running over the valued field and no other free variables. Now set  $W_x := \{y \in \mathcal{O}_x^m : \varphi(y) \text{ holds}\}$  and define

$$I_{\varphi,f,x}(s) = \int_{W_x} |f|_x^s |dx|_x,$$

for  $x$  a closed point of  $\text{Spec } \mathcal{O}$ . One should remark these integrals are of much more general nature than the previously considered cone integrals, since quantifiers are now allowed in  $\varphi$ . In this more general setting we still want a geometrical object which specializes, for almost all  $x$  to the local integrals  $I_{\varphi,f,x}(s)$ . In [9] this aim is achieved, at the cost of replacing the “naive” ring  $\mathcal{M}_{\text{loc}}$  by a Grothendieck group of Chow motives<sup>2</sup>. More precisely, it is shown in [9], that there exists a canonical rational function  $I_{\varphi,f,\text{mot}}(T)$  with coefficients in an appropriate Grothendieck ring of Chow motives, such that, for almost all closed points  $x$  in  $\text{Spec } \mathcal{O}$ ,  $I_{\varphi,f,\text{mot}}(T)$  specializes - after taking the trace of the Frobenius on an étale realization and after setting  $T = q_x^{-s}$  - to the  $p$ -adic integral  $I_{\varphi,f,x}(s)$ .

<sup>2</sup>For an indication of why Chow motives arise naturally in these questions, see 7.11

7.11. To what extent are the constructible sets  $E_{I_k}^\circ$  in Theorem 7.5 unique? The left hand side of (7.5.1) being canonical, it follows that the class of the right hand side is independent of the resolution. On the other hand it is unclear how one can deduce the motivic zeta function from the  $p$ -adic ones. The first problem is that distinct varieties over  $\mathbb{Q}$  may have the same number of points in  $\mathbb{F}_p$ , which is in particular the case for isogenous elliptic curves over  $\mathbb{Q}$ . Much deeper is the fact, due to Faltings [15], that if, for almost all  $p$ ,  $n_p(E) = n_p(E')$ , with  $E$  and  $E'$  elliptic curves over  $\mathbb{Q}$ , then  $E$  and  $E'$  are isogenous. Since isogenous elliptic curves define the same Chow motive, one sees Chow motives appearing in a natural way in the play. Let us recall (see [25] for more details), that for  $k$  a field, a Chow motive over  $k$  is just a triple  $(S, p, n)$  with  $S$  proper and smooth over  $k$ ,  $p$  an idempotent correspondence with coefficients in  $\mathbb{Q}$  on  $X$ , and  $n$  in  $\mathbb{Z}$ . One endows Chow motives over  $k$  with the structure of a pseudo-abelian category which we denote by  $\text{Mot}_k$  and we denote by  $K_0(\text{Mot}_k)$  its Grothendieck group which may also be defined as the abelian group associated to the monoid of isomorphism classes of objects in  $\text{Mot}_k$  with respect to the natural sum  $\oplus$ . There is also a tensor product on  $\text{Mot}_k$  inducing a product on  $K_0(\text{Mot}_k)$  which provides  $K_0(\text{Mot}_k)$  with a natural ring structure. Assume now that  $k$  is of characteristic zero. By a result of Gillet and Soulé [16] and Guillén and Navarro Aznar [18] there exists a unique morphism of rings

$$\chi_c : \mathcal{M} \longrightarrow K_0(\text{Mot}_k)$$

such that  $\chi_c([S]) = [h(S)]$  for  $S$  projective and smooth, where  $h(S)$  denotes the Chow motive associated to  $S$ , *i.e.*  $h(S) = (S, \text{id}, 0)$ . Let us still denote by  $\mathbf{L}$  the image of  $\mathbf{L}$  by  $\chi_c$ . Since  $\mathbf{L} = [(\text{Spec } k, \text{id}, -1)]$ , it is invertible in  $K_0(\text{Mot}_k)$ , hence  $\chi_c$  can be extended uniquely to a ring morphism

$$\chi_c : \mathcal{M}_{\text{loc}} \longrightarrow K_0(\text{Mot}_k).$$

**7.12. Problem.** Assume  $k = \mathbb{Q}$  (one can ask a similar question when  $k$  is a number field, or more generally of finite type over  $\mathbb{Q}$ ). Is it true that for  $\alpha$  in  $\mathcal{M}_{\text{loc}}$  if  $n(\alpha) = 0$  then  $\chi_c(\alpha) = 0$ ?

A positive answer to this question would imply that the knowledge of the motivic integrals may be deduced from that of the corresponding  $p$ -adic integrals.

7.13. Since in the simplest examples, as the ones given in §9, the zeta functions  $\zeta_{L_p}(s)$  are just rational functions of  $p$  and  $p^{-s}$  and the corresponding motivic zeta functions rational functions of  $\mathbf{L}$  and  $\mathbf{L}^{-s}$ , we give an example showing that non trivial motives may indeed appear.

Let  $L$  be the class two nilpotent Lie algebra over  $\mathbb{Z}$  of dimension 9 as a free  $\mathbb{Z}$ -module given by the following presentation:

$$L = \left\langle \begin{array}{l} x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3 : (x_1, x_4) = y_3, (x_1, x_5) = y_1, (x_1, x_6) = y_2 \\ (x_2, x_4) = y_2, (x_2, x_6) = y_1, (x_3, x_4) = y_1, (x_3, x_5) = y_3 \end{array} \right\rangle$$

where all other commutators are defined to be 0. This Lie algebra was discovered by the first author to answer negatively the following question asked by Grunewald, Segal and Smith in [17]: if  $L$  is a Lie algebra over  $\mathbb{Z}$ , do there exist finitely many

rational function  $\Phi_1(X, Y), \dots, \Phi_r(X, Y)$  such that for each prime  $p$  there exists  $i \in \{1, \dots, r\}$  and  $\zeta_{L_p}^{\leq}(s) = \Phi_i(p, p^{-s})$ ? Let  $E$  be the elliptic curve  $Y^2 = X^3 - X$ . One can show, by a direct calculation (see [14]), that there exist two non zero rational functions  $P_1(X, Y)$  and  $P_2(X, Y) \in \mathbb{Q}(X, Y)$  such that, for almost all primes  $p$ ,

$$\zeta_{L_p}^{\leq}(s) = P_1(p, p^{-s}) + |E(\mathbb{F}_p)|P_2(p, p^{-s}).$$

By the result of Faltings already alluded to, it follows that the Chow motive of the curve  $E$  is canonically attached to the Lie algebra  $L$ . It seems also quite natural to guess, but this has to be checked, that the calculation of the associated motivic cone integral gives

$$P_1(\mathbf{L}, \mathbf{L}^{-s}) + [E]P_2(\mathbf{L}, \mathbf{L}^{-s}).$$

To see where the elliptic curve is hidden in this Lie algebra, consider  $\det((x_i, x_{i+3})) = 0$  which gives an equation for the projective version of  $E$ .

7.14. There is one case in which one can deduce an expression for the motivic zeta function from the corresponding expression for the  $p$ -adic zeta functions:

**7.15. Theorem.** *Let  $L$  be a Lie algebra over  $\mathbb{Z}$  additively isomorphic to  $\mathbb{Z}^d$ . Suppose that there exist two rational functions  $\Phi(X, Y)$  and  $\Psi(X, Y) \in \mathbb{Q}(X, Y)$  such that, for almost all primes  $p$ ,  $\zeta_{L_p}(s) = \Phi(p, p^{-s})$  and  $P_{L \otimes_{\mathbb{Q}} \mathbb{Q}[[t]], \mathcal{X}_t^{\leq}}(\mathbf{L}^{-s}) = \Psi(\mathbf{L}, \mathbf{L}^{-s})$ . Then  $\Phi(X, Y) = \Psi(X, Y)$ .*

*Proof.* This follows from Corollary 7.9 and the fact that if  $\Phi(p, p^{-s}) = \Psi(p, p^{-s})$  for almost all primes  $p$  then  $\Phi(X, Y) = \Psi(X, Y)$ .  $\square$

## 8. TOPOLOGICAL ZETA FUNCTIONS

8.1. In [5] a topological zeta function was associated to each polynomial  $f \in \mathbb{C}[X]$ . The function was defined from the explicit formula in terms of a resolution of singularities for the associated local Igusa zeta functions of  $f$ . It can be interpreted as considering the explicit formula as  $p \mapsto 1$ . The heart of the paper [5] was the proof that this definition was independent of the resolution. The proof given there depended essentially on using the  $p$ -adic analysis of these local zeta functions and the associated explicit formulae. However the current motivic setting gives a proof of the independence of the resolution without going to the local zeta functions as we shall explain.

In the setting of groups and Lie algebras it was unclear that such a topological zeta function could be associated to every nilpotent group or  $\mathbb{Z}$ -Lie algebra. This depended on producing an explicit formula. Since [13] now provides such an explicit formula for zeta functions of nilpotent groups and Lie algebras and more generally for any cone integral, we can define the associated topological zeta function and use the setting of the current paper to prove that it is independent of the resolution and subdivision of the cone.

We now proceed similarly as in section 2.3 of [6] to define topological zeta functions. Let  $\mathcal{M}[\mathbf{L}^{-s}]'_{\text{loc}}$  be the subring of  $\mathcal{M}[\mathbf{L}^{-s}]_{\text{loc}}$  generated by the ring of polynomials  $\mathcal{M}_{\text{loc}}[\mathbf{L}^{-s}]$  and by the quotients  $(\mathbf{L} - 1)(1 - \mathbf{L}^{-Ns-n})^{-1}$  for  $(N, n) \in \mathbb{N}^2 \setminus (0, 0)$ .

Recall we denoted by  $\overline{\mathcal{M}}_{\text{loc}}$  the image of  $\mathcal{M}_{\text{loc}}$  in  $\widehat{\mathcal{M}}$ . In particular  $\overline{\mathcal{M}}_{\text{loc}}$  is a quotient without  $(\mathbf{L} - 1)$ -torsion of  $\mathcal{M}_{\text{loc}}$ .

By expanding  $\mathbf{L}^{-s}$  and  $(\mathbf{L} - 1)(1 - \mathbf{L}^{-Ns-n})^{-1}$  into series in  $\mathbf{L} - 1$ , one gets a canonical morphism of algebras

$$\varphi : \mathcal{M}[\mathbf{L}^{-s}]'_{\text{loc}} \rightarrow \overline{\mathcal{M}}_{\text{loc}}[s][[(Ns + n)^{-1}]_{(N,n) \in \mathbb{N}^2 \setminus (0,0)}][[\mathbf{L} - 1]],$$

where  $[[\mathbf{L} - 1]]$  denotes completion with respect to the ideal generated by  $\mathbf{L} - 1$ . Taking the quotient of  $\overline{\mathcal{M}}_{\text{loc}}[s][[(Ns + n)^{-1}]_{(N,n) \in \mathbb{N}^2 \setminus (0,0)}][[\mathbf{L} - 1]]$  by the ideal generated by  $\mathbf{L} - 1$ , one obtains the evaluation morphism:

$$\begin{aligned} \text{ev}_{\mathbf{L}=1} : \overline{\mathcal{M}}_{\text{loc}}[s][[(Ns + n)^{-1}]_{(N,n) \in \mathbb{N}^2 \setminus (0,0)}][[\mathbf{L} - 1]] &\longrightarrow \\ &(\overline{\mathcal{M}}_{\text{loc}}/\mathbf{L} - 1)[s][[(Ns + n)^{-1}]_{(N,n) \in \mathbb{N}^2 \setminus (0,0)}]. \end{aligned}$$

We denote by  $\chi_{\text{top}}(X)$  the usual Euler characteristic of a smooth, projective  $k$ -scheme (say in étale  $\overline{\mathbb{Q}_l}$ -cohomology). By §6.1 of [7], this factorises through a morphism  $\chi_{\text{top}} : \overline{\mathcal{M}}_{\text{loc}} \rightarrow \mathbb{N}$ . Since  $\chi_{\text{top}}(\mathbf{L}) = 1$ , this induces then a morphism

$$\chi_{\text{top}} : (\overline{\mathcal{M}}_{\text{loc}}/\mathbf{L} - 1)[s][[(Ns + n)^{-1}]_{(N,n) \in \mathbb{N}^2 \setminus (0,0)}] \rightarrow \mathbb{Z}[s][[(Ns + n)^{-1}]_{(N,n) \in \mathbb{N}^2 \setminus (0,0)}].$$

**8.2. Definition.** The *topological cone zeta function*  $Z_{\mathcal{D},\text{top}}(s)$  associated to the cone data  $\mathcal{D}$  is defined to be

$$Z_{\mathcal{D},\text{top}}(s) := (\chi_{\text{top}} \circ \text{ev}_{\mathbf{L}=1} \circ \varphi)(Z_{\mathcal{D},\text{geom}}(s)).$$

**8.3. Definition.** Let  $L$  be a  $k[[t]]$ -algebra additively isomorphic to  $(k[[t]])^d$  or a  $\mathbb{Z}$ -Lie algebra additively isomorphic to  $\mathbb{Z}^d$ . Let  $L$  be an algebra additively isomorphic to  $\mathbb{Z}^d$ . We define the *topological zeta function*  $Z_{L,\text{top}}(s)$  of  $L$  to be

$$Z_{L,\text{top}}(s) := (\chi_{\text{top}} \circ \text{ev}_{\mathbf{L}=1} \circ \varphi)(Z_{L,\text{geom}}(s)).$$

These definitions make sense since  $Z_{\mathcal{D},\text{geom}}(s)$  and  $Z_{L,\text{geom}}(s)$  belong to  $\mathcal{M}[\mathbf{L}^{-s}]'_{\text{loc}}$  by Theorem 7.5.

We can get an explicit formula for the topological zeta functions from that for the associated motivic zeta function. We use the notation defined prior to the statement of Theorem 7.5. We define the subset  $W_{\text{top}} \subset \{0, \dots, w\}$  such that  $k \in W_{\text{top}}$  if and only if  $|I_k| = |M_k|$ . These are the indices of pieces of the cone whose dimension is equal to the dimension of the vector space spanned by all those basis elements with non-zero coefficients somewhere in the piece.

**8.4. Proposition.** *Let  $\mathcal{D}$  be a set of cone integral data. Then with the notation above:*

$$Z_{\mathcal{D},\text{top}}(s) = \sum_{k \in W_{\text{top}}} \chi_{\text{top}}(E_{I_k}^\circ) \prod_{j \in M_k} \frac{1}{(A_j s + B_j)}.$$

Some pieces  $R_j$  of the cone may be missing in this expression. Note that in the special case of the Igusa zeta function we do see all the pieces  $R_j$  since the cone there is the full positive quadrant. Nevertheless, in the more general setting of a cone integral, the constants  $A_j$  and  $B_j$  ( $j \in \{1, \dots, q\}$ ) will appear in this formula somewhere whenever the one-dimensional edge  $R_j$  sits on an open simplicial piece of the cone of dimension corresponding to the number of non-zero entries of  $R_j$ .

## 9. EXAMPLES

9.1. **Free abelian.** Let  $L = \mathbb{Z}^d$ . Then

$$\begin{aligned} P_{L \otimes k[[t]], \mathcal{X}_t^{\leq}}(s) &= (1 - \mathbf{L}^{-s})^{-1} \dots (1 - \mathbf{L}^{-s+d-1})^{-1} \\ Z_{L, \text{geom}}(s) &= \frac{(1 - \mathbf{L}^{-1})}{(1 - \mathbf{L}^{-s-d})} \cdots \frac{(1 - \mathbf{L}^{-d})}{(1 - \mathbf{L}^{-s-1})} \\ Z_{L, \text{top}}(s) &= \frac{1}{(s+d) \dots (s+1)}. \end{aligned}$$

The calculation of the integral in the  $p$ -adic setting translates immediately to a calculation of the motivic integral, proving the above identities.

9.2. **Heisenberg.** Let

$$L = \begin{pmatrix} 0 & \mathbb{Z} & \mathbb{Z} \\ & 0 & \mathbb{Z} \\ & & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} P_{L \otimes k[[t]], \mathcal{X}_t^{\leq}}(s) &= (1 - \mathbf{L}^{-s})^{-1} (1 - \mathbf{L}^{1-s})^{-1} (1 - \mathbf{L}^{2-2s})^{-1} (1 - \mathbf{L}^{3-2s})^{-1} (1 - \mathbf{L}^{3-3s}) \\ Z_{L, \text{geom}}(s) &= \frac{(1 - \mathbf{L}^{-1})(1 - \mathbf{L}^{-2})(1 - \mathbf{L}^{-3})(1 - \mathbf{L}^{-3s-6})}{(1 - \mathbf{L}^{-s-3})(1 - \mathbf{L}^{-s-2})(1 - \mathbf{L}^{-2s-4})(1 - \mathbf{L}^{-2s-3})} \\ Z_{L, \text{top}}(s) &= \frac{3}{2(s+3)(s+2)(2s+3)}. \end{aligned}$$

Using Proposition 5.3,

$$Z_{L, \text{geom}}(s) = \frac{(1 - \mathbf{L}^{-1})(1 - \mathbf{L}^{-2})(1 - \mathbf{L}^{-3})}{(1 - \mathbf{L}^{-1})^3} \int_A \mathbf{L}^{-\text{ord}_t(x)(s+2)} \mathbf{L}^{-\text{ord}_t(y)(s+1)} \mathbf{L}^{-\text{ord}_t(z)s} d\mu$$

where

$$A = \{(x, y, z) \in k[[t]]^3 : \text{ord}_t(xy) \geq \text{ord}_t(z)\}.$$

The polynomial associated to the cone integral data already defines a non-singular space consisting only of normal crossings. So no resolution of singularities is required. The associated cone  $\overline{D_T}$  is the span of the four one-dimensional pieces where we record the associated  $A_j$  and  $B_j$ :

$$\begin{aligned} R_1 &= (1, 0, 1), A_1 = 2, B_1 = 4 \\ R_2 &= (1, 0, 0), A_2 = 1, B_2 = 3 \\ R_3 &= (1, 0, 1), A_3 = 2, B_3 = 3 \\ R_4 &= (0, 1, 0), A_4 = 1, B_4 = 2. \end{aligned}$$

The following is a table of the pieces  $R_k$   $k \in \{0, \dots, w\}$  corresponding to a suitable decomposition of this cone where  $R_{i_1} \dots R_{i_r}$  denotes the open piece spanned by  $R_{i_1}, \dots, R_{i_r}$ :

$R_k$	$ I_k $	$ M_k $	$E_{I_k}^\circ$
0	0	0	$(\mathbf{L} - 1)^3$
$R_1$	2	1	$(\mathbf{L} - 1)$
$R_2$	1	1	$(\mathbf{L} - 1)^2$
$R_3$	2	1	$(\mathbf{L} - 1)$
$R_4$	1	1	$(\mathbf{L} - 1)^2$
$R_1R_2$	2	2	$(\mathbf{L} - 1)$
$R_1R_3$	3	2	1
$R_2R_3$	3	2	1
$R_2R_4$	2	2	$(\mathbf{L} - 1)$
$R_3R_4$	2	2	$(\mathbf{L} - 1)$
$R_1R_2R_3$	3	3	1
$R_2R_3R_4$	3	3	1

Using our explicit formula it is possible then to derive the expression for  $Z_{L,\text{geom}}(s)$  given above.

To derive the expression for  $Z_{L,\text{top}}(s)$  note that  $\chi_{\text{top}}((\mathbf{L} - 1)^i) = 0$  for  $i > 0$ . Hence those pieces  $R_k$  for which  $|I_k| = |M_k|$  and  $\chi_{\text{top}}(E_{I_k}^\circ) \neq 0$  are  $R_1R_2R_3$  and  $R_2R_3R_4$ . This gives the following expression for  $Z_{L,\text{top}}(s)$  :

$$\begin{aligned} Z_{L,\text{top}}(s) &= \frac{1}{(2s+4)(s+3)(2s+3)} + \frac{1}{(s+3)(2s+3)(s+2)} \\ &= \frac{3}{2(s+3)(2s+3)(s+2)}. \end{aligned}$$

Of course one could directly deduce this by applying the function  $\chi_{\text{top}} \circ \text{ev}_{\mathbf{L}=1} \circ \varphi$  to the final expression for  $Z_{L,\text{geom}}(s)$ . However it is instructive to see the explicit formula for  $Z_{L,\text{top}}(s)$  at work.

9.3.  $\mathfrak{sl}_2$ . Let

$$L = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : (a, b, c) \in \mathbb{Z}^3 \right\}.$$

Then

$$\begin{aligned} P_{L \otimes k[[t]], \mathcal{X}_t^{\leq}}(s) &= (1 - \mathbf{L}^{-s})^{-1} (1 - \mathbf{L}^{1-s})^{-1} (1 - \mathbf{L}^{2-2s})^{-1} (1 - \mathbf{L}^{1-2s})^{-1} (1 - \mathbf{L}^{1-3s}) \\ Z_{L,\text{geom}}(s) &= \frac{(1 - \mathbf{L}^{-1})(1 - \mathbf{L}^{-2})(1 - \mathbf{L}^{-3})(1 - \mathbf{L}^{-3s-8})}{(1 - \mathbf{L}^{-s-3})(1 - \mathbf{L}^{-s-2})(1 - \mathbf{L}^{-2s-4})(1 - \mathbf{L}^{-2s-5})} \\ Z_{L,\text{top}}(s) &= \frac{3s+8}{2(s+3)(s+2)^2(2s+5)}. \end{aligned}$$

Using Proposition 5.3 we have the following description of  $Z_{L,\text{geom}}(s)$

$$Z_{L,\text{geom}}(s) = \frac{(1 - \mathbf{L}^{-1})(1 - \mathbf{L}^{-2})(1 - \mathbf{L}^{-3})}{(1 - \mathbf{L}^{-1})^3} \int_{\text{W}} \mathbf{L}^{-\text{ord}_t(a)(s+2)} \mathbf{L}^{-\text{ord}_t(x)(s+1)} \mathbf{L}^{-\text{ord}_t(z)s} d\mu$$

where

$$W = \left\{ \left( \begin{array}{ccc} a & b & c \\ 0 & x & y \\ 0 & 0 & z \end{array} \right) \in \mathrm{Tr}_3(R) : \begin{array}{l} v(x) \leq v(4zb) \\ v(x) \leq v(4by) \\ v(zx) \leq v(ax^2 + 4cxy - 4by^2) \end{array} \right\}.$$

The polynomial associated to the cone integral data is singular in this case. There have been several calculations of the zeta function of  $\mathfrak{sl}_2(\mathbb{Z}_p)$  for  $p > 2$  (see [20] and [11]) which give the answer above with  $\mathbf{L}$  replaced by  $p$ . However, it is not possible to deduce directly that the above expression is a correct formula for the corresponding motivic zeta function, cf. the discussion at the end of Section 7. However, recently, the first author and Ph.D. student Gareth Taylor [12], have used the approach of applying resolution of singularities to the non-singular polynomial associated to the cone integral data of the integral above to deduce the expression for the  $p$ -adic integrals. The advantage of this approach is that it formally translates into a calculation of the corresponding motivic zeta function. Alternatively, one can observe that the resolution of the polynomial gives rise to  $E_I$  which are generated out of  $\mathbf{L}$ , *i.e.* we don't get any exotic varieties, so that there exists a rational function  $\Psi(X, Y) \in \mathbb{Q}(X, Y)$  such that the motivic zeta function is  $\Psi(\mathbf{L}, \mathbf{L}^{-s})$ . Hence, by Corollary 7.9, one is able to deduce that the rational function describing the  $p$ -adic expressions is the same as  $\Psi(X, Y)$ .

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