INDEX CLASSES OF HILBERT MODULES WITH BOUNDARY

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Abstract. Cet article présente une description systématique des extensions autoadjointes graduées d'un module de Hilbert à bord sur une C*-algèbre B, paramétrée par les projecteurs d'une grasmannienne généralisée qui remplace les projecteurs spectraux de l'opérateur du bord. Le théorème principal énonce que les classes d'indices obtenues dans $K_*(B)$ sont additives. Il s'en déduit une réponse positive à une conjecture de J. Lott sur la théorie de l'indice à coefficients dans $\mathbf{R}/\mathbf{Z}$ et des résultats inédits sur l'invariance des hautes signatures par SK-équivalence.

1. Introduction

Let V be a smooth compact riemannian manifold with boundary W, $\dim V = 2m$, and $\mathcal{U} \simeq W \times [0,1]$ a tubular neighboorhood. Suppose that V and W are equipped with compatible riemannian and spin structures (*i.e.* the restriction to $\mathcal{U}$ of the riemannian structure of V is $dt^2 \oplus h$). Then on $\mathcal{U}$, the spinor bundle S_V reduces to $S_W \otimes \mathbf{C}^2$, and one has isomorphism :

$$L^2(\mathcal{U}, S_V) \simeq L^2(W, S_W) \otimes L^2([0,1]) \otimes \mathbf{C}^2 \tag{1.1}$$

Let D and A the Dirac operators on V and W. The operator $D \subset D^*$ is symmetric but not selfadjoint, and for $\xi, \eta \in C^\infty(\mathcal{U}, S_W)$ one has :

$$D \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial}{\partial u} + A \\ \frac{\partial}{\partial u} + A & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{1.2}$$

In [1], Atiyah-Patodi-Singer have defined global boundary conditions for D and obtain a selfadjoint extension $\hat{D}$ of D as follows : let p_0 is the orthogonal projection of $L^2(W, S_W)$ onto the spectral subspace of A for the positive real numbers. Then the boundary condition $(p_0 \oplus 1 - p_0)u = 0$ give a graded selfadjoint extension with compact resolvent, and thus a well defined index given by [1, theorem 5] :

$$\operatorname{Ind}\hat{D} = \int_V \hat{A}(\nabla) - \frac{1}{2}(\eta(W) + \dim\ker A) \tag{1.3}$$

It is shown in [8] that any spectral projector p_Y where $Y \subset \mathbf{R}$ differs from $\mathbf{R}_+$ by a compact set gives graded selfadjoint extensions with compact resolvent. It appeared also that the really important data for building selfadjoint extension comes from relations (1.1) and (1.2) between operators, regardless to the precise nature of them.

In this paper, we extend these constructions within the framework of Hilbert modules with boundary over a C*-algebra B introduced in [11]. Such objects appear naturally whith geometric operators on (possibly singular) quotient spaces of manifolds with boundary, such as submersions of manifolds with boundary, group actions or foliation on a manifold with boundary.

Thus one has a symmetric operator $D \subset D^*$ acting on a Hilbert module $\mathcal{E}$ over a C*-algebra B with a unitary involution τ, and $A = A^*$ a selfadjoint operator with compact resolvent acting on a Hilbert module $\mathcal{E}_b$, such that $(\mathcal{E}, D, \tau)$ and $(\mathcal{E}_b, A)$ satisfy the same relations as (1.1), (1.2). We describe the possible graded selfadjoint extensions $\hat{D}$ of D, and under the additional assumption that $(1 + D^*D)^{-1} \in \mathcal{C}(E)$, we caracterise those extensions with compact resolvent. The existence of such extensions rely on the cobordism invariance of the index with value in $K_*(B)$ [11, theorem 4.3].

These operators defines element $[\mathcal{E}, \hat{D}, \tau]$ in the group $K_0(B)$.

The main toolused here is a Hilbert module $\mathcal{G}(A)$ over B, on which A acts naturally, which has the property that graded selfadjoint extension of D are in a one-to-one correspondance with hermitian projectors in $\mathcal{G}(A)$. We show that there is a nonempty set $\mathcal{X}_+(A)$ of projectors of $\mathcal{G}(A)$ such that for any $p \in \mathcal{X}_+(A)$, the corresponding selfadjoint extension D^p has compact resolvent.

As manifolds with boundary can be glued to obtain closed manifolds, one can paste together two Hilbert modules with isomorphic boundary, and one obtains a module without boundary. Then we establish the main theorem of this article which expresses that the indices classes behave additively under this construction (theorem 9.6). This can be viewed as a generalization of the Novikov additivity for the signature of manifolds with boudary. This latter is a fairly obvious consequence of the index relation (1.3), but in our situation there is no analogous formula, and one has to deal directly with K-theory classes.

Selfadjoint extensions of this kind of operator have already been defined in particular cases by several authors. When $B = C(T)$ is a commutative C*-algebra R. Melrose and P. Piazza [21] have defined families of selfadjoint extensions associated to a smooth fibration $Z \to T$ by manifolds with boundaries. F. B. Wu [27] has considered the Hilbert modules associated to a principal covering of a manifold with boundary, B being now the C*-algebra of the structural group of the covering. In these articles, selfadjoint extensions are built-up from boundary values as in [1], and rely on the notion of *spectral section*.

We were led to a different method for two reasons : the existence of spectral sections is not established in general but for the case considered in [21], which means that B is commutative; the space of spectral sections is too "small" and we were only able to solve the technical difficulties arising from analysis in Hilbert modules in this larger space $\mathcal{G}(A)$. The link between the two constructions is made in section 8.

Here we have only developped the even dimensional case : the module is graded over $\mathbf{Z}/2\mathbf{Z}$ and its the boundary is ungraded. The odd dimensional case can be treated by using the appropriate formalism, and would give similar results.

We then explain two consequences of these results concerning topological questions.

Let V be a compact spin manifold. Then the index theorem for flat bundles of Atiayh-Patodi-Singer [1] states the equality of the topological index $K^1(V, \mathbf{R}/\mathbf{Z}) \to \mathbf{R}/\mathbf{Z}$ with the analytical index which can be computed by means of the η-invariant. J. Lott has asked whether it would be possible to generalize this result for families of operators [19].

Given $f : V \to W$ a smooth K-oriented submersion of compact smooth manifolds, the topological index is induced by the Gysin homomorphism

$$f! : K^1(V, \mathbf{R}/\mathbf{Z}) \to K^{1+i}(W, \mathbf{R}/\mathbf{Z}),$$

with $i \equiv \dim V + \dim W \bmod 2$. When the dimension of the fiber of f is even, J. Lott defined the analitical index involving the η-differential form of J-M Bismuth and J. Cheeger [4] and proved the equality between the two indices.

The question was left open when $\dim V + \dim W$ is odd [19, section 7]. Here we define the analytical index in that case and prove the corresponding index formula. The proof use the presentation by [2] of $K^1(V, \mathbf{R}/\mathbf{Z})$ as a quotient of $K^1(V, \mathbf{Q}/\mathbf{Z}) \oplus H^1(V, \mathbf{R})$ and relies on a $\mathbf{Q}/\mathbf{Z}$-index formula (proposition 10.1).

Finally let $B\pi$ be the classifying space of a countable group π, and (V, f) a smooth singular manifold over $B\pi$ which means that $f : V \to B\pi$ is a continuous map. By a construction of Fomenko and Miščenko [7], one can associate to (V, f) a Hilbert module $\Omega_\pi(V)$ over $B = C^*(\pi)$ the C*-algebra of π. If V is a compact oriented manifold with boundary, then this module together with the signature operator is an example of Hilbert module with boundary over $C^*(\pi)$.

Suppose then that V is a closed partitioned by a codimension one submanifold W. Then the signature operator on $\Omega_\pi(V)$ gives a class in $K_0(C^*(\pi))$ and we compute how it changes when (V, f) is modified by a diffeomorphism over W.

Then these results are applied to a question raised by J. Lott and S. Weinberger on the cut-and-paste invariance of higher signature of singular manifolds over $B\pi$. If $c \in H^*(B\pi, \mathbf{R})$, then the higher signature associated is the number :

$$< L(V) \cup f^*(c), [V] > .$$

Cut-and-paste equivalence has been defined by U. Karras, M. Kreck, W. Neumann, E. Ossa [13] and appears as a particular case of the previous situation.

Various criteria ensuring the invariance of higher signature are obtained. some of them extend results of E. Leichtnam, J. Lott and P. Piazza [17, corollary 0.4], and by E. Leichtnam and W. Lueck in [18, corollary 0.7]. In particular, we relax the condition $\operatorname{im} d$ closed involved in [17, 18], and establish the invariance under the weaker condition $\overline{\operatorname{im} d}$ complemented in $\Omega^m_\pi(V)$ $(2m = \dim V)$.

We would like to express our thanks to J. Lott for helpful conversations on this last problem.

2. Hilbert modules

Let B a separable C*-algebra. Recall that a *C*-module over B* or a *Hilbert B-module* is a right Banach module $\mathcal{E}$ over B equipped with a sesquilinear form, or a *B-product* from $\mathcal{E} \times \mathcal{E}$ to B such that for all $\xi \in \mathcal{E}$ and $a, b \in B$, we have: $< \xi, \xi > \geq 0$ and $< \xi a, \eta b > = a^* < \xi, \eta > b$ such that the norm $\xi \to \sqrt{\| < \xi, \xi > \|}$ is equivalent to the Banach norm of B.

A continuous operator $a : \mathcal{E} \to \mathcal{E}$ possesses a B-adjoint for this sesquilinear form if there exists continuous b on $\mathcal{E}$ such that for all $\xi, \eta \in \mathcal{E}$, one has $< a\xi, \eta > = < \xi, b\eta >$. Such an adjoint is denoted by a^*, and the set of adjointable operators is a C*-algebra denoted by $\mathcal{L}(E)$. For $\xi, \eta \in \mathcal{E}$, let $\theta(\xi, \eta)$ the operator given by the formula: $\theta(\xi, \eta)\zeta = < \xi, \zeta > \eta$. These operators generate a closed ideal $\mathcal{C}(\mathcal{E}) \subset \mathcal{L}(\mathcal{E})$, called usually the *algebra of compact operators of $\mathcal{E}$*.

For example, the standard stable module over B is the Hilbert module $\mathcal{H}_B = l^2(\mathbf{N}) \otimes B$, and we have $\mathcal{C}(\mathcal{H}_B) = \mathcal{C} \otimes B$ where the second $\mathcal{C}$ is the usual algebra of compact operator on a $l^2(\mathbf{N})$. According to the stabilization theorem of G. Kasparov, every countable generated Hilbert module is isomorphic to the image of a hermitian projector $e \in \mathcal{L}(\mathcal{H}_B)$. This gives a map from K-theory groups (independant of the choice of e):

$$K_*(\mathcal{C}(\mathcal{E})) \to K_*(B).$$

In the sequel, we shall assume, without any loose of genrality, that the above map in K-theory is an isomorphism. Indeed, this happens if the Hilbert module is *full*, *i.e.* the closed ideal in B generated by $< \mathcal{E}, \mathcal{E} >$ is equal to B [26].

A B-linear operator T of $\mathcal{E}$, closed and densely defined, has an adjoint T^* the graph of which is given by the set of elements $(\eta, \zeta) \in \mathcal{E} \oplus \mathcal{E}$ such that $< T\xi, \eta > = < \xi, \zeta >$ for all $\xi \in \operatorname{dom} T$ and then we set $T^*\eta = \zeta$. The domain of T is denoted by $\operatorname{dom} T$. A linear subspace $Y \subset \operatorname{dom} T$ is a *core* or an *essential domain* for the closed operator T if it is the closure of its restriction to Y. An operator T is symmetric if $\operatorname{dom} T \subset \operatorname{dom} T^*$ and $T\xi = T^*\xi$ for $\xi \in \operatorname{dom} T$, and selfadjoint if $T = T^*$.

The operator is said *regular* if $(1 + T^*T)$ and $(1 + TT^*)$ are the inverse of injective elements of $\mathcal{L}(\mathcal{E})$ with dense range. In this case, we have then $T(1 + T^*T)^{-1} \in \mathcal{E}$ and $(1 + TT^*)^{-1}T \in \mathcal{E}$ [16]. If T is selfadjoint, then T is regular if an only if $\operatorname{im}(T \pm i) = \mathcal{E}$. If T is regular selfadjoint, then the usual functional calculus can be performed on T.

Given a closed densely defined operator T on $\mathcal{E}$, we define the graph of T and we denote by $\mathcal{W}(T)$ the Hilbert module obtained from $\operatorname{dom} T$ with the B-product $< \xi, \eta >^T = < \xi, \eta > + < T\xi, T\eta >$. The canonical injection $I : \mathcal{W}(T) \to \mathcal{E}$ is continuous and we have $I \in \mathcal{L}(\mathcal{W}(T), \mathcal{E})$ if an only if T is regular, and $I \in \mathcal{C}(\mathcal{W}(T), \mathcal{E})$ if and only if $(1 + T^*T)^{-1} \in \mathcal{C}(\mathcal{E})$.

We precise now some terms which will be used in the sequel.

i) An *unbounded module over* B is the data of a Hilbert module $\mathcal{E}$ over B with an unbounded closed densely defined regular operator T commuting with B. We shall call $(\mathcal{E}, T, \tau)$ an *even unbounded module* if τ is a unitary involution such that $\tau T + T\tau = 0$.
ii) A *symmetric module over* B is an unbounded module $(\mathcal{E}, T)$ with $T \subset T^*$.
iii) A *closed module over* B is an unbounded module $(\mathcal{E}, T)$ with $T + T^*$ and $(i + T)^{-1} \in \mathcal{C}(\mathcal{E})$.

Example 2.1. It will be useful to define a model in each of these classes, playing the same role as the stable module $\mathcal{H}_B$. Let $\bar{\partial}$ be the closure on $L^2(S^1)$ of the Dirac operator with essential domain $C^\infty(S^1)$, which is self-adjoint with compact resolvent. Set $\mathcal{B}_B = L^2(S^1) \otimes B$ and $A_B = \bar{\partial} \otimes Id$. By Fourier transform, one has $\mathcal{B}_B \simeq l^2(\mathbf{Z}) \otimes B$ and A_B is conjugate to the usual multiplication operator $(A_B\xi)(n) = n\xi(n)$. The unbounded module $(\mathcal{B}_B, A_B)$ is closed.

Now, let V be an oriented compact two dimensional riemannian manifold with boundary $\partial V = S^1$ and $\mathcal{C}_B = L^2(V, \Lambda_{\mathbf{C}}(T^*V)) \otimes B$. Let D_0 be the closure of the signature operator with essential domain $C_c^\infty(V - \partial V, \Lambda_{\mathbf{C}}(T^*V))$. Then with $D_B = D_0 \otimes Id$, and τ_B the ampliation to B of the Hodge involution on $\Lambda_{\mathbf{C}}(T^*V)$, the triple $(\mathcal{C}_B, D_B, \tau_B)$ is a symmetric module with $(1 + D_B^* D_B)^{-1}$ compact.

We fix a normal neighboorhood $U \simeq S^1 \times [0,1]$, and suppose that the restriction to U of the riemannian metric r on V has the form $r = ds^2 \oplus dt^2$ where s is the standard coordinate on S^1 and t the normal coordinate. We shall refer to these unbounded modules as the *stable* closed or symmetric module, although it is clear that these are not the only examples.

Recall that if $(\mathcal{E}, T)$ is a closed module, we can associate an element $[\mathcal{E}, T] \in K_1(B)$ as follows [26]: as the operator $F = T(1 + T^2)^{-\frac{1}{2}}$ satisfies $F^* = F$ and $F^2 - 1 \in \mathcal{C}(\mathcal{E})$, its image in the quotient C*-algebra $\mathcal{L}(\mathcal{E})/\mathcal{C}(\mathcal{E})$

is a unitary involution , and gives a class in $K_0(\mathcal{L}(\mathcal{E})/\mathcal{C}(\mathcal{E}))$. From the exact sequence :

$$\{0\} \longrightarrow \mathcal{C}(\mathcal{E}) \longrightarrow \mathcal{L}(\mathcal{E}) \longrightarrow \mathcal{L}(\mathcal{E})/\mathcal{C}(\mathcal{E}) \longrightarrow \{0\}$$

one gets a six-term exact sequence in K-theory, and particularly the index map : $K_0(\mathcal{L}(\mathcal{E})/\mathcal{C}(\mathcal{E})) \mapsto K_1(\mathcal{C}(\mathcal{E}))$. Then $[\mathcal{E}, T]$ is given by the image of this class by the index map $K_0(\mathcal{L}(\mathcal{E})/\mathcal{C}(\mathcal{E})) \mapsto K_1(\mathcal{C}(\mathcal{E})) \simeq K_1(B)$.

Analogously, to a closed even module $(\mathcal{E}, T, \tau)$ corresponds a class in $K_0(B)$. The involution τ determines a $\mathbf{Z}_2$-grading with $\mathcal{E}^{\pm} = \ker(\tau \mp 1)$. The restriction of $T(1+T^2)^{-\frac{1}{2}}$ to $\mathcal{E}^+ \to \mathcal{E}^-$ is a unitary modulo compact operators, and thus determines a class in $K_1(\mathcal{L}(\mathcal{E})/\mathcal{C}(\mathcal{E}))$.

This class will be denoted $[\mathcal{E}, T, \tau]$, and if there is no ambiguity $[\mathcal{E}, T]$ or simply $[T]$.

3. Selfadjoint extensions

Let $(\mathcal{E}, D)$ be a symmetric module over B. One may ask whether there exists selfadjoint extensions of D. The classical theory of J. von Neumann [6] has been successfully extended in our context by K. Napiorkowski and S. Woronowicz [22] and can be summarized as follows.

Let denote the deficiency subspaces $\mathcal{J}_+(D) = \ker(D^* - i)$ and $\mathcal{J}_-(D) = \ker(D^* + i)$. Let $U : \mathcal{J}_+ \to \mathcal{J}_-(D)$ be a B-isometry ; one defines an operator D^U on $\mathcal{E}$, with $\operatorname{dom} D^U = \operatorname{dom} D \oplus \text{Graphe(U)}$ and $D^U(\xi + \eta + U\eta) = \xi + i\eta - iU\eta$ pour $\xi \in \operatorname{dom} D$, $\eta \in \mathcal{J}_+(D)$. One checks that D^U is a symmetric extension of D.

Proposition 3.1. *The operator D^U is a regular selfadjoint extension of D if and only if U is a unitary. Every regular selfadjoint extension is of this form.*

The map $U \to (i + D^U)^{-1}$ from unitaries in $\mathcal{L}(\mathcal{J}_+(D), \mathcal{J}_-(D))$ to $\mathcal{L}(\mathcal{E})$ is norm continuous.

Proof. The first assertion is mainly the contents of theorem 5.2 of [22]. For the second assertion, one notes that $\mathcal{E} = \mathrm{im}(D+i) \oplus \mathcal{J}_+(D)$. As the restriction of $(D^U+i)^{-1}$ to $\mathrm{im}(D+i)$ is independent of U and as for $\xi \in \mathcal{J}_+(D)$, one has $(D^U+i)^{-1}\xi = \frac{1}{2}(1+U)\xi$, the continuity of $U \to (D^U+i)^{-1}$ for U unitary follows.

□

Let $(\mathcal{E}, D, \tau)$ be an even symmetric module : it is important to know whether the selfadjoint extensions are also anticommuting with τ.

Lemma 3.2. *Let $U \in \mathcal{L}(\mathcal{J}_+(D), \mathcal{J}_-(D))$ be a unitary. The selfadjoint extension D^U associated with U anticommutes with τ if and only if τU is a unitary involution.*

If $(1+D^*D)^{-1} \in \mathcal{C}(\mathcal{E})$, one has $(i+D^U)^{-1} \in \mathcal{C}(\mathcal{E})$ if and only if $(1+\tau)(1-p)$ and $(1-\tau)p$ are elements of $\mathcal{C}(\mathcal{J}_+(D), \mathcal{E})$, where p is a hermitian projection of $\mathcal{J}_+(D)$ defined by $\tau U = 1-2p$.

Proof. The operator $-\tau D^U \tau$ is always a selfadjoint extension of D, and $-\tau D^U \tau = D^V$ where $V = \tau U^* \tau$. If D^U is graded, then $U = -\tau U^* \tau$. One can then write :

$$\mathrm{dom}\, D^U = \{\xi + (1+\tau)\eta + (1-\tau)\zeta; \xi \in \mathrm{dom}\, D, \eta \in \ker p, \zeta \in \mathrm{im}\, p\}.$$

Then, the canonical injection $\mathcal{W}(D^U) \to \mathcal{E}$ is compact if and only if the map $1+U = (1+\tau)(1-p) + (1-\tau)p$ is compact.

□

It follows that if $p \in \mathcal{L}(\mathcal{J}_+(D))$ is projector which satisfies the last conditions of the lemma above, we have a well defined class $[\mathcal{E}, D^p, \tau] \in K_0(B)$, where D^p is the selfadjoint extension associated with $\tau(1-2p)$.

Example 3.3. Let us illustrate this with the stable symmetric module $(\mathcal{C}_B, D_B, \tau_B)$ as in example 2.1. One knows that if $Y \subset \mathbf{R}$, then the spectral projector on Y of $\bar{\partial}$ acting on $L^2(S^1)$ defines a selfadjoint extension D_Y of D_0 which has a compact resolvent if and only if the symmetric difference of Y with

$\mathbf{R}_+$ is compact [10]. Then $D^Y \otimes Id$ is a selfadjoint extension of D_B with compact resolvent, and it is associated to a projector p_Y of $\mathcal{J}_+(D_B)$. One can write the projector $p_Y = \tilde{p}_Y \otimes 1$ where $\tilde{p}_Y$ is the spectral projector of $\bar{\partial}$ for Y. From [10], lemme 2.3, one deduces then that p_Y *and* $1 - p_Y$ are stable Hilbert module. Let denote $\mathbf{p}_0$ the projector obtained for $Y = \mathbf{R}_+$. Then, by [1, Theorem 4.14], the index of this selfadjoint extension of the signature operator on $V$is equal to the relative signature of V, which is zero as $\dim V = 2$. Thus we have :

$$[\mathcal{C}_B, D_B^{\mathbf{p}_0}, \tau_B] = 0 \tag{3.1}$$

Let p a projector in a Hilbert module $\mathcal{E}$ and define the *Grasmannian* $\mathcal{G}_p$ be the set of projectors q such that $q - p \in \mathcal{C}(\mathcal{E})$. Recall that given a projector $q \in \mathcal{G}_p$, one can define a class $[p-q]$ in $K_0(B)$: the operator qp viewed as an element of $\mathcal{L}(\operatorname{im} p, \operatorname{im} q)$ is invertible modulo compact operators, and gives the class in $K_0(B)$ [26].

Lemma 3.4. *For any $q, r \in \mathcal{G}_p$, one has equality in $K_0(B)$:*

$$[r - q] + [q - p] = [r - p] \tag{3.2}$$

Let $\mathcal{E} = \mathcal{H}_B \oplus \mathcal{H}_B$ and $p : \mathcal{E} \mapsto \mathcal{H}_B \oplus 0$ be the projection onto the first summand. Then the map $q \mapsto [q - p]$ induces a bijection from $\pi_0(\mathcal{G}_p)$ to $K_0(B)$.

Proof. We have $rp - rqp = r(1 - q)p \in \mathcal{C}(\operatorname{im} p, \operatorname{im} r)$, which shows that, modulo compacts operators, rp is the product of rq by qp, and thus we have $[rp] = [rq] + [qp]$ in $K_0(B)$, showing the first assertion. Assuming now $\mathcal{E} = \mathcal{H}_B \oplus \mathcal{H}_B$, let us show that the map is surjective. We use the fact that every class $x \in K_0(B)$ can be represented by a difference $[e] - [f]$ where e, f are finite rank projectors in $\mathcal{H}_B$. Then, the projector $q = (p-e) \oplus f$ satisfies $[q-p] = x$. For the injectivity on $\pi_0(\mathcal{G}_p)$, it suffices to prove, that if $[q-p] = 0$ then q is homotopic to p within $\mathcal{G}_p$. As $\operatorname{im} p = \mathcal{H}_B$, by a classical result of Fomenko-Mishchenko [26] there exists finite rank projectors $e \subset p$ and $f \subset q$, such that $qp(\operatorname{im} e) \subset \operatorname{im} f$ and that qp maps $\operatorname{im} p - e$ isomorphically onto $\operatorname{im} q - f$. As $p - e$ is orthogonal to f and $q - f$ is orthogonal to e, one has a homotopy within $\mathcal{G}_p$ from q to $f + p - e$. By the stabilization theorem of G. Kasparov, the module $\operatorname{im}(1 - p) \oplus \operatorname{im}(e)$ is stable, and there the projectors e and f, which are supposed equivalent are then connected by a homotopy $e(t), t \in [0, 1]$ of finite projectors. The family $q_t = e(t) + p - e$ gives then the homotopy between q and p.

□

We finally resume the result of this section :

Proposition 3.5. *Let $(\mathcal{E}, D, \tau)$ be an even symmetric module over B, such that $(1 + D^*D)^{-1} \in \mathcal{C}(\mathcal{E})$, and $p \in \mathcal{L}(\mathcal{J}_+(D))$ be a hermitian projector such that the corresponding selfadjoint extension D^p of D has compact resolvent. Then for any hermitian projector $q \in \mathcal{L}(\mathcal{J}_+(D))$, the selfadjoint extension D^q associated has compact resolvent if and only if $q \in \mathcal{G}_p$.*

In that case, the difference $[\mathcal{E}, D^q] - [\mathcal{E}, D^p] \in K_0(B)$ only depends of $[q - p] \in K_0(B)$.

Proof. The first part is a straightforward application of lemma 3.2. Suppose that $[q - p] = 0$. Let $(\mathcal{C}_B, D_B)$ the stable symmetric module defined in exemple 2.1, and $\mathbf{p}_0 \in \mathcal{J}_+(D_B)$ the projector defining the selfadjoint extension of D_B corresponding to $Y = \mathbf{R}_+$, as in example 3.3. One can add $(\mathcal{C}_B, D_B)$ without modifying the K-theory class, and replace p by $p \oplus \mathbf{p}_0 \in \mathcal{J}_+(D) \oplus \mathcal{J}_+(D_B)$. By the stabilization theorem of G. Kasparov, $\operatorname{im} p$ and $\operatorname{im}(1-p)$ are isomorphic to $\mathcal{H}_B$, and thus, by the preceding lemma, $q \oplus \mathbf{p}_0$ is homotopic to $p \oplus \mathbf{p}_0$ and $[\mathcal{E}, D^q] = [\mathcal{E}, D^p]$.

□

4. The Grasmannian of a closed module

Let $(\mathcal{E}, A)$ be a closed module. Then A is a selfadjoint operator and we can perform continuous functional calculus. However, spectral projections need not to exist. We define here a useful substitute for them. Define $C_c(A) \subset \mathcal{E}$ to be the dense submodule algebraically generated by $\operatorname{im} f(A)$, for $f \in C_c(\mathbf{R})$.

Let f be a continuous function on $\mathbf{R}$ whith value in $\mathbf{R}_+^*$. Denote by $\mathcal{E}(f(A))$ the Hilbert module completion of $C_c(A)$ with the B-product $< \xi, \eta >_f = < f(A)\xi, \eta >$. Observe that $\mathcal{W}(A) = \mathcal{E}(\sqrt{1 + A^2})$.

Let f, g be continuous real functions on $\mathbf{R}$ with $f, g > 0$, and χ be a real continuous function on $\mathbf{R}$. The operator $\xi \to \chi(A)\xi$ for $\xi \in C_c(A)$ extends to a closed densely defined operator denoted $\bar{\chi}(A)$ from $\mathcal{E}(f(A))$ to $\mathcal{E}(g(A))$.

Lemma 4.1. *The following properties hold :*

a) The operator $\bar{\chi}(A)$ is selfadjoint regular.

b) If $\chi^2 g = O(f)$ over $\mathbf{R}$, then we have $\bar{\chi}(A) \in \mathcal{L}(\mathcal{E}(f(A)), \mathcal{E}(g(A)))$.

c) If $\chi^2 g = o(f)$ when $u \to \pm\infty$, and if $(i + A)^{-1} \in \mathcal{C}(\mathcal{E})$, then $\bar{\chi}(A)$ belongs to $\mathcal{C}(\mathcal{E}(f(A)), \mathcal{E}(g(A)))$.

Proof. If $\chi^2 g = O(f)$, for $\eta \in C_c(A)$, one has, for some $C > 0$:

$$\|\bar{\chi}(A)\eta\|_{\mathcal{E}(g(A))} = \|\chi(A)\sqrt{g}(A)\eta\|_{\mathcal{E}} \leq C\|\sqrt{f}(A)\eta\| = \|\eta\|_{\mathcal{E}(f(A)}$$

This equality shows that $\bar{\chi}(A)$ is everywhere defined and continuous. By a similar computation, one sees that with $\psi = \chi\sqrt{fg^{-1}}$, the operator $\bar{\psi}(A)$ is everywhere defined and continuous from $\mathcal{E}(g(A))$ to $\mathcal{E}(f(A))$ and is the adjoint of $\bar{\chi}(A)$. This proves *b)*.

If $\chi^2 g = o(f)$, it follows from *b)* that the map $t : \xi \to \sqrt{f}(A)\xi$ belongs to $\mathcal{L}(\mathcal{E}(f(A)), \mathcal{E})$ and the map $s : \xi \to \frac{1}{\sqrt{g(A)}}\xi$ belongs to $\mathcal{L}(\mathcal{E}, \mathcal{E}(g(A)))$. The following diagram is commutative :

$$\begin{tikzcd}
\mathcal{E}(f(A)) \arrow[r, "\bar{\chi}"] \arrow[d, "t"'] & \mathcal{E}(g(A)) \\
\mathcal{E} \arrow[r, "\chi\frac{\sqrt{g}}{\sqrt{f}}(A)"] & \mathcal{E} \arrow[u, "s"']
\end{tikzcd}$$

By hypothesis, the bottom map is compact, and so is the top one, which proves *c)*.

Finally, *a)* follows from *b)* by taking the function $u = (i+\chi^2)^{-\frac{1}{2}}$.

□

Example 4.2. i) If $\chi \equiv 1$, then $\bar{\chi} : \mathcal{E}(f(A)) \to \mathcal{E}(g(A)$ is the closure of the identity $C_c(A) \to C_c(A)$.

ii) If $\chi(u) = u$, then $\bar{\chi}$ comes from the operator A itself.

iii) If $\chi = \frac{f}{g}$, then $\bar{\chi}$ is a unitary $\mathcal{E}(f(A) \to \mathcal{E}(g(A))$.

Let now $\mathcal{G}(A) = \mathcal{E}(g_0(A))$, where :

$$g_0(t) = \sqrt{2}(\sqrt{1+t^2} - t)^{\frac{1}{2}} \tag{4.1}$$

By the preceding lemma, the operator $f(A)$ is in $\mathcal{L}(\mathcal{G}(A))$

Definition 4.3. *Let* $f : \mathbf{R} \to \mathbf{R}$ *be a continuous map with* $\lim_{u \longrightarrow \pm\infty} f(u) = \pm 1$. *The Grasmannian of* A *is the set denoted* $\mathcal{X}^0_+(A)$ *of projections* $p \in \mathcal{L}(\mathcal{G}(A))$ *such that* $2p - 1 - f(A) \in \mathcal{C}(\mathcal{G}(A))$.

Let $\pi : \mathcal{L}(\mathcal{E}) \to \mathcal{L}(\mathcal{E})/\mathcal{C}(\mathcal{E})$ the quotient map onto the Calkin algebra of A. Then, as the resolvent of A is compact, $\pi(f(A))$ is a unitary involution independent of the choice of f. One has $\pi(f(A)) = 2\mathbf{r} - 1$ for a projection $\mathbf{r}$, and p is in the Grasmannian of A if and only if $\pi(p) = \mathbf{r}$.

Example 4.4. Let $\lambda \in \mathbf{R}$; then we shall say that the spectral projector of A exists at λ if there is a projector $p_\lambda \in \mathcal{L}(\mathcal{E})$ and a regular selfadjoint operator A_1 (resp. A_2) on $\operatorname{im} p_\lambda$ (resp. $\operatorname{im}(1-p_\lambda)$ such that A is the direct sum of A_1 and A_2 and spectrum$(A_1) \subset]-\infty, \lambda]$, and spectrum$(A_2) \subset [\lambda, +\infty[$.

Then if it exists, the spectral projetor p_λ of A is in $\mathcal{G}(A)$.

One may asks when the Grasmannian space is nonempty. We keep the notations defined in examples 2.1 and 3.3. Using the stable module $\mathcal{B}_B$ one has the following proposition :

Proposition 4.5. *Let $(\mathcal{E}, A)$ be closed module over B. If $\mathcal{X}_+^0(A) \neq \emptyset$, then $[\mathcal{E}, A] = 0$ in $K_1(B)$. Inversely, if $[\mathcal{E}, A] = 0$, then $\mathcal{X}_+^0(A \oplus A_B) \neq \emptyset$.*

Proof. The first assertion follows directly from the six-term exact sequence associated to the exact sequence

$$\mathcal{C}(\mathcal{E}) \to \mathcal{L}(\mathcal{E}) \to \mathcal{Q}(\mathcal{E}).$$

Conversely, suppose that $[\mathcal{E}, A] = 0$. Then let $\mathbf{r} = \pi(\frac{1}{2}(A(1+|A|)^{-1} + 1)$ is a projector the class of which in $K_0(\mathcal{Q}(\mathcal{E}))$ is in the image of $K_0(\mathcal{L}(\mathcal{E}))$. Then the class of $\mathbf{r} \oplus 0$ in $K_0(\mathcal{Q}(\mathcal{E} \oplus \mathcal{B}_B)$ is null, as $K_0(\mathcal{L}(\mathcal{E} \oplus \mathcal{B}_B) = \{0\}$. By definition, there exists a unitary $\mathbf{U} \in \mathcal{Q}(\mathcal{E} \oplus \mathcal{B}_B)$ such that :

$$\mathbf{U}^*(\mathbf{r} \oplus \mathbf{P}_0)\mathbf{U} = 1 \oplus \mathbf{P}_0. \tag{4.2}$$

Here $\mathbf{P}_0$ is the image of $\mathbf{p}_0$ in $\mathcal{Q}(\mathcal{B}_B)$, and we use the property that $\operatorname{im} \mathbf{p}_0$ and $\operatorname{im} 1 - \mathbf{p}_0$ are stable modules.

As $\mathcal{E} \oplus \operatorname{im} \mathbf{p}_0$ is stable, one can then lift $\mathbf{U}(1 \oplus \mathbf{P}_0)$ to a partial isometry $V \in \mathcal{L}(\mathcal{E} \oplus \operatorname{im} \mathbf{p}_0, \mathcal{E} \oplus \mathcal{B}_B)$ [26]; then VV^* is a projector in $\mathcal{X}_+^0(A \oplus A_B)$.

□

5. The cylinder of a module

In this section, $(\mathcal{E}_b, A)$ will be a closed module, *i.e.* an unbounded module and A selfadjoint with compact resolvent. For $f \in C_c(\mathbf{R})$, the operator $f(A)$ is compact. Let as before $C_c(A) \subset \mathcal{E}_b$ be the dense submodule generated by $\operatorname{im} f(A)$, for $f \in C_c(\mathbf{R})$.

Let $\mathcal{E} = \mathcal{E}_b \otimes L^2([0, +\infty[) \otimes \mathbf{C}^2$, endowed with the obvious graduation $\tau = 1 \otimes 1 \otimes \tau_0$ where τ_0 is the involution of $\mathbf{C}^2$ given by the matrix :

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\Psi(A)$ be the closure on $\mathcal{E}$ with essential domain $C_c^\infty(]0, +\infty[) \otimes \mathcal{E}_b \otimes \mathbf{C}^2$ and such that for all $\xi, \eta \in C_c^\infty(]0, +\infty[) \otimes \mathcal{E}_b$:

$$\Psi(A)\begin{pmatrix}\xi\\ \eta\end{pmatrix}=\begin{pmatrix}0 & -\frac{\partial}{\partial u}+A\\ \frac{\partial}{\partial u}+A & 0\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}=\begin{pmatrix}-\frac{\partial \eta}{\partial u}+A\eta\\ \frac{\partial \xi}{\partial u}+A\xi\end{pmatrix} \tag{5.1}$$

Lemma 5.1. *The operator $\Psi(A)$ is symmetric and for all $\varphi \in C_c([0,+\infty[)$, we have $\varphi(1+|\Psi(A)|^2)^{-1} \in \mathcal{C}(\mathcal{E})$.*

Proof. This comes from the fact that this an external tensor product [11]. ☐

We look now for the graded selfadjoint extensions of $\Psi(A)$, and we describe explicitely the submodule $\mathcal{J}_+(\Psi(A)) = \ker(\Psi(A)^* - i)$.

Let f_i, $i = 0, 1, 2$ be the continuous functions on $\mathbf{R}$ given by :

$$\begin{aligned} f_1(t) &= (1+t^2)^{-\frac{1}{4}} \\ f_2(t) &= ((1+t^2)^{\frac{1}{2}} - t)(1+t^2)^{-\frac{1}{4}} \end{aligned} \tag{5.2}$$

Then one has $g_0(t) = (f_1(t)^2 + f_2(t)^2)^{\frac{1}{2}}$ where g_0 is defined in (4.1), and recall that $\mathcal{G}(A) = \mathcal{E}_b(g_0(A))$. As we have $g_0 \geq f_j$, for $j = 1, 2$, the canonical continuous injections $e_j : \mathcal{G}(A) \to \mathcal{E}_b(f_j(A))$, $j = 1, 2$ are adjointable , by lemma 4.1. By the same lemma, for fixed $u > 0$, the maps $\xi \to \exp(-u\sqrt{1+A^2})\xi$ and $\xi \to (\sqrt{1+A^2} - A)\exp(-u\sqrt{1+A^2})\xi$ are in $\mathcal{L}(\mathcal{E}_b(f_1(A)), \mathcal{E})$. For $j = 1, 2$ there are continuous maps Φ_j from $\mathcal{E}_b(f_j(A))$ to $\mathcal{E}$ uniquely caracterized for $\xi \in C_c(A)$:

$$\begin{aligned} \Phi_1(\xi)(u) = &\quad (\exp(-u\sqrt{1+A^2})\xi, 0) \\ \Phi_2(\xi)(u) = &\quad (0, i(\sqrt{1+A^2} - A)\exp(-u\sqrt{1+A^2})\xi) \end{aligned}$$

and let $\Phi_0 = (\Phi_1 \circ e_1) + (\Phi_2 \circ e_2)$ the map from $\mathcal{G}(A)$ to $\mathcal{E}$.

Lemma 5.2. *The maps Φ_0, Φ_1, Φ_2 are continuous and adjointable. The map Φ_0 is an isometry such that* $\operatorname{im}\Phi_0 = \mathcal{J}_+(\Psi(A))$.

Proof. The map Φ has an adjoint uniquely determined by the formula, for $\eta \in C_c^\infty(]0,+\infty[) \otimes C_c(A)$:

$$\Phi_1^*(\eta) = \int_0^{+\infty} \sqrt{1+A^2}\exp(-u\sqrt{1+A^2})\eta(u)du.$$

Analogously, $\Phi_2 \in \mathcal{L}(\mathcal{E}_b(f_2(A)), \mathcal{E})$, and so is Φ_0. For all $\xi \in C_c(A) \subset \mathcal{G}(A)$, one readily checks that $\Phi_0(\xi) \in \mathcal{J}_+(\Psi(A))$. As $C_c(A)$ is dense in $\mathcal{G}(A)$, and $\mathcal{J}_+(\Psi(A))$ is closed, one gets $\operatorname{im}\Phi_0 \subset \mathcal{J}_+(\Psi(A))$.

We prove the reverse inclusion and let $(\xi,\eta) \in \mathcal{J}_+(\Psi(A))$. For any $f \in C_c(\mathbf{R})$, the tensor product $f(A)\otimes 1 \otimes 1$ leaves stable $\mathcal{J}_+(\Psi(A))$, and thus by density one can assume that $\xi(u), \eta(u) \in C_c(A)$ for almost every $u \in \mathbf{R}_+$. By a little convolution argument, we are reduced to the case where $\xi, \eta \in \mathcal{E}_b \otimes_\pi C^\infty(\mathbf{R}_+)$, and that for all $u \geq 0$, $\xi(u), \eta(u) \in C_c(A)$. We are then led to the following system :

$$\begin{cases} -\frac{\partial \eta}{\partial u} + A\eta = & i\xi \\ \frac{\partial \xi}{\partial u} + A\xi = & i\eta \end{cases}$$

from what it follows:

$$\begin{cases} \xi = & \Phi_1(\xi(0)) \\ \eta = & \Phi_2(\xi(0)) \end{cases}$$

□

We know then that the graded selfadjoint extensions of $\Psi(A)$ are parametrized by the projections in $\mathcal{L}(\mathcal{G}(A))$. Let denote by $\Psi(A)^p$ the selfadjoint extension of $\Psi(A)$ defined by the projection $p \in \mathcal{L}(\mathcal{G}(A))$.

Theorem 5.3. *Let $(\mathcal{E}_b, A)$ be a closed module, and $(\mathcal{E}, \Psi(A), \tau)$ the even symmetric module associated as above. The deficiency kernel $\mathcal{J}_+(\Psi(A))$ is canonically isomorphic to $\mathcal{G}(A)$. For any projection $p \in \mathcal{L}(\mathcal{G}(A))$ and $\varphi \in C_c(\mathbf{R}_+)$, one has $\varphi(i+\Psi(A)^p)^{-1} \in \mathcal{C}(\mathcal{E})$ if and only if $p \in \mathcal{X}_+^0(A)$.*

Proof. Recall the orthogonal decomposition :

$$\mathcal{E}(i+\Psi(A)^p) = \mathcal{E}(i+\Psi(A)) \oplus \operatorname{Gr}(\tau(1-2p))$$

where $\mathrm{Gr}(\tau(1-2p)) \subset \mathcal{J}_+(\Psi(A)) \oplus \mathcal{J}_-(\Psi(A))$ is the graph of $\tau(1-2p)$. By hypothesis, the composition of the canonical injection $\mathcal{E}(i+\Psi(A))$ into $\mathcal{E}$ by multiplication by φ is compact, and thus all the problem remains in finding the p for which the map $\xi \to \varphi(1-2p)\xi$ is in $\mathcal{C}(\mathcal{G}(A), \mathcal{E})$. However, if the composition of inclusion of $\mathrm{Gr}(\tau(1-2p))$ into $\mathcal{E}$ by the operator multiplication by φ is compact, then the inclusion alone is compact. Indeed, let $\varphi_k \in C_c(\mathbf{R})$ be a sequence such that $0 \le \varphi_k \le 1$ and converging to 1 uniformly on every compact of $\mathbf{R}_+$. Then it is sufficient to prove that the sequence of multiplication operators restricted to $\operatorname{im}\Phi_0$ converges in norm to the identity. For $\xi \in \mathcal{G}(A) \cap \mathcal{E}_b$, we have :

$$\|(1-\varphi_k)\Phi_0(\xi)\| = \left\{\int_0^\infty (1-\varphi_k(u))^2 \|\exp(-u\sqrt{1+A^2})T\xi\|^2 du\right\}^{\frac{1}{2}},$$

where $T^2 = 1 + (\sqrt{A^2+1} + A)^2$ and thus this assertion follows from the equality for $M \ge 1$:

$$\int_M^{+\infty} \|\exp(-u\sqrt{1+A^2})T\xi\|^2 du \le \exp(-M)\|\xi\|^2_{\mathcal{G}(A)}.$$

We are reduced to find the projections p such that the $\xi \to (1-2p)\xi$ is compact. Let now χ be a increasing continuous function on $\mathbf{R}$ such that $\chi(u) = 0$ if $u \le 0$, and $\lim \chi(u) = 1$ whenever $u \to +\infty$. Let $\chi_+ = \chi(A) \in \mathcal{L}(\mathcal{E}_0)$, and $\chi_- = 1 - \chi_-$. The image of χ_+ in the quotient algebra $\mathcal{L}(\mathcal{E}_0)/\mathcal{C}(\mathcal{E}_0)$ does not depend of the choice of the function χ subject to that conditions. We use lemma 3.2 and first show that the maps from $\mathcal{J}_+(\Psi(A))$ to $\mathcal{E}$ given by $\xi \to (1+\tau)\chi_-\xi$, and $\xi \to (1-\tau)\chi_+\xi$ are compact maps. One has a commutative diagram :

$$\begin{array}{ccc}
\mathcal{G}(A) & \xrightarrow{e_1} & \mathcal{E}_b(f_1(A)) \\
{\scriptstyle\Phi_0}\downarrow & & \downarrow{\scriptstyle\Phi_1} \\
\mathcal{E} & \xrightarrow{(1+\tau)} & \mathcal{E}
\end{array}$$

We apply the lemma 4.1 to $f = g_0$, $g = f_1$, and χ. For $x \to -\infty$, we have that $\dfrac{f_1}{g_0}(x) = (1 + (\sqrt{1+x^2} - x)^2)^{-\frac{1}{2}} \simeq \dfrac{1}{2x}$, and thus we have $e_1 \circ \chi_- \in \mathcal{C}(\mathcal{G}(A), \mathcal{E}_b(f_1(A)))$, and so $(1+\tau)\circ\Phi_0\circ\chi_-$ is compact.

Similarly, we have $\dfrac{f_2}{g_0}(x) \simeq \dfrac{1}{2x}$ whenever $x \simeq +\infty$, which implies that $e_2 \circ \chi_+ \in \mathcal{C}(\mathcal{G}(A), \mathcal{E}_b(f_2(A)))$, and thus the operator $(1-\tau) \circ \Phi_0 \circ \chi_+ = \Phi_2 \circ e_2 \circ \chi_+$ is compact.

For p a projection, the assertion $2p - 1 + A(1+|A|)^{-1} \in \mathcal{C}(\mathcal{G}(A))$ is equivalent to $p - \chi_+ \in \mathcal{C}(\mathcal{G}(A))$, from what it follows immediately that the condition is sufficient.

We show now that the condition is necessary. If the maps $(1-\tau) \circ \Phi_0 \circ p$ and $(1+\tau) \circ \Phi_0 \circ (1-p)$ are compact, then the operators $\Phi_1 \circ e_1 \circ (p - \chi_+)$ and $\Phi_2 \circ e_2 \circ (1 - p - \chi_-)$ are compact, and thus $\Phi_0 \circ (p - \chi_+)$ and $p - \chi_+$ are too.

□

6. The pasting projection

We compute the Grasmannian in a basic example that will be used later. We need first a lemma :

Lemma 6.1. *Let $p \in \mathcal{X}^0_+(A)$. Then there exists $h \in C_c(]0, +\infty[)$, $k \in C_c(]-\infty, 0[)$ and $r \in \mathcal{X}^0_+(A)$ such that $rh(A) = h(A)$, $rk(A) = 0$, and such that p and r can be joined in $\mathcal{X}^0_+(A)$ by a norm continuous path.*

Proof. Let $a < b$ and χ be a continuous function on $\mathbf{R}$ such that $\chi(u) = 0$ whenever $u \leq a$ and $\chi(u) = 1$ whenever $u \geq b$, for some $a, b \in \mathbf{R}$. As $p - \chi(A)$ is compact, for any ε, we can find a finite rank selfadjoint operator $k = \sum \theta(\xi_i, \eta_i)$ such that $\|\chi(A) + k - p\| \leq \varepsilon$, and moreover, one can suppose that there exists $h \in C_c(\mathbf{R})$ such that $\xi_i, \eta_i \in \operatorname{im} h(A)$. If ε is small enough, 0 and 1 are separated in spectrum$(\chi(A) + k)$ and there exists a simple loop $\gamma : S^1 \to \mathbf{C} -$ spectrum$(\chi(A) + k)$ around the connected component of 1. Let r be the projector given by the Cauchy integral :

$$r = \frac{i}{2\pi} \int_\gamma (\chi(A) + k - z)^{-1} dz.$$

One has :

$$r-p=\frac{i}{2\pi}\int_\gamma(\chi(A)+k-z)^{-1}(\chi(A)+k-p)(p-z)^{-1}dz.$$

This shows that $r-p\in\mathcal{C}(\mathcal{G}(A))$ and $\|r-p\|\leq\varepsilon_1$. Thus $r\in\mathcal{X}_+^0(A)$ and r is homotopic to p. As there exists $l\in C_c(]-\infty,+\infty[)$ such that $(\chi(A)+k)l(A)=\chi(A)+k$, one has $rl(A)=l(A)r=r$. Take any function g such that $gh=0$, then $rg(A)=rh(A)g(A)=(hg)(A)=0$.

□

By lemma 4.1, there is a canonical unitary $U_A:\mathcal{G}(A)\to\mathcal{G}(-A)$, uniquely determined by the equation for $\xi\in C_c(A)$:

$$U_A\xi=g_0(-A)^{-1}g_0(A)\xi=(\sqrt{A^2+1}-A)\xi. \tag{6.1}$$

It is easy to see $U_A^*\mathcal{X}_+^0(-A)U_A$ is exactly the set of projectors $q=1-p$,whenever $p\in\mathcal{X}_+^0(A)$.

By the lemma 4.1, the operator $T=\sqrt{A^2+1}+A$ is selfadjoint regular on $\mathcal{G}(A)$, and thus its graph is orthocomplemented in $\mathcal{G}(A)\oplus\mathcal{G}(A)$. The following will be needed later :

Lemma 6.2. *The hermitian projector Q of $\mathcal{G}(A)\oplus\mathcal{G}(A)$ onto the graph of T belongs to $\breve{\mathcal{X}}=(U_A\oplus 1)^*\mathcal{X}_+^0((-A)\oplus A)(U_A\oplus 1)$. For any $p\in\mathcal{X}_+(A)$, the projector $(1-p)\oplus p$ is homotopic to Q in $\breve{\mathcal{X}}$.*

Proof. By proposition 3.5, we can replace p by any other projector in $\mathcal{X}_+^0(A)$. Let $r\in\mathcal{X}_+^0(A)$ given by the last lemma. Then $r(\operatorname{dom}A)\subset\operatorname{dom}A$.

As $T-(1-r)T(1-r)\in\mathcal{C}(\mathcal{G}(A))$, the operator $(1-r)T(1-r)$ is regular and selfadjoint.

For $\xi\in\mathcal{G}(A)$, one has :

$$\| < Tk(A)\xi, k(A)\xi >\geq\|k(A)\xi\|^2,$$

which shows that $(1-r)T(1-r)$ is bounded above on $\operatorname{im}(1-r)$, and is surjective. Thus $(1-r)T(1-r)$ is the inverse of some element $a\in\mathcal{L}((1-r)\mathcal{G})$. As $T^{-1}=\sqrt{1+A^2}-A$, one gets similarly that $(1-r)T^{-1}(1-r)$ and that

$(1-r)T^{-1}(1-r)T(1-r)-(1-r)$ are compact, which shows that a is compact too.

One sees then that $\operatorname{im} Q$ is homotopic to the direct sum of $\operatorname{im} r$ and of the graph of $(1-r)T(1-r)$. The graph of $(1-r)T(1-r)$ is the same as the graph of a, and is homotopic to $\operatorname{im}(1-r)$.

$\square$

Remark 6.3. Observe that if $a = a^* \in \mathcal{L}(E)$, then $\mathcal{G}(A+a) = \mathcal{G}(A)$ as B-module, but the norms differs. However, there is canonical map $\mathcal{X}_+^0(A) \to \mathcal{X}_+^0(A+a)$, defined as follows : as projector $p \in \mathcal{X}_+^0(A)$ is an idempotent in $\mathcal{L}(\mathcal{G}(A+a))$, there is an hermitian projection$\tilde{p}$ onto $\operatorname{im} p$ in $\mathcal{L}(\mathcal{G}(A+a)$. Then $\tilde{p} \in \mathcal{X}_+^0(A+a)$, and this map $p \to \tilde{p}$ settles a bijection between the two grasmannians.

Definition 6.4. *Let $(\mathcal{E}, A)$ be a closed module and $a = a^* \in \mathcal{L}(E)$. Let Q the hermitain projector defined in the preceding lemma and $\tilde{Q}$ be the image of Q in $\mathcal{X}_+^0((-A-a)\oplus A)$ by the map defined above.*

Then define $\mathcal{X}_+^\sigma(A, A+a) \subset \mathcal{X}_+^0((-A-a)\oplus A)$ as the connected component of $\tilde{Q}$.

Let $L^2(\mathbf{R}) \otimes \mathcal{E}_b \otimes \mathbf{C}^2$, and D_0 is the symmetric operator given by the formula (5.1) with essential domain $C_c^\infty(\mathbf{R}^*) \otimes \mathcal{E}_b \otimes \mathbf{C}^2$, and D is the selfadjoint extension of D_0 with essential domain $C_c^\infty(\mathbf{R}) \otimes \mathcal{E}_b \otimes \mathbf{C}^2$. By an easy change of variable as in *example 7.2*, one has an identification of $L^2(\mathbf{R})$ with $L^2(\mathbf{R}_+) \otimes \mathbf{C}^2$ and of D_0 with $\Psi((-A)\oplus A)$, and one has $\mathcal{J}_+(D_0) = \mathcal{G}(-A) \oplus \mathcal{G}(A)$.

Let $\mathcal{R} : \mathcal{G}(-A) \to \mathcal{G}(A)$ be the closed densely defined unique regular extension of the identity map of $C_c(A)$, so that for $\xi \in C_c(A)$, one has $\mathcal{R}\xi = \xi$. One checks easily that $T = U_A^* \mathcal{R} U_A$, where $T = \sqrt{1+A^2} + A$ and U_A is the unitary of (6.1).

Proposition 6.5. *The projector corresponding to the selfadjoint extension D is equal to the orthogonal projection of $\mathcal{G}(-A) \oplus \mathcal{G}(A)$ onto the graph of $\mathcal{R}$.*

Equivalently, under the canonical identification of $\mathcal{J}_+(D_0)$ with $\mathcal{G}(A)\otimes\mathbf{C}^2$, it corresponds to the projection Q onto the graph of T.

Proof. By the lemma 4.1, we have two continuous injection $i_\pm : \mathcal{W}(A) \to \mathcal{G}(\pm A)$ and the direct sum $i_- \oplus i_+$ embeds $\mathcal{W}(A)$ as a closed subpace of $\mathcal{G}(-A)\oplus\mathcal{G}(A)$. Clearly, the projector associated to D is the orthogonal projection of $\mathcal{G}(-A)\oplus\mathcal{G}(A) \to \operatorname{im} i_- \oplus i_+$ One sees easily that $\operatorname{im} i_- \oplus \operatorname{im} i_+$ is the graph of $\mathcal{R}$.

$\square$

Remark 6.6. To simplify, for $p \in \mathcal{X}^0_+(A)$, we shall often denote by $\Psi(-A)^{1-p}$ the selfadjoint extension corresponding to the projector $q \in \mathcal{X}^0_+(-A)$ such that $U_A^* q U_A = 1-p$.

7. Hilbert modules with boundary

We recall now the definition of boundary modules, as introduced in [11]. Let $(\mathcal{E}, D, \tau)$ be an even symmetric module and $(\mathcal{E}_b, A)$ be a closed module over B and suppose that there exists projections $r, s \in \mathcal{L}(\mathcal{E})$ such that :

i) $r + s = 1$
ii) $\operatorname{im} s$ is isomorphic to $\mathcal{E}_b \otimes L^2([0,1]) \otimes \mathbf{C}^2$
iii) By this isomorphism, the restriction to $\operatorname{im} s$ of the involution τ is $1\otimes 1\otimes \tau_0$

Using *ii)*, we identify these two spaces. For $\psi \in C([0,1])$, set $b(\psi) \in \mathcal{L}(\mathcal{E})$ to be the operator : $b(\psi) = \psi(1)s + 1 \otimes m(\psi) \otimes 1$, where $m(\psi)$ is the multiplication by ψ on $L^2([0,1])$.

Definition 7.1. *The closed module $(\mathcal{E}_b, A)$ is the boundary of the symmetric module $(\mathcal{E}, D, \tau)$ if the following properties hold :*

a. *For every $\varphi \in C^\infty_c(]0,1])$, one has $b(\varphi)\operatorname{dom} D^* \subset \operatorname{dom} D$.*
b. *For every $\varphi, \psi \in C^\infty([0,1])$ such that* $\operatorname{support}(\varphi)\cap\operatorname{support}(\psi) \neq \emptyset$, *one has $b(\varphi) D b(\psi) = 0$.*

c. *For every* $\varphi \in C_c^\infty([0,1[)$, *we have* $Db(\varphi) = \Psi(A)b(\varphi)$, *and for* $\xi \in \operatorname{dom} D$ *or* $\xi \in \operatorname{dom}\Psi(A)$, *one has :*

$$Db(\varphi)\xi = \Psi(A)b(\varphi)\xi.$$

Property *c.* of this definition implies that D is not selfadjoint, otherwise $\Psi(A)$ would be too and that the question of computing selfadjoint extensions of D is localized at the boundary.

As proven in [11], these conditions imply that if $\varphi \in C^\infty([0,1])$, then the commutator $[D, b(\varphi)]$ is bounded, and is equal to the operator acting in $\operatorname{im} p$ and given by the matrix :

$$\begin{pmatrix} \varphi' & 0 \\ 0 & -\varphi' \end{pmatrix}$$

Another consequence of the definition is that the subspace of $\xi \in \operatorname{dom} D$ such that $r\xi \in C_c^\infty(]0,1]) \otimes \mathcal{E}_b \otimes \mathbf{C}^2$ is an essential domain for D.

Example 7.2. *i)* The most simple construction is given by $\mathcal{E} = \mathcal{E}_b \otimes L^2(\mathbf{R}_+, \mathbf{C}^2)$ with $D = \Psi(A)$ as defined in the last section by the formula (5.1).

ii) Let $\mathcal{E} = \mathcal{E}_b \otimes L^2(\mathbf{R}_-, \mathbf{C}^2)$ with D defined by the formula (5.1), and q the hermitian projector onto $\mathcal{E}_b \otimes L^2(]-\infty, -1], \mathbf{C}^2)$ and $p = 1 - q$. Then the isomorphism θ of $\operatorname{im} p$ with $L^2([0,1]) \otimes \mathcal{E}_b \otimes \mathbf{C}^2$ given by:

$$\theta \begin{pmatrix} \xi(s) \\ \eta(s) \end{pmatrix} = \begin{pmatrix} \xi(-s) \\ -\,\eta(-s) \end{pmatrix},$$

gives to $(\mathcal{E}, D, \tau)$ the structure of a Hilbert module with boundary equal to $(\mathcal{E}_b, -A)$.

iii) Another example is the Hilbert module $\mathcal{E} = \mathcal{E}_b \otimes L^2([0,1]) \otimes \mathbf{C}^2$ with the operator given by the matrix (5.1) and in that case, the boundary is equal to $(\mathcal{E}_b, A) \oplus (\mathcal{E}_b, -A)$.

iv) Given an even symmetric module $(\mathcal{E}, D, \tau)$ with boundary $(\mathcal{E}_b, A)$ as above, we can form the module with opposite grading $(\mathcal{E}, D, -\tau)$. This opposite module has a boundary equal to $(\mathcal{E}_b, -A)$, by using the isomorphism κ of $L^2([0,1]) \otimes \mathcal{E}_b \otimes \mathbf{C}^2$ given by :

$$\kappa \begin{pmatrix} \xi(s) \\ \eta(s) \end{pmatrix} = \begin{pmatrix} -\eta(1-s) \\ \xi(1-s) \end{pmatrix}.$$

Let $(\mathcal{E}_1, D_1, \tau_1)$, be a module with boundary $(\mathcal{E}_b, -A)$ and $(\mathcal{E}_2, D_2, \tau_2)$ another one with boundary $(\mathcal{E}_b, A)$, and let s_j, r_j, b_j be the corresponding objects for $j = 1, 2$ of definition 7.1. Let again denote by θ the unitary $\operatorname{im} r_1 \to \operatorname{im} r_2$ which intertwines τ_1 and τ_2, $\Psi(-A)$ and $\Psi(A)$. Then one can form the module $\mathcal{E}_1 \# \mathcal{E}_2 = \operatorname{im} s_2 \oplus \operatorname{im} r_1 \oplus \operatorname{im} s_1$, with an obvious graduation denoted $\tau = \tau_1 \# \tau_2$.

Let $U = 1 \oplus \theta : \mathcal{E}_1 \to \operatorname{im} s_1 \oplus \operatorname{im} r_2$, and $\varphi \in C_c^\infty([0,1[)$ such that $\varphi(0) = 1$, and define on $\mathcal{E}_1 \# \mathcal{E}_2$ the operator $D_1 \# D_2$, with domain $U(b_1(1-\varphi) \operatorname{dom} D_1) + b_2(\varphi) \operatorname{dom} D_2$, by the obvious formula :

$$D_1 \# D_2 \xi = U(b_1(1-\varphi) D_1 \xi_1) + b_2(\varphi)\ D_2 \xi_2. \tag{7.1}$$

Proposition 7.3. *The operator $D_1 \# D_2$ on $\mathcal{E}$ is regular selfadjoint, and does not depend of the choice of φ.*

If for $i = 1, 2$, $(1 + D_i^* D_i)^{-1}$ is compact, then $(i + D)^{-1} \in \mathcal{C}(\mathcal{E})$.

Proof. This is proven in [11], proposition 3.9.

□

We describe now the selfadjoint extensions of an even symmetric module $(\mathcal{E}, D, \tau)$ property with boundary $(\mathcal{E}_b, A)$. We keep the notations of the preceding section and of the definition 7.1 and let $\varphi \in C_c^\infty([0,1[)$ such that $\varphi(0) = 1$. For every projection p in $\mathcal{J}_+(\Psi(A))$ and $\Psi(A)^p$ the corresponding selfadjoint extension, we define

$$D^p = b(1-\varphi) D + \varphi \Psi(A)^p$$

with domain $\operatorname{dom} D^p = b(1-\varphi) \operatorname{dom} D + \varphi \operatorname{dom} \Psi(A)^p$ where $\psi \in C_c^\infty([0,1[)$ is equal to 1 onto the support of ϕ.

Proposition 7.4. *The operator D^p is selfadjoint, regular and does not depend of the choice of φ. Every selfadjoint graded extension of D is equal to D^p for some projection p. If $(1+D^*D)^{-1} \in \mathcal{C}(\mathcal{E})$, it satisfies $b(\psi)(i+D^p)^{-1} \in \mathcal{C}(\mathcal{E})$ for any $\psi \in C_c^\infty([0,1[)$ if and only if p belongs to the Grasmannian of A in $\mathcal{G}(A)$.*

Proof. We have a continuous map $\Theta : \mathcal{W}(\Psi(A)^p) \oplus \mathcal{W}(D) \to \mathcal{W}(D^p)$ given by $\Theta(\xi, \eta) = \psi\xi + (1-\psi)\eta$. Let $\tilde{\Theta} : \mathcal{W}(\Psi(A)^p) \oplus \mathcal{W}(D) \to \mathcal{E}$ given also by $\Theta(\xi, \eta) = \psi\xi + (1-\psi)\eta$, so that $\tilde{\Theta} = \mathcal{I} \circ \Psi$ where $\mathcal{I}$ is the canonical injection $\mathcal{W}(D^p) \to \mathcal{E}$. By the lemma 2.2 of [11], Θ and $\tilde{\Theta}$ are adjointable. As by definition, the map Θ is surjective, it has a right inverse in $\mathcal{L}(\mathcal{W}(D^p), \mathcal{W}(\Psi(A)^p) \oplus \mathcal{W}(D))$, which shows that $\mathcal{I}$ is adjointable.

The self-adjointness of D^p and the independance of the definition from the choice of φ is proven as in [11], proposition 3.9.

The last assertion is the traduction of proposition 5.3.

□

Let $(\mathcal{E}, D, \tau)$ symmetric with $(1 + D^*D)^{-1} \in \mathcal{C}(\mathcal{E})$ and boundary $(\mathcal{E}_b, A)$. For any p in the Grasmannian space of A, then one obtains a closed module and a class in $K_0(B)$ denoted :

$$[\mathcal{E}, D^p, \tau]. \tag{7.2}$$

By proposition 3.1, $[\mathcal{E}, D^p, \tau] = [\mathcal{E}, D^q, \tau]$ if p and q are in the same norm component in $\mathcal{X}_+^0(A)$.

8. Boundary values

The usual way of determining selfadjoint extensions of differential operators on manifolds with boundary go through conditions on boundary values of the functions in the domain.

In order to get graded extensions of Dirac operators, Atiyah, Patodi and Singer were lead to global boundary values. Selfadjoint extensions have been defined in this way for some examples of Hilbert modules by R. Melrose and P. Piazza [21] and F.B. Wu [27], by introducing spectral sections of an operator.

Let recall that if $(\mathcal{E}_b, A)$ is a closed module, then a projector p in $\mathcal{E}$ is a spectral section for A if there are increasing function continuous f_1, f_2 on $\mathbf{R}$, such that $f_i \equiv 0$ near $-\infty$ and $f_i \equiv 1$ near $+\infty$, we have $\operatorname{im} f_1(A)) \subset \operatorname{im} p$, and $\operatorname{im}(1 - f_2(A)) \subset \operatorname{im} 1 - p$.

Given a spectral section p, one sees that p defines an idempotent everywhere defined e in $\mathcal{G}(A)$. One checks that e is a closed operator and thus is continuous. Let $\bar{p}$ be the associated hermitian projection in $\mathcal{G}$.

Lemma 8.1. *One has $\bar{p} = UpU^*$ and $\bar{p} \in \mathcal{X}_+^0(A)$*

This shows that spectral sections of A acting on $\mathcal{E}$ are in bijective correspondance with spectral sections of A as an operator on $\mathcal{G}(A)$. In particular, if there exists a spectral section for A, then $\mathcal{X}_+^0(A)$ is nonempty. Inversely, if $\mathcal{X}_+^0(A) \neq \emptyset$, then by lemma 6.1, there exists a spectral section.

Spectral sections are well adapted for boundary value conditions. Recall that by corollary 4.2, the map $\xi \to f_0(A)\xi$ implements a unitary $U : \mathcal{G}(A) \to \mathcal{E}$.

Let S_0^p be the operator on $\mathcal{E} \otimes L^2(\mathbf{R}_+) \otimes \mathbf{C}^2$ with domain $\xi = (\xi_1, \xi_2)$, with $\xi_i \in C_c(A) \otimes C_c([0, +\infty[)$, and :

$$p\xi_1(0) = 0$$

$$(1 - p)\xi_2(0) = 0,$$

and $S_0^p \xi = \Psi(A)\xi$. One has readily :

Proposition 8.2. *The operator S_0^p is essentially selfadjoint and its closure is exactly the selfadjoint extension $\Psi(A)^{\bar{p}}$.*

9. Additivity of the index

We investigate now some properties of these K-theory classes. For the following, one uses the convention of remark 6.6.

Proposition 9.1. *Let $\mathcal{E} = \mathcal{E}_b \otimes L^2([0,1], \mathbf{C}^2)$ and $p, q \in \mathcal{X}^0_+(A)$. Let $D(p,q)$ be the selfadjoint extension associated with $p \oplus 1-q$. Then one has in $K_0(B)$:*

$$[\mathcal{E}, D(p,q), \tau] = [p-q].$$

Proof. First case : $p = q$. It follows from lemma 6.2 and proposition 6.5 that $D(p,p) = 0$. Indeed, $[\mathcal{E}, D(p,p), \tau]$ is then in $K_0(B)$ the external cup-product of $[\mathcal{E}_b, A]$ with the K-theory class of the Dirac operator on S^1.

Second case. When $\mathcal{E}_b$ finite, $A = 0$ and $p = 1$, it is straighforward that $D(1,0) = [\mathcal{E}_b]$.

In the general case, after eventually a stabilization, we can assume that $\mathcal{E}_b = \mathcal{E}_{b,0} \oplus B^N$, and that there exist $p_0 \in \mathcal{L}(\mathcal{E}_{b,0})$ and $e, f \in M_N(B)$ with $[p-q] = [e] - [f]$ and $p = p_0 \oplus e$, $q = (1-p_0) \oplus f$. The first and second steps imply then $[\mathcal{E}, D(p,q), \tau] = [e] - [f]$.

□

We come now to the principal result of this section. For the following, one uses the convention of remark 6.6.

Theorem 9.2 (Pasting)**.** *Let $(\mathcal{E}_1, D_1, \tau_1)$ be a Hilbert module over B with boundary $(\mathcal{E}_b, -A)$ and $(\mathcal{E}_2, D_2, \tau_2)$ a Hilbert module over B with boundary $(\mathcal{E}_b, A)$. For any projectors $p, q \in \mathcal{X}^0_+(A)$, we have equality in $K_0(B)$:*

$$[\mathcal{E}_1, D_1^{1-q}, \tau_1] + [\mathcal{E}_2, D_2^p, \tau_2] = [\mathcal{E}_1 \# \mathcal{E}_2, D_1 \# D_2, \tau_1 \# \tau_2] + [p-q].$$

Proof. In the case $p = q$, this is a consequence of lemma 6.2 and proposition 6.5. From the proposition 9.1, one has then $[\mathcal{E}_2, D_2^p] = [\mathcal{E}_2, D_2^q] + [p-q]$ and thus :

$$[\mathcal{E}_1, D_1^{1-q}] + [\mathcal{E}_2, D_2^r] = [\mathcal{E}_1, D_1^{1-q}] + [\mathcal{E}_2, D_2^q] + [p-q]$$

$$= [\mathcal{E}_1 \# \mathcal{E}_2, D_1 \# D_2] + [p-q].$$

□

Given a spin manifold V with a collared boundary, then the Dirac operator acting on the Hilbert space of L^2-sections of the spinor bundle is a Hilbert module with boundary. The index of this operator is zero [?]. One has the generalization of this invariance property of the index by bordism.

Theorem 9.3. *Let $(\mathcal{E}_b, A)$ be a closed module, which is the boundary of a symmetric module $(\mathcal{E}, D, \tau)$. If $(1 + D^*D)^{-1} \in \mathcal{C}(\mathcal{E})$, then $[\mathcal{E}_b, A] = 0$ in $K_1(B)$.*

Proof. Cf. [11]. □

As we are interested here in computing such classes, it is important to know if $\mathcal{X}_+^0(A) \neq \emptyset$. By theorem 9.3, one has $[\mathcal{E}_b, A] = 0$, which means that the K-theoretical obstruction of proposition 4.5 does not hold. As the converse is not necessarily true, we are led to the following definition.

Definition 9.4. *Given a closed module$(\mathcal{E}_b, A)$, set $\mathcal{X}_+(A) = \mathcal{X}_+^0(A \oplus A_B)$ be the Grasmannian space of $A \oplus A_B$, where $(\mathcal{B}_B, A_B)$ be the stable closed module defined in example 2.1. Thus a projector $p \in \mathcal{L}(\mathcal{G}(A \oplus A_B))$ belongs to $\mathcal{X}_+(A)$ if and only if $2p-1$ is equal to $A(1+A^2)^{-1} \oplus A_B(1+A_B^2)^{-1}$modulo compact operators.*

By proposition 4.5, if $(\mathcal{E}_b, A)$ is the boundary of $(\mathcal{E}, D, \tau)$ with $(1+D^*D)^{-1}$ compact, then $\mathcal{X}_+(A)$ is nonempty. However, when $\mathcal{X}_+^0(A)$ is itself nonempty, the map $\mathcal{X}_+^0(A) \to \mathcal{X}_+(A)$ given by $p \to p \oplus \mathbf{p}_0$, where $\mathbf{p}_0$ is the projector defined in exemple 3.3, gives rise to a bijection between the connected components of $\mathcal{X}_+^0(A)$ and of $\mathcal{X}_+(A)$.

Remark 9.5. Let $(\mathcal{C}_B, D_B, \tau_B)$ the stable symmetric even module defined in exemple 2.1. This module has a boundary equal to the stable closed module $(\mathcal{B}_B, A_B)$. We see that $(\mathcal{E}_b \oplus \mathcal{B}_B, A \oplus A_B)$ is the boundary of $(\mathcal{E} \oplus \mathcal{C}_B, D \oplus D_B, \tau \oplus \tau_B)$.

Given $p \in \mathcal{X}_+(A)$, we have now a well defined graded selfadjoint extension of $D \oplus D_B$, and one obtains an element in $K_0(B)$. If $\mathcal{X}_+^0(A) \neq \emptyset$, we obtain essentially the same K-theory classes : indeed, by exemple 3.3, one has

$[\mathcal{C}_B, D_B^{\mathbf{p}_0}, \tau_B] = 0$, and thus $[\mathcal{E}, D^p, \tau] = [\mathcal{E} \oplus \mathcal{C}_B, (D \oplus D_B)^{p \oplus \mathbf{p}_0}, \tau \oplus \tau_B)$ for any $p \in \mathcal{X}_+(A)$.

However, given $p \in \mathcal{X}_+(A)$, we shall keep the notation $[\mathcal{E}, D^p, \tau]$ for this element of $K_0(B)$.

One has the translation of the pasting theorem :

Theorem 9.6 (Pasting)**.** *Let $(\mathcal{E}_1, D_1, \tau_1)$ be a Hilbert module over B with boundary $(\mathcal{E}_b, -A)$ and $(\mathcal{E}_2, D_2, \tau_2)$ a Hilbert module over B with boundary $(\mathcal{E}_b, A)$. For any projectors $p, q \in \mathcal{X}_+(A)$, we have equality in $K_0(B)$:*

$$[\mathcal{E}_1, D_1^{1-q}, \tau_1] + [\mathcal{E}_2, D_2^p, \tau_2] = [\mathcal{E}_1 \# \mathcal{E}_2, D_1 \# D_2, \tau_1 \# \tau_2] + [p - q].$$

Corollary 9.7. *Let $(\mathcal{E}, D, \tau)$ be a symmetric module with boundary $(\mathcal{E}_b, A)$. For any projectors $p, q \in \mathcal{X}_+(A)$, one has equality in $K_0(B)$:*

$$[\mathcal{E}, D^p, \tau] - [\mathcal{E}, D^q, \tau] = [p - q].$$

Corollary 9.8. *Let $(\mathcal{E}, D, \tau)$ be a symmetric module with boundary. One has $[\mathcal{E} \# \mathcal{E}, D \# D, -\tau \# \tau] = 0$ in $K_0(B)$.*

Proof. Consider the module with opposite grading $(\mathcal{E}, D, -\tau)$; it is again a module with boundary $(\mathcal{E}_b, -A)$. For any $p \in \mathcal{X}_+(A)$, the K-theory class $[\mathcal{E}, D^{1-p}, -\tau]$ is the inverse of $[\mathcal{E}, D^p, \tau]$. By the preceding theorem, one has $[\mathcal{E} \# \mathcal{E}, D \# D, -\tau \# \tau] = [\mathcal{E}, D^{1-p}, -\tau] + [\mathcal{E}, D^p, \tau] = 0$.

□

10. Example : a cup-product in $K(., \mathbf{Q}/\mathbf{Z})$

The K-theory group $K(., \mathbf{Q}/\mathbf{Z})$ where intoduced in [1] from which we recall the definition. Let M_k be the reduced mapping cone of the map $f_k(z) = z^k$ of S^1. If A is a C*-algebra, then set :

$$K_i(A, \mathbf{Z}_k) = K_i(A \otimes C(M_k)), \tag{10.1}$$

where $\mathbf{Z}_k = \mathbf{Z}/k\mathbf{Z}$. It is proven in [24] that $K_i(A, \mathbf{Z}_k) \simeq K_i(A \otimes D_k)$ for any separable C*-algebra such that $K_0(D_k) = \mathbf{Z}_k$ and $K_1(D_k) = \{0\}$.

By the Puppe theorem, we have exact sequences :

$$\{0\} \to K^0(S^1) \xrightarrow{\mathrm{Id}} K^0(S^1) \to K^1(M_k) \to \{0\}$$
$$\{0\} \to K^1(S^1) \xrightarrow{\times k} K^1(S^1) \to K^0(M_k) \to \{0\}$$

One obtains a direct system by introducing the map $f_l^* : K^0(M_k) \to K^0(M_{kl})$ induced from f_l, and one may set :

$$K_*(A, \mathbf{Q}/\mathbf{Z}) = \varinjlim K_*(A, \mathbf{Z}_k).$$

The element $H_k \in K^0(M_k)$ image of $1 \in K^1(S^1)$ in the Puppe sequence satisfies $f_l^*(H_k) = lH_{kl}$. There exists a complex vector bundle L_k over the (unreduced) mapping cone $\hat{M}_k$ such that $H_k = [L_k] - [\hat{M}_k \times \mathbf{C}]$. One can fix a linear isomorphism $\alpha_k : L_k \times \mathbf{C}^k \simeq \hat{M}_k \times \mathbf{C}^k$, which is unique up to isotopy as $K^1(\hat{M}_k)$ is null.

By composing $z \to z \otimes H_k$ with the canonical map $K_i(A, \mathbf{Z}_k) \to K_i(A, \mathbf{Q}/\mathbf{Z})$, one gets a morphism :

$$j_k : K_i(A) \to K_i(A, \mathbf{Q}/\mathbf{Z}) \tag{10.2}$$

The group $K_1(A, \mathbf{Z}_k)$ admits the following geometrical description. Let (E_1, E_2, θ) a triple where E_i is a projective finitely generated Hilbert module over A, and $\theta : E_1 \times \mathbf{C}^k \to E_2 \times \mathbf{C}^k$ is an isomorphism. To such a triple corresponds two isomorphisms θ_0, θ_1 from $E_1 \otimes \mathbf{C}^k \otimes L_k$ to $E_2 \otimes \mathbf{C}^k$, with $\theta_0 = (1 \otimes \alpha_k) \circ (\theta \otimes 1)$ and $\theta_1 = (\theta \otimes 1) \circ (1 \otimes \alpha_k)$. The quadruple $(E_1 \otimes \mathbf{C}^k \otimes L_k, E_2 \otimes \mathbf{C}^k, \theta_0, \theta_1)$ defines then an element denoted $[E_1, E_2, \theta] \in K_1(A \otimes C(\hat{M}_k))$. In the exact sequence :

$$K_0(A \otimes C(M_k)) \to K_0(A \otimes C(\hat{M}_k)) \to K_0(A),$$

the pushforward of this element in $K_0(A)$ is zero, and thus $[E_1, E_2, \theta] \in K_1(A \otimes C(M_k))$ as the left arrow is injective. It is shown in [2] that $K_1(A, \mathbf{Z}/k\mathbf{Z})$ can be alternatively defined in this geometrical way, by setting the appropriate equivalence relations.

Let now A and B two C*-algebras, and $KK^j(A,B)$ the bivariant K-theory group of G. Kasparov [15]. One has a well defined cup-product

$$K_i(A,\mathbf{Q}/\mathbf{Z})\times KK^j(A,B)\to K_{i+j}(B,\mathbf{Q}/\mathbf{Z}),$$

uniquely determined by the relation, for $x\in K_j(A)$ and $y\in KK(A,B)$:

$$j_k(x)\otimes_A y=j_k(x\otimes_A y) \tag{10.3}$$

An element of $KK^1(A,B)$ can be represented by an unbounded cycle which is a closed B-module $(\mathcal{E},T)$ such that $\mathcal{E}$ is also a left A-module with $\mathcal{A}\subset A$ a dense subalgebra such that $[T,a]$ is bounded and densely defined for any $a\in\mathcal{A}$ [3].

Let $y=[\mathcal{E},T]\in KK^1(A,B)$ and $x=[E_1,E_2,\theta]\in K_1(A,\mathbf{Q}/\mathbf{Z})$. One may choose for $i=1,2$ a dense $\mathcal{A}$-submodule $E_i^0\subset E_i$ with $<E_i^0,E_i^0>\subset\mathcal{A}$, and an unbounded connection T^i for T on $\mathcal{E}_i=E_i\otimes_{\mathcal{A}}\mathcal{E}$ as in [25], theorem 1.8. Thus T_i is selfadjoint regular operator with essential domain $E_i^0\otimes\operatorname{dom}T$ and for any $\xi\in E_i^0$, the map :

$$\eta\to T_i(\xi\otimes\eta)-\xi\otimes T\eta$$

extends to a bounded adjointable map from E_i to $\mathcal{E}_i$.

The connections kT^1 and $(\theta\otimes 1)^*(kT^2)$ on $\mathbf{C}^k\otimes\mathcal{E}_1$ differ by a bounded element and thus are homotopic. Let $R(t)$, $t\in[0,1]$ be such a homotopy trivial near 0 and 1, and R the operator acting on $\mathcal{J}=\mathbf{C}^k\otimes\mathcal{E}_1\otimes L^2([0,1],\mathbf{C}^2)$ by the formula analogous to (5.1) :

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix}\to\begin{pmatrix}-\frac{\partial\eta}{\partial u}+R(t)\eta\\ \frac{\partial\xi}{\partial u}+R(t)\xi\end{pmatrix} \tag{10.4}$$

Then $(\mathcal{J},R)$ is a Hilbert module with boundary over B with boundary equal to $(\mathcal{E}_1\oplus\mathcal{E}_2,kT^1\oplus(-kT^2))$. By proposition 6.2, the space of projections $\mathcal{X}_+^\sigma(kT^1,kT^2)$ of definition 6.4 is nonempty.

Proposition 10.1. *Let $x=[E_1,E_2,\theta]\in K_1(A,\mathbf{Q}/\mathbf{Z})$ and $y=[\mathcal{E},T]\in KK^1(A,B)$ and $(\mathcal{J},R)$ the Hilbert module with boundary as defined above. Then $[\mathcal{J},R^P]\otimes H_k\in K_0(B,\mathbf{Z}_k)$ is independant of the choice of T_1,T_2 , of* *the homotopy $R(t)$, and of the choice of $P \in \mathcal{X}_+^\sigma(kT^1, kT^2)$. One has equality in $K_0(B, \mathbf{Q}/\mathbf{Z})$:*

$$x \otimes_A y = j_k([\mathcal{J}, R^P]).$$

Proof. The class of $[E_1, E_2, \theta]$ in $K_1(B, \mathbf{Z}_k)$ is represented by the quadruple $(E_1 \otimes \mathbf{C}^k \otimes L_k, E_2 \otimes \mathbf{C}^k, \theta_0, \theta_1)$ as above. Let K the projective module on $A \otimes C(M_k) \otimes C(S^1)$ constructed from $E_1 \otimes \mathbf{C}^k \otimes L_k \otimes C([0,1])$ and $E_2 \otimes \mathbf{C}^k \otimes C([0,1])$ and by identifying the fiber at $\{0\}$and $\{1\}$ using θ_0 and θ_1.

The product $[E_1, E_2, \theta] \otimes_A [\mathcal{E}, T]$ is now equal to the cup product $K_1(A \otimes C(S^1), \mathbf{Z}_k) \times KK(A \otimes C(S^1), B) \to K_0(B, Z_k)$ given by $[K] \otimes_{S^1} [T \otimes 1 + 1 \otimes \bar{\partial}]$ where $\bar{\partial}$ is the Dirac operator on S^1.

Theorem 9.6 and the associativity of the intersection product show that in $K_0(B \otimes C(\hat{M}_k))$, this last element is precisely equal to

$$[\mathcal{J}, R^P] \otimes [L_k] - [\mathcal{J}, R^P] = [\mathcal{J}, R^P] \otimes H_k,$$

which ends the proof.

□

11. R/Z homology theory

Given a smooth manifold Atiyah-Patodi-Singer have defined a K-theory group with coefficients in $\mathbf{R}/\mathbf{Z}$ [2]. Using the map j_k of (10.2), one defines :

$$j : K^0(V, Q) \simeq \varinjlim K^0(V) \to K^0(V, \mathbf{Q}/\mathbf{Z}),$$

and one take then $K^*(V, \mathbf{R}/\mathbf{Z})$ as the coimage of :

$$j \oplus \mathrm{ch} : K^0(V, Q) \to K^0(V, \mathbf{Q}/\mathbf{Z}) \oplus H^*(V, \mathbf{R}),$$

where $H^*(V, \mathbf{R})$ is the de Rham cohomology, and $\mathrm{ch} : K^*(V, \mathbf{R}) \simeq H^*(V, \mathbf{R})$ is the Chern caracter which is an isomorphism.

This K-theory where extended by M. Karoubi for Frechet algebras [12]. It appears to be a generalized cohomology theory and one the exact sequence :

$$K^1(V) \to K^1(V) \otimes \mathbf{R} \xrightarrow{\partial} K^1(V, \mathbf{R}/\mathbf{Z}) \to K^0(V) \to K^0(V) \times \mathbf{R}$$

In the presentation of [2], the map ∂ in this exact sequence becomes simply the inclusion $\omega \to 0 \oplus \omega$ followed by the canonical inclusion.

Given a unitary representation ρ of the fundamental group $\pi_1(V)$ into $\mathbf{C}^N$, one can associate a complex vector bundle E_ρ on V with flat connection ∇. There exists a positive integer k and a trivialisation $\theta : E_\rho \otimes \mathbf{C}^k \simeq V \times \mathbf{C}^{Nk}$, to which in turn corresponds a closed differential form

$$\mathrm{tch}(\nabla, \theta) = \frac{1}{k} \mathrm{Tch}(k\nabla, \theta^*(kd_N)),$$

where this latter is the *transgressed* caracteristic form between $k\nabla$ and the trivial connection d_N on $V \times \mathbf{C}^N$.

One defines then $[\rho] \in K^1(V, \mathbf{R}/\mathbf{Z})$ as the image of $[E_\rho, V \times \mathbf{C}^n, \theta] \oplus \omega(\nabla, \theta)$.

Suppose that V is spin and let A be the Dirac operator. The *index theorem for flat bundles* of [2] states that the cup product $[\rho] \otimes A \in \mathbf{R}/\mathbf{Z}$ is equal to the mod $\mathbf{Z}$ class of :

$$\frac{1}{2}(\eta(A^\rho) - N\eta(\rho) + \dim\ker A^\rho - \dim\ker A), \tag{11.1}$$

where $\eta(A^\rho)$ is the eta invariant of [1].

In order to extend this relation to all the group, J. Lott where then lead to introduce $K^1(V, \mathbf{R}/Z)$ for a manifold V, as follows [19, definition 5]: a cycle is quadruple $\mathcal{E} = (E, h, \nabla, \omega)$ where E is a complex vector bundle, h is an hermitian metric on E, ∇ is an euclidian connection and ω is a differential form on E defined up to $\operatorname{im} d$ and such that $d\omega = \mathrm{ch}(\nabla)$.

These cycles generate a group together wiht the appropriate equivalence relation ; we shall denote momently $\tilde{K}(V)$ this group. Given such a cycle $\mathcal{E} = (E, h, \nabla, \omega)$, then the relation $d\omega = \mathrm{ch}(\nabla)$ shows that there exist an isomorphism $\theta : E \otimes \mathbf{C}^k \simeq V \times \mathbf{C}^{Nk}$. Let $\mathrm{Tch}(\nabla, d_N) \in \Omega/\operatorname{im} d$ the transgresion form defined using θ. Then the assignment $\mathcal{E} \to [E, V \times \mathbf{C}^N, \theta] + [0, -\mathrm{Tch}(\nabla, d_N]$ gives rises to an isomorphism $K^1(V, R/Z) \to \tilde{K}(V)$.

Lemma 11.1. *Let V, W, Z be smooth manifolds. Then the intersection products $K^*(V, A) \times KK^*(V, W) \to K^*(W, A)$ for $A = \mathbf{Q}/\mathbf{Z}$ or $A = \mathbf{R}$ induce a* *well defined product :*

$$K^*(V,\mathbf{R}/\mathbf{Z}) \times KK^*(V,W) \to K^*(W,\mathbf{R}/\mathbf{Z})$$

For any $x \in K^*(V,\mathbf{R}/\mathbf{Z})$, $y \in KK(V,W)$ and $z \in KK(W,Z)$ one has (associativity):

$$x \otimes_V (y \otimes_W z) = (x \otimes_V y) \otimes_W z$$

Proof. Let $x \in K^*(V,\mathbf{Q})$ and $t \in KK(V,W)$. By (10.3), one has $j(x) \otimes t = j(x \otimes t)$, and $\mathrm{ch}(x) \otimes t = \mathrm{ch}(x \otimes t)$.

Thus the kernel of the map from $K^0(V,\mathbf{Q}/\mathbf{Z}) \oplus H^*(V,\mathbf{R}) \to K^*(W,\mathbf{R}/\mathbf{Z})$ contains $j \oplus \mathrm{ch}(K^0(V,Q))$.

□

12. $\mathbf{R}/\mathbf{Z}$-index for submersion

Let V,W two compact smooth manifolds and $f : V \to W$ a K-oriented submersion, which means that the fibers of f are spin manifolds. Suppose that the fibers of f are odd dimensional, or equivalently $\dim V = \dim W \equiv 1 \bmod 2$.

As $K^1(.,\mathbf{R}/\mathbf{Z})$ is oriented, the map f gives rise to a Gysin homomophism, $i = \dim\ker f + 1$:

$$f! : K^1(V,\mathbf{R}/\mathbf{Z}) \to K^i(W,\mathbf{R}/\mathbf{Z}),$$

which can be interpred as a topological index. As exposed in the introduction, we define an analytical index for f and show the equality between both indexes, and thus we solve a problem raised by J. Lott [19].

Let E be an hermitian complex vector bundle over V with ∇ an hermitian connection along the fibers. We can form the family A^E of the Dirac operators along the fibers of f, with coefficients in E, acting in the continuous field over $C(W)$ the fiber of which is the Hilbert space of L^2 sections of the spinor bundle with coefficients in E.

Under the hypothesis that $\{0\}$ is isolated in the spectrum of A^E, J.-M. Bismuth and J. Cheeger [4] and then X. Dai [?] have defined a smooth differential form $\bar{\eta}(A^E)$ on W, defined modulo $\operatorname{im} d$ as a generalization of the (classical) eta invariant.

This construction has been extended by R. Melrose and P. Piazza [21] by using pseudodifferential spectral sections. For such a projection P one can define a smooth differential form on W, defined up to $\operatorname{im} d$, denoted $\check{\eta}(A^E, P)$. If $\{0\}$ is isolated in the spectrum of the family, then with P_0 being the projector onto the spectral subspace generated by positive eigenvalues :

$$\check{\eta}(A^E, P_0) = \bar{\eta}(A^E) + \operatorname{ch} \ker A^E$$

Let (E, h, ∇, ω) be a $\mathbf{R}/\mathbf{Z}$-cycle . Then it defines a class $[E, \nabla, \omega] \in K^1(V, \mathbf{R}/\mathbf{Z})$ and its topological index is (we skip on the hermitian structure) :

$$\operatorname{Ind}_{\mathrm{t}}([E, \nabla, \omega]) = f!([E, \nabla, \omega])$$

Let A_k^E be the Dirac operator along the fibers of f acting in $E \otimes \mathbf{C}^k$ and A_{Nk} be the tensor product of the Dirac operator (without coefficients) along f with $\mathbf{C}^{Nk}$ and θ an isomorphism of $E \times \mathbf{C}^k$ with the trivial vector bundle of rank Nk. Then $\theta^*(A_{Nk})$ differs from A_k^E by a bounded operator and there exists a pseudodifferential spectral section P as defined in [21], and one has $P \in \mathcal{X}_+^\sigma(A_k^E, \theta^*(A_{Nk}))$ (definition 6.4).

If there exists a pseudodifferential spectral section R for A_k^E, then one can take $P = R \oplus \tilde{R}$ where $\tilde{R}$ is the canonical image of R in $\mathcal{X}_+^0(A_{Nk})$ defined in remark 6.3.

Lemma 12.1. *The differential form :*

$$\frac{1}{2}\check{\eta}(A_k^E \oplus \theta^*(-A_{Nk}), P) - f_*(\hat{A}(\ker f) \wedge \omega)$$

is closed and its image in $H^*(W, \mathbf{R})$ does not depend of the choice of P.

Here $\ker f$ is the tangent bundle along the fibers of f, or equivalently the kernel of the differential $df : TV \to TW$, and $\hat{A}(\ker f)$ its so-called $\hat{A}$-genus, a universal polynomial in the Pontryagin classes, and $f_*(\omega)$ is the integration of ω alongs the fiber of f. Let $\alpha = \frac{1}{2}\check{\eta}(A_k^E \oplus \theta^*(-A_{Nk}), P)$.

Proof. Let $(\mathcal{J}, R)$ be as in the last section the Hilbert module with boundary constructed from $(E, V \times \mathbf{C}^k, \theta)$ and P. As P comes from $\mathcal{X}^{\sigma}_{+}(A^E_k, \theta^*(A_{Nk}))$, the K-theory class is independant of P by theorem 9.6 and corollary 9.7. The index formula of [21, theorem 2] gives now :

$$\mathrm{ch}[\mathcal{J}, R^P] = kf_*(\hat{A}(\ker f) \wedge \mathrm{tch}(\nabla, \theta)) - k\alpha. \tag{12.1}$$

This shows that $\alpha - f_*(\hat{A}(\ker f) \wedge \mathrm{tch}(\nabla, \theta)$ is closed and so is $\alpha - f_*(\hat{A}(\ker f) \wedge \omega)$.

□

Then the analitycal index of (E, h, ∇, ω) is defined as :

$$\mathrm{Ind_a}([E, \nabla, \omega]) = \partial[\frac{1}{2}\check{\eta}(A^E_k \oplus \theta^*(-A_{Nk}), P) - f_*(\hat{A}(\ker f) \wedge \omega)]$$

The next theorem shows that this definition is independant of the choice of θ.

Theorem 12.2. *Let $f : V \to W$ be a smooth K-oriented submersion of smooth compact manifols, (E, ∇, ω) a geometric $\mathbf{R}/\mathbf{Z}$-cycle. Then $\mathrm{Ind}_a([E, h, \nabla, \omega])$ does not depend on θ and one has :*

$$\mathrm{Ind}_t([E, \nabla, \omega]) = \mathrm{Ind}_a([E, \nabla, \omega]).$$

Proof. Let ω a differential form representing a class $z \in H^*(V, \mathbf{R})$. The image $f!(z)$ is then represented by $f!(\omega \wedge \hat{A}(\ker f))$.

The module $(\mathcal{E}, A_1)$ defines a class in $KK^1(V, W)$. By the functoriality theorem of A. Connes and G. Skandalis [5], the cup-product by $[\mathcal{E}, A_1]$ induce a linear map $K^1(V, \mathbf{R}/\mathbf{Z}) \to K^i(W, \mathbf{R}/\mathbf{Z})$ which is Gysin map $f!$.

Keeping the notations of the last lemma, then by proposition 10.1 the image of $[E, \nabla, \omega]$ by $(\mathcal{E}, A_1)$ is $[\mathcal{J}, R^P] \oplus f_*(\hat{A}(\ker f) \wedge (\omega - \mathrm{Tch}(\nabla, d_N))$. The formula (12.1) shows that it is equivalent $\mathrm{mod} K_0(V, \mathbf{Q})$, to $0 \oplus [\alpha - f_*(\hat{A}(\ker f) \wedge \omega)]$.

□

Remark 12.3. One can check that $\mathrm{ch}(\mathrm{Ind_a}([E, \nabla, \omega])$ in $H^*(W; \mathbf{R}/\mathbf{Q})$ is given by the same formula of Definition 15 of [19].

13. Hilbert modules of coverings

Let V be a smooth riemannian oriented manifold, π a countable group and $\pi \to \tilde{V} \to V$ a Galois covering. We shall denote by B either the envelopping C*-algebra of $\mathbf{C}[\pi]$, either the reduced C*-algebra of π, which is the completion of the group algebra $\mathbf{C}[\pi]$ for the left representation of π in $\mathcal{L}(l^2(\pi))$. One defines on $C_c(\tilde{V}, \Lambda_{\mathbf{C}}(T^*(\tilde{V}))$ a product with value in B by the formula [7] :

$$< \xi, \eta > (t) = \int_{\tilde{V}} \bar{\xi(x)} \wedge \eta(t^{-1}x) > .$$

The completion of $C_c(\tilde{V}, \Lambda_{\mathbf{C}}(T^*(\tilde{V}))$ for this product is a Hilbert module denoted $\Omega_\pi(V)$. The signature operator D on $\tilde{V}$ acts on the space $C_c^\infty(\tilde{V}, \Lambda_{\mathbf{C}}(T^*(\tilde{V}))$ and extends to a closed symmetric regular operator on $\Omega_\pi(V)$, again denoted by D. If V is a complete riemannian manifold, then D is selfadjoint. If V is compact, then D is symmetric and $(1 + D^*D)^{-1}$ is a compact operator of $\Omega_\pi(V)$; if V is closed, then one gets a well defined class $[\Omega_\pi(V), D] \in K_*(B)$.

There is an alternative way of defining this class. Using the left representation of π in $C^*(\pi)$, one can form the flat bundle $\mathcal{V} = \tilde{V} \times_\pi C^*(\pi)$ over V. This is a Hilbert module over $C(V) \otimes C^*(\pi)$ which gives a class in $KK(\mathbf{C}, C(V) \otimes C^*(\pi))$. Let $\Sigma_V \in KK(C(V), \mathbf{C}) \simeq K_*(C(V))$ be the class defined by the signature operator. Then J. Rosenberg proved the following equality [23, theorem 3.3] :

$$[\Omega_\pi(V), D] = \mathcal{V} \otimes_{C(V)} \Sigma_V \tag{13.1}$$

A particular case of this construction occurs when V is a manifold with oriented boundary W, with $\dim V$ even. We get by the same construction a Hilbert module $\Omega_\pi(W)$ and we shall denote by A the signature operator acting on it. Let $\mathcal{U} \simeq [0,1] \times W$ be a normal neighboorhood, and suppose that the riemannian structure g on V satisfies:

$$g = dt^2 \oplus h$$

where h is the riemannian structure on W. Let τ the unitary involution coming from the grading of $\tilde{S}$.

Associated to $\mathcal{U}$ there is a projector s of $\Omega_\pi(V)$ and a canonical isomorphism of $\operatorname{im} s$ with $\Omega_\pi(W)\otimes L^2([0,1])\otimes \mathbf{C}^2$. For every $\omega \in C_c(\tilde{V}, \Lambda_{\mathbf{C}}(T^*(\tilde{V}))$ with support in the preimage of $\mathcal{U}$, under this isomorphim, one has $D\omega = \Psi(A)\omega$ [1, 10].

Lemma 13.1. *With respect to the canonical isomorphism* $\operatorname{im} s \simeq \Omega_\pi(W) \otimes L^2([0,1])\otimes \mathbf{C}^2$*, the closed module* $(\Omega_\pi(W), A)$ *has the structure of a boundary of the unbounded module* $(\Omega_\pi(V), D, \tau)$*.*

Proof. Let $\hat{V} = W\times]-\infty, 0] \cup V$ be the "elongated" manifold endowed with the riemannian metric $\hat{g}$ which is g on V and $dt^2 \oplus h$ on $W\times]-\infty,0]$. This is a complete manifold and the covering extends to a covering $\tilde{\hat{V}} \to \hat{V}$. The associated unbounded selfadjoint module $(\Omega_\pi(\hat{V}), \hat{D})$ is separated by $(\Omega_\pi(W), A)$, and the proof now follows from [11, prop. 3.9].

□

14. Higher classes of partitioned manifolds

Here we consider a singular oriented manifold (V, f) over $B\pi$, *i.e.* V is an oriented closed smooth manifold and $f : V \to B\pi$ is a continuous map from V to a classifying space of a countable group π. Let $M \subset V$ be a codimensional one manifold which partitions V into two manifolds with boundary Y_+ and Y_-. Thus one has an identification $\partial Y_\pm \simeq W$ and $V = Y_- \cup_W Y_+$.

Let $\theta : W \to W$ be a smooth oriented diffeomorphism and let $V_\theta = Y_- \cup_\theta Y_+$ be the oriented manifold obtained by identifying $x \in W$ with $\theta(x)$.

Let $U_+ \simeq [0,1[\times W$ (resp. $U_- \simeq]-1,0] \times W$) be a tubular neighboorhood of W in Y_+ (resp. Y_-), and h be a riemannian metric on W. Then one can fix a riemannian metric g_1 on V such that the restriction on U_+ or U_- is $dt^2 \oplus h$ and a riemannian metric g_2 on V_θ such that the restriction on U_+ is $dt^2 \oplus h$ and to U_- is $dt^2 \oplus \theta^*(h)$.

Let $\tilde{W} \to W$ be the covering of W corresponding to the restriction of f to W. We suppose that θ lift to a transformation $\tilde{\theta}$ of $\tilde{W}$ commuting with the action of π. Thus we get a covering $\tilde{V_\theta} \to V_\theta$ of structural group π.

The closed module $(\Omega_\pi(V), D)$ (resp. $(\Omega_\pi(V_\theta, D_\theta)$, $(\Omega_\pi(W), A))$ associated to the signature operator on V (resp. V_θ, W) defines element of $K_*(C^*(\pi))$.

Our aim here is to compare the K-theory class defined by $(\Omega_\pi(V), D)$ and $(\Omega_\pi(V_\theta), D_\theta)$.

Let $M_\theta = M \times_\theta S^1$ be the mapping torus of θ. We endow now $[0,1] \times W$ with a metric of the form $g_3 = dt^2 \oplus h(t)$ where $h(t)$ is a norm continous family of metric constant near 0 and 1 with $h_0 = h$ and $h(1) = \theta^*(h)$. This defines a metric on M_θ too. The mapping torus Z of $\tilde{W}$ by $\tilde{\theta}$ is a covering of M_θ. In particular, one obtains a class $[\Omega_\pi(Z), D_Z] \in K_0(C^*(\pi))$, where D_Z is the signature operator associated to g_3.

Theorem 14.1. *One has equality in $K_0(C^*(\pi))$:*

$$[\Omega_\pi(V), D] + [\Omega_\pi(Z), D_Z] = [\Omega_\pi(V_\theta), D_\theta].$$

Proof. As $\Omega_\pi(W)$ is a boundary, there exists $p \in \mathcal{X}_+(A)$ by proposition 4.5. Let S (resp. T) the symmetric signature operator on Y_+ (resp. Y_-) for the metric induced by g_1 and T_θ the symmetric signature operator on Y_- for the metric induced by g_2 (the symmetric signature operator induced on Y_+ by g_2 is equal to S). One has by theorem 9.6, $[D] = [S^p] + [T^{1-p}]$ and $[D_\theta] = [FAS^p] + [T_\theta^{1-\theta^*(p)}]$ and thus $[D_\theta] - [D] = [T_\theta^{1-\theta^*(p)}] - [T^{1-p}]$.

Let R be the selfadjoint extension of the symmetric signature operator on $[0,1] \times \tilde{W}$ associated to the projector $p \oplus \theta^*(1-p)$. Then by theorem 9.6 et proposition 9.1, $[R] = [\Omega_\pi(Z), D_Z]$, and by corollary 9.8 one has :

$$[T_\theta^{1-\theta^*(p)}] - [T^{1-p}] - [R] = 0,$$

which ends the proof.

$\square$

Now we look for sufficient conditions to have $[D_Z] = 0$.

Let $Z_t \to W$, for $t \in S^1$ the covering restriction of Z to $W \times \{t\}$. Let $\mathcal{E}$ be the Hilbert module over $C^*(\pi) \otimes C(S^1)$ which is the continuous field of Hilbert module over $C^*(\pi)$ with fiber $\Omega_\pi(Z_t)$ at $t \in S^1$. There is associated a continuous field of Dirac operators $\mathcal{A} = (A_t)_{t \in S^1}$. Using the bivariant K-theory [15], one gets :

Lemma 14.2. *The class of* $(\Omega_\pi(Z), D_Z)$ *in* $K_0(C^*(\pi))$ *is the intersection product of the class of* $(\mathcal{E}, \mathcal{A})$ *in* $KK(\mathbf{C}, C^*(\pi) \otimes C(S^1))$ *with the class of the signature operator* $[\bar{\partial}] \in KK(C(S^1), \mathbf{C})$.

Proof. The class of $(\Omega_\pi(Z), D_Z)$ in $K_0(C^*(\pi))$ is the intersection product of $\mathcal{V}$ with the signature class $\Sigma \in KK(C(M_\theta), \mathbf{C})$ (13.1).

By [5, 9], Σ is itself the intersection product of $\Sigma_p \in KK(C(M_\theta), C(S^1))$ with $[\bar{\partial}] \in KK(C(S^1), \mathbf{C})$, where Σ_p is defined by the continuous field of signature operators along the fibers of the canonical projection $p : M_\theta \to S^1$.

Thus, by associativity of intersection product [15]:

$$[\Omega_\pi(Z), D_Z] = (\mathcal{V} \otimes_{C(M_\theta)} \Sigma_p) \otimes_{C(S^1)} [\bar{\partial}].$$

Finally, to prove $\mathcal{V} \otimes_{C(M_\theta)} \Sigma_p = [\mathcal{E}, \mathcal{A}]$, it suffices to adapt the argument of the proof of (13.1) given in [23] in the context of the functor $RKK(C(V); ., .)$ developped by G. Kasparov [14].

□

A sufficient condition then to have invariance of higher class is thus given by the nullity of the class of $[\mathcal{E}, \mathcal{A}]$.

We have a graduation $\Omega_\pi(W) = \sum \Omega_\pi^k(W)$ corresponding to the degree of differential forms. Let d_b the lift to $\Omega_\pi(W)$ of the de Rham exterior derivative on W, and γ the Hodge involution [10]. As the signature operator $A = \gamma d_b + d_b \gamma$ leaves invariant the parity of degree, we shall work with the restriction A^0 of A to he subspace of differential forms of even degree $\Omega_\pi^{(0)} = \sum_{2k \geq 0} \Omega_\pi^{2k}(W)$. Without any loss of generality, one may assume that m is even, where $\dim W = 2m - 1$.

Let A_m the selfadjoint operator equal to $d_b\gamma$ on $\Omega^m_\pi(W)$. Suppose then that

$$\Omega^m_\pi(W) = \overline{\operatorname{im} A_m} \oplus \ker A_m. \tag{14.1}$$

This is equivalent to suppose that the closure of $\operatorname{im} A_m$ is the image of a selfadjoint projection $e \in \mathcal{L}(\Omega^m_\pi(V))$. This property is independant of the choice of the metric.

Remark 14.3. This assumption is verified if $\operatorname{im} A_m$ is closed.

Let $\theta \in \mathcal{L}(\Omega_\pi(W))$ be the unitary involution defined by $\theta(u) = u$ if $\partial u \leq m-1$ and $\theta(u) = -u$ if $\partial u \geq m$

By the computation of [1, p.67], the operator splits as a direct sum $A^0 = A_m + R$ where we identify A_m with a selfadjoint operator of $\operatorname{im} e$ with compact resolvent and R is a selfadjoint operator of $\ker A_m \oplus \sum_{2k\neq m} \Omega^{2k}_\pi(W)$ with compact resolvent. Moreover, one has :

$$R\theta + \theta R = 0 \tag{14.2}$$

Then one can associate an element of $\mathcal{X}^0_+(A^0)$ as follows : the spectral projection e_+ for the positive spectrum of A_m is defined. Let ζ be an *odd* smooth function on $]-\infty, +\infty[$ such that $\zeta(0) = 0$, $\zeta(u) \leq 1$ for all u and $\zeta(u) = 1$ for $u \geq u_0$ and some $u_0 > 0$, and let Q_0 be the following projector :

$$Q_0 = e_+ + \frac{1}{2}(1 - e + \zeta(R) + \theta\sqrt{1 - e - \zeta(R)^2}) \tag{14.3}$$

Lemma 14.4. *The operator Q_0 is a hermitian projector of $\Omega^{(0)}_\pi$ which belongs to $\mathcal{X}^0_+(A^0)$, and does not depend up to homotopy of the choice of ζ.*

Proof. One has :

$$Q_0^2 = p + \frac{1}{4}(1 - e + \zeta(R)^2 + 1 - e - \zeta(R)^2 + 2\zeta(R) + 2\theta\sqrt{1 - e - \zeta(R)^2} + \{\zeta(R), \theta\}\sqrt{1 - e - \zeta(R)^2} = p + \frac{1}{2}((1 - e + \zeta(R) + \theta\sqrt{1 - e - \zeta(R)^2}) = Q_0 \tag{14.4}$$

Thus Q_0 is an hermitian projector. If ζ_1 is another function satisfying the condition above, then $H(u,t) = (1-t)\zeta(u) + t\zeta(u)$ gives a homotopy between the corresponding projectors.

Finally, one checks that the image of Q_0 in the Calkin algebra is the direct sum of the image of e_+ and of the positive part of the image of R.

□

Now one can form $Q = Q_0 + \gamma Q_0 \gamma$: this is a projector in $\mathcal{X}_+^0(A)$.

Lemma 14.5. *If the closure* $\operatorname{im} A_m$ *is complemented in* $\Omega_\pi^m(W)$*, then* $[\mathcal{E}, \mathcal{A}] = 0$ *in* $K_1(B)$.

Proof. The operator $\mathcal{A}$ is a continuous family over S^1 the value at $t \in S^1$ being the signature operator associated to the riemannian metric $h(t)$. Thus one can form the projector Q_t by the same formula as (14.3). This gives a projector $\mathcal{Q}$ of the Hilbert module $\mathcal{E}$ which satisfies by construction $\mathcal{Q} \in \mathcal{X}_+^0(\mathcal{A})$. Thus $[\mathcal{E}, \mathcal{A}] = 0$ by proposition 4.5.

□

From the last theorem and the preceeding lemmas, we have now :

Corollary 14.6. *If the closure of* $\operatorname{im} A_m$ *is complemented in* $\Omega_\pi^m(W)$*, then one has equality :*

$$[\Omega_\pi(V), D] = [\Omega_\pi(V_\theta), D_\theta].$$

14.1. **The isometric case.** We make the assumption in this subsection that there exists a riemannian metric on W obeying $\theta^*(h) = h$. Thus θ is an isometry and there are easier criterias to get the nullity of $[\mathcal{E}, \mathcal{A}]$.

Lemma 14.7. *If there exists $p \in \mathcal{X}_+^0(A)$ such that $\theta^*(p) = p$, then $[\mathcal{E}, \mathcal{A}] = 0$.*

Proof. If there exists such a p, then one construct a projector $\mathcal{P} \in \mathcal{X}_+^0(\mathcal{A})$, and the conclusion follows from proposition 4.5.

□

Now we use the notion defined in example 4.4.

Corollary 14.8. *If there exists a spectral projector for A at λ, then*

$$[\Omega_\pi(V), D] = [\Omega_\pi(V_\theta), D_\theta].$$

Corollary 14.9. *If the relative commutant of π in the group of deck transformations of $\tilde{W}$ is reduced to the neutral element, then*

$$[\Omega_\pi(V), D] = [\Omega_\pi(V_\theta), D_\theta].$$

This holds in particular if the covering is galoisian and π without center.

15. Extensible cohomology of $B\pi$

Let V be a smooth riemannian oriented closed manifold, π a countable group and $\pi \to \tilde{V} \to V$ a Galois covering and c be a singular cohomology class on $B\pi$. Then the higher signature associated to c is defined as the number :

$$\mathcal{HS}(V, f, c) =< f^*[c] \cup L(V), [V] > \tag{15.1}$$

Let $\beta : K_0(B\pi) \to K_0(C^*(\pi))$ be the analytic assembly map of G. Kasparov [14].

Definition 15.1. *A cohomology class $c \in H^*(B\pi, \mathbf{R})$ is extensible if there exist an additive morphism $\varphi_c : K_0(B) \to \mathbf{C}$ such that for any $x \in K_0(B\pi)$ one has :*

$$\varphi_c(\beta(x)) =< \mathrm{ch}(x), c > .$$

Recall that π satisfies the strong Novikov conjecture if $\beta \otimes 1 : K_0(B\pi) \otimes \mathbf{Q} \to K_0(C^*(\pi)) \otimes \mathbf{Q}$ is injective. For a countable group π, the following assertions are equivalent:

i.- All cohomology classes on $B\pi$ are extensible.

ii.- The group π satisfies the strong Novikov conjecture.

As one can take for φ_c any map which extends c on $\operatorname{im} \beta_{\mathbf{Q}}$, one has *i.* implies *ii.* Inversely, if $x \in K_0(B\pi) \otimes \mathbf{Q}$ satisfies $\beta(x) = 0$, then $\mathrm{ch}(x) = 0$ and $x = 0$ in $K_0(B\pi) \otimes \mathbf{Q}$ as the chern caracter is an isomorphism.

Proposition 15.2. *If c is extensible, then one has the equality :*

$$< \varphi_c, [\Omega_\pi(V), D] >= \mathcal{HS}(V, f, c) \tag{15.2}$$

Proof. This equality follows from relation (13.1).

□

Proposition 15.3. *If $\Omega_\pi^m(W) = \ker A_m \oplus \overline{\operatorname{im} A_m}$, then for any extensible cohomology class in $B\pi$, one has :*

$$\mathcal{HS}(V, f, c) = \mathcal{HS}(V_\theta, f_\theta, c)$$

Proposition 15.4. *If $\theta^*(h) = h$, and if the condition of corollaries 14.6, 14.8 or 14.9 are fullfilled, for any extensible cohomology class in $B\pi$, one has :*

$$\mathcal{HS}(V, f, c) = \mathcal{HS}(V_\theta, f_\theta, c)$$

16. Cut and past invariance of higher signatures

Let (V_1, f_1) and (V_2, f_2) two closed oriented manifolds over $B\pi$. Then we say that (V_1, f_1) and (V_2, f_2) are cut-and-paste equivalent if the following conditions are fullfilled :

a.- There exists two manifolds with boundary A and B and an oriented diffeomorphism $\varphi_i : \partial A \to \partial B$ such that $V_i = A \cup_{\varphi_i} B$.

b.- The restrictions of f_1 and f_2 to A (resp. B) are homotopic.

This notion were studied by U. Karras, M. Kreck, W. Neumann, E. Ossa [13] and lead to the definition of a group $SK(B\pi)$. If (V_1, f_1) and (V_2, f_2) are cut-and-paste equivalent, then they define the same element in $SK(B\pi)$.

J. Lott and S. Weinberger [20] raised the question of finding condition under which higher signatures are invariant under cut-and-paste equivalence.

Let $W = \partial V_1$ and $\tilde{W}$ the covering associated to the restriction of f_1 to W. The map $\theta = \varphi_2^{-1} \circ \varphi_1$ is an oriented diffeomorphism of W.

Let $\tilde{A} \to A$ (resp. $\tilde{B} \to B$) be the coverings determined by the restrictions of f_1 to A (resp. B).

Lemma 16.1. *There exists a π-equivariant lift $\tilde{\theta}$ of θ such that V_2 is isomorphic to $A \cup_\theta B$ and such that the covering $\tilde{A} \cup_{\tilde{\theta}} \tilde{B}$ is isomorphic to the covering associated to f_2.*

Proof. Consider for $i = 1, 2$ the restrictions of f_i to W (resp. ∂B). By hypothesis, there exists an homotopy $H(z, u)$, $z \in W$, $t \in [0, 1]$ with $H(z, 0) = f_1(z)$ and $H(z, 1) = f_2(z)$. There exists another homotopy $K : \partial B \times [0, 1] \to B\pi$ with $K(z, 0) = f_1(\varphi_1(z))$ and $K(z, 1) = f_2(\varphi_2(z))$.

By pasting these homotopies, one gets a continuous map $W \times_\theta S^1 \to B\pi$, and thus a covering of $W \times_\theta S^1$. By parallel transport along S^1, one obtains the automorphism $\tilde{\theta}$ of $\tilde{W}$.

$\square$

Thus we are in the situation of section 14. By translating the results of the previous sections, and keeping the same notations, one then can asserts that extensible higher signatures are cut-an-paste invariant :

A - If the closure of $\operatorname{im} A_m$ is complemented in $\Omega_\pi^m(W)$.

B - If $\theta^*(h) = h$, and there exists a spectral projector for A.

C - If $\theta^*(h) = h$ and the relative commutant of π in the group of deck transformations of $\tilde{W}$ is reduced to the neutral element.

In particular, if π satisfies the strong Novikov conjecture then in the previous conditions, all the higher signatures of V_1 and V_2 with coefficients in $B\pi$ are the same.

Remark 16.2. The cut-and-paste invariant was established in [17, corollary 0.4] under the hypothesis that $\operatorname{im} A_m$ closed in $\Omega_\pi^m(W)$ and that π is hyperbolic group, and in [18, corollary 0.7] under the hypothesis that $\operatorname{im} A_m$ closed in $\mathcal{E}_\pi^m(W)$ and that the assembly map $K_*(B\pi) \to K_*(C^*(\pi))$ is injective.

We refer to [18] for counter-examples to the cut-and paste invariance of higher signature when the above conditions are not satisfied.

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