ON THE DISTRIBUTION OF POINTS OF BOUNDED HEIGHT ON EQUIVARIANT COMPACTIFICATIONS OF VECTOR GROUPS

by Antoine Chambert-Loir & Yuri Tschinkel

ABSTRACT. — We prove asymptotic formulas for the number of rational points of bounded height on smooth equivariant compactifications of the affine space.

RÉSUMÉ. — Nous établissons un développement asymptotique du nombre de points rationnels de hauteur bornée sur les compactifications équivariantes lisses de l'espace affine.

0. Introduction

A theorem of Northcott asserts that for any real number B there are only finitely many rational points in the projective space $\mathbf{P}^n$ with *height* smaller than B. An asymptotic formula for this number (as B tends to infinity) has been proved by Schanuel [**16**]. Naturally, it is interesting to consider more general projective varieties. There is already a number of results in this direction. The techniques employed can be grouped in three main classes:

– the classical circle method in analytic number theory permits to treat complete intersections of small degree in projective spaces of large dimension (cf. for example [**4**]);

– harmonic analysis on adelic points of reductive groups leads to results for toric varieties [**2**], flag varieties [**9**] and horospherical varieties [**17**];

– elementary (but nontrivial) methods for del Pezzo surfaces of degree 4 or 5, and for cubic surfaces (Salberger, Swinnerton-Dyer, Heath-Brown, cf. [**15**] and the references therein; cf. also de la Bretèche [**5**]).

Submitted on the arXiv, May 2, 2000

ANTOINE CHAMBERT-LOIR, Institut de mathématiques de Jussieu, Boite 247, 4 place Jussieu, F-75252 Paris Cedex 05 • *E-mail :* chambert@math.jussieu.fr

YURI TSCHINKEL, Department of Mathematics, Princeton University, Princeton, (NJ) 08544-1000, U.S.A. *E-mail :* ytschink@math.princeton.edu

2000 Mathematics Subject Classification. — 11G50 (11G35, 14G05).

Key words and phrases. — Heights, Poisson formula, Manin's conjecture, Tamagawa measure.

This research has been stimulated by a conjecture put forward by Batyrev and Manin. They proposed in [**1**] an interpretation of the growth rate in terms of the mutual positions of the class of the line bundle giving the projective embedding, the anticanonical class and the cone of effective divisors in the Picard group of the variety. Peyre refined this conjecture in [**14**] by introducing an adelic Tamagawa-type number which appears as the leading constant in the expected asymptotic formula for the anticanonical embedding. Batyrev and the second author proposed an interpretation of the leading constant for arbitrary ample line bundles, see [**3**].

In this paper we consider a new class of varieties, namely *equivariant compactifications of vector groups*. On the one hand, we can make use of harmonic analysis on the adelic points of the group. On the other hand, such varieties have a rich geometry. In particular, in contrast to flag varieties and toric varieties, they admit geometric deformations, and in contrast to the complete intersections treated by the circle method their Picard group can have arbitrarily high rank. Their geometric classification is a difficult open problem already in dimension 3 (see [**10**]).

A basic example of such algebraic varieties is of course $\mathbf{P}^n$ endowed with the action of $\mathbf{G}_a^n \subset \mathbf{P}^n$ by translations. A class of examples may be provided by the following geometric construction: Take $X_0 = \mathbf{P}^n$ endowed with the translation action of $\mathbf{G}_a^n$. Let Y be a smooth subscheme of X_0 which is contained in the hyperplane at infinity. Then, the blow-up $X = \mathrm{Bl}_Y(X_0)$ contains the isomorphic preimage of $\mathbf{G}_a^n$ and the action of $\mathbf{G}_a^n$ lifts to X. The rank of $\mathrm{Pic}(X)$ is equal to the number of irreducible components of Y. Using equivariant resolutions of singularities, one can produce even more complicated examples, although less explicitly.

Our first steps towards this paper are detailed in [**6**] and [**7**]. There we studied the cases $n = 2$ with Y a finite union of $\mathbf{Q}$-rational points and $n > 2$ with Y a smooth hypersurface contained in the hyperplane at infinity.

We now describe the main theorem of this article. Let X be an equivariant compactification of the additive group $\mathbf{G}_a^n$ over a number field F. We assume that X is smooth, projective and that the boundary divisor $D = X \setminus \mathbf{G}_a^n$ has strict normal crossings: this means that over an algebraic closure of F, D is a sum of smooth divisors meeting transversally. However, we do not assume that the irreducible components D_α ($\alpha \in \mathcal{A}$) of D are geometrically irreducible.

The Picard group of X is free and has a canonical basis given by the classes of D_α. The cone of effective divisors consists of the divisors $\sum_{\alpha\in\mathcal{A}} d_\alpha D_\alpha$ with $d_\alpha \geqslant 0$ for all α. Denote by K_X^{-1} the anticanonical line bundle on X and by $\rho = (\rho_\alpha)$ its class in $\mathrm{Pic}(X)$.

Let $\lambda = (\lambda_\alpha)$ be a class contained in the interior of the cone of effective divisors $\Lambda_{\mathrm{eff}}(X) \subset \mathrm{Pic}(X)$ and $\mathcal{L}_\lambda$ the corresponding line bundle, equipped with a smooth adelic metric (see 3.1 for the definition). With the above notations, we have $\lambda_\alpha > 0$ for all α. Denote by $H_{\mathcal{L}_\lambda}$ the associated exponential height on $X(F)$. Let $a_\lambda = \max(\rho_\alpha/\lambda_\alpha)$ and let b_λ be the cardinality of

$$\mathcal{B}_\lambda = \{\alpha \in \mathcal{A}\,;\, \rho_\alpha = a_\lambda\lambda_\alpha\}.$$

Put

$$c_\lambda = \prod_{\alpha \in \mathcal{B}_\lambda} \lambda_\alpha^{-1}.$$

For example, one has $a_\rho = 1$ and $b_\rho = \operatorname{rk}\operatorname{Pic}(X)$. We also denote by $\tau(\mathcal{K}_X)$ the Tamagawa number as defined by Peyre in [**14**].

THEOREM 0.1. — a) *The series*

$$Z_\lambda(s) = \sum_{x \in \mathbf{G}_a^n(F)} H_{\mathcal{L}_\lambda}(x)^{-s}$$

converges absolutely and uniformly for $\mathrm{Re}(s) > a_\lambda$ *and has a meromorphic continuation to* $\mathrm{Re}(s) > a_\lambda - \delta$ *for some* $\delta > 0$*, with a unique pole at* $s = a_\lambda$ *of order* b_λ*. Moreover,* Z_λ *has polynomial growth in vertical strips in this domain.*

b) *There exist positive real numbers* τ_λ, δ' *and a polynomial* $P_\lambda \in \mathbf{R}[x]$ *of degree* $b_\lambda - 1$ *with leading coefficient*

$$c_\lambda \tau_\lambda / (b_\lambda - 1)!$$

such that the number $N(\mathcal{L}_\lambda, B)$ *of* F*-rational points in* $\mathbf{G}_a^n$ *with height* $H_{\mathcal{L}_\lambda}$ *smaller than* B *satisfies*

$$N(\mathcal{L}_\lambda, B) = B^{a_\lambda} P_\lambda(\log B) + O(B^{a_\lambda - \delta'}),$$

for $B \to \infty$*. For* $\lambda = \rho$ *we have* $\tau_\rho = \tau(\mathcal{K}_X)$*.*

REMARK 0.2. — When the compactification X is smooth but its boundary is not a divisor with strict normal crossings our theorem still implies an asymptotic for the number of rational points of bounded height. Namely, there exists a proper modification $\pi : \tilde{X} \to X$ which is a composition of equivariant blow-ups with smooth centers and we can apply the theorem to $(\tilde{X}, \pi^*\lambda)$. The arguments of Lemma 11.2 show that $a_{\pi^*\lambda} = a_\lambda$ and $b_{\pi^*\lambda} = b_\lambda$.

REMARK 0.3. — Granted the smoothness assumption on the adelic metric, note that the normalization of the height in Theorem 0.1 can be arbitrary. By a theorem of Peyre [**14**, § 5], this means that the points of bounded anticanonical height are equidistributed with respect to the Tamagawa measure (compatible with the choice of the height function) in the adelic space $X(\mathbf{A}_F)$. Let us explain this briefly. Fix a smooth adelic metric on the anticanonical line bundle, this defines a height function H. Let $d\tau_H$ be the renormalized Tamagawa measure on $X(\mathbf{A}_F)$. A smooth positive function f on $X(\mathbf{A}_F)$ determines another height function, namely $H' = fH$. Applied to such H', the main theorem implies that

$$\lim_{B \to +\infty} \frac{(r-1)!}{c_\rho} \frac{1}{B(\log B)^{r-1}} \sum_{\substack{x \in \mathbf{G}_a^n(F) \\ H(x) \leqslant B}} f(x) = \int_{X(\mathbf{A}_F)} f(\mathbf{x}) d\tau_H(\mathbf{x}).$$

REMARK 0.4. — Theorem 0.1 implies that the open subset $\mathbf{G}_a^n$ does not contain any accumulating subvarieties.

The proof proceeds as follows. First we extend the height function to the adelic space $\mathbf{G}_a^n(\mathbf{A})$. Next we apply the additive Poisson formula and find a representation of $Z_\lambda(s)$ as a sum over the characters of $\mathbf{G}_a^n(\mathbf{A})/\mathbf{G}_a^n(F)$ of the Fourier transforms of H.

The Fourier transforms of H decompose as products over all places v of "global" integrals on $X(F_v)$ which are reminiscent of Igusa zeta functions. At almost all nonarchimedean places, we compute explicitely the local Fourier transforms in terms of the reduction of X modulo the corresponding prime. The obtained formulas resemble Denef's formula in [**8**] for Igusa's local zeta function. For the remaining nonarchimedian places we find estimates. This leads to a proof of the meromorphic continuation of the Fourier transforms at each character.

Invariance properties of the height reduce the summation over $\mathbf{G}_a^n(F)$ to one over a lattice. The meromorphic continuation in part a) of the theorem follows then from additional estimates for the Fourier transforms at the infinite places.

At this stage one has a meromorphic continuation of $Z_\lambda(s)$ to the domain $\mathrm{Re}(s) > a_\lambda - \delta$ for some $\delta > 0$, with a single pole at $s = a_\lambda$ whose order is less or equal than b_λ. It remains to check that the order of the pole is exactly b_λ; we need to prove that the limit

$$\lim_{s \to a_\lambda} Z_\lambda(s)(s - a_\lambda)^{b_\lambda}$$

is strictly positive. For $\lambda = \rho$ this is more or less straightforward: the main term is given by the summand corresponding to the trivial character and the Tamagawa number defined by Peyre appears naturally in the limit. For other λ we use the Poisson formula again and relate the limit to an integral of the height over some subspace which is shown to be strictly positive.

Part b) follows by a Tauberian theorem.

Sections 1 and 2 are devoted to geometric facts about equivariant compactifications of vector groups. In Sections 3 and 4 we give basic results concerning height functions on such varieties. We also introduce the height zeta function $Z_\lambda(s)$ relative to the open set $\mathbf{G}_a^n \subset X$ and establish for each $\lambda \in \Lambda_{\mathrm{eff}}(X)$ the existence of a $\sigma_\lambda > 0$ such that $Z_\lambda(s\lambda)$ converges absolutely for $\mathrm{Re}(s) > \sigma_\lambda$. In Section 5 we state the Poisson formula and set up some notations.

In Section 6 we give general estimates for the Fourier transforms of local heights. They are used at the places of bad reduction and at the archimedean places. In Sections 7 and 8 we analyse the Fourier transforms at the trivial character and at nontrivial characters, respectively. We found it convenient to restrict at first to the case when all D_α are geometrically irreducible. In Section 9 we explain how to extend the formulas to the general case.

In Section 10 we prove our main theorem for the anticanonical line bundle (*i.e.* $\lambda = \rho$). The case of arbitrary effective line bundles is addressed in Section 11.

Acknowledgements. — The work of the second author was partially supported by the NSA.

1. Geometry

Let X be an equivariant compactification of the additive group $\mathbf{G}_a^n$ of dimension n over a field F: X is an algebraic variety endowed with an action of $\mathbf{G}_a^n$ with a dense orbit isomorphic to $\mathbf{G}_a^n$. We will always assume that X is smooth, projective, and that over a separable closure $\bar{F}$ of F, the boundary $X \setminus \mathbf{G}_a^n$ is a divisor with strict normal crossings (otherwise, we pass to a desingularization with these properties). This means that over $\bar{F}$, we have a decomposition in irreducible components

$$X_{\bar{F}} \setminus \mathbf{G}_a^n = \bigcup_{\alpha \in \mathcal{A}} D_\alpha$$

where $\mathcal{A}$ is a finite set and for each $\alpha \in \mathcal{A}$, D_α is a smooth integral divisor in $\bar{X} = X_{\bar{F}}$. Moreover, for any $A \subset \mathcal{A}$, $D_A := \bigcap_{\alpha \in A} D_\alpha$ is either empty or smooth purely of codimension $\#A$ in $\bar{X}$. For any $A \subset \mathcal{A}$, we shall write

$$D_A^\circ = D_A \setminus \bigcup_{A' \supsetneq A} D_{A'}$$

so that the sets D_A° form a partition of X by locally closed subsets as A varies through all subsets of $\mathcal{A}$.

The natural action of $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$ on $\bar{X}$ induces an action of Γ_F on $\mathcal{A}$ such that for any $A \subset \mathcal{A}$, $g(D_A) = D_{g(A)}$. By Galois descent, a nonempty stratum D_A (or D_A°) is defined over a subfield F' if and only if the subgroup $\Gamma_{F'}$ of Γ_F stabilizes A.

PROPOSITION 1.1. — *One has natural isomorphisms of Γ_F-modules (resp. Γ_F-monoids)*

$$\bigoplus_{\alpha \in \mathcal{A}} \mathbf{Z} D_\alpha \to \mathrm{Pic}(\bar{X}) \quad \textit{and} \quad \bigoplus_{\alpha \in \mathcal{A}} \mathbf{N} D_\alpha \to \Lambda_{\mathrm{eff}}(\bar{X}),$$

where $\Lambda_{\mathrm{eff}}(X)$ is the monoid of classes of effective divisors of X.

Subsequently, we identify the divisors D_α (and the corresponding line bundles) with their classes in the Picard group.

Proof. — These maps are equivariant under the action of Γ_F. Hence, it remains to show injectivity and surjectivity.

Let $\mathcal{L}$ be a line bundle on X. As X is smooth and $\mathrm{Pic}(\mathbf{G}_a^n) = 0$, there is a divisor D in X not meeting $\mathbf{G}_a^n$ such that $\mathcal{L} \simeq \mathcal{O}_X(D)$. Such a divisor D is a sum $\sum_{\alpha \in A} n_\alpha D_\alpha$. Moreover, such a D is necessarily unique. If $\mathcal{O}_X(D) \simeq \mathcal{O}_X(D')$ for $D = \sum_\alpha n_\alpha D_\alpha$ and $D' = \sum_\alpha n'_\alpha D_\alpha$, then the canonical rational section $s_D/s_{D'}$ of $\mathcal{O}_X(D - D') = \mathcal{O}_X$ is a rational function on X without zeroes nor poles on $\mathbf{G}_a^n$. Rosenlicht's lemma (see lemma 1.2 below) implies that $s_D/s_{D'}$ is constant, so that $D = D'$.

We remark that any effective cycle Z on X is rationally equivalent to a cycle that does not meet $\mathbf{G}_a^n$. Indeed, if t is the parameter of a subgroup of $\mathbf{G}_a^n$ isomorphic to $\mathbf{G}_a$, we can consider the specialization of the cycles $t+Z$ when $t\to\infty$. $\square$

LEMMA 1.2 (Rosenlicht). — *If* $f\in F(X)$ *has neither zeroes nor poles on* $\mathbf{G}_a^n$*, then* $f\in F^*$.

COROLLARY 1.3. — 1) *The* Γ_F*-module* $\mathrm{Pic}(\bar{X})$ *is a permutation module. In particular,* $H^1(\Gamma_F,\mathrm{Pic}(\bar{X}))=0$.

2) $\mathrm{Pic}(X)=\mathrm{Pic}(\bar{X})^{\Gamma_F}$ *is a free* $\mathbf{Z}$*-module of finite rank equal to the number of* Γ_F*-orbits in* $\mathcal{A}$.

Let f be a nonzero linear form on $\mathbf{G}_a^n$ viewed as an element of $F(X)$. Its divisor can be written as

$$\mathrm{div}(f)=E(f)-\sum_{\alpha\in\mathcal{A}} d_\alpha(f)D_\alpha$$

where $E(f)$ is the unique irreducible component of $\{f=0\}$ that meets $\mathbf{G}_a^n$ and $d_\alpha(f)$ are integers. (The divisor $E(f)$ can also be seen as the closure in X of the hypersurface of $\mathbf{G}_a^n$ defined by f.)

Since $E(f)$ is rationally equivalent to $\sum_{\alpha\in A} d_\alpha(f)D_\alpha$, the preceding proposition implies the following lemma:

LEMMA 1.4. — 1) *For any nonzero linear form* f*, one has* $d_\alpha(f)\geqslant 0$.

2) *If* α *and* $\beta\in\mathcal{A}$ *are conjugate under* Γ_F*, then* $d_\alpha(f)=d_\beta(f)$.

PROPOSITION 1.5. — *Let* X *be a normal equivariant compactification of* $\mathbf{G}_a^n$ *over* F*. Every line bundle* $\mathcal{L}$ *on* X *admits a unique* $\mathbf{G}_a^n$*-linearization. If* $\mathcal{L}$ *is effective then* $H^0(X,\mathcal{L})$ *has a unique line of* $\mathbf{G}_a^n$*-invariant sections.*

Proof. — Since $\mathbf{G}_a^n$ has no nontrivial characters, it follows from Proposition 1.4 and from the proof of Proposition 1.5 in **[13]**, Chapter 1, that any line bundle on X admits a unique $\mathbf{G}_a^n$-linearization.

Assume that $H^0(X,\mathcal{L})\neq 0$ and consider the induced action of $\mathbf{G}_a^n$ on the projectivization $\mathbf{P}(H^0(X,\mathcal{L}))$. Borel's fixed point theorem implies that there exists a nonzero section $s\in H^0(X,\mathcal{L})$ such that the line Fs is fixed under this action. As $\mathbf{G}_a^n$ has no nontrivial characters, s itself is fixed. The divisor $\mathrm{div}(s)$ is $\mathbf{G}_a^n$-invariant; therefore, $\mathrm{div}(s)$ does not meet $\mathbf{G}_a^n$ and is necessarily a sum $\sum d_\alpha D_\alpha$ such that $\mathcal{L}\simeq\mathcal{O}_X(\sum d_\alpha D_\alpha)$. Because of Proposition 1.1, every other such section will be proportional to s. $\square$

REMARK 1.6. — We observe that if $D=\sum d_\alpha D_\alpha$ for integers $d_\alpha\geqslant 0$, then the canonical section s_D of $\mathcal{O}_X(D)$ is $\mathbf{G}_a^n$-invariant.

2. Vector fields

We now recall some facts concerning vector fields on equivariant compactifications of algebraic groups. Let G be a connected algebraic group over F and $\mathfrak{g}$ its Lie algebra of invariant vector fields. Let X be a smooth equivariant compactification of G. Denote by $D = X \setminus G$ the boundary. We assume that D is a divisor with strict normal crossings. Let $\mathcal{T}_X$ be the tangent bundle of X. Evaluating a vector field at the neutral element $\mathbf{1}$ of G induces a "restriction map"

$$H^0(X, \mathcal{T}_X) \to \mathcal{T}_{X,\mathbf{1}} = \mathfrak{g}.$$

Conversely, given $\partial \in \mathfrak{g}$, there is a unique vector field ∂^X such that for any open subset U of X and any $f \in \mathcal{O}_X(U)$, $\partial^X(f)(x) = \partial_g f(g \cdot x)|_{g=\mathbf{1}}$. The map $\partial \mapsto \partial^X$ is a section of the restriction map. Note however that the vector field $\partial^X|_G$ on G is in general not G-invariant.

LEMMA 2.1. — *For* $G = \mathbf{G}_a^n$ *and for any* $\partial \in \mathfrak{g}$ *the restriction* $\partial^X|_G$ *is invariant under* G.

PROPOSITION 2.2. — *Let* $x \in D$ *and fix a local equation* s_D *of* D *in a neighborhood* U *of* x. *Then, for any* $\partial \in \mathfrak{g}$, $\partial^X \log s_D = \frac{\partial^X(s_D)}{s_D}$ *is a regular function in* U.

Proof. — We can choose (étale) local coordinates $x_1, \dots, x_n$ in U, i.e., elements of $\mathcal{O}_X(U)$, such that $dx_1, \dots, dx_n$ generate $\Omega^1_X|_U$ and such that $s_D = x_1 \dots x_r$ for some integer $r \in \{1, \dots, n\}$. We can then write uniquely

$$\partial^X = \sum_{j=1}^n f_j \frac{\partial}{\partial x_j}$$

for some functions $f_j \in \mathcal{O}_X(U)$. By purity, we can even assume that $r = 1$ (since X is smooth, a function which is regular in the complement to a codimension 2 subscheme is regular everywhere). Let D_x be the irreducible component of D containing x ; necessarily, G stabilizes D_x. Therefore, the function $(g, y) \mapsto x_1(g \cdot y)$ is identically 0 in a Zariski neighborhood of $(\mathbf{1}, x) \in G \times X$, hence $\partial^X(x_1)$ vanishes on U. As $\partial^X(x_1) = f_1$, f_1 is a multiple of x_1: there exists a unique $g_1 \in \mathcal{O}_X(U)$ such that $f_1 = x_1 g_1$, hence

$$\partial^X \log x_1 = g_1 \in \mathcal{O}_X(U).$$

The lemma is proved. □

EXAMPLE 2.3. — For $G = \mathbf{G}_a = \operatorname{Spec} F[x]$ or $G = \mathbf{G}_m = \operatorname{Spec} F[x, x^{-1}]$, the Lie algebra $\mathfrak{g}$ has a canonical basis ∂, given by the local parameter x at the neutral element (respectively 0 and 1) of G. If we embed G in $\mathbf{P}^1$ equivariantly, we get $\partial^X = \partial/\partial x$ for $G = \mathbf{G}_a$ and $x\partial/\partial x$ for $G = \mathbf{G}_m$. We see that it vanishes at infinity (being $\{\infty\}$ or $\{0; \infty\}$, accordingly). This is a general fact, as the following lemma shows.

LEMMA 2.4 (Hassett/Tschinkel). — *There exist integers* $\rho_\alpha \geqslant 1$ *such that*

$$\omega_X^{-1} \simeq \mathcal{O}_X(\sum \rho_\alpha D_\alpha).$$

If $G = \mathbf{G}_a^n$*, then for each* α*,* $\rho_\alpha \geqslant 2$*.*

Proof. — Let $\partial_1, \dots, \partial_n$ be a basis of $\mathfrak{g}$. Then, $\rho := \partial_1^X \wedge \dots \wedge \partial_n^X$ is a global section of $\det \mathcal{T}_X = \omega_X^{-1}$. Moreover, ρ does not vanish on G. Therefore, we can write $\operatorname{div}(\rho) = \sum \rho_\alpha D_\alpha$. Let D be one of the components in $\operatorname{div}(\rho)$ and $x \in D$. Fix local coordinates $x_1, \dots, x_n$ in a neighborhood U of x and assume that in U, D is defined by the equation $x_1 = 0$. Write $\partial_i^X = \sum f_{ij} \frac{\partial}{\partial x_j}$, with $f_{ij} \in \mathcal{O}_X(U)$. We have seen in Proposition 2.2 that for any i, $\partial_i^X(x_1) = f_{i1} \in (x_1)$. Hence, $\mu \in (x_1) \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_n}$.

We will not use the sharper bound which is valid in the case of $G = \mathbf{G}_a^n$. We refer to **[10]**, Theorem 2.7 for its proof. □

3. Metrizations

Let F be a number field and $\mathfrak{o}_F$ its ring of integers. Denote by F_v the completion of F at a place v, by $\mathfrak{o}_v$ the ring of integers in F_v if v is nonarchimedean, by $\mathbf{A}$ the ring of adeles of F and by $\mathbf{A}_{\text{fin}}$ the (restricted) product of F_v over the nonarchimedean v. The valuation in F_v is normalized in such a way that for any Haar measure μ_v on F_v and any measurable subset $I \subset F_v$, $\mu_v(aI) = |a|_v \, \mu_v(I)$. In particular, it is the usual absolute value for $F_v = \mathbf{R}$, its square for $F_v = \mathbf{C}$, it satisfies $|p|_p = 1/p$ if $F_v = \mathbf{Q}_p$ and if $\mathcal{N}_v : F_v \to \mathbf{Q}_p$ is the norm map, $|x|_v = |\mathcal{N}_v(x)|_p$. Let X be an equivariant compactification of $\mathbf{G}_a^n$ as above and $\mathcal{L}$ a line bundle on X, endowed with its canonical linearization.

DEFINITION 3.1. — *A* smooth adelic metric *on* $\mathcal{L}$ *is a family of* v*-adic norms* $\|\cdot\|_v$ *on* $\mathcal{L}$ *for all places* v *of* F *satisfying the following properties:*

(a) *if* v *is archimedean, then* $\|\cdot\|_v$ *is* $\mathcal{C}^\infty$*;*

(b) *if* v *is nonarchimedean, then* $\|\cdot\|_v$ *is locally constant (i.e., the norm of any local basis is locally constant for the* v*-adic topology);*

(c) *there exists an open dense subset* $U \subset \operatorname{Spec}(\mathfrak{o}_F)$*, a flat projective* U*-scheme* $\mathcal{X}_{/U}$ *extending* X *together with an action of* $\mathbf{G}^n_{a/U}$ *extending the action of* $\mathbf{G}_a^n$ *on* X *and a linearized line bundle* $\mathcal{L}$ *on* $\mathcal{X}_{/U}$ *extending the linearized line bundle on* X*, such that for any place* v *lying over* U*, the* v*-adic metric on* $\mathcal{L}$ *is given by the integral model.*

LEMMA 3.2. — *Let* v *be a nonarchimedean valuation of* F *and* $\|\cdot\|_v$ *a locally constant* v*-adic norm on* $\mathcal{L}$*. Then the stabilizer of* $(\mathcal{L}, \|\cdot\|_v)$*, i.e., the set of* $g \in \mathbf{G}_a^n(\mathfrak{o}_v)$ *which act isometrically on* $(\mathcal{L}, \|\cdot\|_v)$*, is a compact open subgroup of* $\mathbf{G}_a^n(\mathfrak{o}_v)$*.*

Proof. — First we assume that $\mathcal{L}$ is effective. By Proposition 1.5, we have a nonzero invariant section s.

If m and p_2 denote the action and the second projection $G \times X \to X$, respectively, endow the trivial line bundle $m^*\mathcal{L} \otimes p_2^*\mathcal{L}$ on $\mathbf{G}_a^n \times X$ on $G \times X$ with a tensor-product metric. On $\mathbf{G}_a^n(F_v) \times \mathbf{G}_a^n(F_v)$, the norm of the canonical section 1 is given by the function

$$(g, x) \mapsto \|s(g + x)\|_v \, \|s(x)\|_v^{-1} .$$

Hence, it extends to a locally constant function on $\mathbf{G}_a^n(F_v) \times X(F_v)$. Its restriction to the compact subset $\mathbf{G}_a^n(\mathfrak{o}_v) \times X(F_v)$ is uniformly continuous. Since it is locally constant and equal to 1 on $\{1\} \times X(F_v)$, there exists a neighborhood of $\{1\} \times X(F_v)$ on which it equals 1. Such a neighborhood contains one of the form $K_v \times X(F_v)$, where K_v is a compact open subgroup in $\mathbf{G}_a^n(\mathfrak{o}_v)$. This proves the lemma in the effective case.

In the general case, we write $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ for two effective line bundles (each having a nonzero global section). We can endow $\mathcal{L}_2$ with any locally constant v-adic metric and $\mathcal{L}_1$ with the unique (necessarily locally constant) v-adic metric such that the isomorphism $\mathcal{L}_1 \simeq \mathcal{L} \otimes \mathcal{L}_2$ is an isometry. By the previous case, there exist two open compact subgroups $K_{1,v}$ and $K_{2,v}$ contained in $\mathbf{G}_a^n(\mathfrak{o}_v)$ which act isometrically on $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Their intersection acts isometrically on $\mathcal{L}$. □

PROPOSITION 3.3. — *If $\mathcal{L}$ is endowed with a smooth adelic metric then for all but finitely many places v of F the stabilizer of $(\mathcal{L}, \|\cdot\|_v)$ is equal to $\mathbf{G}_a^n(\mathfrak{o}_v)$. Therefore, their product over all finite places of F is a compact open subgroup of $\mathbf{G}_a^n(\mathbf{A}_{\mathrm{fin}})$.*

Proof. — It suffices to note that if v lies over the open subset $U \subset \mathrm{Spec}(\mathfrak{o}_F)$ given by the definition of an adelic metric then the stabilizer of $(\mathcal{L}, \|\cdot\|_v)$ equals $\mathbf{G}_a^n(\mathfrak{o}_v)$. □

From now on we choose adelic metrics on the line bundles $\mathcal{L}$ in such a way that the tensor product of two line bundles is endowed with the product of the metrics (this can be done by fixing smooth adelic metrics on a $\mathbf{Z}$-basis of $\mathrm{Pic}(X)$ and extending by linearity). We shall denote by $\mathbf{K}$ the compact open subgroup of $\mathbf{G}_a^n(\mathbf{A}_{\mathrm{fin}})$ stabilizing all these metrized line bundles on X . We also fix an open dense $U \subset \mathrm{Spec}(\mathfrak{o}_F)$, a flat and projective model $\mathcal{X}_{/U}$ over U and models of the line bundles $\mathcal{L}$ such that the chosen v-adic metrics for all line bundles on X are given by these integral models.

4. Heights

Let X be an algebraic variety over F and $(\mathcal{L}, \|\cdot\|_v)$ an adelically metrized line bundle on X. The associated *height function* is defined as

$$H_{\mathcal{L}} : X(F) \to \mathbf{R}_{>0} \qquad H_{\mathcal{L}}(x) = \prod_v H_{\mathcal{L},v}(x) := \prod_v \|s\|_v (x)^{-1},$$

where s is any F-rational section of $\mathcal{L}$ not vanishing at x. The product formula ensures that $H_{\mathcal{L}}$ does not depend on the choice of the F-rational section s (though the local heights $H_{\mathcal{L},v}(x)$ do).

In Section 3 we have defined simultaneous metrizations of line bundles on equivariant compactifications of $\mathbf{G}_a^n$. This allows to define compatible systems of heights. We define a height pairing

$$H = \prod_v H_v \,:\, \mathbf{G}_a^n(\mathbf{A}) \times \mathrm{Pic}(X)_{\mathbf{C}} \to \mathbf{C}$$

by

$$(\mathbf{x}; \mathbf{s}) = ((\mathbf{x}_v); \sum s_\alpha D_\alpha)) \mapsto \prod_v \prod_\alpha \|s_\alpha\|_v (\mathbf{x}_v)^{-s_\alpha}.$$

Here s_α is a fixed $\mathbf{G}_a^n$-invariant local section of $\mathcal{O}_X(D_\alpha)$. We have a natural restriction

$$H \,:\, \mathbf{G}_a^n(F) \times \mathrm{Pic}(X) \to \mathbf{R}_{>0}.$$

PROPOSITION 4.1. — *The height pairing* H *is* $\mathbf{K}$*-invariant in the first component (where* $\mathbf{K}$ *is the stabilizer of all line bundles on* X *with the chosen metrization) and (exponentially) linear in the Picard component. The restriction of* H *to any class* $\mathcal{L} \in \mathrm{Pic}(X)$ *is a height with respect to some smooth adelic metric on the line bundle* $\mathcal{L}$.

PROPOSITION 4.2. — *Assume that* $\mathcal{L}$ *is in the interior of the effective cone of* $\mathrm{Pic}(X)$ *(i.e.,* $\mathcal{L} \simeq \mathcal{O}_X(\sum d_\alpha D_\alpha)$ *for some positive integers* $d_\alpha > 0$*). Then, for any real* B*, there are only finitely many* $x \in \mathbf{G}_a^n(F)$ *such that* $H(x; \mathcal{L}) \leqslant B$.

In view of Lemma 2.4, this applies to the anticanonical line bundle K_X^{-1}.

Proof. — Let $\mathcal{M}$ be an ample line bundle and ν a sufficiently large integer such that $\mathcal{L}^\nu \otimes \mathcal{M}^{-1}$ is effective. It follows from the preceding section that $\mathcal{L}^\nu \otimes \mathcal{M}^{-1}$ has a section s which does not vanish on $\mathbf{G}_a^n$. This implies that the function $x \mapsto -\log H(x; \mathcal{L}^\nu \otimes \mathcal{M}^{-1})$ is bounded from above on $\mathbf{G}_a^n(F)$. Therefore, there exists a constant $C > 0$ such that for any $x \in \mathbf{G}_a^n(F)$,

$$H(x; \mathcal{L}) \geqslant C H(x; \mathcal{M})^{1/\nu}.$$

We may now apply Northcott's theorem and obtain the desired finiteness. □

REMARK 4.3. — The same argument shows that the rational map given by the sections of a sufficiently high power of $\mathcal{L}$ is an embedding on $\mathbf{G}_a^n$.

The main tool in the study of asymptotics for the number of points of bounded height is the *height zeta function*

$$Z(\mathbf{s}) = \sum_{x \in \mathbf{G}_a^n(F)} H(x; \mathbf{s})^{-1}, \qquad \mathbf{s} = (s_\alpha) \in \mathrm{Pic}(X)_{\mathbf{C}}.$$

PROPOSITION 4.4. — *There exists a nonempty open subset* $\Omega \subset \mathrm{Pic}(X)_{\mathbf{R}}$ *such that* $Z(\mathbf{s})$ *converges absolutely to a holomorphic function in the tube domain* $\Omega + i\,\mathrm{Pic}(X)_{\mathbf{R}} \subset \mathrm{Pic}(X)_{\mathbf{C}}$.

Proof. — Fix a basis $(\mathcal{L}_j)_j$ of $\mathrm{Pic}(X)$ consisting of (classes of) line bundles lying in the interior of the effective cone of $\mathrm{Pic}(X)$. Let Ω_t denote the open set of all linear combinations $\sum t_j\mathcal{L}_j \in \mathrm{Pic}(X)_{\mathbf{R}}$ such that for some j, $t_j > t$. Fix some ample line bundle $\mathcal{M}$. It is well known that the height zeta function of X relative to $\mathcal{M}$ converges for $\mathrm{Re}(s)$ big enough, say $\mathrm{Re}(s) > \sigma_0$. In the proof of Proposition 4.2 we may choose some ν which works for any $\mathcal{L}_j$, so that for any j

$$H(x;\mathcal{L}_j)^{-1} \ll H(X;\mathcal{M})^{-1/\nu}.$$

Since $H(\cdot;\mathcal{L}_j)$ is bounded from below on $\mathbf{G}_a^n(F)$ it follows that for any $\mathcal{L} \in \Omega_1$

$$H(x;\mathcal{L})^{-1} \ll H(X;\mathcal{M})^{-1/\nu}$$

Therefore, the height zeta function converges absolutely and uniformly on the tube domain $\Omega_{\nu\sigma_0} + i\,\mathrm{Pic}(X)_{\mathbf{R}}$. □

The following sections are devoted to the study of analytic properties of $Z(\mathbf{s})$.

5. The Poisson formula

We recall basic facts concerning harmonic analysis on the group $\mathbf{G}_a^n$ over the adeles $\mathbf{A} = \mathbf{A}_F$ (cf., for example, **[18]**). For any prime number p, we can view $\mathbf{Q}_p/\mathbf{Z}_p$ as the p-Sylow subgroup of $\mathbf{Q}/\mathbf{Z}$ and we can define a local character ψ_p of $\mathbf{G}_a(\mathbf{Q}_p)$ by setting

$$\psi_p\colon x_p \mapsto \exp(2\pi i x_p).$$

At the infinite place of $\mathbf{Q}$ we put

$$\psi_\infty\colon x_\infty \mapsto \exp(-2\pi i x_\infty),$$

(here x_∞ is viewed as an element in $\mathbf{R}/\mathbf{Z}$). The product of local characters gives a character ψ of $\mathbf{G}_a(\mathbf{A}_{\mathbf{Q}})$ and, by composition with the trace, a character of $\mathbf{G}_a(\mathbf{A})$.

If $\mathbf{a} \in \mathbf{G}_a^n(\mathbf{A})$, let $f_{\mathbf{a}} = \langle\cdot,\mathbf{a}\rangle$ be the corresponding linear form on $\mathbf{G}_a^n(\mathbf{A})$ and $\psi_{\mathbf{a}} = \psi\circ f_{\mathbf{a}}$. This defines a map $\mathbf{G}_a^n(\mathbf{A}) \to \mathbf{G}_a^n(\mathbf{A})^*$. It is well known that this map is a Pontryagin duality. The subgroup $\mathbf{G}_a^n(F) \subset \mathbf{G}_a^n(\mathbf{A})$ is discrete, cocompact and we have an induced Pontryagin duality

$$\mathbf{G}_a^n(\mathbf{A}) \to (\mathbf{G}_a^n(\mathbf{A})/\mathbf{G}_a^n(F))^*$$

(see **[18]**). We fix selfdual Haar measures dx_v on $\mathbf{G}_a(F_v)$ for all v. We refer to **[18]** for an explicit normalization of these measures. We will use the fact that for all but finitely many v the volume of $\mathbf{G}_a(\mathfrak{o}_v)$ with respect to dx_v is equal to 1. Thus we have an induced selfdual Haar measure dx on $\mathbf{G}_a(\mathbf{A})$ and the product measure $d\mathbf{x}$ on $\mathbf{G}_a^n(\mathbf{A})$. The Fourier transform (in the adelic component) of the height pairing on $\mathbf{G}_a^n(\mathbf{A}) \times \mathrm{Pic}(X)_{\mathbf{C}}$ is defined by

$$\hat{H}(\mathbf{s};\psi_{\mathbf{a}}) = \int_{\mathbf{G}_a^n(\mathbf{A})} H(\mathbf{x};\mathbf{s})^{-1}\psi_{\mathbf{a}}(\mathbf{x})\,d\mathbf{x}.$$

We will use the Poisson formula in the following form (cf. **[12]**, p. 280).

THEOREM 5.1. — *Let* H *be a continuous function on* $\mathbf{G}_a^n(\mathbf{A})$ *such that both* H *and its Fourier transform* $\hat{\mathsf{H}}$ *are integrable and such that the series*

$$\sum_{\mathbf{x}\in\mathbf{G}_a^n(F)} \mathsf{H}(\mathbf{x}+\mathbf{b})$$

converges absolutely and uniformly when $\mathbf{b}$ *belongs to any compact subset in* $\mathbf{G}_a^n(\mathbf{A})/\mathbf{G}_a^n(F)$. *Then,*

$$\sum_{\mathbf{x}\in\mathbf{G}_a^n(F)} \mathsf{H}(\mathbf{x}) = \sum_{\mathbf{a}\in\mathbf{G}_a^n(F)} \hat{\mathsf{H}}(\psi_{\mathbf{a}}).$$

The following lemma (a slight strengthening of Proposition 4.4) verifies the two hypotheses of the Poisson formula 5.1 concerning H. The hypotheses concerning $\hat{\mathsf{H}}$ will be verified in subsequent sections.

LEMMA 5.2. — *Let* X *be a smooth projective equivariant compactification of* $\mathbf{G}_a^n$ *and* H *the height pairing defined in Section 4. There exists a nonempty open subset* $\Omega \subset \operatorname{Pic}(X)_{\mathbf{R}}$ *such that for any* $\mathbf{s} \in \Lambda + \mathrm{i}\operatorname{Pic}(X)_{\mathbf{R}}$ *the series*

$$\sum_{x\in\mathbf{G}_a^n(F)} \mathsf{H}(x+\mathbf{b};\mathbf{s})^{-1}$$

converges absolutely, uniformly in $\mathbf{b} \in \mathbf{G}_a^n(\mathbf{A})/\mathbf{G}_a^n(F)$ *and locally uniformly in* $\mathbf{s}$.

Proof. — Since the natural action of $\mathbf{G}_a^n$ on $\operatorname{Pic}(X)$ is trivial, for any $\mathbf{b} \in \mathbf{G}_a^n(\mathbf{A})$ the function $x \mapsto \mathsf{H}(x+\mathbf{b};\mathcal{L})$ is a height function for $\mathcal{L}$, induced by a "twisted" adelic metric. When $\mathbf{b}$ belongs to some compact subset of $\mathbf{G}_a^n(\mathbf{A})$ these adelic metrics are comparable: they differ only for a finite number of places and for these places the comparability follows from the projectivity of X. This implies the lemma. $\square$

PROPOSITION 5.3. — *For all characters* $\psi_{\mathbf{a}}$ *which are nontrivial on the compact subgroup* $\mathbf{K}$ *of* $\mathbf{G}_a^n(\mathbf{A}_{\mathrm{fin}})$ *and all* $\mathbf{s}$ *such that* $\mathsf{H}(\cdot;\mathbf{s})^{-1}$ *is integrable, we have*

$$\hat{\mathsf{H}}(\mathbf{s};\psi_{\mathbf{a}}) = 0.$$

Proof. — This follows from the invariance of the height under the action of the compact $\mathbf{K}$. $\square$

Consequently, we have a *formal* identity for the height zeta function:

$$Z(\mathbf{s}) = \sum_{\mathbf{a}\in\mathfrak{d}_X} \hat{\mathsf{H}}(\mathbf{s};\psi_{\mathbf{a}}). \tag{5.4}$$

Here we denoted by $\mathfrak{d}_X \subset \mathbf{G}_a^n(F)$ the set of all $\mathbf{a}$ such that $\psi_{\mathbf{a}}$ is trivial on $\mathbf{K}$. This set is a sub-$\mathfrak{o}_F$-module of $\mathbf{G}_a^n(F)$, commensurable with $\mathbf{G}_a^n(\mathfrak{o}_F)$.

In the remaining sections we will justify the Poisson formula. First we define the set of bad valuations S: as in Section 3, we fix a good model $\mathcal{X}_{/U}$ (flat projective U-scheme) over an open dense subset $U \subset \operatorname{Spec}(\mathfrak{o}_F)$ such that the local height functions restricted to $\mathbf{G}_a^n(F_v)$ are invariant under $\mathbf{G}_a^n(\mathfrak{o}_v)$. We still denote by D_α the schematic closure of D_α in $\mathcal{X}_{/\mathrm{U}}$.

We define a set of valuations S by saying a valuation $v \in S$ if either

- v is archimedean, or
- $v \notin U$, or
- the residual characteristic of v is 2 or 3, or
- the volume of $\mathbf{G}_a^n(\mathfrak{o}_v)$ with respect to $d\mathbf{x}_v$ is not equal to 1, or
- over the local ring $\mathfrak{o}_v$, the union $\bigcup_\alpha D_\alpha$ is not a transverse union of smooth relative divisors over U.

Let $f_{\mathbf{a}}$ be a linear form on $\mathbf{G}_a^n$, considered as an element of $F(X)$ and $\mathrm{div}(f_{\mathbf{a}})$ its divisor on X. It can be extended to $\mathcal{X}_{/\mathrm{U}}$. We denote by $S(\mathbf{a})$ the set of all valuations v such that either

- v is contained in S, or
- $\mathbf{a} \equiv 0 \pmod{\mathfrak{m}_v}$.

In Section 6 we prove estimates for $\prod_{v\in S(\mathbf{a})} \hat{H}_v(\mathbf{s};\psi_{\mathbf{a}})$. In Sections 7 and 8 we compute explicitely the local Fourier transforms $\hat{H}_v(\mathbf{s};\psi_{\mathbf{a}})$ for all $v \notin S(\mathbf{a})$. This leads to the identifications of the poles for each $\hat{H}(\mathbf{s};\psi_{\mathbf{a}})$ and to a meromorphic continuation of each summand in Equation 5.4. Combining with the estimates at archimedian places we obtain a meromorphic continuation of $Z(\mathbf{s})$. The pole of highest order will be given by the contribution from the trivial character $\mathbf{a} = 0$.

6. General estimates

Let X be a smooth equivariant compactification of $\mathbf{G}_a^n$ as above. On Pic(X) we fixed coordinates, given by the classes D_α (where $\alpha \in \mathcal{A}$). Let $\rho = (\rho_\alpha)$ be the class of the anticanonical line bundle K_X^{-1} in these coordinates. For any $\kappa \in \mathbf{R}$ we denote by Ω_κ the tube domain in $\mathrm{Pic}(X)_{\mathbf{C}}$ given by $\mathrm{Re}(s_\alpha) > \rho_\alpha + \kappa$ for all $\alpha \in \mathcal{A}$. We define the (nonrenormalized) Tamagawa measure $d\mu_v$ on $X(F_v)$ by

$$d\mathbf{x}_v = H_v(\mathbf{x};\rho)d\mu_v.$$

This is a measure obtained from a metrization on the canonical line bundle by a standard procedure (see **[19]** for its definition in the context of linear algebraic groups and **[14]** for its definition and basic properties in the context of smooth algebraic varieties).

In the following three sections we temporarily assume that the irreducible components of the boundary of X *are geometrically irreducible.* In Section 9, we shall explain how our results extend to the general case.

PROPOSITION 6.1. — *Let* S *be any finite set of valuations. For every* $\varepsilon > 0$ *and any* $N > 0$ *there exists a constant* $C(S,\varepsilon,N)$ *such that for all* $\mathbf{a} \in \mathfrak{d}_X$ *and all* $\mathbf{s} = (s_\alpha) \in \Omega_{-1+\varepsilon}$ *one has the estimate*

$$\left|\prod_{v\in S} \hat{H}_v(\mathbf{s};\psi_{\mathbf{a}})\right| \leqslant C(S,\varepsilon,N)\frac{(1+\|\mathbf{s}\|)^{N[F:\mathbf{Q}]}}{\prod_{v|\infty}(1+\|\mathbf{a}\|_v)^N},$$

where $\|\mathbf{s}\| = \|\mathrm{Re}(\mathbf{s})\| + \|\mathrm{Im}(\mathbf{s})\|$.

We subdivide the proof of this proposition into a sequence of lemmas.

LEMMA 6.2. — *The function* $H_v(\cdot;\mathbf{s})^{-1}$ *is integrable on* $\mathbf{G}_a^n(F_v)$ *if and only if for all* $\alpha \in \mathcal{A}$ *we have* $\mathrm{Re}(s_\alpha) > \rho_\alpha - 1$. *Moreover, for all* $\varepsilon > 0$ *and all nonarchimedean places* v, *there exists a constant* $C_v(\varepsilon)$ *such that for all* $\mathbf{s} \in \Omega_{-1+\varepsilon}$ *and all* $\mathbf{a} \in \mathfrak{d}_X$ *one has the estimate*

$$\left|\hat{H}_v(\mathbf{s};\psi_{\mathbf{a}})\right| \leqslant C_v(\varepsilon).$$

Proof. — First of all, we replace $\psi_{\mathbf{a}}(\mathbf{x})$ by 1. Now we use an analytic partition of unity on $X(F_v)$ and the assumption that the boundary $D = \sum_{\alpha\in\mathcal{A}} D_\alpha$ is a strict normal crossings divisor. It suffices to compute the integral over a relatively compact neighborhood of 0 in F_v^n (denoted by $\mathcal{K}$) on which we have a set coordinates $x_1, \ldots, x_n$ so that in $\mathcal{K}$ the divisor D is defined by the equations $x_1 \cdots x_a = 0$ for some $a \in \{1, \ldots, n\}$. In $\mathcal{K}$ there exist continuous bounded functions h_α (for $\alpha \in \mathcal{A}$) such that

$$\int_{\mathcal{K}} H_v(\mathbf{x};\mathbf{s})^{-1}\,d\mathbf{x}_v = \int_{\mathcal{K}} \prod_{i=1}^{a} |x_i|_v^{s_{\alpha(i)}-\rho_{\alpha(i)}} \exp\Big(\sum s_\alpha h_\alpha(x)\Big)\,d\mu_v.$$

The integral over $\mathcal{K}$ is comparable to an integral of the same type with functions h_α replaced by 0. The lemma is now a consequence of the following well-known lemma. □

LEMMA 6.3. — *Let* K *be a local field. Denote by* $\mathcal{B}$ *the unit ball in* K. *The function* $x \mapsto |x|^s$ *is integrable on* $\mathcal{B}$ *if and only if* $\mathrm{Re}(s) > -1$.

Proof. — We may assume $s \in \mathbf{R}$. Choose $c \in \,]0;1[$ such that the annulus $c < |x| \leqslant 1$ has positive measure in K and let $I_0 = \int_{c<|x|\leqslant 1} |x|^s\,dx$. Then, we have

$$I_n = \int_{c^{n+1}<|x|\leqslant c^n} |x|^s\,dx = c^{n(s+1)} I_0.$$

It follows that the integral over K converges if and only if the geometric series $\sum c^{n(s+1)}$ converges, that is if $s+1>0$. □

PROPOSITION 6.4. — *Let* v *be an archimedean place of* F *and let* $\varepsilon_v = [F_v : \mathbf{R}]$. *For any* $N > 0$ *and any* $\varepsilon > 0$, *there exists a constant* $C_v(\varepsilon, N)$ *such that for any* $\mathbf{s} =\in \Omega_{-1+\varepsilon}$ *and all* $\mathbf{a} \in \mathfrak{d}_X$ *(and* $\neq 0$*), we have*

$$\hat{H}_v(\mathbf{s};\psi_{\mathbf{a}}) \leqslant C_v(\varepsilon, N)\frac{(1+\|\mathbf{s}\|)^{\varepsilon_v N}}{(1+\|\mathbf{a}\|_v)^N}.$$

Proof. — Let us assume for the moment that $F_v = \mathbf{R}$. We shall write $\mathbf{a}_v = (a_1, \ldots, a_n)$, $\mathbf{x}_v = (x_1, \ldots, x_n)$ and $d\mathbf{x}_v = d\mathbf{x}$. Then, in the domain Ω_{-1} one has

$$\hat{H}_v(\mathbf{s};\psi_{\mathbf{a}}) = \int_{\mathbf{R}^n} H_v(\mathbf{x};\mathbf{s})^{-1} \exp(-2i\pi\langle \mathbf{a}, \mathbf{x}\rangle)\,d\mathbf{x}.$$

Now we use integration by parts: for $j \in \{1, \dots, n\}$ we have

$$2i\pi a_j \hat{H}_v(\mathbf{s}; \psi_{\mathbf{a}}) = \int_{\mathbf{R}^n} \frac{\partial}{\partial x_j} H_v(\mathbf{x}; \mathbf{s})^{-1} \exp(-2i\pi\langle \mathbf{a}, \mathbf{x}\rangle)\, d\mathbf{x}$$

and by induction

$$(2i\pi a_j)^N \hat{H}_v(\mathbf{s}; \psi_{\mathbf{a}}) = \int_{\mathbf{R}^n} \left(\frac{\partial^N}{\partial x_j^N} H_v(\cdot; \mathbf{s})^{-1} \right)(\mathbf{x}) \exp(-2i\pi\langle \mathbf{a}, \mathbf{x}\rangle)\, d\mathbf{x}.$$

For any α, let us define $h_\alpha = \log \|s_{D_\alpha}\|$; it is a $\mathcal{C}^\infty$ function on $\mathbf{G}_a^n(\mathbf{R})$.

Let x be a point of $X(\mathbf{R})$ and let A be the set of $\alpha \in \mathcal{A}$ such that $x \in D_\alpha$. If t_α is a local equation of D_α in a neighborhood U of some $\mathbf{R}$-point of D_α, then there is a $\mathcal{C}^\infty$ function φ_α on U such that for $\mathbf{x} \in U \cap \mathbf{G}_a^n(\mathbf{R})$ we have

$$h_\alpha(\mathbf{x}) = \log |t_\alpha(\mathbf{x})| + \varphi_\alpha(\mathbf{x}).$$

It then follows from Propositions 2.1 and 2.2 that for each α,

$$\frac{\partial}{\partial x_j} h_\alpha(\mathbf{x}) = \frac{1}{2}\frac{\partial}{\partial x_j} \log |t_\alpha(\mathbf{x})| + \frac{\partial}{\partial x_j} \varphi_\alpha(\mathbf{x})$$

extends to a $\mathcal{C}^\infty$ function on $X(\mathbf{R})$.

From the equality

$$H_v(\mathbf{x}; \mathbf{s})^{-1} = \prod_{\alpha \in \mathcal{A}} \exp(-s_\alpha h_\alpha(\mathbf{x}))$$

we deduce the existence of a homogeneous polynomial $P_N \in \mathbf{R}[X_\alpha^{(1)}, \dots, X_\alpha^{(N)}]$ of degree N (with the convention that each $X_\alpha^{(p)}$ has degree p) and such that

$$(\partial/\partial x_j)^N H_v(\mathbf{x}; \mathbf{s})^{-1} = H_v(\mathbf{x}; \mathbf{s})^{-1} P_N(s_\alpha \partial_j h_\alpha, s_\alpha \partial_j^2 h_\alpha, \dots, s_\alpha \partial_j^N h_\alpha).$$

This implies that there exists a constant $C_v(\varepsilon, N, j)$ such that

$$\left|(2i\pi a_j)^N\right| \left|\hat{H}_v(\mathbf{s}; \psi_{\mathbf{a}})\right| \leqslant C_v(\varepsilon, N, j)(1 + \|\mathbf{s}\|)^N \int_{\mathbf{R}^n} |H_v(\mathbf{x}; \mathbf{s})|^{-1}\, d\mathbf{x}.$$

Choosing j such that $|a_j|$ is maximal gives $|a_j| \geqslant \|\mathbf{a}\| / \sqrt{n}$, hence an upper bound

$$\left|\hat{H}_v(\mathbf{s}; \psi_{\mathbf{a}})\right| \leqslant C'_v(\varepsilon, N) \frac{(1 + \|\mathbf{s}\|)^N}{(1 + \|\mathbf{a}\|)^N} \hat{H}_v(\mathrm{Re}(\mathbf{s}); \psi_{\mathbf{0}}),$$

where $\psi_{\mathbf{0}}$ is the trivial character and $C'_v(\varepsilon, N)$ some positive constant. To conclude the proof it suffices to remark that for any $\varepsilon > 0$, $\hat{H}_v$ is bounded on the set $\Omega_{-1+\varepsilon}$ (but the bound depends on ε).

The case $F_v = \mathbf{C}$ is treated using a similar integration by parts. (The exponent 2 on the numerator comes from the fact that for a complex place v, $\|\cdot\|_v$ is the square of a norm.) □

7. Non archimedean computation at the trivial character

In this section we consider only $v \notin S$. Let $\mathfrak{m}_v \subset \mathfrak{o}_v$ be the maximal ideal, $k_v = \mathfrak{o}_v/\mathfrak{m}_v$ and $q = \#k_v$. Recall that we have fixed a good model $\mathcal{X}_{/U}$ over some $U \subset \operatorname{Spec} \mathfrak{o}_F$. To simplify notations we will write $\mathbf{x} = \mathbf{x}_v$, $d\mathbf{x} = d\mathbf{x}_v$ etc.

The following formula is an analogue of Denef's formula in [**8**, Thm 3.1] for Igusa's local zeta function.

THEOREM 7.1. — *For all $v \notin S$ and all $\mathbf{s} \in \Omega_{-1} \subset \operatorname{Pic}(X)_{\mathbf{C}}$ we have*

$$\hat{H}_v(\mathbf{s}; \psi_{\mathbf{0}}) = q^{-\dim X} \sum_{A \subset \mathcal{A}} \#D_A^\circ(k_v) \prod_{\alpha \in A} \frac{q-1}{q^{1+s_\alpha - \rho_\alpha} - 1}.$$

REMARK 7.2. — For $\mathbf{s} = \rho$ we get $\#\mathcal{X}_{/U}(k_v)/q^{\dim X}$, the expected local density at v.

We split the integral along residue classes mod $\mathfrak{m}_v$. Let $\tilde{x} \in \mathcal{X}(k_v)$ and $A = \{\alpha \in \mathcal{A}\,;\, \tilde{x} \in D_\alpha\}$ so that $\tilde{x} \in D_A^\circ$.

We can introduce local (étale) coordinates x_α ($\alpha \in A$) and y_β ($\beta \in B$, $\#A + \#B = \dim X$) around $\tilde{x}$ such that locally, the divisor D_α is defined by the vanishing of x_α. Using the normal crossing property of the boundary divisor, one checks the formula

$$\begin{aligned}\int_{\mathrm{red}^{-1}(\tilde{x})} H_v(\mathbf{x};\mathbf{s})^{-1}\, d\mathbf{x} &= \int_{\mathfrak{m}_v^A \times \mathfrak{m}_v^B} q^{-\sum_{\alpha\in A}(s_\alpha - \rho_\alpha)v(x_\alpha)} \prod_{\alpha\in A} dx_\alpha \prod_{\beta \in B} dy_\beta \\ &= \frac{1}{q^{\#B}} \prod_{\alpha \in A} \int_{\mathfrak{m}_v} q^{-(s_\alpha - \rho_\alpha)v(x)}\, dx = \frac{1}{q^{\dim X}} \prod_{\alpha\in A} \frac{q-1}{q^{1+s_\alpha-\rho_\alpha} - 1}\end{aligned}$$

where the last equality follows from:

LEMMA 7.3. —

$$\int_{\mathfrak{m}_v} q^{-sv(x)}\, dx = \frac{1}{q} \frac{q-1}{q^{1+s} - 1}.$$

Proof. — Indeed,

$$\int_{\mathfrak{m}_v} q^{-sv(x)}\, dx = \sum_{n=1}^\infty q^{-sn} \operatorname{vol}(\mathfrak{m}_v^n \setminus \mathfrak{m}_v^{n+1}) = \sum_{n=1}^\infty q^{-sn} q^{-n} \big(1 - \frac{1}{q}\big) = \frac{1}{q}\frac{q-1}{q^{1+s}-1}.$$

□

We need to estimate the number of k_v-points in D_α, uniformly over v.

LEMMA 7.4. — *There exists a constant $C(X)$ such that for all $v \notin S$ and all $A \subset \mathcal{A}$ we have the estimates:*

- *if* $\#A = 1$, $\left|\#D_A(k_v) - q_v^{\dim X - 1}\right| \leqslant C(X) q_v^{\dim X - 3/2}$;
- *if* $\#A \geqslant 2$, $\#D_A(k_v) \leqslant C(X) q_v^{\dim X - \#A}$.

Proof. — We use the fact that for any projective variety Y of dimension $\dim Y$ and degree $\deg Y$ the number of k_v-points is estimated as

$$\#Y(k_v) \leqslant (\deg Y)\#\mathbf{P}^{\dim Y}(k_v),$$

(see, e.g., Lemma 3.9 in [**7**]). Since X is projective, all D_A can be realized as subvarieties in some projective space. This proves the second part. To prove the first part, we apply Lang-Weil's estimate [**11**] to the geometrically irreducible smooth U-scheme D_α. □

PROPOSITION 7.5. — *For all $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that for any $\mathbf{s} \in \Omega_{-\frac{1}{2}+\varepsilon}$ and any finite place $v \notin S$,*

$$\left| \hat{H}_v(\mathbf{s};\psi_\mathbf{0}) \prod_{\alpha\in\mathcal{A}} (1 - q^{-(1+s_\alpha-\rho_\alpha)}) - 1 \right| \leqslant C(\varepsilon) q^{-1-\varepsilon}.$$

Proof. — Using the uniform estimates from Lemma 7.4 we see that in the formula for $\hat{H}_v$, each term with $\#A \geqslant 2$ is $O(q^{-(\frac{1}{2}+\varepsilon)\#A}) = O(q^{-1-2\varepsilon})$, with uniform constants in O. Turning to the remaining terms, we get

$$\begin{aligned} & 1 + \sum_{\alpha\in\mathcal{A}} (\frac{1}{q} + O(1/q^2)) \frac{q-1}{q^{1+s_\alpha-\rho_\alpha} - 1} \\ &= 1 + \sum_{\alpha\in\mathcal{A}} q^{-(1+s_\alpha-\rho_\alpha)}(1 - \frac{1}{q})(1 - q^{-(1+s_\alpha-\rho_\alpha)})^{-1} + O(q^{-3/2}) \\ &= 1 + \sum_{\alpha\in\mathcal{A}} \frac{q^{-(1+s_\alpha-\rho_\alpha)}}{1 - q^{-(1+s_\alpha-\rho_\alpha)}} + O(q^{-3/2}) \\ &= \prod_{\alpha\in\mathcal{A}} (1 - q^{-(1+s_\alpha-\rho_\alpha)})^{-1}(1 + O(q^{-1-2\varepsilon})) + O(q^{-3/2}). \end{aligned}$$

Finally, we have the desired estimate. □

For X as above and $\mathbf{s} = (s_\alpha) \in \mathrm{Pic}(X)_\mathbf{C}$, the (multi-variable) Artin L-function is given by

$$L(\mathrm{Pic}(X); \mathbf{s}) = \prod_{\alpha\in\mathcal{A}} \zeta_F(s_\alpha) = \prod_{v \text{ finite}} \prod_{\alpha\in\mathcal{A}} (1 - q^{-s_\alpha})^{-1}.$$

From the properties of ζ_F we conclude that $L(\mathrm{Pic}(X); \mathbf{s})$ admits a meromorphic continuation and that it has polynomial growth in vertical strips. For $\mathbf{s} = (s, \ldots, s)$ we get the usual Artin L-function of $\mathrm{Pic}(X)$—here a power of the Dedekind zeta function—which is used in the regularization of the Tamagawa measure, see [**14**].

COROLLARY 7.6. — *For all $\varepsilon > 0$ there exists a holomorphic bounded function φ on $\Omega_{-1/2+\varepsilon}$ such that for any $\mathbf{s} \in \Omega_0$ one has*

$$\hat{H}(\mathbf{s};\psi_\mathbf{0}) = \varphi(\mathbf{s}) L(\mathrm{Pic}(X); \mathbf{s} - \rho + 1).$$

Moreover, the Tamagawa number is given by

$$\tau(\mathcal{K}_X) = \varphi(\rho)/L^*(\mathrm{Pic}(X);\mathbf{1}).$$

Proof. — The previous proposition shows that in $\Omega_{-1/2+\varepsilon}$, the function φ is defined by an absolutely convergent Euler product $\prod_v \varphi_v$, hence the holomorphy and boundedness. Moreover, $\varphi_v(\rho) = \hat{H}_v(\rho;\psi_{\mathbf{0}})L_v(\mathrm{Pic}(X);1)^{-1}$. Hence,

$$\varphi(\rho) = \prod_v \varphi_v(\rho) = \prod_v \hat{H}_v(\rho;\psi_{\mathbf{0}})L_v(\mathrm{Pic}(X);1)^{-1}.$$

It remains to identify $\varphi(\rho)/L^*(1,\mathrm{Pic}(\bar{X}))$ with the Tamagawa number $\tau(X)$ corresponding to a metrized anticanonical line bundle, as defined by Peyre in **[14]**. This is immediate from the definitions. Note however that Peyre apparently doesn't use the selfdual measure for his definition in *loc. cit.*, but inserts the appropriate correcting factor $\Delta_F^{-\dim X/2}$. □

8. Other characters

In this section we consider only $v \notin S(\mathbf{a})$. Our aim here is to compute as explicitely as possible the Fourier transforms of local heights with character $\psi_{\mathbf{a}}$. In general, this will be only possible up to some error term.

The calculations in the preceding Section allow us to strengthen Proposition 6.1. For $\mathbf{a} \in \mathbf{G}_a^n(F)$, we denote by $\|\mathbf{a}\|_\infty$ the norm of $\mathbf{a}$ in the diagonal embedding $F \hookrightarrow F \otimes_{\mathbf{Q}} \mathbf{R} = \prod_{v|\infty} F_v$.

PROPOSITION 8.1. — *For any $\varepsilon > 0$, there exist an integer $\kappa \geqslant 0$ and for any $N \geqslant 0$, a constant $C(\varepsilon, N)$ such that for any $\mathbf{s} \in \Omega_{-1/2+\varepsilon}$ and all $\mathbf{a} \in \mathfrak{d}_X$ we have*

$$\prod_{v \in S(\mathbf{a})} \left|\hat{H}_v(\mathbf{s};\psi_{\mathbf{a}})\right| \leqslant C(\varepsilon,N)(1+\|\mathbf{s}\|)^N(1+\|\mathbf{a}\|_\infty)^{\kappa-N}.$$

Proof. — For $v \in S$ the local integrals converge absolutely in the domain under consideration and are bounded as in Proposition 6.1. For $v \notin S$, we have shown that there exists a constant c such that for all $\mathbf{a} \in \mathfrak{d}_X$ and all $\mathbf{s} \in \Omega_{-1/2+\varepsilon}$ one has the estimate

$$\left|\hat{H}_v(\mathbf{s};\psi_{\mathbf{a}}) - 1\right| \leqslant \frac{c}{q_v^{\varepsilon}}.$$

This implies that $\left|\hat{H}_v(\mathbf{s};\psi_{\mathbf{a}})\right|$ is bounded independently of $\mathbf{a}$, $v \in S(\mathbf{a}) \setminus S$ and $\mathbf{s} \in \Omega_{-1/2+\varepsilon}$ and $\mathbf{a}$. For $a \in \mathfrak{o}_F$, there is a trivial estimate

$$\sum_{\mathfrak{p} \supset (a)} 1 \ll \log(1+\mathcal{N}(a)),$$

which implies that for some constant κ,

$$\prod_{v \in S(\mathbf{a})\setminus S} \left|\hat{H}_v(\mathbf{s};\psi_{\mathbf{a}})\right| \ll (1+\mathcal{N}(a))^{\kappa} \ll \prod_{v|\infty}(1+\|\mathbf{a}\|_v)^{\kappa}.$$

Using Proposition 6.1, we have for all $N > 0$,

$$\prod_{v\in S(\mathbf{a})} \left|\hat{H}_v(\mathbf{s};\psi_{\mathbf{a}})\right| \ll C(S,\varepsilon,N)(1+\|s\|)^{N[F:\mathbf{Q}]}\prod_{v|\infty}(1+\|\mathbf{a}\|_v)^{\kappa-N}$$
$$\leqslant C'(S,\varepsilon,N)(1+\|s\|)^{N[F:\mathbf{Q}]}(1+\|\mathbf{a}\|_\infty)^{(\kappa-N)[F:\mathbf{Q}]}.$$

Replacing N by $N[F:\mathbf{Q}]$ concludes the proof of the proposition. □

For the explicit calculation at good places, let us recall some notations: $\mathfrak{m}_v \subset \mathfrak{o}_v$ is the maximal ideal, $\pi = \pi_v$ a uniformizing element, k_v the residue field, $q = q_v = \#k_v$, $\psi = \psi_v = \psi_{\mathbf{a},v}$, $\mathbf{x} = \mathbf{x}_v$, $\mathbf{a} = \mathbf{a}_v$, $d\mathbf{x} = d\mathbf{x}_v$. As in Section 1, let $f = f_{\mathbf{a}}$ be a linear form on $\mathbf{G}_a^n$ and write $\operatorname{div}(f) = E - \sum_\alpha d_\alpha D_\alpha$. We also let $A_0 = \{\alpha;\ d_\alpha = 0\}$ and $A_1 = \{\alpha;\ d_\alpha = 1\}$.

PROPOSITION 8.2. — *There exists a constant $C(\varepsilon)$ independent of $\mathbf{a} \in \mathfrak{d}_X$ such that for any $v \notin S(\mathbf{a})$,*

$$\left|\hat{H}_v(\mathbf{s};\psi_{\mathbf{a}}) \prod_{\alpha\in A_0(\mathbf{a})} (1-q^{-(1+s_\alpha-\rho_\alpha)}) - 1\right| \leqslant C(\varepsilon)q^{-1-\varepsilon}.$$

Similarly to the proof of Proposition 7.5 in the preceding Section, this proposition is proved by computing the integral on residue classes.

Let $\tilde{x} \in X(k_v)$ and $A = \{\alpha;\ \tilde{x} \in D_\alpha\}$. We now consider three cases:

Case 1. $A = \varnothing$. — The sum of all these contribution is equal to the integral over $\mathbf{G}_a^n(\mathfrak{o}_v)$:

$$\int_{\mathbf{G}_a(\mathfrak{o}_v)} H_v(\mathbf{x};\mathbf{s})^{-1}\psi_{\mathbf{a}}(\mathbf{x})\,d\mathbf{x} = \int_{\mathfrak{o}_v^n} \psi(\langle \mathbf{a},\mathbf{x}\rangle)\,d\mathbf{x} = 1.$$

Case 2. $A = \{\alpha\}$ *and* $\tilde{x} \notin E$. — We can introduce analytic coordinates x_α and y_β around $\tilde{x}$ such that locally $f(x) = ux_\alpha^{-d_\alpha}$ with $u \in \mathfrak{o}_v^*$. Then, we compute the integral of $H_v(\mathbf{x};\mathbf{s})^{-1}\psi_{\mathbf{a}}(\mathbf{x})$ as

$$\int_{\mathrm{red}^{-1}(\tilde{x})} = \int_{\mathfrak{m}_v\times\mathfrak{m}_v^{n-1}} q^{-(s_\alpha-\rho_\alpha)v(x_\alpha)}\psi(ux_\alpha^{-d_\alpha})\,dx_\alpha d\mathbf{y}$$
$$= \frac{1}{q^{n-1}}\sum_{n_\alpha\geqslant 1} q^{-(1+s_\alpha-\rho_\alpha)n_\alpha}\int_{\mathfrak{o}_v^*}\psi(u\pi^{-n_\alpha d_\alpha}u_\alpha^{-d_\alpha})\,du_\alpha.$$

LEMMA 8.3. — *For all integers $d \geqslant 0$ and $n \geqslant 1$ and all $u \in \mathfrak{o}_v^*$,*

$$\int_{\mathfrak{o}_v^*}\psi(u\pi^{-nd}t^d)\,dt = \begin{cases} 1-1/q & \text{if } d = 0;\\ -1/q & \text{if } n = d = 1;\\ 0 & \text{else.}\end{cases}$$

Proof. — If $d=1$, the computation runs as follows

$$\int_{\mathfrak{o}_v^*} \psi(u\pi^{-n}t)\,dt = \int_{\mathfrak{o}_v} \psi(u\pi^{-n}t)\,dt - \frac{1}{q}\int_{\mathfrak{o}_v} \psi(u\pi^{-n+1}t)\,dt = \begin{cases} 0 & \text{if } n \geqslant 2; \\ -1/q & \text{if } n=1. \end{cases}$$

For $d \geqslant 2$, let $r = v_p(d)$. Since we assumed F_v to be unramified over $\mathbf{Q}_p$, $r = v_\pi(d)$. We will integrate over disks $D(\xi,\pi^e) \subset \mathfrak{o}_v^*$ for $e \geqslant 1$ suitably chosen. Indeed, if $v \in \mathfrak{o}_v$ and $t = \xi + \pi^e v$,

$$t^d = \xi^d + d\pi^e\xi^{d-1}v \pmod{\pi^{2e}}$$

hence, if a is chosen such that

$$e - nd + r < 0 \quad \text{and} \quad 2e - nd \geqslant 0,$$

$$\int_{D(\xi,\pi^e)} \psi(u\pi^{-nd}t^d)\,dt = q_v^{-e}\psi(u\pi^{-nd}\xi^d)\int_{\mathfrak{o}_v} \psi(d\pi^{e-nd}u\xi^{d-1}v)\,dv = 0.$$

We can find such an e if and only if $2(nd-r-1) \geqslant nd$, *i.e.* $nd \geqslant 2r+2$. If $r=0$, this is true since $d \geqslant 2$. If $r \geqslant 1$, one has $nd \geqslant p^r \geqslant 2r+2$ since we assumed $p \geqslant 5$. □

This lemma implies the following trichotomy:

$$\begin{aligned} \int_{\mathrm{red}^{-1}(\tilde{x})} H(\mathbf{x};\mathbf{s})^{-1}\psi_{\mathbf{a}}(\mathbf{x})\,d\mathbf{x} &= \frac{q-1}{q^n}\frac{1}{q^{1+s_\alpha-\rho_\alpha}-1} && \text{if } d_\alpha = 0; \\ &= -\frac{1}{q^n}q^{-(1+s_\alpha-\rho_\alpha)} && \text{if } d_\alpha = 1; \\ &= 0 && \text{if } d_\alpha \geqslant 2. \end{aligned}$$

Case 3. $\#A \geqslant 2$ or $\#A = 1$ and $\tilde{x} \in E$. — Under some transversality assumption, we could compute explicitely the integral as before. We shall however content ourselves with the estimate obtained by replacing ψ by 1 in the integral.

The total contribution of these points will therefore be smaller than

$$q^{-\dim X}\sum_{\#A\geqslant 2} \#D_A^\circ(k_v)\prod_{\alpha\in A}\frac{q-1}{q^{1+s_\alpha-\rho_\alpha}-1} + q^{-\dim X}\sum_{A=\{\alpha\}}\#(D_\alpha\cap E)(k_v)\frac{q-1}{q^{1+s_\alpha-\rho_\alpha}-1}. \tag{8.4}$$

Finally, the Fourier transform is estimated as follows:

$$\hat{H}_v(\mathbf{s};\mathbf{a}) = 1 + q^{-n}\sum_{\alpha\in A_0(\mathbf{a})}\#D_\alpha^\circ(k_v)\frac{q-1}{q^{1+s_\alpha-\rho_\alpha}-1} - q^{-n}\sum_{\alpha\in A_1(\mathbf{a})}\#D_\alpha^\circ(k_v)\frac{1}{q^{1+s_\alpha-\rho_\alpha}} + ET$$

with an "error term" *ET* smaller than the expression in (8.4). It is now a simple matter to rewrite this estimate as in the statement of 8.2. □

We deduce from these estimates that $\hat{H}(\mathbf{s};\mathbf{a})$ has a meromorphic continuation:

COROLLARY 8.5. — *For any* $\varepsilon > 0$ *and* $\mathbf{a} \in \mathfrak{d}_X \setminus \{0\}$ *there exists a holomorphic bounded function* $\varphi(\cdot;\mathbf{a})$ *on* $\Omega_{-1/2+\varepsilon}$ *such that for any* $\mathbf{s} \in \Omega_0$

$$\hat{H}(\mathbf{s};\mathbf{a}) = \prod_{v} \hat{H}_v(\mathbf{s};\psi_{\mathbf{a}}) = \varphi(\mathbf{s};\psi_{\mathbf{a}}) \prod_{\alpha\in A_0(\mathbf{a})} \zeta_F(1+s_\alpha-\rho_\alpha).$$

Moreover, for any $N > 0$ *there exist constants* $N' > 0$ *and* $C(\varepsilon, N)$ *such that for any* $\mathbf{s} \in \Omega_{-1/2+\varepsilon}$*, one has the estimate*

$$|\varphi(\mathbf{s};\mathbf{a})| \leqslant C(\varepsilon,N)(1+\|\mathrm{Im}(\mathbf{s})\|)^{N'}(1+\|\mathbf{a}\|_\infty)^{-N}.$$

9. The nonsplit case

In this section; we extend the previous calculations of the Fourier transform to the nonsplit case, *i.e.* when the geometric irreducible components of $X \setminus \mathbf{G}_a^n$ are no longer assumed to be defined over F.

LEMMA 9.1. — *Let* $x \in X(F)$ *and* $A = \{\alpha \in \mathcal{A}\,;\, x \in D_\alpha\}$. *Fix for any orbit* $\bar{\alpha} \in A/\Gamma_F$ *some element* α *and let* F_α *be the field of definition of* D_α. *Then there exist an open neighborhood* U *of* x*, étale coordinates around* x *over* $\bar{F}$, x_α *(*$\alpha \in A$*) and* y_β *such that* x_α *is a local equation of* D_α *and such that the induced morphism* $U_{\bar{F}} \to \mathbf{A}^n_{\bar{F}}$ *descends to an étale morphism*

$$U \to \prod_{\bar{\alpha}\in A/\Gamma_F} \mathrm{Res}_{F_\alpha/F}\mathbf{A}^1 \times \mathbf{A}^{n-a} \qquad (a = \#A).$$

An analogous result holds over the local fields F_v *and also on* $\mathfrak{o}_v$*,* v *being any finite place of* F *such that* $v \notin S$.

Proof. — For any orbit α, we have chosen some element (also denoted α). Fix first a local equation x_α for the corresponding D_α which is already a local equation over F_α. Then if $\alpha' = g\alpha$ (for some $g \in \Gamma_F$) is another element of the orbit of α set $x_{\alpha'} = g \cdot x_\alpha$. This is well defined since we assumed that the equation x_α is invariant under Γ_{F_α}.

Finally, add local F-rational étale coordinates corresponding to a basis of the subspace in $\Omega^1_{X,x}$ which is complementary to the span of dx_α for $\alpha \in A$. ☐

Let v be a place of F. The above lemma allows us to identify a neighborhood of x in $X(F_v)$ (for the analytic topology) with a neighborhood of 0 in the product $\prod_{\alpha\in A/\Gamma_v} F_{v,\alpha}\times F_v^{n-a}$, the local heights $\prod_{\alpha\in\alpha} H_{\alpha,v}(\xi)$ being replaced by

$$\mathcal{N}_{F_{v,\alpha}/F_v}(\xi) \times h_{\alpha,v}(\xi)$$

where $h_{\alpha,v}$ is a smooth function.

Similarly, for good places v we identify $\mathrm{red}^{-1}(\tilde{x})$ with $\prod \mathfrak{m}_{v,\alpha} \times \mathfrak{m}^{n-a}$ and the functions $h_{\alpha,v}$ are equal to 1.

The assertions of Section 6 still hold in this more general case and the proofs require only minor modifications. However the calculations of Sections 7 and 8 have to be redone.

THEOREM 9.2 (cf. Thm. 7.1). — *One has*

$$\hat{H}_v(\mathbf{s};\psi_0) = q_v^{-\dim X} \sum_{A\subset \mathcal{A}/\Gamma_v} \#D_A^\circ(k_v) \prod_{\alpha\in A/\Gamma_v} \frac{q_v^{f_\alpha}-1}{q_v^{f_\alpha(1+s_\alpha-\rho_\alpha)}-1}$$

where f_α *is degree of* $F_{v,\alpha}$ *over* F_v.

COROLLARY 9.3 (cf. Prop. 7.5). — *One has*

$$\hat{H}_v(\mathbf{s};\psi_0) = \prod_{\alpha\in\mathcal{A}/\Gamma_v} \big(1-q_v^{-f_\alpha(1+s_\alpha-\rho_\alpha)}\big)^{-1}\big(1+O(q_v^{-1-\varepsilon})\big).$$

In the general case, we introduce the multi-variable Artin L-function of $\mathrm{Pic}(\bar{X})$ as

$$L(\mathrm{Pic}(\bar{X});\mathbf{s}) = \prod_{\alpha\in\mathcal{A}/\Gamma_F} \zeta_{F_\alpha}(s_\alpha).$$

Its restriction to the line $(s,\dots,s)$ is the usual Artin L-function of $\mathrm{Pic}(\bar{X})$. It has a pole of order $\#(\mathcal{A}/\Gamma_F) = \mathrm{rk}(\mathrm{Pic}\,X)$ at $s=1$.

COROLLARY 9.4 (cf. Cor. 7.6). — *For all* $\varepsilon>0$*, there exists a holomorphic bounded function* φ *on* $\Omega_{-1/2+\varepsilon}$ *such that for any* $\mathbf{s}\in\Omega_0$ *one has*

$$\hat{H}(\mathbf{s};\psi_0) = \varphi(\mathbf{s})L(\mathrm{Pic}(\bar{X});\mathbf{s}-\rho+1)$$

and the Tamagawa measure of $X(\mathbf{A}_F)$ *is equal to*

$$\tau(\mathcal{K}_X) = \varphi(\rho)/L^*(\mathrm{Pic}(\bar{X});\mathbf{1}).$$

At nontrivial characters, the calculations are modified analogously and we have

$$\hat{H}(\mathbf{s};\mathbf{a}) = \varphi(\mathbf{s};\mathbf{a}) \prod_{\alpha\in A_0(\mathbf{a})/\Gamma_F} \zeta_{F_\alpha}(s_\alpha-\rho_\alpha+1)$$

for some holomorphic function $\varphi(\cdot;\mathbf{a})$ as in Corollary 8.5.

10. Anticanonical asymptotics

We combine the computations of the previous sections. It will be convenient to set $\varphi(s;\mathbf{0})=\varphi(s)$ and $A_0(\mathbf{0})=\mathcal{A}$. Then for any $\mathbf{s}\in\Omega_0$ one has

$$Z(\mathbf{s}) = \sum_{\mathbf{a}\in\partial_X} \hat{H}(\mathbf{s};\psi_{\mathbf{a}}) = \sum_{\mathbf{a}\in\partial_X} \varphi(\mathbf{s};\mathbf{a}) \prod_{\alpha\in A_0(\mathbf{a})/\Gamma_F} \zeta_{F_\alpha}(1+s_\alpha-\rho_\alpha).$$

Hence,

$$Z(\mathbf{s}) \prod_{\alpha\in\mathcal{A}/\Gamma_F}(s_\alpha-\rho_\alpha) = \sum_{\mathbf{a}\in\partial_X} \varphi(\mathbf{s};\mathbf{a}) \prod_{\alpha\in A_0(\mathbf{a})/\Gamma_F} \zeta_{F_\alpha}(1+s_\alpha-\rho_\alpha)(s_\alpha-\rho_\alpha) \prod_{\substack{\beta\in\mathcal{A}/\Gamma_F\\ \beta\notin A_0(\mathbf{a})/\Gamma_F}} (s_\beta-\rho_\beta).$$

This is a sum of holomorphic functions on $\Omega_{-1/2}$ and the estimate in 8.5 (taking $N > n[F:\mathbf{Q}]$ there) implies that the series converges uniformly in that domain. Therefore, we have shown that the function

$$\mathbf{s} \mapsto Z(\mathbf{s}) \prod_{\alpha \in \mathcal{A}/\Gamma_F} (s_\alpha - \rho_\alpha)$$

has a holomorphic continuation to $\Omega_{-1/2}$. Moreover, this continuation has a polynomial growth in vertical strips. The restriction of $Z(\mathbf{s})$ to a line $(\lambda_\alpha)\mathbf{C}$ (with $\lambda_\alpha > 0$ for all α) gives

$$Z(\lambda s) = h_\lambda(s) \times \prod_{\alpha \in \mathcal{A}/\Gamma_F} (\lambda_\alpha s - \rho_\alpha)$$

where h_λ is a holomorphic function for $\operatorname{Re}(s) > (\rho_\alpha - \frac{1}{2})/\lambda_\alpha$, providing a meromorphic continuation to this domain. The pole of highest order is at

$$a = a_\lambda = \max_\alpha(\rho_\alpha/\lambda_\alpha).$$

This number is less or equal than the number of $\alpha \in \mathcal{A}/\Gamma_F$ such that $\rho_\alpha/\lambda_\alpha = \sigma$. For $\lambda = \rho$, $a_\rho = 1$, $b_\rho = \operatorname{rk}\operatorname{Pic}(X) = r$ and we have

$$\lim_{s \to 1} Z(\rho s)(s-1)^r = \frac{\tau(\mathcal{K}_X)}{\prod_{\alpha \in \mathcal{A}/\Gamma_F} \rho_\alpha},$$

hence the order of the pole is exactly r. Using a standard Tauberian theorem, we deduce an asymptotic development for the number of rational points in $\mathbf{G}_a^n$ of bounded anticanonical height.

THEOREM 10.1. — *There exists a real number $\delta > 0$ and a polynomial $P \in \mathbf{R}[X]$ of degree $r-1$ such that the number of F-rational points on $\mathbf{G}_a^n \subset X$ of anticanonical height $\leqslant B$ satisfies*

$$N(\mathcal{K}_X^{-1}, B) = BP(\log B) + O(B^{1-\delta}).$$

The leading coefficient of P is given by

$$\frac{1}{(r-1)!}\tau(\mathcal{K}_X) \prod_{\alpha \in \mathcal{A}/\Gamma_F} \rho_\alpha^{-1}.$$

11. The general case

In this section we establish our main theorem. Let $\lambda \in \Lambda_{\text{eff}}(X)$ be an effective class and $\mathcal{L}_\lambda$ the corresponding line bundle on X equipped with a smooth adelic metric as in Section 3. The function $s \mapsto Z(\lambda s)$ is holomorphic for $\operatorname{Re}(s) > a_\lambda = \max(\rho_\alpha/\lambda_\alpha)$ and admits a meromorphic extension to the left of a_λ. Recall from the previous section that it has a pole of order $\leqslant b_\lambda$ at a_λ. Here, b_λ is the cardinality of

$$\mathcal{B}_\lambda = \{\alpha \in \mathcal{A}/\Gamma_F\,;\ \rho_\alpha = a_\lambda \lambda_\alpha\}.$$

The integer b_λ is the codimension of the face of the effective cone $\Lambda_{\text{eff}}(X)$ containing the class $a_\lambda\lambda - \rho$.

Denote by $Z_\lambda(s)$ the series

$$Z_\lambda(s) = \sum_{\substack{\mathbf{a} \text{ such that} \\ \mathcal{A}_0(\mathbf{a}) \supset \mathcal{B}_\lambda}} \hat{H}(\lambda s; \mathbf{a}).$$

It follows from the calculations above that

$$\lim_{s \to a_\lambda} Z(\lambda s)(s - a_\lambda)^{b_\lambda} = \lim_{s \to a_\lambda} Z_\lambda(s)(s - a_\lambda)^{b_\lambda} \tag{11.1}$$

since all other terms converge (uniformly) to zero when $s \to a_\lambda$.

The set V_λ of $\mathbf{a} \in \mathbf{G}_a^n(F)$ such that $\mathcal{A}_0(\mathbf{a})$ contains $\mathcal{B}_\lambda$ is a sub-vector space. Let $\mathbf{G}_\lambda \subset \mathbf{G}_a^n$ be the F-subvector space defined by the corresponding linear forms $\langle \mathbf{a}, \cdot \rangle$. Then, the autoduality on $\mathbf{G}_a^n(\mathbf{A}_F)$ identifies V_λ with the dual of $\mathbf{G}_\lambda(\mathbf{A}_F)\mathbf{G}_a^n(F)$. We apply the Poisson summation formula and obtain

$$Z_\lambda(s) = \int_{\mathbf{G}_\lambda(\mathbf{A}_F)\mathbf{G}_a^n(F)} H(\mathbf{x}; \lambda s)^{-1}\, d\mathbf{x} = \sum_{\mathbf{x} \in (\mathbf{G}_a^n/\mathbf{G}_\lambda)(F)} \int_{\mathbf{G}_\lambda(\mathbf{A}_F)} H(\mathbf{x} + \mathbf{y}; \lambda s)^{-1}\, d\mathbf{y}.$$

(The justification of Poisson summation formula is as in Section 5)

The last sum consists of integrals of the type studied in Section 7. However, the Zariski closure Y of $\mathbf{G}_\lambda$ in X need not be smooth.

LEMMA 11.2. — *Without loss of generality we may assume that $Y \subset X$ is a smooth compactification whose boundary is a divisor with strict normal crossings given by the $D_\alpha \cap Y$.*

Proof. — Equivariant resolution of singularities (in char. 0) implies that there exists a proper modification $\pi : \tilde{X} \to X$ which is a composition of a equivariant blow-ups along smooth centers not meeting $\mathbf{G}_a^n$ such that the Zariski closure $\tilde{Y} \subset \tilde{X}$ of $\mathbf{G}_\lambda$ is a smooth equivariant compactification whose boundary is a divisor with strict normal crossings obtained by intersecting the components of the boundary of $\tilde{X}$ with $\tilde{Y}$.

We replace $\mathcal{L}_\lambda$ by $\pi^*\mathcal{L}_\lambda = \mathcal{L}_{\tilde{\lambda}}$. Unless π is an isomorphism, π is not smooth and $\omega_{\tilde{X}} \otimes \pi^*\omega_X^{-1}$ has a nonzero coefficient at each exceptional divisor. This implies that $\tilde{\lambda} - a_\lambda \tilde{\rho}$ lies on the boundary of $\Lambda_{\text{eff}}(\tilde{X})$; it follows that $a_{\tilde{\lambda}} = a_\lambda$ and $b_{\tilde{\lambda}} = b_\lambda$. □

From now on we assume that the conclusion is satisfied for $Y \subset X$.

LEMMA 11.3. — *There exist integers $\tilde{\rho}_\alpha \leqslant \rho_\alpha$ for $\alpha \in \mathcal{A}$ such that*

$$\omega_Y^{-1} = \sum \tilde{\rho}_\alpha (D_\alpha \cap Y)$$

and $\tilde{\rho}_\alpha = \rho_\alpha$ if $\alpha \in B(\lambda)$.

Proof. — We prove this by induction on the codimension of Y in X. If $\mathbf{G}_\lambda = \operatorname{div}(f_{\mathbf{a}})$, the adjunction formula shows that

$$\omega_Y^{-1} = \omega_X^{-1}(-Y)|_Y = \sum (\rho_\alpha - d_\alpha(\mathbf{a})) D_\alpha \cap Y.$$

For $\mathbf{a} \in V_\lambda$ the nonnegative integer $d_\alpha(\mathbf{a})$ is equal to 0 if $\alpha \in \mathcal{B}_\lambda$. The lemma is proved. □

PROPOSITION 11.4. — *For each* $\mathbf{x} \in (\mathbf{G}_a^n/\mathbf{G}_\lambda)(F)$ *there exists a strictly positive integer* $\tau_\lambda(\mathbf{x}) > 0$ *such that*

$$\lim_{s\to a_\lambda^+} (s-a_\lambda)^{b_\lambda} \int_{\mathbf{G}_\lambda(\mathbf{A}_F)} H(\mathbf{x}+\mathbf{y};\lambda s)^{-1}\,d\mathbf{y} = \tau_\lambda(\mathbf{x}).$$

Proof. — It is sufficient to prove the proposition when $\mathbf{x}=0$ as the integrals for different values of $\mathbf{x}$ are comparable. Let $d\tau$ be the (nonrenormalized) Tamagawa measure on $Y(\mathbf{A}_F)$: by definition, on $\mathbf{G}_\lambda(\mathbf{A}_F)$, $d\tau = H(\mathbf{y};\tilde\rho)^{-1}d\mathbf{y}$. We have to estimate

$$\lim_{s\to a_\lambda} (s-a_\lambda)^{b_\lambda} = \int_{Y(\mathbf{A}_F)} H(\mathbf{y};\lambda s-\tilde\rho)^{-1}\,d\tau.$$

For $\mathrm{Re}(s) > a_\lambda$ the adelic integral splits as a product of local integrals over all places of F. By the results of Section 7, for almost all finite places v one has

$$\int_{Y(F_v)} H_v(\mathbf{y};\lambda s-\tilde\rho)^{-1}\,d\tau = q_v^{-\dim Y} \sum_{A\subset\mathcal{A}/\Gamma_F} \#(D_A^\circ\cap Y)(k_v) \prod_{\alpha\in\mathcal{A}/\Gamma_F} \frac{q_v^{f_\alpha}-1}{q_v^{f_\alpha(1+s\lambda_\alpha-\tilde\rho_\alpha)}-1}$$

and

$$\left(\int_{Y(F_v)} H_v(\mathbf{y};s\lambda-\tilde\rho)^{-1}\,d\tau\right) \prod_{\alpha\in\mathcal{A}/\Gamma_v} \left(1-q_v^{-f_\alpha(1+s\lambda_\alpha-\tilde\rho_\alpha)}\right) = 1 + O(q_v^{-1-\varepsilon}).$$

Let φ_λ be the absolutely convergent product of such factors over all finite places v. It defines a bounded and nonvanishing holomorphic function on the half-plane $\mathrm{Re}(s) > a$ for some $a < a_\lambda$. Moreover, for $\mathrm{Re}(s) > a_\lambda$,

$$\int_{Y(\mathbf{A}_F)} H(\mathbf{y};s\lambda-\tilde\rho)^{-1}\,d\tau = \varphi_\lambda(s) \prod_{\alpha\in\mathcal{A}/\Gamma_F} \zeta_{F_\alpha}(1+s\lambda_\alpha-\tilde\rho_\alpha).$$

Then,

$$\lim_{s\to a_\lambda} (s-a_\lambda)^{b_\lambda} = \varphi_\lambda(a_\lambda)\Big(\prod_{\alpha\in\mathcal{B}_\lambda/\Gamma_F} \lambda_\alpha^{-1}\mathrm{Res}_{s=1}\zeta_{F_\alpha}(1)\Big) \prod_{\alpha\notin\mathcal{B}_\lambda/\Gamma_F} \zeta_{F_\alpha}(1+a_\lambda\lambda_\alpha-\tilde\rho_\alpha) > 0.$$

Note that for $\alpha\notin\mathcal{B}_\lambda/\Gamma_F$,

$$a_\lambda\lambda_\alpha-\tilde\rho_\alpha \geqslant a_\lambda\lambda_\alpha-\rho_\alpha > 0.$$

The proposition is proved. □

We now can conclude as in the case of the anticanonical line bundle:

THEOREM 11.5. — *Let* $\lambda\in\Lambda_{\mathrm{eff}}(X)$ *be an effective class and* $\mathcal{L}_\lambda$ *the corresponding line bundle equipped with a smooth adelic metric as in Section 3. There exist a polynomial* $P_\lambda\in\mathbf{R}[X]$ *of degree* $b_\lambda-1$ *and a real number* $\varepsilon>0$ *such that the number* $N(\mathcal{L}_\lambda,B)$ *of* F*-rational points on* $\mathbf{G}_a^n$ *of* $\mathcal{L}_\lambda$*-height* $\leqslant B$ *satisfies*

$$N(\mathcal{L}_\lambda,B) = B^{a_\lambda}P_\lambda(\log B) + O(B^{a_\lambda-\varepsilon}).$$

The leading term of P_λ is equal to

$$\frac{1}{(b_\lambda - 1)!}\left(\sum_{\mathbf{x}\in(\mathbf{G}_a^n/\mathbf{G}_\lambda)(F)} \tau_\lambda(\mathbf{x})\right)\left(\prod_{\alpha\in\mathcal{B}_\lambda/\Gamma_F} \lambda_\alpha^{-1}\right).$$

It is compatible with the description by Batyrev and Tschinkel in [**3**].

References

[1] V. V. BATYREV & YU. I. MANIN – "Sur le nombre de points rationnels de hauteur bornée des variétés algébriques", *Math. Ann.* **286** (1990), p. 27–43.

[2] V. V. BATYREV & YU. TSCHINKEL – "Manin's conjecture for toric varieties", *J. Algebraic Geometry* **7** (1998), no. 1, p. 15–53.

[3] ______, "Tamagawa numbers of polarized algebraic varieties", in *Nombre et répartition des points de hauteur bornée* [**15**], p. 299–340.

[4] B. J. BIRCH – "Forms in many variables", *Proc. London Math. Soc.* **265A** (1962), p. 245–263.

[5] R. DE LA BRETÈCHE – "Nombre de points de hauteur bornée sur les surfaces de Del Pezzo de degré 5", Tech. Report 6, Université Paris Sud (Orsay), 2000.

[6] A. CHAMBERT-LOIR & YU. TSCHINKEL – "Points of bounded height on equivariant compactifications of vector groups, I", E-print, math.NT/9902042, 1999, To appear in *Compositio Math.*

[7] ______, "Points of bounded height on equivariant compactifications of vector groups, II", E-print, math.NT/9904008, 1999, To appear in *J. Number Theory*.

[8] J. DENEF – "On the degree of Igusa's local zeta function", *Amer. J. Math.* **109** (1987), p. 991–1008.

[9] J. FRANKE, YU. I. MANIN & YU. TSCHINKEL – "Rational points of bounded height on Fano varieties", *Invent. Math.* **95** (1989), no. 2, p. 421–435.

[10] B. HASSETT & YU. TSCHINKEL – "Geometry of equivariant compactifications of $\mathbf{G}_a^n$", *Internat. Math. Res. Notices* **22** (1999), p. 1211–1230.

[11] S. LANG & A. WEIL – "Number of points of varieties in finite fields", *Amer. J. Math.* **76** (1954), p. 819–827.

[12] YU. I. MANIN & A. PANCHISHKIN – *Number theory I. Introduction to number theory*, Springer, Berlin, 1995.

[13] D. MUMFORD, J. FOGARTY & F. KIRWAN – *Geometric invariant theory*, Ergeb., no. 34, Springer Verlag, 1994.

[14] E. PEYRE – "Hauteurs et mesures de Tamagawa sur les variétés de Fano", *Duke Math. J.* **79** (1995), p. 101–218.

[15] ______ (ed.) – *Nombre et répartition des points de hauteur bornée*, Astérisque, no. 251, 1998.

[16] S. SCHANUEL – "Heights in number fields", *Bull. Soc. Math. France* **107** (1979), p. 433–449.

[17] M. STRAUCH & YU. TSCHINKEL – "Height zeta functions of toric bundles over flag varieties", *Selecta Math. (N.S.)* **5** (1999), no. 3, p. 325–396.

[18] J. T. TATE – "Fourier analysis in number fields, and Hecke's zeta-functions", *Algebraic Number Theory* (Proc. Instructional Conf., Brighton, 1965), Thompson, Washington, D.C., 1967, p. 305–347.

[19] A. WEIL – *Adeles and algebraic groups*, Progr. Math., no. 23, Birkhäuser, 1982.