# Open Diophantine Problems

by

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## Content



## 0. Introduction

The International Year of Mathematics 2000 offered an opportunity to recall Hilbert's twenty three problems of 1900 and to consider some of the main conjectures in different branches of mathematics. We focus here on Diophantine Analysis, to which Hilbert 7th and 10th problem belong.

We collect a number of open questions concerning Diophantine equations (including Catalan and Pillai Conjectures), Diophantine approximation (featuring the $abc$ conjecture) and transcendental number theory (with, for instance, Schanuel's Conjecture). Some questions related with Mahler's measure and Weil absolute logarithmic height are then considered (e.g. Lehmer's Problem). We also discuss Mazur's question on the density of rational points on a variety, especially in the special case of algebraic groups, in connection with transcendence problems in several variables. We say only a few words on metric problems, equidistribution questions, Diophantine approximation on manifolds and Diophantine analysis on function fields.

`http://www.dm.unipi.it/hilbertoday/abs2`
`http://www.math.jussieu.fr/~miw/articles/odp.ps`

# 1. Diophantine Equations

## 1.1. Points on Curves

Among the 23 problems of Hilbert [Hi 1900], [Gu 2000] the tenth one is the shortest statement:

*Given a Diophantine equation with any number of unknown quantities and with integral numerical coefficients: To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.*

An equation of the form $f(\underline{x}) = 0$, where $f \in \mathbb{Q}[X_1, \ldots, X_n]$ is a given polynomial, while the unknowns $\underline{x} = (x_1, \ldots, x_n)$ are rational integers, is a *Diophantine equation*. To solve this equation amounts to determine the integer points on the corresponding hypersurface of the affine space. Hilbert's tenth problem is to give an algorithm which tells whether such a given Diophantine equation has a solution or not.

There are other types of Diophantine equations. First of all one may consider rational solutions instead of integer ones: in this case one considers rational points on a hypersurface. Next one may consider integer or rational points over a number field. There is a situation which is intermediate between integer and rational points, where the unknowns are *$S$-integral points*. This means that $S$ is a fixed finite set of prime numbers (rational primes, or prime ideals in the number field), and the denominators of the solutions are restricted to belong to $S$. Examples are the Thue-Mahler equation

$$F(x, y) = p_1^{z_1} \cdots p_k^{z_k}$$

where $F$ is a homogeneous polynomial and $p_1, \ldots, p_k$ fixed primes (the unknowns are $x, y, z_1, \ldots, z_k$) and the generalized Ramanujan-Nagell equation $x^2 + D = p^n$ where $D$ is a fixed integer, $p$ a fixed prime, and the unknowns are $x$ and $n$ (see for instance [ShT 1986], [Ti 1998] [Sh 1999] and [BuSh 2001] for these and similar other questions).

Also, it is interesting to deal with simultaneous Diophantine equations, i.e. to investigate rational or integer points on algebraic varieties.

The final answer to Hilbert original 10th problem has been given in 1970 by Yu. Matiyasevich, after the works of M. Davis, H. Putnam and J. Robinson. This was the culminating stage of a rich and beautiful theory (see [DMR 1976], [Mat 1999] and [Mat 2001]). The solution is negative: there is no hope nowadays to achieve a complete theory of the subject. But one may still hope that there is a positive answer if one restricts Hilbert's initial question to equations in few variables, say $n = 2$, which amounts to considering integer points on a plane curve. In this case deep results have been achieved during the 20th century and many results are known, but much more remains unveiled.

The most basic results are Siegel's (1929) and Faltings' (1980) ones. Siegel's Theorem deals with integer points and produces an algorithm to decide whether the set of solutions form a finite or an infinite set. Faltings result, solving Mordell's Conjecture, does the same for rational solutions, i.e. rational points on curves. To these two outstanding achievements of the 20th century, one may add Wiles' contribution which not only settles Fermat's last Theorem, but also provides a quantity of similar results for other curves [K 1999].

Some natural questions occur:

a) To answer Hilbert's tenth problem for this special case of plane curve, which means to give an algorithm to decide whether a given Diophantine equation $f(x, y) = 0$ has a solution (in $\mathbb{Z}$, and the same problem for $\mathbb{Q}$).

b) To give an upper bound for the number of either rational or integral points on a curve.

c) To give an algorithm for solving explicitly a given Diophantine equation.

Further questions may be asked. For instance in question b) one might ask for the exact number of solutions; it may be more relevant to consider more generally the number of points on any number field, or the number of points of bounded degree.

Our goal here is not to describe in detail the state of the art on these questions (see for instance [L 1991]). It will suffice to say that a complete answer to question a) is not yet available, that a number of results are known on question b) (the latest work on this topic being due to G. Rémond [Re 2000]), and that question c) is unanswered even for integer points, and even for the special case of curves of genus 2. We do not require a practical algorithm, but only (to start with) for a theoretical one. So our first open problem will be an effective refinement to Siegel's Theorem.

**Problem 1.1.** *Let $f \in \mathbb{Z}[X,Y]$ be a polynomial such that the equation $f(x,y)=0$ has only finitely many solutions $(x,y) \in \mathbb{Z}\times\mathbb{Z}$. Give an upper bound for* max$\{|x|,|y|\}$ *when $(x,y)$ is such a solution, in terms of the degree of $f$ and of the maximal absolute value of the coefficients of $f$.*

That such a bound exists is part of the hypothesis, but the question is to give it explicitly (and, if possible, in a close form).

Further similar questions might also be asked for more variables (rational points on varieties), for instance Schmidt's norm form equations. We refer the reader to [L 1974] and [L 1991] for such questions, including the Lang-Vojta Conjectures.

Even the simplest case of quadratic forms gives rise to open problems: the determination of all positive integers which are represented by a given binary form is far from being solved. Also it is expected that infinitely many real quadratic fields have class number one, but it is not even known that there are infinitely many number fields (without restriction) with class number one. Recall that the first complete solution of Gauß' class number 1 and 2 problems (for imaginary quadratic fields) have been reached by transcendence methods (Baker and Stark) so it has been a Diophantine problem! Nowadays more efficient methods (Goldfeld, Gross-Zagier,...– see [L 1991] Chap. V § 5) are available.

A related open problem is the determination of Euler *numeri idonei* [Ri 2000]. Fix a positive integer $n$. If $p$ is an odd prime for which there exist integers $x \geq 0$ and $y \geq 0$ with $p = x^2 + ny^2$, then

(i) gcd$(x, ny) = 1$

(ii) the equation $p = x_1^2 + ny_1^2$ in integers $x_1 \geq 0$ and $y_1 \geq 0$ has the only solution $x_1 = x$ and $y_1 = y$.

Now let $p$ be an odd integer such that there exist integers $x \geq 0$ and $y \geq 0$ with $p = x^2 + ny^2$ and such that the conditions (i) and (ii) above are satisfied. If these properties imply that $p$ is prime, then the number $n$ is called *numeri idonei*. Euler found 65 such integers $n$:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 16, 18, 21, 22, 24, 25, 28, 30, 33, 37, 40, 42, 45, 48, 57, 58, 60, 70, 72, 78, 85, 88, 93, 102, 105, 112, 120, 130, 133, 165, 168, 177, 190, 210, 232, 240, 253, 273, 280, 312, 330, 345, 357, 385, 408, 462, 520, 760, 840, 1320, 1365, 1848.

It is known that there is at most one more number in the list, but one expects there is no other one.

Here is just one example ([Sie 1964] problem 58 p. 112; [Guy 1994] D18) of an open problem dealing with simultaneous Diophantine quadratic equations: *Is there a perfect integer cuboid?* The existence of a box with integer edges $x_1, x_2, x_3$, face diagonals $y_1, y_2, y_3$ and body diagonal $z$, amounts to solving the system of four simultaneous Diophantine equations in seven unknowns

$$\begin{aligned} x_1^2 + x_2^2 &= y_3^2 \\ x_2^2 + x_3^2 &= y_1^2 \\ x_3^2 + x_1^2 &= y_2^2 \\ x_1^2 + x_2^2 + x_3^2 &= z^3 \end{aligned}$$

in $\mathbb{Z}$. We don't know whether there is a solution, but it is known that there is no perfect integer cuboid with the smallest edge $\leq 2^{31}$.

### 1.2. Exponential Diophantine Equations

In a Diophantine equation, the unknowns occur as the variables of polynomials, while in an *exponential Diophantine equation* (see [ShT 1986]), some exponents also are variables. One may consider the above-mentioned Ramanujan-Nagell equation $x^2+D=p^n$ as an exponential Diophantine equation.

A famous open problem is Catalan's one which dates back to 1844 [Cat 1844], the same year where Liouville constructed the first examples of transcendental numbers (see also [Sie 1964] problem 77 p. 116; [Sie 1970] n° 60 p. 42; [ShT 1986] Chap. 12; [N 1986] Chap. 11; [Ri 1994]; [Guy 1994] D9; [Ri 2000] Chap. 7). The «Note extraite d'une lettre adressée à l'Éditeur par Monsieur E. Catalan, Répétiteur à l'école polytechnique de Paris», published in Crelle Journal [Cat 1844], reads:

> «*Je vous prie, Monsieur, de bien vouloir énoncer, dans votre recueil, le théorème suivant, que je crois vrai, bien que je n'aie pas encore réussi à le démontrer complètement: d'autres seront peut-être plus heureux:*
>
> *Deux nombres entiers consécutifs, autres que* 8 *et* 9*, ne peuvent être des puissances exactes; autrement dit: l'équation $x^m - y^n = 1$, dans laquelle les inconnues sont entières et positives, n'admet qu'une seule solution.*»

**Conjecture 1.2** (Catalan). *The equation*

$$x^p - y^q = 1,$$

*where the unknowns $x$, $y$, $p$ and $q$ are integers all $\geq 2$, has only one solution $(x,y,p,q)=(3,2,2,3)$.*

In other terms, the only example of consecutive numbers which are perfect powers $x^p$, $p \geq 2$, is $(8,9)$. Further information on this topic is available in Ribenboim's book [Ri 1994]. Tijdeman's result [Ti 1976b] in 1976 shows that there are only finitely many solutions. More precisely, for any solution $x,y,p,q$, the number $\max\{p,q\}$ can be bounded by an effectively computable absolute constant. Once $\max\{p,q\}$ is bounded, only finitely many exponential Diophantine equations remain to be considered, and there are algorithms to complete the solution (based on Baker's method). Such a bound has been computed, but it is somewhat large: according to M. Mignotte, any solution $x,y,p,q$ to Catalan's equation should satisfy

$$\max\{p,q\} < 10^{18}.$$

Catalan asked for integral solutions, like in Siegel's Theorem, while Falting's Theorem deals with rational points. Dipendra Prasad asked me whether it is reasonable to expect that the set of tuple $(x,y,p,q)$ in $\mathbb{Q}^2 \times \mathbb{N}^2$ satisfying the conditions

$x^p - y^q = 1$, *and the curve* $X^p - Y^q = 1$ *has genus* $\geq 1$

should be finite. This indeed would follow from the $abc$ Conjecture (see § 2.1).

The fact that the right hand side in Catalan's equation is 1 is crucial: nothing is known if one replaces it by another positive integer. The next conjecture was proposed by S.S. Pillai [Pi 1945] at a conference of the Indian Mathematical Society held in Aligarh (see also [Sie 1964] problem 78 p. 117; [ShT 1986]; [Ti 1998]; [Sh 1999]).

**Conjecture 1.3** (Pillai). *Let $k$ be a positive integer. The equation*

$$x^p - y^q = k,$$

*where the unknowns $x$, $y$, $p$ and $q$ are integers all $\geq 2$, has only finitely many solutions $(x, y, p, q)$.*

This means that in the increasing sequence of perfect powers $x^p$, with $x \geq 2$ and $p \geq 2$:

$$4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, 144, 169, \ldots,$$

the difference between two consecutive terms tends to infinity. It is not even known that for, say, $k = 2$, Pillai's equation has only finitely many solutions. A related open question is whether the number 6 occurs as a difference between two perfect powers: *Is there a solution to the diophantine equation $x^p - y^q = 6$?* (see [Sie 1970] problem 238a p. 116).

A conjecture which implies Pillai's one has been suggested by T.N. Shorey in [Sh 2001]. Let $f \in \mathbb{Z}[X]$ be a polynomial of degree $n$ with at least two distinct roots and $f(0) \neq 0$. Let $L$ be the number of nonzero coefficients of $f$: write

$$f(X) = b_1 X^{n_1} + \cdots + b_{L-1} X^{n_{L-1}} + b_L$$

with $n = n_1 > n_2 > \cdots > n_{L-1} > 0$ and $b_i \neq 0$ $(1 \leq i \leq L)$. Set $H = \mathrm{H}(f) = \max_{1 \leq i \leq L} |b_i|$.

**Conjecture 1.4** (Shorey). *There exists a positive number $C$ which depends only on $L$ and $H$ with the following property. Let $m$, $x$ and $y$ be rational integers with $m \geq 2$ and $|y| > 1$ satisfying*

$$y^m = f(x).$$

*Then either $m \leq C$, or else there is a proper subsum in*

$$y^m - b_1 x^{n_1} - \cdots - b_{L-1} x^{n_{L-1}} - b_L$$

*which vanishes.*

Consider now the positive integers which are perfect powers $y^q$, with $q \geq 2$, and such that all digits in some basis $x \geq 2$ are 1's. Examples are 121 in basis 3, 400 in basis 7 and 343 in basis 18. To find all solutions amounts to solve the exponential Diophantine equation

$$\frac{x^n - 1}{x - 1} = y^q,$$

where the unknown $x, y, n, q$ are positive rational integers with $x \geq 2$, $y \geq 1$, $n \geq 3$ and $q \geq 2$. Only 3 solutions are known:

$$(x, y, n, q) = (3, 11, 5, 2),\ (7, 20, 4, 2),\ (18, 7, 3, 3),$$

corresponding to

$$\frac{3^5 - 1}{2} = 11^2, \quad \frac{7^4 - 1}{6} = 20^2, \quad \frac{18^3 - 1}{17} = 7^3$$

and one does not know whether these are the only solutions (see [ShT 1986]; [Guy 1994] D10; [Ti 1998]; [Sh 1999], [Sh 2001] and [BuM 1999]) and it is expected that there is no other one.

The next question is to determine all integers with identical digits in some basis, which amounts to solve the equation

$$z\frac{x^n - 1}{x - 1} = y^q,$$

where the unknown $x, y, n, q, z$ are positive rational integers with $x \geq 2$, $y \geq 1$, $n \geq 3$, $1 \leq z < y$ and $q \geq 2$.

Another type of exponential Diophantine equation has been studied in a joint paper by H.P. Schlickewei and W.M. Schmidt [ScV 2000] where they state the following conjecture.

**Conjecture 1.5.** *Let $k \geq 2$ be an integer and $\alpha_1, \ldots, \alpha_n$ be non-zero elements in a field $K$ of zero characteristic, such that no quotient $\alpha_i/\alpha_j$ with $j \neq i$ is a root of unity. Consider the function*

$$F(X_1, \ldots, X_k) = \det \begin{pmatrix} \alpha_1^{X_1} & \cdots & \alpha_k^{X_1} \\ \vdots & \ddots & \vdots \\ \alpha_1^{X_k} & \cdots & \alpha_k^{X_k} \end{pmatrix}.$$

*Then the equation*

$$F(0, x_2, \ldots, x_k) = 0$$

*has only finitely many solutions $(x_2, \ldots, x_k) \in \mathbb{Z}^{k-1}$ such that in the corresponding determinant of Conjecture 1.5, all $(k-1) \times k$ and all $k \times (k-1)$ submatrices have rank $k-1$.*

Further exponential Diophantine equations are worth of study. See for instance [N 1986] Chap. III, [ShT 1986], [Ti 1998] and [Sh 1999].

Among the numerous applications of Baker's transcendence method are several questions related with the greatest prime factors of certain numbers. In this connection we mention Grimm's Conjecture ([Gri 1969], [N 1986] Chap. III § 3, [Guy 1994] B32):

**Conjecture 1.6** (Grimm). *Given $k$ consecutive composite integers $n+1, \ldots, n+k$, there exist $k$ distinct primes $p_1, \ldots, p_k$ such that $n+j$ is divisible by $p_j$, $1 \leq j \leq k$.*

M. Langevin [La 1977] rephrased this conjecture as follows: given an increasing sequence of positive integers $n_1 < \cdots < n_\ell$ for which the product $n_1 \cdots n_\ell$ has less than $\ell$ distinct prime factors, there is a prime $p$ in the range $n_1 \leq p \leq n_\ell$.

Following P. Erdős and R.G. Selfridge, a consequence is Conjecture 1.6 is that each interval $[N^2, (N+1)^2]$, $N \geq 1$, should contain a prime number.

A weaker form of Conjecture 1.6, also open, is:

**Conjecture 1.7.** *If there is no prime in the interval $[n+1, n+k]$, then the product $(n+1)\cdots(n+k)$ has at least $k$ distinct prime divisors.*

M. Langevin [La 1977] extended Grimm's Conjecture to arithmetical progressions, and also suggested a stronger statement than Conjecture 1.6:

**Conjecture 1.8** (Langevin). *Given an increasing sequence $n_1 < n_2 < \cdots < n_k$ of positive integers such that $n_1, n_2, \ldots, n_k$ are multiplicatively dependent, there exists a prime number in the interval $[n_1, n_k]$.*

Even if they may not be classified as Diophantine questions, the following open problems (see [L 1996]) are related to this topic: the twin prime conjecture, Goldbach problem (*is every even integer $\geq 4$ the sum of two primes?*), Bouniakovsky's conjecture, Schinzel's hypothesis (H) (see also [Sie 1964] § 29) and Bateman-Horn's Conjecture.

The Diophantine equation

$$x^p + y^q = z^r$$

has also a long history in relation with Fermat's last Theorem ([K 1999], [Ri 2000] § 9.2.D). If we look at the solutions in positive integers $(x, y, z, p, q, r)$ for which

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$$

and such that $x$, $y$, $z$ are relatively prime, then only 10 solutions are known, and the $abc$ Conjecture (see § 2.1) predicts anyway that the set of such solutions is finite. For all known solutions, one of $p$, $q$, $r$ is 2; this led R. Tijdeman and D. Zagier to conjecture that there is no solution with the further restriction that each of $p$, $q$ and $r$ is $\geq 3$.

### 1.3. Markoff Spectrum

The original Markoff(*) equation (1879) is $x^2+y^2+z^2=3xyz$ (see [Ca 1957] Chap. II; [CuFl 1989] Chap. 2; [Guy 1994] D12 and [Ri 2000] § 10.5.B). Here is an algorithm which produces all solutions in positive integers. Given any solution $(x,y,z)=(m,m_1,m_2)$, we fix two of the three coordinates; then we get a quadratic equation in the third coordinate, for which we already know a solution. By the usual process of cutting with a rational line we deduce another solution. This produces from one solution $(m,m_1,m_2)$ three other solutions

$$(m',m_1,m_2), \quad (m,m_1',m_2), \quad (m,m_1,m_2'),$$

where

$$m'=3m_1m_2-m, \quad m_1'=3mm_2-m_1, \quad m_2'=3mm_1-m_2.$$

These three solutions are called *neighbours* of the original one. Apart from the two so-called *singular* solutions $(1,1,1)$ and $(2,1,1)$, the three components of $(m,m_1,m_2)$ are pairwise distinct, and the three neighbours of $(m,m_1,m_2)$ are pairwise distinct. Assuming $m>m_1>m_2$, then one checks

$$m_1'>m_2'>m>m'.$$

Hence there is one neighbour of $(m,m_1,m_2)$ with maximum component less than $m$, and two neighbours with maximum component greater than $m$. It easily follows that one produces all solutions, starting from $(1,1,1)$, by taking successively the neighbours of the known solutions. Here is the Markoff tree, with the notation of Harvey Cohn [Co 1993], where $(m_1',m,m_2)$ is written on the left and $(m_2',m,m_1)$ on the right:

- $(1,1,1)$
  - $(2,1,1)$
    - $(5,1,2)$
      - $(29,5,2)$
        - $(433,29,5)$
          - ⋮
          - ⋮
        - $(169,29,2)$
          - ⋮
          - ⋮
      - $(13,5,1)$
        - $(194,13,5)$
          - ⋮
          - ⋮
        - $(34,13,1)$
          - ⋮
          - ⋮

The main open problem on this topic ([Ca 1957] p. 33, [CuFl 1989] p. 11 and [Guy 1994] D12) is to prove that each largest component occurs only once in a triple of this tree:

(*) His name is spelled *Markov* in probability theory.

**Conjecture 1.9.** *Fix a positive integer $m$ for which the equation*

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2$$

*has a solution in positive integers $(m_1, m_2)$ with $0 < m_1 \leq m_2 \leq m$. Then such a pair $(m_1, m_2)$ is unique.*

This conjecture has been checked for $m \leq 10^{105}$.

The sequence

$$1,\ 2,\ 5,\ 13,\ 29,\ 34,\ 89,\ 169,\ 194,\ 233,\ 433,\ 610,\ 985,\ 1325,\ 1597, \ldots$$

of integers $m$ satisfying the hypotheses of Conjecture 1.9 is closely related with the question of best possible rational approximation to quadratic irrational numbers: for each $m$ in this sequence there is an explicit quadratic form $f_m(x, y)$ such that $f_m(x, 1) = 0$ has a root $\alpha_m$ for which

$$\limsup_{q \to \infty} \bigl| q(q\alpha_m - p) \bigr| = \frac{m}{\sqrt{9m^2 - 4}}. \tag{1.10}$$

The sequence of $(m, f_m, \alpha_m, \mu_m)$ with $\mu_m = \sqrt{9m^2 - 4}/m$ starts as follows:

| $m$ | 1 | 2 | 5 | 13 |
|---|---|---|---|---|
| $f_m(x,1)$ | $x^2 + x - 1$ | $x^2 + 2x - 1$ | $5x^2 + 11x - 5$ | $13x^2 + 29x - 13$ |
| $\alpha_m$ | $\overline{1}$ | $\overline{2}$ | $\overline{2211}$ | $\overline{221111}$ |
| $\mu_m$ | $\sqrt{5}$ | $\sqrt{8}$ | $\sqrt{221}/5$ | $\sqrt{1517}/13$ |

The third row gives the continued fraction expansion for $\alpha_m$, where $\overline{2211}$, for instance, stands for $[2, 2, 1, 1, 2, 2, 1, 1, 2, 2, 1, 1, \ldots]$. Conjecture 1.9 amounts to claim that there is no ambiguity in the notation $f_m$: given $m$, two quadratic numbers $\alpha_m$ satisfying (1.10) should be roots of equivalent quadratic forms.

Hence Markoff spectrum is closely related to rational approximation to a single real number. A generalization to simultaneous approximation is considered in the § 2.2.

## 2. Diophantine Approximation

In this section we restrict ourself to problems in Diophantine approximation which do not require introducing a notion of height for algebraic numbers: these will be discussed in § 4 only.

### 2.1. *abc* Conjecture

For a positive integer $n$, we denote by

$$R(n) = \prod_{p|n} p$$

the *radical* or *squarefree part* of $n$.

The $abc$ conjecture is born from a discussion between D.W. Masser and J. Œsterlé ([Œ 1988] p. 169; see also [Ma 1990], as well as [L 1990], [L 1991] Chap. II § 1; [L 1993] Ch. IV § 7; [Guy 1994] B19; [Br 1999]; [Ri 2000] § 9.4.E; [V 2000]; [Maz 2000] and [2]).

**Conjecture 2.1** ($abc$ Conjecture). *For each $\epsilon > 0$ there exists a positive number $\kappa(\epsilon)$ which has the following property: if $a$, $b$ and $c$ are three positive rational integers which are relatively prime and satisfy $a + b = c$, then*

$$c < \kappa(\epsilon) R(abc)^{1+\epsilon}.$$

Conjecture 2.1 implies a previous conjecture by L. Szpiro on the conductor of elliptic curves: *Given any $\epsilon > 0$, there exists a constant $C > 0$ such that, for every elliptic curve with minimal discriminant $\Delta$ and conductor $N$, we have $|\Delta| < N^{6+\epsilon}$.*

When $a$, $b$ and $c$ are three positive relatively prime integers satisfying $a + b = c$, define

$$\lambda(a, b, c) = \frac{\log c}{\log R(abc)}.$$

and

$$\varrho(a, b, c) = \frac{\log abc}{\log R(abc)}.$$

Here are the six largest known values for $\lambda(abc)$ (in [Br 1999] p. 102–105 as well as in [2], one can find all the 140 known values of $\lambda(a, b, c)$ which are $\geq 1.4$).

| | $a + b = c$ | $\lambda(a, b, c)$ | author(s) |
|---|---|---|---|
| 1 | $2 + 3^{10} \cdot 109 = 23^5$ | $1.629912\ldots$ | É. Reyssat |
| 2 | $11^2 + 3^2 5^6 7^3 = 2^{21} \cdot 23$ | $1.625991\ldots$ | B. de Weger |
| 3 | $19 \cdot 1307 + 7 \cdot 29^2 \cdot 31^8 = 2^8 \cdot 3^{22} \cdot 5^4$ | $1.623490\ldots$ | J. Browkin – J. Brzezinski |
| 4 | $283 + 5^{11} \cdot 13^2 = 2^8 \cdot 3^8 \cdot 17^3$ | $1.580756\ldots$ | J. Browkin – J. Brzezinski, A. Nitaj |
| 5 | $1 + 2 \cdot 3^7 = 5^4 \cdot 7$ | $1.567887\ldots$ | B. de Weger |
| 6 | $7^3 + 3^{10} = 2^{11} \cdot 29$ | $1.547075\ldots$ | B. de Weger |

Here are the six largest known values for $\varrho(abc)$, according to [2], where one can find the complete list of 46 known triples $(a, b, c)$ with $0 < a < b < c$, $a + b = c$ and $\gcd(a, b) = 1$ satisfying $\varrho(a, b, c) > 4$.

| | $a + b = c$ | $\varrho(a, b, c)$ | author(s) |
|---|---|---|---|
| 1 | $13 \cdot 19^6 + 2^{30} \cdot 5 = 3^{13} \cdot 11^2 \cdot 31$ | 4.41901 . . . | A. Nitaj |
| 2 | $2^5 \cdot 11^2 \cdot 19^9 + 5^{15} \cdot 37^2 \cdot 47 = 3^7 \cdot 7^{11} \cdot 743$ | 4.26801 . . . | A. Nitaj |
| 3 | $2^{19} \cdot 13 \cdot 103 + 7^{11} = 3^{11} \cdot 5^3 \cdot 11^2$ | 4.24789 . . . | B. de Weger |
| 4 | $2^{35} \cdot 7^2 \cdot 17^2 \cdot 19 + 3^{27} \cdot 107^2 = 5^{15} \cdot 37^2 \cdot 2311$ | 4.23069 . . . | A. Nitaj |
| 5 | $3^{18} \cdot 23 \cdot 2269 + 17^3 \cdot 29 \cdot 31^8 = 2^{10} \cdot 5^2 \cdot 7^{15}$ | 4.22979 . . . | A. Nitaj |
| 6 | $17^4 \cdot 79^3 \cdot 211 + 2^{29} \cdot 23 \cdot 29^2 = 5^{19}$ | 4.22960 . . . | A. Nitaj |

As noticed by M. Langevin [La 1992], a consequence of the $abc$ Conjecture 2.1 is the following open problem [E 1980], arising from questions asked by logicians [Wo 1981], [Guy 1994] B29 and B35, [BLSW 1996]:

**Conjecture 2.2** (Erdős-Woods). *There exists a positive integer $k$ such that, for $m$ and $n$ positive integers, the conditions*

$$R(m + i) = R(n + i) \quad (i = 0, \ldots, k - 1)$$

*imply $m = n$.*

One suspects that $k = 3$ is an admissible value: this would mean that if $m$ and $n$ have the same prime divisors, $m+1$, $n+1$ have the same prime divisors and $m+2$, $n+2$ have the same prime divisors, then $m = n$.

That $k = 2$ is not an admissible value is easily seen: 75 and 1215 have the same prime divisors, and this is true also for 76 and 1216.

$$R(75) = 15 = R(1215), \qquad R(76) = 2 \cdot 19 = R(1216).$$

Apart from this sporadic example, there is also a sequence of examples: for $m = 2^h - 2$ and $n = 2^h m$ we have

$$R(m) = R(n) \quad \text{and} \quad R(m + 1) = R(n + 1)$$

because $n + 1 = (m + 1)^2$.

A generalization of the Erdős-Woods problem to arithmetic progressions has been suggested by T.N. Shorey:

(?) *Does there exist a positive integer $k$ such that, for any $m$, $n$, $d$ and $d'$ positive integers satisfying* $\gcd(m, d) = \gcd(n, d') = 1$, *the conditions*

$$R(m + id) = R(n + id') \quad (i = 0, \ldots, k - 1)$$

*imply $m = n$ and $d = d'$?*

If the answer is positive the integer $k$ is greater than 3, as shown by several examples of quadruples $(m, n, d, d')$, like $(2, 2, 1, 7)$, $(2, 8, 79, 1)$ or $(4, 8, 23, 1)$:

$$R(2) = R(2), \quad R(3) = R(2 + 7), \quad R(4) = R(2 + 2 \cdot 7),$$

$$R(2) = R(4) = R(8), \quad R(2+79) = R(4+23) = R(9), \quad R(2+2\cdot 79) = R(4+2\cdot 23) = R(10).$$

Another related problem of T.S. Motzkin and E.G. Straus ([Guy 1994] B19) is to determine the pairs of integers $m, n$ such that $m$ and $n+1$ have the same prime divisors, and also $n$ and $m+1$ have the same set of prime divisors. The known examples are

$$m = 2^k + 1, \qquad n = m^2 - 1 \qquad (k \geq 0)$$

and the sporadic example $m = 35 = 5 \cdot 7$, $n = 4374 = 2 \cdot 3^7$, which gives $m + 1 = 2^2 \cdot 3^2$ and $n + 1 = 5^4 \cdot 7$.

We also quote another conjecture attributed to P. Erdős in [La 1992] and to R.E. Dressler in [2].

**Conjecture 2.3** (Erdős–Dressler) . *If $a$ and $b$ are two positive integers with $a < b$ and $R(a) = R(b)$ then there is a prime $p$ with $a < p < b$.*

The first estimates in the direction of the $abc$ Conjecture 2.1 have been achieved by Tijdeman, Stewart and Yu Kunrui (see [StY 1991]), using ($p$-adic) lower bounds for linear forms in logarithms:

$$\log c \leq \kappa(\epsilon) R(abc)^{(1/3)+\epsilon}.$$

Further connections between the $abc$ Conjecture 2.1 and measures of linear independence of logarithms of algebraic numbers have been pointed out by Baker [B 1998] and Philippon [P 1999a] (see also [W 2000b] exercise 1.11). We reproduce here the main conjecture of the addendum of [P 1999a]. For a rational number $a/b$ with relatively prime integers $a, b$, we denote by $\mathrm{h}(a/b)$ the number $\log \max\{|a|, |b|\}$.

**Conjecture 2.4** (Philippon). *There exist real numbers $\epsilon$, $\alpha$ and $\beta$ with $0 < \epsilon < 1/2$, $\alpha \geq 1$ and $\beta \geq 0$, and a positive integer $B$, such that for any nonzero rational numbers $x$, $y$ satisfying $xy^B \neq 1$, if $S$ denotes the set of prime numbers for which $|xy^B + 1|_p < 1$, then*

$$-\sum_{p \in S} \log |xy^B + 1|_p \leq B\Big(\alpha \mathrm{h}(x) + \epsilon \mathrm{h}(y) + \big(\alpha B + \epsilon\big)\big(\beta + \sum_{p \in S} \log p\big)\Big).$$

The conclusion is a lower bound for the $p$-adic distance between $-xy^B$ and 1; the main point is that several $p$'s are involved.

Examples of optimistic archimedean estimates related with measures of linear independence of logarithms of algebraic numbers are the Lang-Waldschmidt Conjectures in [L 1978b] (introduction to Chap. X and XI, p. 212–217). Here is a simple example:

**Conjecture 2.5** (Lang-Waldschmidt). *For any $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$ such that, for any nonzero rational integers $a_1, \ldots, a_m$, $b_1, \ldots, b_m$ with $a_1^{b_1} \cdots a_m^{b_m} \neq 1$,*

$$\left| a_1^{b_1} \cdots a_m^{b_m} - 1 \right| \geq \frac{C(\epsilon)^m B}{(|b_1| \cdots |b_m| \cdot |a_1| \cdots |a_m|)^{1+\epsilon}},$$

*where $B = \max_{1 \leq i \leq m} |b_i|$.*

Similar questions related with Diophantine approximation on toruses are discussed in [L 1991] Chap. IX, § 7.

Conjecture 2.5 deals with rational integers; we shall consider more generally algebraic numbers in § 4, once we have defined a notion of height in § 3.

From either Conjecture 2.1 or Conjecture 2.5 one deduces a quantitative refinement to Pillai's Conjecture 1.3:

**Conjecture 2.6.** *For any $\epsilon > 0$, there is a constant $C(\epsilon) > 0$ such that, for any positive integers $x$, $y$, $p$, $q$ satisfying $x^p \neq y^q$, the inequality*

$$|x^p - y^q| \geq C(\epsilon) \max\{x^p, y^q\}^{1-(1/p)-(1/q)-\epsilon}$$

*holds.*

We consider two special cases of Conjecture 2.6: first $(p, q) = (2, 3)$, which gives rise to Hall's Conjecture [H 1971] (also [L 1991Chap. II, § 1):

**Conjecture 2.7** (Hall). *If $x$ and $y$ are positive integers with $y^2 \neq x^3$, then*

$$|y^2 - x^3| \geq C \max\{y^2, x^3\}^{1/6}.$$

In this statement there is no $\epsilon$ – maybe Conjecture 2.7 will be true by a sort of accident, but one may also expect that the estimate is too strong to be true.

The second special case is $(x, y) = (3, 2)$. The question of how small $3^n - 2^m$ can be in comparison with $2^m$ has been raised by J.E. Littlewood [Guy 1994] F23. The example

$$\frac{3^{12}}{2^{19}} = 1 + \frac{7153}{524288} = 1.013\ldots$$

is related with music scales.

For further questions dealing with exponential Diophantine equations, we refer to Chap. 12 of the book of Shorey and Tijdeman [ShT 1986], as well as to the more recent surveys [Ti 1998] and [Sh 1999].

### 2.2. Thue-Siegel-Roth-Schmidt

One of the main open problems in Diophantine approximation is to produce an effective version of the Thue-Siegel-Roth Theorem: *For any $\epsilon > 0$ and any irrational algebraic number $\alpha$, there is a positive constant $C(\alpha, \epsilon) > 0$ such that, for any rational number $p/q$,*

$$\left|\alpha - \frac{p}{q}\right| > C(\epsilon) q^{-2-\epsilon}. \tag{2.8}$$

In analogy with the negative answer to Hilbert's 10th problem by Matiyasevich, it has been suggested by M. Mignotte that such an effective version may be impossible. If this turns out to be the case, then, according to E. Bombieri (see [2]), an efffective version of the $abc$ Conjecture would also be out of reach. Michel Langevin noticed that the $abc$ Conjecture yields a stronger inequality than Roth's one:

$$\left|\alpha - \frac{p}{q}\right| > \frac{C(\epsilon)}{R(pq)q^\epsilon}\cdot$$

So far, only effective improvements of Liouville's lower bound are known, and to improve them is still already a big challenge.

Another goal would be to improve the estimate in Roth Theorem: in the lower bound (2.8) one would like to replace $q^{-2-\epsilon}$ by, say, $q^{-2}(\log q)^{-1-\epsilon}$. It is expected that for any irrational real number $\alpha$ of degree $\geq 3$, in inequality (2.8) the term $q^{-2-\epsilon}$ cannot be replaced by $q^{-2}$, but the set of $\alpha$ for which the answer is known is empty! This question is often asked for the special case of the number

$\sqrt[3]{2}$, but another example (due to Ulam – see for instance [Guy 1994] F22) is the real algebraic number $\xi$ defined by

$$\xi = \frac{1}{\xi + y} \quad \text{with} \quad y = \frac{1}{1+y}.$$

Essentially nothing is known on the continued fraction expansion of an algebraic number of degree $\geq 3$; one does not know the answer to any of the following two questions:

(2.9 ?) *Does there exist one with bounded partial quotients?*

(2.10 ?) *Does there exist one with unbounded partial quotients?*

It is usually expected is that the continued fraction expansion of a real algebraic number of degree at least 3 has always unbounded partial quotients. More precisely one expects that real algebraic numbers of degree $\geq 3$ behave like "almost all" real numbers (see § 5.1).

Let $\psi(q)$ be a continuous positive real valued function. Assume that the function $q\psi(q)$ is nonincreasing. Consider the inequality

$$\left|\theta - \frac{p}{q}\right| > \frac{\psi(q)}{q}. \tag{2.11}$$

**Conjecture 2.12.** *Let $\theta$ be real algebraic number of degree at least 3. Then inequality (2.11) has infinitely many solutions in integers $p$ and $q$ with $q > 0$ if and only if the integral*

$$\int_1^\infty \psi(x)dx$$

*diverges.*

A far reaching generalization of Roth Theorem to simultaneous approximation is W.M. Schmidt Subspace Theorem. Here are two special cases:

- *Given real algebraic numbers $\alpha_1, \dots, \alpha_n$ such that $1, \alpha_1, \dots, \alpha_n$ are linearly independent over $\mathbb{Q}$, for any $\epsilon > 0$ the inequality*
$$\max_{1\leq i\leq n} \left|\alpha_i - \frac{p_i}{q}\right| < \frac{1}{q^{1+(1/n)+\epsilon}}$$
*has only finitely many solutions $(p_1, \dots, p_n, q)$ in $\mathbb{Z}^{n+1}$ with $q > 0$.*
- *Given real algebraic numbers $\alpha_1, \dots, \alpha_n$ such that $1, \alpha_1, \dots, \alpha_n$ are linearly independent over $\mathbb{Q}$, for any $\epsilon > 0$ the inequality*
$$|q_1\alpha_1 + \cdots + q_n\alpha_n - p| < \frac{1}{q^{n+\epsilon}}$$
*has only finitely many solutions $(q_1, \dots, q_n, p)$ in $\mathbb{Z}^{n+1}$ with $q = \max\{|q_1|, \dots, |q_n|\} > 0$.*

These two types of Diophantine statements are parallel to the two types of Padé Approximants. It would be interesting to consider the analog of Schmidt's Subspace Theorem in case of Padé Approximants, and also to investigate a corresponding analogue of Khinchine transference principle [Ca 1957].

One of the most important consequences of Schmidt's Subspace Theorem is the finiteness of nondegenerate solutions of the equation

$$x_1 + \cdots + x_n = 1,$$

where the unknowns are integers (or $S$-integers) in a number field. Here, non-degenerate means that no proper subsum vanishes. One main open question is to prove an effective version of this result.

Schmidt's' Theorem, which is a generalization of Roth's Theorem, is not effective. Only for $n = 2$ one knows bounds for the solutions of the $S$-unit equation $x_1 + x_2 = 1$, thanks to Baker's method (see [B 1975] Chap. 5; [L 1978b] Chap. VI; [ShT 1986] Chap 1; [Se 1989] and [L 1991]). One would like to extend Baker's method (or any other effective method) to the higher dimensional case.

A generalization of Markoff spectrum to simultaneous approximation is not yet available: even the first step is missing. Given a positive integer $n$ and real numbers $(\xi_1, \ldots, \xi_n)$, not all of which are rational, define $c_n = c_n(\xi_1, \ldots, \xi_n)$ to be the infimum of all $c$ in the range $0 < c \leq 1$ for which the inequality

$$q|q\xi_i - p_i|^n < c$$

has infinitely many solutions. Then define the *$n$-dimensional simultaneous Diophantine approximation constant* $\gamma_n$ to be the supremum of $c_n$ over tuples $(\xi_1, \ldots, \xi_n)$ as above. Following [1], here is a summary of what is known about the first values of the approximation constants:

$$\begin{array}{rcccccccl} & & & \gamma_1 & = & \dfrac{1}{\sqrt{5}} & = & 0.4472135955\ldots & \text{(Hurwitz)} \\ 0.2857142857 & = & \dfrac{2}{7} & \leq \gamma_2 & \leq & \dfrac{64}{169} & = & 0.3786982249\ldots & \text{(Cassels and Nowak)} \\ 0.1206045378 & = & \dfrac{2}{5\sqrt{11}} & \leq \gamma_3 & \leq & \dfrac{1}{2(\pi-2)} & = & 0.4379845985\ldots & \text{(Cusick and Spohn)} \end{array}$$

We illustrate now with Waring's problem the importance of proving effective Roth-type inequalities for irrational algebraic numbers.

In 1770, a few months before J.L. Lagrange proved that every positive integer is the sum of at most four squares of integers, E. Waring ([Wa 1770] Chap. 5, Theorem 47 (9)) wrote:

> *"Every integer is a cube or the sum of two, three, ... nine cubes; every integer is also the square of a square, or the sum of up to nineteen such; and so forth. Similar laws may be affirmed for the correspondingly defined numbers of quantities of any like degree."*

See also Note 15 of the translator in [Wa 1770].

For $k \geq 2$ define $g(k)$ as the smallest positive integer such that any integer is the sum of $g$ elements of the form $x^k$ with $x \geq 0$. In other terms for each positive integer $n$ the equation

$$n = x_1^k + \cdots + x_m^k$$

has a solution if $m = g(k)$, while there is a $n$ which is not the sum of $g(k) - 1$ such $k$-th powers.

Lagrange's Theorem, which solved a conjecture of Bachet and Fermat, is $g(2) = 4$. Following Chap. IV of [N 1986], here are the values of $g(k)$ for the first integers $k$, with the author(s) name(s) and date:

| $k =$ 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|
| $g(k) =$ 4 | 9 | 19 | 37 | 73 | 143 |
| J.J. Lagrange | A. Wieferich | R. Balasubramanian<br>J-M. Deshouillers<br>F. Dress | J. Chen | S.S. Pillai | L.E. Dickson |
| 1770 | 1909 | 1986 | 1964 | 1940 | 1936 |

For each integer $k \geq 2$, define

$$I(k) = 2^k + [(3/2)^k] - 2.$$

It is easy to check $g(k) \geq I(k)$: write

$$3^k = 2^k q + r \quad \text{with} \quad 0 < r < 2^k, \quad q = [(3/2)^k]$$

and consider the integer

$$N = 2^k q - 1 = (q-1)2^k + (2^k - 1)1^k.$$

Since $N < 3^k$, writing $N$ as a sum of $k$-th powers can involve no term $3^k$, and since $N < 2^k q$ it involves at most $(q-1)$ terms $2^k$, all others being $1^k$; hence it requires a total number of at least $(q-1) + (2^k - 1) = I(k)$ terms.

It has been checked that $g(k) = I(k)$ for $2 \leq k \leq 471\,600\,000$, and K. Mahler proved that $g(k) = I(k)$ for any sufficiently large $k$. The problem is that Mahler's proof relies on a $p$-adic version of the Thue-Siegel-Roth Theorem, and therefore is not effective. So there is a gap, of which we don't even know the size. The conjecture, dating back to 1853, is $g(k) = I(k)$ for any $k \geq 2$, and this would follow from the estimate (see [N 1986] p. 226):

$$\left\| \left(\frac{3}{2}\right)^k \right\| \geq 2 \cdot \left(\frac{3}{4}\right)^k,$$

where $\|\cdot\|$ denote the distance to the nearest integer. As remarked by Sinnou David, such an estimate (for sufficiently large $k$) also follows from the $abc$ Conjecture 2.1!

In [M 1968] Mahler defined a *$Z$-number* as a real number $\alpha$ such that the fractional part $r_n$ of $\alpha(3/2)^n$ satisfies $0 \leq r_n < 1/2$ for any positive integer $n$. It is not known whether $Z$-numbers exist (see [FLP 1995]). A related remark by J.E. Littlewood ([Guy 1994] E18) is that we are not yet able to prove that the fractional part of $e^n$ does not tend to 0 as $n$ tends to infinity.

A well known conjecture of Littlewood ([B 1975] Chap. 10, § 1 and [PoV 2000]) asserts that *for any pair $(x, y)$ of real numbers and any $\epsilon > 0$, there exists a positive integer $q$ such that*

$$q\|qx\| \cdot \|qy\| < \epsilon.$$

According to G. Margulis (communication of G. Lachaud), the proofs in a 1988 paper by B.F. Skubenko (see M.R. 94d:11047) are not correct and cannot be fixed.

There are several open questions known as "view obstruction problem". One of them is the following. *Given $n$ positive integers $k_1, \ldots, k_n$, there exists a real number $x$ such that*

$$\|k_i x\| \geq \frac{1}{n+1} \quad \textit{for} \quad 1 \leq i \leq n.$$

It is known that $1/(n+1)$ cannot be replaced by a larger number [CuP 1984].

### 2.3. Irrationality and Linear Independence Measures

Given a real number $\theta$, the first Diophantine question is to decide whether $\theta$ is rational or not. This is a qualitative question, and it is remarkable that an answer is provided by a quantitative property of $\theta$: it depends ultimately on the quality of rational Diophantine approximations to $\theta$. Indeed, on one hand, if $\theta$ is rational, then there exists a positive constant $c = c(\theta)$ such that

$$\left|\theta - \frac{p}{q}\right| > \frac{c}{q}$$

for any $p/q \in \mathbb{Q}$. An admissible value for $c$ is $1/b$ when $\theta = a/b$. On the other hand, if $\theta$ is irrational, then there are infinitely many rational numbers $p/q$ such that

$$0 < \left|\theta - \frac{p}{q}\right| < \frac{1}{q^2}\cdot$$

Hence, in order to prove that $\theta$ is irrational, it suffices to prove that for any $\epsilon > 0$ there is a rational number $p/q$ such that

$$0 < \left|\theta - \frac{p}{q}\right| < \frac{\epsilon}{q}\cdot$$

This is a rather weak requirement: there are rational approximations in $1/q^2$, and we need only to produce rational approximations better than the trivial ones in $1/q$. Accordingly one should expect that it rather easy to prove the irrationality of a given real number. In spite of that, the class of "interesting" real numbers which are known to be irrational is not as large as one would expect. For instance no proof of irrationality has been given so far for Euler's constant

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) = 0.5772157\ldots,$$

neither for Catalan's constant

$$G = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^2} = \frac{\pi}{4}\int_0^1 {}_2F_1\left(\begin{matrix}1/2\ ,\ 1/2\\ 1\end{matrix}\middle| t\right)\frac{dt}{\sqrt{4t}} = 0.915965594\ldots,$$

nor for

$$\Gamma(1/5) = \int_0^\infty e^{-t}t^{-4/5}dt = 4.59084371\ldots$$

or for numbers like

$$e + \pi = 5.8598744\ldots, \qquad e^\gamma = 1.781072\ldots$$

and

$$\sum_{n\geq 1}\frac{\sigma_k(n)}{n!} \quad (k = 1, 2) \quad \text{where} \quad \sigma_k(n) = \sum_{d|n} d^k$$

(see [Guy 1994] B14).

Here is another irrationality question raised by P. Erdős and E. Straus in 1975 (see [E 1988] and [Guy 1994] E24). Define an *irrationality sequence* as an increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that, for any sequence $(t_k)_{k\geq 1}$ of positive integers, the real number

$$\sum_{k\geq 1}\frac{1}{n_k t_k}$$

is irrational. It has been proved by Erdős that $(2^{2^k})_{k\geq 1}$ is an irrationality sequence. On the other hand the sequence $(k!)_{k\geq 1}$ is not, since

$$\sum_{k\geq 1} \frac{1}{k!(k+2)} = \frac{1}{2}.$$

An open question is whether an irrationality sequence must increase very rapidly. No irrationality sequence $(n_k)_{k\geq 1}$ is known for which $n_k^{1/2^k}$ tends to 1 as $k$ tends to infinity. Also it is not known whether the sequence $2, 3, 7, 43, \ldots$ satisfying the recurrence relation

$$n_{k+1} = n_k^2 - n_k + 1$$

is an irrationality sequence or not.

Many further open irrationality questions are raised in [E 1988]. Another related example is Conjecture 5.4 below.

Assume now that the first step has been completed and that we know our number $\theta$ is irrational. Then there are (at least) two directions for further investigation:

1) Considering several real numbers $\theta_1, \ldots, \theta_n$, a fundamental question is to decide whether or not they are linearly independent over $\mathbb{Q}$. One main example is to start with the successive powers of one number: $1, \theta, \theta^2, \ldots, \theta^{n-1}$; the goal is to decide whether $\theta$ is algebraic of degree $< n$. If $n$ is not fixed, the question is whether $\theta$ is transcendental. This question, which is relevant also for complex numbers, will be considered in the next section. Notice also that the problem of algebraic independence is included here: it amounts to the linear independence of monomials.
2) Another direction of research is to consider a quantitative refinement of the irrationality statement, namely an *irrationality measure*: we wish to bound from below the nonzero number $|\theta - (p/q)|$ when $p/q$ is any rational number; this lower bound will depend on $\theta$ as well as the denominator $q$ of the rational approximation. In case where a statement weaker than an irrationality result is known, namely if one can prove only that one at least of $n$ numbers $\theta_1, \ldots, \theta_n$ is irrational, then a quantitative refinement will be a lower bound (in terms of $q$) for

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \ldots \left|\theta_n - \frac{p_n}{q}\right|\right\},$$

when $p_1/q, \ldots, p_n/q$ are $n$ rational numbers and $q > 0$ a common denominator.

The study of rational approximation of real numbers is achieved in a satisfactory way for numbers whose regular continued fraction expansion is known. This is the case for rational numbers (!), for quadratic numbers, as well as for a small set of transcendental numbers, like

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, \ldots] = [2, \{\overline{1, 2m, 1}\}_{m\geq 1}]$$

$$e^2 = [7, 2, 1, 1, 3, 18, 5, 1, 1, 6, 30, 8, 1, 1, 9, 42, 11, \ldots] = [7, \{\overline{3m-1, 1, 1, 3m, 12m+6}\}_{m\geq 1}]$$

and

$$e^{1/n} = [1, n-1, 1, 1, 3n-1, 1, 1, 5n-1, 1, 1, \ldots] = [\{\overline{1, (2m-1)n-1, 1}\}_{m\geq 1}]$$

for $n > 1$. On the other hand, even for a real number $x$ for which an irregular continued fraction expansion is known, like

$$\log 2 = \left[\frac{1}{1+}\;\frac{1}{1+}\;\frac{4}{1+}\;\frac{9}{1+}\;\cdots\;\frac{n^2}{1+}\cdots\right]$$

or

$$\frac{\pi}{4} = \left[ \frac{1}{1+} \, \frac{9}{2+} \, \frac{25}{2+} \, \frac{49}{2+} \, \cdots \, \frac{(2n+1)^2}{2+} \, \cdots \right]$$

one does not know how well $x$ can be approximated by rational numbers. No regular pattern is expected from the regular continued fraction of $\pi$:

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4, 2, 6, 6, 6, 1, \ldots]$$

nor from any number "easily" related to $\pi$.

One expects that for any $\epsilon > 0$ there are constants $C(\epsilon) > 0$ and $C'(\epsilon) > 0$ such that

$$\left| \log 2 - \frac{p}{q} \right| > \frac{C(\epsilon)}{q^{2+\epsilon}} \quad \text{and} \quad \left| \pi - \frac{p}{q} \right| > \frac{C'(\epsilon)}{q^{2+\epsilon}}$$

hold for any $p/q \in \mathbb{Q}$, but this is known only with larger exponents than $2+\epsilon$, namely $3.8913998\ldots$ and $8,0161\ldots$ respectively (Rukhadze and Hata). The best known exponent for an irrationality measure of

$$\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3} = 1.202056903\ldots$$

is $5.513891\ldots$, while for $\pi^2$ (or for $\zeta(2) = \pi^2/6$) it is $5.441243\ldots$ (both results due to Rhin and Viola). For a number like $\Gamma(1/4)$, the existence of absolute positive constants $C$ and $\kappa$ for which

$$\left| \Gamma(1/4) - \frac{p}{q} \right| > \frac{C}{q^\kappa}$$

has been proved only recently [P 1999b]. The similar problem for $e^\pi$ is not yet solved: there is no proof so far that $e^\pi$ is not a Liouville number.

Earlier we distinguished two directions for research once we know the irrationality of some given numbers: either, on the qualitative side, one studies the linear dependence relations, or else, on the quantitative side, one investigates the quality of rational approximation. One can combine both: a quantitative version of a result of $\mathbb{Q}$-linear independence of $n$ real numbers $\theta_1, \ldots, \theta_n$, is a lower bound, in terms of $\max\{|p_1|, \ldots, |p_n|\}$, for

$$|p_1\theta_1 + \cdots + p_n\theta_n|$$

when $(p_1, \ldots, p_n)$ is in $\mathbb{Z}^n \setminus \{0\}$.

For some specific classes of transcendental numbers, A.I. Galochkin [G 1983], A.N. Korobov (Th. 1.22 of [FN 1998] Chap. 1 § 7) and more recently P. Ivankov proved extremely sharp measures of linear independence (see [FN 1998] Chap. 2 §§ 6.2 and 6.3).

A general and important problem is to improve the known measures of linear independence for logarithms of algebraic numbers, as well as elliptic logarithms, abelian logarithms, and more generally logarithms of algebraic points on commutative algebraic groups. For instance the conjecture that $e^\pi$ is not a Liouville number should follow from improvements of known linear independence measures for logarithms of algebraic numbers.

The next step, which is to get sharp measures of algebraic independence for transcendental numbers, will be considered later (see § 4.3).

The so-called Mahler's Problem (see [W 2001] § 4.1) is related with linear combination of logarithms $|b - \log a|$:

**Conjecture 2.13** (Mahler). *There exists an absolute constant $c > 0$ such that for $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ with $a \geq 2$,*

$$|a - e^b| > a^{-c}.$$

A stronger conjecture is suggested in [W 2001] (4.1):

$$|a - e^b| > b^{-c}.$$

So far the best known estimate is

$$|a - e^b| > e^{-c(\log a)(\log b)},$$

so the problem is to replace in the exponent the product $(\log a)(\log b)$ by the sum $\log a + \log b$.

Another topic which belongs to Diophantine approximation is the theory of *equidistributed sequences*. For a positive integer $r \geq 2$, a *normal number* in base $r$ is a real number such that the sequence $(xr^n)_{n\geq 1}$ is equidistributed modulo 1. Almost all numbers for Lebesgue measure are normal (i.e. normal in basis $r$ for any $r > 1$), but it is not known whether there are integers $r$ for which numbers like $\sqrt{2}$, $e$, $\pi$ are normal in basis $r$ (see [Ra 1976]).

## 3. Transcendence

When $K$ is a field and $k$ a subfield, we denote by $\mathrm{trdeg}_k K$ the transcendence degree of the extension $K/k$. In the case $k = \mathbb{Q}$ we write simply $\mathrm{trdeg} K$ (see [L 1993] Chap. VIII, § 1).

### 3.1. Schanuel's Conjecture

We concentrate here on problems related with transcendental number theory. To start with we consider the classical exponential function $e^z = \exp(z)$. A recent reference on this topic is [W 2000b].

Schanuel's Conjecture is a simple but far reaching statement – see the historical note to Chap. III of [L 1966].

**Conjecture 3.1** (Schanuel). *Let $x_1, \ldots, x_n$ be $\mathbb{Q}$-linearly independent complex numbers. Then the transcendence degree over $\mathbb{Q}$ of the field $\mathbb{Q}\big(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}\big)$ is at least $n$.*

According to S. Lang ([L 1966] p. 31): "From this statement, one would get most statements about algebraic independence of values of $e^t$ and $\log t$ which one feels to be true". See also [L 1971] p. 638–639 and [Ri 2000] § 10.7.G. For instance the following statements [Ge 1934] are consequences of Conjecture 3.1.

(?) *Let $\beta_1, \ldots, \beta_n$ be $\mathbb{Q}$-linearly independent algebraic numbers and let $\log \alpha_1, \ldots, \log \alpha_m$ be $\mathbb{Q}$-linearly independent logarithms of algebraic numbers. Then the numbers*

$$e^{\beta_1}, \ldots, e^{\beta_n}, \ \log \alpha_1, \ldots, \log \alpha_m$$

*are algebraically independent.*

(?) *Let $\beta_1, \ldots, \beta_n$ be algebraic numbers with $\beta_1 \neq 0$ and let $\log \alpha_1, \ldots, \log \alpha_m$ be logarithms of algebraic numbers with $\log \alpha_1 \neq 0$ and $\log \alpha_2 \neq 0$. Then the numbers*

$$e^{\beta_1 e^{\beta_2 e^{\cdot^{\cdot^{\cdot^{\beta_{n-1} e^{\beta_n}}}}}}} \quad \text{and} \quad \alpha_1^{\alpha_2^{\cdot^{\cdot^{\cdot^{\alpha_m}}}}}$$

*are transcendental, and there is no nontrivial algebraic relation between such numbers.*

A quantitative refinement of Conjecture 3.1 is suggested in [W 1999b] Conjecture 1.4.

A quite interesting approach to Schanuel's Conjecture is given in [Ro 2001a] where D. Roy states the next conjecture which he shows to be equivalent to Schanuel's one. Let $\mathcal{D}$ denote the derivation

$$\mathcal{D} = \frac{\partial}{\partial X_0} + X_1 \frac{\partial}{\partial X_1}$$

over the ring $\mathbb{C}[X_0, X_1]$. The *height* of a polynomial $P \in \mathbb{C}[X_0, X_1]$ is defined as the maximum of the absolute values of its coefficients.

**Conjecture 3.2** (Roy). *Let $\ell$ be a positive integer, $y_1, \ldots, y_\ell$ complex numbers which are linearly independent over $\mathbb{Q}$, $\alpha_1, \ldots, \alpha_\ell$ nonzero complex numbers and $s_0, s_1, t_0, t_1, u$ positive real numbers satisfying*

$$\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\} \quad \textit{and} \quad \max\{s_0, s_1 + t_1\} < u < \frac{1}{2}(1 + t_0 + t_1).$$

*Assume that, for any sufficiently large positive integer $N$, there exists a nonzero polynomial $P_N \in \mathbb{Z}[X_0, X_1]$ with partial degree $\le N^{t_0}$ in $X_0$, partial degree $\le N^{t_1}$ in $X_1$ and height $\le e^N$ which satisfies*

$$\left| (\mathcal{D}^k P_N)\Big( \sum_{j=1}^{\ell} m_j y_j, \prod_{j=1}^{\ell} \alpha_j^{m_j} \Big) \right| \le \exp(-N^u)$$

*for any nonnegative integers $k_1, \ldots, k_\ell$ with $k \le N^{s_0}$ and $\max\{m_1, \ldots, m_\ell\} \le N^{s_1}$. Then, we have*

$$\mathrm{trdeg}\,\mathbb{Q}(y_1, \ldots, y_\ell, \alpha_1, \ldots, \alpha_\ell) \ge \ell.$$

This work of Roy also provides an interesting connection with other open problems related to Schwarz' Lemma for complex functions of several variables (see [Ro 2001c] Conjectures 6.1 and 6.3).

The most important special case of Schanuel's Conjecture is the *Conjecture of algebraic independence of logarithms of algebraic numbers:*

**Conjecture 3.3** (Algebraic Independence of Logarithms of Algebraic Numbers). *Let $\lambda_1, \ldots, \lambda_n$ be $\mathbb{Q}$-linearly independent complex numbers. Assume that the numbers $e^{\lambda_1}, \ldots, e^{\lambda_n}$ are algebraic. Then the numbers $\lambda_1, \ldots, \lambda_n$ are algebraically independent.*

An interesting reformulation of Conjecture 3.3 is due to D. Roy [Ro 1995]. Denote by $\mathcal{L}$ the set of complex numbers $\lambda$ for which $e^\lambda$ is algebraic. Hence $\mathcal{L}$ is a $\mathbb{Q}$-vector subspace of $\mathbb{C}$. Roy's statement is:

(?) *For any algebraic subvariety $V$ of $\mathbb{C}^n$ defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers, the set $V \cap \mathcal{L}^n$ is the union of the sets $E \cap \mathcal{L}^n$, where $E$ ranges over the set of vector subspaces of $\mathbb{C}^n$ which are contained in $V$.*

Such a statement is reminiscent of several of Lang's Conjectures in Diophantine geometry (e.g. [L 1991] Chap. I § 6 Conjectures 6.1 and 6.3).

Not much is known on the algebraic independence of logarithms of algebraic numbers, apart from the work of D. Roy on the rank of matrices whose entries are either logarithms of algebraic numbers, or more generally linear combinations of logarithms of algebraic numbers. We refer to [W 2000b] for a detailed study of this question as related ones.

Conjecture 3.3 has many consequences. The next three ones are suggested by the work of D. Roy ([Ro 1989] and [Ro 1990]) on matrices whose entries are linear combinations of logarithms of algebraic numbers (see also [W 2000b] Conjecture 11.17, § 12.4.3 and Exercise 12.12).

Consider the $\overline{\mathbb{Q}}$-vector space $\widetilde{\mathcal{L}}$ spanned by 1 and $\mathcal{L}$. In other words $\widetilde{\mathcal{L}}$ is the set of complex numbers which can be written

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n,$$

where $\beta_0, \beta_1, \ldots, \beta_n$ are algebraic numbers, $\alpha_1, \ldots, \alpha_n$ are nonzero algebraic numbers, and finally $\log \alpha_1, \ldots, \log \alpha_n$ are logarithms of $\alpha_1, \ldots, \alpha_n$ respectively.

**Conjecture 3.4** (Strong Four Exponentials Conjecture). *Let $x_1, x_2$ be two $\overline{\mathbb{Q}}$-linearly independent complex numbers and $y_1, y_2$ be also two $\overline{\mathbb{Q}}$-linearly independent complex numbers. Then at least one of the four numbers $x_1y_1$, $x_1y_2$, $x_2y_1$, $x_2y_2$ does not belong to $\widetilde{\mathcal{L}}$.*

The following special case is also open:

**Conjecture 3.5** (Strong Five Exponentials Conjecture). *Let $x_1, x_2$ be two $\mathbb{Q}$-linearly independent complex numbers and $y_1, y_2$ be also two $\mathbb{Q}$-linearly independent complex numbers. Further let $\beta_{ij}$ $(i = 1, 2,\ j = 1, 2)$, $\gamma_1$ and $\gamma_2$ be six algebraic numbers with $\gamma_1 \neq 0$. Assume that the five numbers*

$$e^{x_1y_1-\beta_{11}},\ e^{x_1y_2-\beta_{12}},\ e^{x_2y_1-\beta_{21}},\ e^{x_2y_2-\beta_{22}},\ e^{(\gamma_1x_1/x_2)-\gamma_2}$$

*are algebraic. Then all five exponents vanish:*

$$x_iy_j = \beta_{ij} \quad (i = 1, 2, \quad j = 1, 2) \quad \textit{and} \quad \gamma_1x_1 = \gamma_2x_2.$$

.

The next conjecture is proposed in [Ro 1995].

**Conjecture 3.6** (Roy). *For any $4 \times 4$ skew-symmetric matrix $\mathtt{M}$ with entries in $\mathcal{L}$ and rank $\leq 2$, either the rows of $\mathtt{M}$ are linearly dependent over $\mathbb{Q}$, or the columns space of $\mathtt{M}$ contains a nonzero element of $\mathbb{Q}^4$.*

Finally a special case of Conjecture 3.6 is the well known Four Exponentials Conjecture due to Schneider [Schn 1957] Chap. V, end of § 4, Problem 1; Lang [L 1966] Chap. II, § 1; [L 1971] p. 638 and Ramachandra [R 1968 II] § 4.

**Conjecture 3.7** (Four Exponentials Conjecture). *Let $x_1, x_2$ be two $\mathbb{Q}$-linearly independent complex numbers and $y_1, y_2$ also two $\mathbb{Q}$-linearly independent complex numbers. Then one at least of the four numbers*

$$\exp(x_iy_j) \qquad (i = 1, 2,\ j = 1, 2)$$

*is transcendental.*

The four exponentials Conjecture can be stated as follows: *consider a $2 \times 2$ matrix whose entries are logarithms of algebraic numbers:*

$$M = \begin{pmatrix} \log\alpha_{11} & \log\alpha_{12} \\ \log\alpha_{21} & \log\alpha_{22} \end{pmatrix};$$

*assume that the two rows of this matrix are linearly independent over $\mathbb{Q}$ (in $\mathbb{C}^2$), and also that the two columns are linearly independent over $\mathbb{Q}$; then the rank of this matrix is 2.*

We refer to [W 2000b] for a detailed discussion of this topic, including the notion of *structural rank of a matrix* and the result, due to D. Roy, that Conjecture 3.3 is equivalent to a conjecture on the rank of matrices whose entries are logarithms of algebraic numbers.

A classical problem on algebraic independence of algebraic powers of algebraic numbers has been raised by A.O. Gel'fond [Ge 1949] and Th. Schneider [Schn 1957] Chap. V, end of § 4, Problem 7. The data are an irrational algebraic number $\beta$ of degree $d$ and a nonzero algebraic number $\alpha$ with a nonzero logarithm $\log\alpha$. We write $\alpha^z$ in place of $\exp\{z\log\alpha\}$. Gel'fond's problem is:

**Conjecture 3.8** (Gel'fond). *The two numbers*

$$\log\alpha \quad \textit{and} \quad \alpha^\beta$$

*are algebraically independent.*

Schneider's question is

**Conjecture 3.9** (Schneider). *The $d-1$ numbers*

$$\alpha^{\beta},\ \alpha^{\beta^2},\dots,\alpha^{\beta^{d-1}}$$

*are algebraically independent.*

The first partial results in the direction of Conjecture 3.9 are due to A.O. Gel'fond [Ge 1952]. For the more recent ones, see [NP 2001], Chap. 13 and 14.

Combining both questions 3.8 and 3.9 yields a stronger conjecture:

**Conjecture 3.10** (Gel'fond-Schneider). *The $d$ numbers*

$$\log\alpha,\ \alpha^{\beta},\ \alpha^{\beta^2},\dots,\alpha^{\beta^{d-1}}$$

*are algebraically independent.*

Partial results are known. They deal, more generally, with the values of the usual exponential function at products $x_iy_j$, when $x_1,\dots,x_d$ and $y_1,\dots,y_\ell$ are linearly independent complex (or $p$-adic) numbers. The six exponentials Theorem states that, in these circumstances, the $d\ell$ numbers $e^{x_iy_j}$ $(1\le i\le d,\ 1\le j\le \ell)$ cannot all be algebraic, provided that $d\ell>d+\ell$. Assuming stronger conditions on $d$ and $\ell$, namely $d\ell\ge 2(d+\ell)$, one deduces that two at least of these $d\ell$ numbers $e^{x_iy_j}$ are algebraically independent. Other results are available involving also either the numbers $x_1,\dots,x_d$ themselves, or $y_1,\dots,y_\ell$, or both. But an interesting point is that, if we wish to get higher transcendence degree, say to obtain that three at least of the numbers $e^{x_iy_j}$ are algebraically independent, one needs so far a further assumption, which is a measure of linear independence for the tuple $x_1,\dots,x_d$ as well as for the tuple $y_1,\dots,y_\ell$. To remove this so-called *technical hypothesis* does not seem to be an easy challenge (see [NP 2001] Chap. 14 §§ 2.2 and 2.3).

The need for such a technical hypothesis seems to be connected with the fact that the actual transcendence methods produce not only a qualititative statement (lower bound for the transcendence degree), but also quantitative statements (see § 4).

Several complex results have not yet been established in the ultrametric situation. Two noticeable instances are:

**Conjecture 3.11** ($p$-adic analog of Lindemann-Weierstraß' Theorem). *Let $\beta_1,\dots,\beta_n$ be $p$-adic algebraic numbers in the domain of convergence of the $p$-adic exponential function $\exp_p$. Then the $n$ numbers $\exp_p\beta_1,\dots,\exp_p\beta_n$ are algebraically independent over $\mathbb{Q}$?*

**Conjecture 3.12** ($p$-adic analog of an algebraic independence result of Gel'fond). *Let $\alpha$ be a non-zero algebraic number inside the domain of convergence of the $p$-adic logarithm $\log_p$, and let $\beta$ be a $p$-adic cubic algebraic number, such that $\beta\log_p\alpha$ is in the domain of convergence of the $p$-adic exponential function $\exp_p$. Then the two numbers*

$$\alpha^{\beta}=\exp_p(\beta\log_p\alpha)\quad\textit{and}\quad \alpha^{\beta^2}=\exp_p(\beta^2\log_p\alpha)$$

*are algebraically independent over $\mathbb{Q}$*

The $p$-adic analog of Conjecture 3.3 would solve Leopoldt's Conjecture on the $p$-adic rank of the units of an algebraic number field [Le 1962] (see also [Gra 2001]), by proving the nonvanishing of the $p$-adic regulator.

Algebraic independence results for the values of the exponential function (or more generally for analytic subgroups of algebraic groups) in several variables have already been established, but they are not yet satisfactory. The conjectures stated p. 292–293 of [W 1986] as well as those of [NP 2001]

Chap. 14 § 2 are not yet proved. One of the main obstacles is the above-mentioned open problem with the technical hypothesis.

The problem of extending the Lindemann-Weierstraß Theorem to commutative algebraic groups is not yet completely solved (see conjectures by Philippon in [P 1987]).

Algebraic independence proofs use elimination theory. Several methods are available; one of them, developped by Masser, Wüstholz and Brownawell, rests on Hilbert Nulstellensatz. In this context we quote the following conjecture of Blum, Cucker, Shub and Smale (see [NP 2001] Chap. 16 § 6.2):

**Conjecture 3.13** (Blum, Cucker, Shub and Smale). *Given an absolute constant $c$ and polynomials $P_1, \ldots, P_m$ with a total of $N$ coefficients and no common complex zeros, there is no program to find, in at most $N^c$ step, the coefficients of polynomials $A_i$ satisfying Bézout's relation*

$$A_1P_1 + \cdots + A_mP_m = 1.$$

In connection with complexity in theoretical computer science, W.D. Brownawell suggests to investigate Diophantine approximation from a new point of view in [NP 2001] Chap. 16 § 6.3.

### 3.2. Polyzeta Values

Many recent papers (see for instance [C 2001]) are devoted to the study of algebraic relations among "multiple zeta values" or "polyzeta"

$$\sum_{n_1 > \cdots > n_k \geq 1} n_1^{-s_1} \cdots n_k^{-s_k},$$

(where $(s_1, \ldots, s_k)$ is a $k$-tuple of positive integers with $s_1 \geq 2$). The main Diophantine conjecture, suggested by the work of D. Zagier, A.B. Goncharov, M. Kontsevich, M. Petitot, Minh Hoang Ngoc and others (see [Z 1994] and [C 2001]) is that all such relations can be deduced from the linear and quadratic ones arising from the *shuffle* and *stuffle* products (including the relations occurring from the study of divergent series – see [W 2000d] for instance). For $p \geq 2$, let $\mathfrak{Z}_p$ denote the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ spanned by the real numbers $\zeta(\underline{s})$ satisfying $\underline{s} = (s_1, \ldots, s_k)$ and $s_1 + \cdots + s_k = p$. Set $\mathfrak{Z}_0 = \mathbb{Q}$ and $\mathfrak{Z}_1 = \{0\}$. Then the $\mathbb{Q}$-subspace $\mathfrak{Z}$ spanned by all $\mathfrak{Z}_p$, $p \geq 0$, is a subalgebra of $\mathbb{R}$ and part of the Diophantine conjecture states:

**Conjecture 3.14** (A.B. Goncharov). *As a $\mathbb{Q}$-algebra, $\mathfrak{Z}$ is the direct sum of $\mathfrak{Z}_p$ for $p \geq 0$.*

In other terms all algebraic relations should be consequences of homogeneous ones, involving values $\zeta(\underline{s})$ with different $\underline{s}$ but with the same weight $s_1 + \cdots + s_k$.

Assuming this conjecture 3.14, the question of *algebraic independence* of the numbers $\zeta(\underline{s})$ is reduced to the question of *linear independence* of the same numbers. The conjectural situation is described by the next conjecture of Zagier [Z 1994] on the dimension $d_p$ of the $\mathbb{Q}$-vector space $\mathfrak{Z}_p$.

**Conjecture 3.15** (D. Zagier). *For $p \geq 3$ we have*

$$d_p = d_{p-2} + d_{p-3}$$

*with $d_0 = 1$, $d_1 = 0$, $d_2 = 1$.*

After the recent work of Écalle and others on this subject, it is known that Conjectures 3.14 and 3.15 in the case $k = 1$ (values of the Riemann zeta function) reduce to the following open conjecture:

**Conjecture 3.16.** *The numbers $\pi$, $\zeta(3), \zeta(5), \ldots, \zeta(2n+1), \ldots$ are algebraically independent.*

So far the only known results on this topic are that

- $\zeta(2n)$ *is transcendental for* $n \geq 1$ (because $\pi$ is transcendental and $\zeta(2n)\pi^{-2n} \in \mathbb{Q}$),
- $\zeta(3)$ *is irrational* (Apéry, 1978),

and

- *The $\mathbb{Q}$-vector space spanned by the $n+1$ numbers $1, \zeta(3), \zeta(5), \ldots, \zeta(2n+1)$ has dimension*

$$\geq \frac{1-\epsilon}{1+\log 2} \log n$$

  *for $n \geq n_0(\epsilon)$* (see [Riv 2000] and [BR 2001]). For instance infinitely many of these numbers $\zeta(2n+1)$ $(n \geq 1)$ are irrational.

It may turn out to be more efficient to work with a larger set of numbers, including special values of multiple polylogarithms

$$\sum_{n_1 > \cdots > n_k \geq 1} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1^{s_1} \cdots n_k^{s_k}}.$$

Probably the right set of points $\underline{z} = (z_1, \ldots, z_k)$ to consider is the set of $k$-tuples consisting of roots of unity. One expects that such a study would yield proofs of the transcendence of numbers like Catalan above mentioned constant, or

$$\sum_{n > k \geq 1} \frac{1}{n^4 k^2} = \zeta(3)^2 - \frac{4\pi^6}{2835},$$

$$\mathrm{Li}_2(1/2) = \sum_{n \geq 1} \frac{1}{n^2 2^n} = \frac{\pi^2}{12} - \frac{1}{2}(\log 2)^2.$$

and

(Ramanujan)
$$\sum_{n \geq k \geq 1} \frac{1}{2^n n^2 k} = \zeta(3) - \frac{1}{12}\pi^2 \log 2,$$

which are not yet known to be irrational.

According to P. Bundschuh [Bun 1979], the transcendence of the numbers

$$\sum_{n=2}^{\infty} \frac{1}{n^s - 1}$$

for even $s \geq 4$ is a consequence of Schanuel's Conjecture 3.1. For $s = 2$ the sum is $3/4$, and for $s = 4$ the value is $(7/8) - (\pi/4)\coth\pi$, which is a transcendental number since $\pi$ and $e^{\pi}$ are algebraically independent.

Nothing is known on the arithmetic nature of the values of Riemann zeta function at rational or algebraic points.

### 3.3. Gamma, Elliptic, Modular, $G$ and $E$-Functions

The transcendence problem of the values of Euler Beta function at rational points has been solved as early as 1940, by Th. Schneider: *for any rational numbers $a$ and $b$ which are not integers and such that $a+b$ is not an integer, the number*

$$\mathrm{B}(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

*is transcendental.* On the other hand transcendence results for the values of the Gamma function itself are not so sharp: apart from Chudnovski's results which imply the transcendence of $\Gamma(1/3)$ and $\Gamma(1/4)$ (and Lindemann's result on the transcendence of $\pi$ which implies that $\Gamma(1/2)=\sqrt{\pi}$ is also transcendental), not so much is known. For instance, as we said earlier, there is no proof so far that $\Gamma(1/5)$ is transcendental. The reason is that the Fermat curve of exponent 5, viz. $x^5+y^5=1$, has genus 2. Its Jacobian is an abelian surface, and the algebraic independence results known for elliptic curves like $x^3+y^3=1$ and $x^4+y^4=1$, which were sufficient for dealing with $\Gamma(1/3)$ and $\Gamma(1/4)$, are not yet known for abelian varieties (see [Grin 2001]).

One might expect that Nesterenko's results (see [NP 2001], Chap. 3) on the algebraic independence of $\pi$, $\Gamma(1/4)$, $e^{\pi}$ and of $\pi$, $\Gamma(1/3)$, $e^{\pi\sqrt{3}}$ should be extended as follows

**Conjecture 3.17.** *Three at least of the four numbers*

$$\pi,\ \Gamma(1/5),\ \Gamma(2/5),\ e^{\pi\sqrt{5}}$$

*are algebraically independent.*

So the challenge is to extend Nesterenko's results on modular functions in one variable (and elliptic curves) to several variables (and abelian varieties).

This maybe one of the easiest question to answer on this topic (but it is yet open). At the opposite one may ask for a general statement which would provide all algebraic relations between Gamma values at rational points. Here is a conjecture of Rohrlich [L 1978a]. Define

$$G(z)=\frac{1}{\sqrt{2\pi}}\Gamma(z).$$

According to the multiplication theorem of Gauss and Legendre, [WW 1902] § 12.15, for each positive integer $N$ and each complex number $x$ such that $Nx\not\equiv 0 \pmod{\mathbb{Z}}$,

$$\prod_{i=0}^{N-1} G\left(x+\frac{i}{N}\right)=N^{(1/2)-Nx}G(Nx).$$

The Gamma function has no zero. Compose it with the canonical map $\mathbb{C}^\times\rightarrow\mathbb{C}^\times/\overline{\mathbb{Q}}^\times$ - which amounts to consider its values modulo the algebraic numbers. The composite map has period 1, and the resulting mapping

$$\overline{G}:\frac{\mathbb{Q}}{\mathbb{Z}}\setminus\{0\}\rightarrow\frac{\mathbb{C}^\times}{\overline{\mathbb{Q}}^\times}$$

is an odd *distribution* on $(\mathbb{Q}/\mathbb{Z})\setminus\{0\}$:

$$\prod_{i=0}^{N-1}\overline{G}\left(x+\frac{i}{N}\right)=\overline{G}(Nx)\quad\text{for}\quad x\in\frac{\mathbb{Q}}{\mathbb{Z}}\setminus\{0\}\quad\text{and}\quad\overline{G}(-x)=\overline{G}(x)^{-1}.$$

Rohrlich's Conjecture ([L 1978a], [L 1978c] Chap. II, Appendix, p. 66) asserts :

**Conjecture 3.18** (Rohrlich). *$\overline{G}$ is the universal odd distribution with values in groups where multiplication by 2 is invertible.*

This leads to the question whether the distribution relations, the oddness relation and the functional equations of the Gamma function generate an ideal over $\overline{\mathbb{Q}}$ of all algebraic relations among the values of $G(x)$ for $x \in \mathbb{Q}$.

Another conjectural extension of his algebraic independence result on Eisenstein series of weight 2, 4 et 6:

$$P(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n},$$

$$Q(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n},$$

$$R(q) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}$$

is conjectured by Nesterenko himself in [NP 2001], Chap. 3 § 1 Conjecture 1.11

**Conjecture 3.19** (Nesterenko). *Let $\tau \in \mathbb{C}$ have positive imaginary part. Assume that $\tau$ is not quadratic. Set $q = e^{2i\pi\tau}$. Then 4 at least of the 5 numbers*

$$\mathbb{Q}\bigl(\tau,\ q,\ P(q),\ Q(q),\ R(q)\bigr)$$

*are algebraically independent.*

Finally we remark that essentially nothing is known on the arithmetic nature of the values of either the Beta or the Gamma function at algebraic irrational points.

A wide range of open problems in transcendental number theory, including not only Schanuel's Conjecture 3.1 and Rohrlich's Conjecture on the values of the Beta functions, but also a conjecture of Grothendieck on the periods of an algebraic variety (see [L 1966] Chap. IV Historical Note; [L 1971] p. 650; [An 1989] p. 6 and [Ch 2001] § 3), are special cases of very general conjectures due to Y. André [An 1997], which deal with periods of mixed motives. A discussion of André's conjectures for certain 1-motives related with products of elliptic curves and their connections with elliptic and modular functions is given in [Be 2000].

A new approach of Grothendieck's Conjecture via Siegel's $G$-functions is initiated in [An 1989] Chap. IX. A development of this method led Y. André to his conjecture on the special points on Shimura varieties [An 1989] Chap. X, § 4, which gave rise to the André–Oort Conjecture [Oo 1997]

**Conjecture 3.20** (André–Oort). *Let $\mathcal{A}_g(\mathbb{C})$ denote the moduli space of principally polarized complex abelian varieties of dimension $g$. Let $Z$ be an irreducible algebraic subvariety of $\mathcal{A}_g(\mathbb{C})$ such that the complex multiplication points on $Z$ are dense for the Zariski topology. Then $Z$ is a subvariety of $\mathcal{A}_g(\mathbb{C})$ of Hodge type.*

Conjecture 3.20 is a far reaching generalization of Schneider's Theorem on the transcendence of $j(\tau)$, where $j$ is the modular invariant and $\tau$ an algebraic point in the Poincaré upper half plane $\mathfrak{H}$, which is not imaginary quadratic ([Schn 1957] Chap. II § 4 Th. 17). We also mention a related conjecture of D. Bertrand which may be viewed as a nonholomorphic analog of Schneider's result (see [NP 2001] Chap. 1 § 4 Conjecture 4.3) and which would answer the following question raised by N. Katz:

(?) *Assume a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ in $\mathbb{C}$ has algebraic invariants $g_2(L)$ and $g_3(L)$ and no complex multiplication. Does this implies that the number*

$$G_2^*(L) = \lim_{s\to 0} \sum_{\omega \in L \setminus \{0\}} \omega^{-2} |\omega|^{-s}$$

*is transcendental?*

Many open transcendence problems dealing with elliptic functions are consequences of André's conjectures (see [Be 2000]), most of which are likely to be very hard. The next one, which is open yet, may be easier, since a number of partial results are already known, after the work of G.V. Chudnovskii and others (see [Grin 2001])

**Conjecture 3.21.** *Given an elliptic curve with Weierstraß equation $y^2 = 4x^3 - g_2x - g_3$, a nonzero period $\omega$, the associated quasi-period $\eta$ of the zeta function and a complex number $u$ which is not a pole of $\wp$,*

$$\text{trdeg}\mathbb{Q}\bigl(g_2, g_3, \pi/\omega, \wp(u), \zeta(u) - (\eta/\omega)u\bigr) \geq 2.$$

Given a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ in $\mathbb{C}$ with invariants $g_2(L)$ and $g_3(L)$, denote by $\eta_i = \zeta_L(z+\omega_i) - \zeta_L(z)$ $(i = 1, 2)$ the corresponding fundamental quasi-periods of the Weierstraß zeta function. Conjecture 3.21 implies that the transcendence degree over $\mathbb{Q}$ of the field $\mathbb{Q}\bigl(g_2(L), g_3(L), \omega_1, \omega_2, \eta_1, \eta_2\bigr)$ is at least 2. Actually, according to André's conjectures, we expect it to be $\geq 3$ in the CM case, and $\geq 5$ otherwise. Moreover the transcendence degree over $\mathbb{Q}$ of the field $\mathbb{Q}\bigl(g_2(L), g_3(L), \omega_1, \omega_2, \eta_1, \eta_2, 2i\pi\bigr)$ should be $\geq 4$ in the CM case, and $\geq 6$ otherwise.

According to [Di 2000] conjectures 1 and 2 p. 187, the following special case can be stated in two equivalent ways: either in terms of values of elliptic functions, or else in terms of values of Eisenstein series $E_2$, $E_4$ and $E_6$ (which are $P$, $Q$ and $R$ in Ramanujan's notation).

(?) *For any lattice $L$ in $\mathbb{C}$ without complex multiplication and for any nonzero period $\omega$ of $L$,*

$$\text{trdeg}\mathbb{Q}\bigl(g_2(L), g_3(L), \omega/\sqrt{\pi}, \eta/\sqrt{\pi}\bigr) \geq 2.$$

(?) *For any $\tau \in \mathfrak{H}$ which is not imaginary quadratic,*

$$\text{trdeg}\mathbb{Q}\bigl(\pi E_2(\tau), \pi^2 E_4(\tau), \pi^6 E_6(\tau)\bigr) \geq 2.$$

Moreover, each of these two statements implies the following one, which is stronger than a conjecture of Lang ([L 1971] p. 652):

(?) *For any $\tau \in \mathfrak{H}$ which is not imaginary quadratic,*

$$\text{trdeg}\mathbb{Q}\bigl(j(\tau), j'(\tau), j''(\tau)\bigr) \geq 2.$$

Further related open problems are proposed by G. Diaz in [Di 1997] and [Di 2000], in connection with conjectures due to D. Bertrand on the values of the modular function $J(q)$, where $j(\tau) = J(e^{2i\pi\tau})$ (see [Ber 1997b] as well as [NP 2001] Chap. 1 § 4 and Chap. 2 § 4).

**Conjecture 3.22** (Bertrand). *Let $q_1, \ldots, q_n$ be nonzero algebraic numbers in the unit open disc such that the $3n$ numbers*

$$J(q_i),\ DJ(q_i),\ D^2J(q_i) \qquad (i = 1, \ldots, n)$$

*are algebraically dependent over $\mathbb{Q}$. Then there exist two indices $i \neq j$ $(1 \leq i \leq n,\ 1 \leq j \leq n)$ such that $q_i$ and $q_j$ are multiplicatively dependent.*

**Conjecture 3.23** (Bertrand). *Let $q_1$ and $q_2$ be two nonzero algebraic numbers in the unit open disc. Suppose that there is an irreducible element $P \in \mathbb{Q}[X, Y]$ such that*

$$P\bigl(J(q_1), J(q_2)\bigr) = 0.$$

*Then there exists a constant $c$ and a positive integer $s$ such that $P = c\Phi_s$, where $\Phi_s$ is the modular polynomial of level $s$. Moreover $q_1$ and $q_2$ are multiplicatively dependent.*

Among Siegel's $G$-functions are the algebraic functions. Transcendence methods produce some information, in particular in connection with Hilbert's Irreducibility Theorem. Let $f \in \mathbb{Z}[X, Y]$ be a polynomial which is irreducible in $\mathbb{Q}(X)[Y]$. According to Hilbert's Irreducibility Theorem, the set of positive integers $n$ such that $P(n, Y)$ is irreducible in $\mathbb{Q}[Y]$ is infinite. No upper bound for an admissible value for $n$ is known.

**Conjecture 3.24.** *Is-there such a bound depending polynomially on the degree and height of $P$?*

Such questions are also related to the *Galois inverse problem* [Se 1989].

Also the polylogarithms

$$\mathrm{Li}_s(z) = \sum_{n\geq 1} \frac{z^m}{n^s}$$

(where $s$ is a positive integer) are $G$-functions; unfortunately the Siegel-Shidlovskii method cannot be used so far to prove irrationality of the values of the Riemann zeta function ([FN 1998] Chap. 5 § 7 p. 247).

With $G$-functions, the other class of analytic functions introduced by C.L. Siegel in 1929 is the class of $E$-functions, which includes the hypergeometric ones. One main open question is to investigate the arithmetic nature of the values at algebraic points of hypergeometric functions with *algebraic* parameters:

$${}_2F_1\left(\begin{matrix}\alpha\ ,\ \beta \\ \gamma\end{matrix}\Big| z\right) = \sum_{n\geq 0} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\cdot\frac{z^n}{n!},$$

defined for $|z| < 1$ and $\gamma \notin \{0, -1, -2, \ldots\}$.

In 1949, C.L. Siegel ([Si 1949] Chap. 2 § 9 p. 54 and 58; see also [FS 1967] p. 62 and [FN 1998] Chap. 5, § 1.2) asked if *any $E$-function satisfying a linear differential equation with coefficients in $\mathbb{C}(z)$ can be expressed as a polynomial in $z$ and a finite number of hypergeometric $E$-functions or functions obtained from them by a change of variables of the form $z \mapsto \gamma z$ with algebraic $\gamma$'s?*

Finally, we quote from [W 1999b]: a conjecture in the "folklore" is that the zeroes of the Riemann zeta function (say their imaginary parts, assuming it $> 0$) are algebraically independent. As suggested by J-P. Serre, one might be tempted to consider also

- The eigenvalues of the zeroes of the hyperbolic Laplacian in the upper half plane modulo $\mathrm{SL}_2(\mathbb{Z})$ (i.e. to study the algebraic independence of the zeroes of Selberg zeta function).
- The eigenvalues of the Hecke operators acting on the corresponding eigenfunctions (Maass forms).

### 3.4. Fibonacci and Miscellanea

Many further open problems arise in transcendental number theory. An intriguing question is to investigate the arithmetic nature of real numbers given in terms of power series involving the Fibonacci sequence

$$F_{n+2} = F_{n+1} + F_n, \qquad F_0 = 0, \quad F_1 = 1.$$

Several results are due to P. Erdős, R. André-Jeannin, C. Badea, J. Sándor, P. Bundschuh, A. Pethő, P.G. Becker, T. Töpfer, D. Duverney, Ku. et Ke. Nishioka, I. Shiokawa... It is known that the number

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = 1$$

is rational, while

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1}} = \frac{1-\sqrt{5}}{2},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}+1} = \frac{\sqrt{5}}{2},$$

are algebraic irrational numbers. Each of the numbers

$$\sum_{n=0}^{\infty} \frac{1}{F_n}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n + F_{n+2}} \quad \text{and} \quad \sum_{n\geq 1} \frac{1}{F_1 F_2 \cdots F_n}$$

is irrational, but it is not known whether they are algebraic or transcendental. The numbers

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n-1}}, \quad \sum_{n=0}^{\infty} \frac{1}{F_n^2}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{F_n^2}, \quad \sum_{n=0}^{\infty} \frac{n}{F_{2n}}, \quad \sum_{n=0}^{\infty} \frac{1}{F_{2^n-1}+F_{2^n+1}} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{F_{2^n+1}}$$

are all transcendental (further results of algebraic independence are known). The main challenge here is to formulate a conjectural statement which would give a satisfactory description of the situation.

There is a similar situation for infinite sums $\sum_n f(n)$ where $f$ is a rational function [Ti 2000]: while

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

and

$$\sum_{n=0}^{\infty} \left( \frac{1}{4n+1} - \frac{3}{4n+2} + \frac{1}{4n+3} - \frac{1}{4n+4} \right) = 0$$

are rational numbers, the sums

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \log 2, \qquad \sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)} = \frac{\pi}{3},$$

$$\sum_{n=0}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi}+e^{-\pi}}{e^{\pi}+e^{-\pi}},$$

and

$$\sum_{n=0}^{\infty} \frac{1}{(6n+1)(6n+2)(6n+3)(6n+4)(6n+5)(6n+6)} = \frac{1}{4320}(192\log 2 - 81\log 3 - 7\pi\sqrt{3})$$

are transcendental. The simplest example of the Euler sums $\sum_n n^{-s}$ (see § 3.2) illustrates the difficulty of the question: here again, even a sufficiently general conjecture is missing.

The arithmetic study of the values of power series gives rise to many open problems. We only mention a few of them.

The next two question are due Mahler [M 1984]:

**Question 3.25** (Mahler). *Are there entire transcendental functions $f(z)$ such that if $x$ is a Liouville number then so is $f(x)$?*

**Conjecture 3.26** (Mahler). *Let $(\epsilon_n)_{n\geq 0}$ be a sequence of elements in $\{0,1\}$. Assume that the real number*

$$\sum_{n\geq 0} \epsilon_n 3^n$$

*is algebraic. Then this number is rational.*

The study of integral valued entire functions gives rise to several open problems; we quote only one of them, which arose in the work of D.W. Masser and F. Gramain on entire functions $f$ of one complex variable which map the ring of Gaussian integers $\mathbb{Z}[i]$ into itself. The initial question (namely to derive an analog of Pólya's Theorem in this setting) has been solved by F. Gramain in [Gr 1981] (after previous work of Fukasawa, Gel'fond, Gruman and Masser): *If $f$ is not a polynomial, then*

$$\limsup_{r\to\infty} \frac{1}{r^2} \log |f|_r \geq \frac{\pi}{2e}\cdot$$

Preliminary works on this estimate gave rise to the following problem, which is still unsolved. For each integer $k \geq 2$, set

$$r_k = \min\bigl\{r > 0\ ;\ \text{ there exists } z \in \mathbb{C} \text{ such that } \mathrm{Card}\bigl(\mathbb{Z}[i] \cap D(z,r)\bigr) \geq k\bigr\},$$

where $D(z,r) = \{w \in \mathbb{C}\ ;\ |w - z| \leq r\}$. For $n \geq 2$, set

$$\delta_n = \left(\sum_{k=2}^{n} \frac{1}{\pi r_k^2}\right) - \log n.$$

The limit $\delta = \lim_{n\to\infty} \delta_n$ exists (it is an analog in dimension 2 of Euler constant), and the best known estimates for it are [GrW 1985]

$$1.811\ldots < \delta < 1.897\ldots$$

Masser conjectures:

$$\delta = 1 + \frac{4}{\pi}\bigl(\gamma L'(1) + L(1)\bigr),$$

where $\gamma$ is Euler's constant and

$$L(s) = \sum_{n\geq 0} (-1)^n (2n+1)^{-s}.$$

Other problems related to the lattice $\mathbb{Z}[i]$ are described in the section "*On the borders of geometry and arithmetic*" of [Sie 1964].

# 4. Height

For a nonzero polynomial $f \in \mathbb{C}[X]$ of degree $d$:

$$f(X) = a_0X^d + a_1X^{d-1} + \cdots + a_{d-1}X + a_d = a_0\prod_{i=1}^{d}(X-\alpha_i),$$

define its *usual height* by

$$\mathsf{H}(f) = \max\{|a_0|, \ldots, |a_d|\}$$

and its *Mahler's measure* by

$$\mathsf{M}(f) = |a_0|\prod_{i=1}^{d}\max\{1, |\alpha_i|\} = \exp\left(\int_0^1 \log|f(e^{2i\pi t})|dt\right).$$

The equality between these two formulae follows from Jensen's formula; it was first noticed by Mahler (see [M 1976] Chap. I § 7, as well as [W 2000b] Chap. 3 and [S 1999]; the latter includes an extension to several variables).

When $\alpha$ is an algebraic number with minimal polynomial $f \in \mathbb{Z}[X]$, define its *Mahler's measure* by $\mathsf{M}(\alpha) = \mathsf{M}(f)$ and its *usual height* by $\mathsf{H}(\alpha) = \mathsf{H}(f)$.

Further, if $\alpha$ has degree $d$, define its *logarithmic height* as

$$\mathsf{h}(\alpha) = \frac{1}{d}\log \mathsf{M}(\alpha).$$

The height of the projective point $(1\colon\alpha_1\colon\cdots\colon\alpha_n) \in \mathbb{P}_n$ is denoted by $\mathsf{h}(1\colon\alpha_1\colon\cdots\colon\alpha_n)$ (see for instance [W 2000b] § 3.2).

Further, if $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is a $n$-tuple of multiplicatively independent algebraic numbers, $\omega(\underline{\alpha})$ denotes the minimum degree of a nonzero polynomial in $\mathbb{Q}[X_1, \ldots, X_n]$ which vanishes at $\underline{\alpha}$.

A side remark is that Mahler's measure of a polynomial in a single variable with algebraic coefficients is an algebraic number. The situation is much more intricate for polynomials in several variables and gives rise to further open problems (C. Deninger, D. Boyd).

## 4.1. Lehmer's Problem

The smallest known value for $d\mathsf{h}(\alpha)$, which was found in 1933 by D. H. Lehmer, is $\log\alpha_0 = 0.1623576\ldots$, where $\alpha_0 = 1.1762808\ldots$ is the real root(*) of the degree 10 polynomial

$$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1.$$

Lehmer asked whether it is true that for every positive $\epsilon$ there exists an algebraic integer $\alpha$ for which $1 < \mathsf{M}(\alpha) < 1 + \epsilon$.

**Conjecture 4.1** (Lehmer's Problem). *There exists a positive absolute constant $c$ such that, for any nonzero algebraic number $\alpha$ of degree at most $d$ which is not a root of unity,*

$$\mathsf{h}(\alpha) \geq \frac{c}{d}\cdot$$

Since $\mathsf{h}(\alpha) \leq \log\overline{\lceil\alpha\rceil}$ , the following statement [SZ 1965] is a weaker assertion than Conjecture 4.1.

(*) Further properties of this smallest known Salem number are described by Don Zagier in his paper: Special values and functional equations of polylogarithms, Appendix A of *Structural properties of polylogarithms*, ed. L. Lewin, Mathematical Surveys and Monographs, vol. **37**, Amer. Math. Soc. 1991, p. 377–400.

**Conjecture 4.2** (Schinzel-Zassenhaus). *There exists an absolute constant $c > 0$ such that, for any nonzero algebraic integer of degree $d$ which is not a root of unity,*

$$\overline{|\alpha|} \geq 1 + \frac{c}{d}\cdot$$

Lehmer's Problem is related to the multiplicative group $\mathbb{G}_{\rm m}$. Generalizations to $\mathbb{G}_{\rm m}^n$ have been considered by many authors (see for instance [Ber 1997a] and [Sch 1999]). In [AD 1999] Conjecture 1.4, F. Amoroso and S. David extend Lehmer's problem 4.1 to simultaneous approximation.

**Conjecture 4.3** (Amoroso-David). *For each positive integer $n \geq 1$ there exists a positive number $c(n)$ having the following property. Let $\alpha_1, \ldots, \alpha_n$ be multiplicatively independent algebraic numbers. Define $D = [\mathbb{Q}(\alpha_1, \ldots, \alpha_n) \colon \mathbb{Q}]$. Then*

$$\prod_{i=1}^{n} \mathsf{h}(\alpha_i) \geq \frac{c(n)}{D}\cdot$$

The next statement ([AD 1999] Conjecture 1.3 and [AD 2000] Conjecture 1.3) is stronger.

**Conjecture 4.4** (Amoroso-David). *For each positive integer $n \geq 1$ there exists a positive number $c(n)$ such that, if $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is a $n$-tuple of multiplicatively independent algebraic numbers, then*

$$\mathsf{h}(1 \colon \alpha_1 \colon \cdots \colon \alpha_n) \geq \frac{c(n)}{\omega(\underline{\alpha})}\cdot$$

Many open questions are related with the height of subvarieties [D 2001]. The next one, dealing with the height of subvarieties of $\mathbb{G}_{\rm m}^n$ and proposed by Amoroso and David in [AD 2000] Conjecture 1.4 (see also Conjecture 1.5 of [AD 2000], which is due to David and Philippon [DP 1999]), is more general than Conjecture 4.4:

**Conjecture 4.5** (Amoroso-David). *For each integer $n \geq 1$ there exists a positive constant $c(n)$ such that, for any algebraic subvariety $V$ of $\mathbb{G}_{\rm m}^n$ which is defined over $\mathbb{Q}$, which is $\mathbb{Q}$-irreducible, and which is not a union of translates of algebraic subgroups by torsion points,*

$$\hat{V} \geq c(n) \deg(V)^{(s-\dim V-1)/(s-\dim V)},$$

*where $s$ is the dimension of the smallest algebraic subgroup of $\mathbb{G}_{\rm m}^n$ containing $V$.*

Let $V$ be an open subset of $\mathbb{C}$. The *Lehmer-Langevin constant* of $V$ is defined as

$$L(V) = \inf \mathsf{M}(\alpha)^{1/[\mathbb{Q}(\alpha):\mathbb{Q}]},$$

where $\alpha$ ranges over the set of nonzero and non-cyclotomic algebraic numbers $\alpha$ lying with all their conjugates outside of $V$. It has been proved by M. Langevin in 1985 that $L(V) > 1$ as soon as $V$ contains a point on the unit circle $|z| = 1$.

**Problem 4.6.** *For $\theta \in (0, \pi)$, define*

$$V_\theta = \{re^{it} \; ; \; r > 0, \; |t| > \theta\}.$$

*Compute $L(V_\theta)$ in terms of $\theta$.*

The solution is known only for a very few values of $\theta$: in 1995 G. Rhin and C. Smyth [RS 1995] computed $L(V_\theta)$ for nine values of $\theta$, including

$$L(V_{\pi/2}) = 1.12\ldots$$

In a different direction, an analog of Lehmer's problem has been raised for elliptic curves, and more generally for abelian varieties. Here is Conjecture 1.4 of [DH 2000]. Let $A$ be an abelian variety defined over a number field $K$ and equipped with a symmetric ample line bundle $\mathcal{L}$. For any $P \in A(\overline{\mathbb{Q}})$ define

$$\delta(Q) = \min \deg(V)^{1/\mathrm{codim}(V)},$$

where $V$ ranges over the proper subvarieties of $A$, defined over $K$, $K$-irreducible and containing $Q$, while $\deg(V)$ is the degree of $V$ with respect to $\mathcal{L}$. Also denote by $\hat{h}_{\mathcal{L}}$ the Néron-Tate canonical height on $A(\overline{\mathbb{Q}})$ associated to $\mathcal{L}$.

**Conjecture 4.7** (David-Hindry). *There exists a positive constant $c$, depending only on $A$ and $\mathcal{L}$, such that for any $P \in A(\overline{\mathbb{Q}})$ which has infinite order modulo any abelian subvariety,*

$$\hat{h}_{\mathcal{L}}(P) \geq c\delta(P)^{-1}.$$

An extension of Conjecture 4.7 to linearly independent tuples is also stated in [DH 2000] Conjecture 1.6.

The dependence on $A$ of these "constants" also raise interesting questions. Just take an elliptic curve $E$ and consider the Néron-Tate height $\hat{h}(P)$ of a nontorsion rational point on a number field $K$. Several invariants are related to $E$: the modular invariant $j_E$, the discriminant $\Delta_E$ and Faltings height $h(E)$. A conjecture of Lang asserts

$$\hat{h}(P) \geq c(K) \max\{1, h(E)\},$$

while S. Lang ([L 1978b] p. 92) and J. Silverman ([Sil 1986] Chap. VIII § 10 Conjecture 9.9) conjecture

$$\hat{h}(P) \geq c(K) \max\{\log |N_{K/\mathbb{Q}}(\Delta_E)|,\ h(j_E)\}.$$

Partial results are known (J. Silverman, M. Hindry and J. Silverman, S. David), but the conjecture is not yet proved.

There is another abelian question related with Mahler's measure. According to D.A. Lind, Lehmer's problem is known to be equivalent to the existence of a continuous endomorphism of the infinite torus $(\mathbb{R}/\mathbb{Z})^{\mathbb{Z}}$ with finite entropy. A similar question has been asked by P. D'Ambros, G. Everest, R. Miles and T. Ward [AEMW 2000] for elliptic curves, and it can be extended to abelian varieties, and more generally to commutative algebraic groups.

## 4.2. Wirsing-Schmidt Conjecture

According to Dirichlet's box principle, for any irrational real number $\theta$ there is an infinite set of rational numbers $p/q$ with $q > 0$ such that

$$\left|\theta - \frac{p}{q}\right| \leq \frac{1}{q^2}. \tag{4.8}$$

There are several extensions of this result. For the first one, we write (4.8) as $|q\theta - p| < 1/q$ and we replace $q\theta - p$ by $P(\theta)$ for some polynomial $P$:

(4.9) *Let $\theta$ be a real number, $d$ and $H$ positive integers. There exists a nonzero polynomial $P \in \mathbb{Z}[X]$, of degree $\leq d$ and usual height $\leq H$, such that*

$$|P(\theta)| \leq cH^{-d}$$

*where* $c = 1 + |\theta| + \cdots + |\theta|^d$.

There is no assumption on $\theta$, but if $\theta$ is algebraic of degree $\leq d$ then there is a trivial solution!

A similar result holds for complex numbers, and more generally for $\theta$ replaced by a $m$-tuple $(\theta_1, \ldots, \theta_m) \in \mathbb{C}^m$ (see for instance [W 2000b] Lemma 15.11). For simplicity, we deal here only with the easiest case.

Another extension of (4.8) is interesting to consider, where $p/q$ is replaced by an algebraic number of degree $\leq d$. If the polynomial $P$ given by (4.9) has a single simple root $\gamma$ close to $\theta$, then

$$|\theta - \gamma| \leq c' H^{-d}$$

where $c'$ depends only on $\theta$ and $d$. However, the root of $P$ which is nearest $\gamma$ may be a multiple root, and may be not unique: this occurs precisely when the first derivative $P'$ of $P$ has a small absolute value at $\theta$. Dirichlet's box principle does not allow us to construct a polynomial $P$ like in (4.8) with a *lower* bound for $|P'(\theta)|$.

However Wirsing [Wi 1961] succeeded to prove the following theorem:

(4.10) *There is an absolute constant $c > 0$ such that, for any transcendental real number $\theta$ and any positive integer $n$, there are infinitely many algebraic numbers $\gamma$ of degree $\leq n$ for which*

$$|\theta - \gamma| \leq \mathsf{H}(\gamma)^{-cn}.$$

Wirsing himself obtained his estimate in 1960 with $cn$ replaced by $(n/2) + 2 - \epsilon_n$, where $\epsilon_n \to 0$ as $n \to \infty$. For $n = 2$, H. Davenport and W.M. Schmidt in 1967 obtained the exponent 3. The conjecture of Wirsing and Schmidt (see [Wi 1961] and [Sch 1980] Chap. VIII § 3; see also [Bu 2000a] and [Bu 2000b]) asserts that the exponent $cn$ in (4.10) can be replaced by $n + 1$:

**Conjecture 4.11** (Wirsing and Schmidt). *For any positive integer $n$, any positive real number $\epsilon$, and any real number $\theta$ which is either transcendental or else is algebraic of degree $> n$, there exists a positive constant $c = c(n, \epsilon, \theta)$ with the following property: there exist infinitely many algebraic numbers $\gamma$ of degree $\leq n$ for which*

$$0 < |\theta - \gamma| < c\mathsf{H}(\gamma)^{-n-1+\epsilon}.$$

There are connections between the problems of algebraic independence we considered in § 1, the question of measures of linear independence of logarithms of algebraic numbers discussed in § 2, and the notion of height which was introduced in § 3.

Consider now the question of simultaneous approximation of complex numbers by algebraic numbers. For a $m$-tuple $\underline{\gamma} = (\gamma_1, \ldots, \gamma_m)$ of algebraic numbers, we define

$$\mu(\underline{\gamma}) = [\mathbb{Q}(\underline{\gamma}) : \mathbb{Q}] \max_{1 \leq j \leq m} \mathsf{h}(\gamma_j),$$

so that for $m = 1$ and $\gamma \in \overline{\mathbb{Q}}$ we have simply $\mu(\gamma) = \log \mathsf{M}(\gamma)$.

So far, connections between simultaneous approximation and algebraic independence have been established only for small transcendence degree. The missing link for large transcendence degree is given by the next statement (see [W 2000b] Conjecture 15.31; [Lau 1998] § 4.2 Conjecture 5; [Lau 1999b] Conjecture 1; [W 2000a] Conjecture 2; as well as [Ro 2001b] Conjectures 1 and 2).

**Conjecture 4.12.** *Let $\underline{\theta} = (\theta_1, \ldots, \theta_m)$ be a $m$-tuple of complex numbers. Define*

$$t = \mathrm{trdeg}\mathbb{Q}(\underline{\theta})$$

*and assume $t \geq 1$. There exist positive constants $c_1$ and $c_2$ with the following property. Let $(D_\nu)_{\nu\geq 0}$ and $(\mu_\nu)_{\nu\geq 0}$ be sequences of real numbers satisfying*

$$c_1 \leq D_\nu \leq \mu_\nu, \quad D_\nu \leq D_{\nu+1} \leq 2D_\nu, \quad \mu_\nu \leq \mu_{\nu+1} \leq 2\mu_\nu \qquad (\nu \geq 0).$$

*Assume also that the sequence $(\mu_\nu)_{\nu\geq 0}$ is unbounded. Then for infinitely many $\nu$ there exists a $m$-tuple $(\gamma_1, \ldots, \gamma_m)$ of algebraic numbers satisfying*

$$[\mathbb{Q}(\underline{\gamma}) : \mathbb{Q}] \leq D_\nu, \quad \mu(\underline{\gamma}) \leq \mu_\nu$$

*and*

$$\max_{1\leq i\leq m} |\theta_i - \gamma_i| \leq \exp\{-c_2 D_\nu^{1/t} \mu_\nu\}.$$

There are two different (related) quantitative refinements to a transcendence result: for a transcendental number $\theta$, either one proves *a transcendence measure*, which is a lower bound for $|P(\theta)|$ when $P$ is a nonzero polynomial with integer coefficients, or else one proves a *measure of algebraic approximation for* $\theta$, which is a lower bound for $|\theta - \gamma|$ when $\gamma$ is an algebraic number. In both cases such a lower bound will depend, usually, on the degree (of the polynomial $P$, or on the algebraic number $\gamma$), and on the height of the same.

Next, given several transcendental numbers $\theta_1, \ldots, \theta_n$, one may consider either a measure of simultaneous approximation by algebraic numbers, namely a lower bound for

$$\max\{|\theta_i - \gamma_i|\}$$

when $\gamma_1, \ldots, \gamma_n$ are algebraic numbers, or a measure of algebraic independence, which is a lower bound for

$$|P(\theta_1, \ldots, \theta_n)|$$

when $P$ is a non-zero polynomial with integer coefficients. The first estimate deals with algebraic points (algebraic sets of zero dimension), the second with hypersurfaces (algebraic sets of codimension 1). There is a whole set of intermediate possibilities which have been investigated by Yu.V. Nesterenko and P. Philippon, and are closely connected.

For instance Conjecture 4.12 deals with simultaneous approximation by algebraic points; M. Laurent and D. Roy ask a general questions for the approximation by algebraic subsets of $\mathbb{C}^m$, defined over $\mathbb{Q}$. For instance Conjecture 2 in [Lau 1999b] as well as the conjecture in § 9 of [Ro 2001d] deal with the more general problem of approximation of points in $\mathbb{C}^n$ by points located on $\mathbb{Q}$-varieties of a given dimension.

For an algebraic subset $Z$ of $\mathbb{C}^m$, defined over $\mathbb{Q}$, denote by $t(Z)$ the size of a Chow form of $Z$.

**Conjecture 4.13** (Laurent-Roy). *Let $\theta \in \mathbb{C}^m$. There is a positive constant $c$, depending only on $\theta$ and $m$, with the following property. Let $k$ be an integer with $0 \leq k \leq m$. For infinitely many integers $T \geq 1$, there exists an algebraic set $Z \subset \mathbb{C}^m$, defined over $\mathbb{Q}$, of dimension $k$, and a point $\alpha \in Z$, such that*

$$t(Z) \leq T^{m-k} \quad \textit{and} \quad |\theta - \alpha| \leq \exp\{-cT^{m+1}\}.$$

Further far reaching open problems in this direction are proposed by Philippon as Problèmes 7, 8 and 10 in [P 1997] § 5.

### 4.3. Logarithms of Algebraic Numbers

We already suggested several questions related with linear independence measures over the field of rational numbers for logarithms of rational numbers (see Conjectures 2.4, 2.5 and 2.13). Now that we have a notion of height for algebraic numbers at our disposal, we extend our study to linear independence measures over the field of algebraic numbers for logarithms of algebraic numbers.

The next statement is Conjecture 14.25 of [W 2000b].

**Conjecture 4.14.** *There exist two positive absolute constants $c_1$ and $c_2$ with the following property. Let $\lambda_1,\ldots,\lambda_m$ be logarithms of algebraic numbers with $\alpha_i = e^{\lambda_i}$ $(1\le i\le m)$, let $\beta_0,\ldots,\beta_m$ be algebraic numbers, $D$ the degree of the number field*

$$\mathbb{Q}(\alpha_1,\ldots,\alpha_m,\beta_0,\ldots,\beta_m)$$

*and finally let $h\ge 1/D$ satisfy*

$$h\ge \max_{1\le i\le m}\mathsf{h}(\alpha_i),\quad h\ge \frac{1}{D}\max_{1\le i\le m}|\lambda_i|\quad\textit{and}\quad h\ge\max_{0\le j\le m}\mathsf{h}(\beta_j).$$

*1) Assume that the number*

$$\Lambda=\beta_0+\beta_1\lambda_1+\cdots+\beta_m\lambda_m$$

*is nonzero. Then*

$$|\Lambda|\ge \exp\bigl\{-c_1 mD^2h\bigr\}.$$

*2) Assume $\lambda_1,\ldots,\lambda_m$ are linearly independent over $\mathbb{Q}$. Then*

$$\sum_{i=1}^m|\lambda_i-\beta_i|\ge\exp\bigl\{-c_2mD^{1+(1/m)}h\bigr\}.$$

Assuming both Conjecture 4.12 and part 2 of Conjecture 4.14, one deduces not only Conjecture 3.3, but also further special cases of Conjecture 3.1 (these connections are described in [W 2000a] as well as [W 2000b] Chap. 15).

As far as part 1 of Conjecture 4.14 is concerned, weaker estimates are available (see [W 2000b] § 10.4). Here is a much weaker (but still open) statement than either Conjecture 2.5 or part 1 of Conjecture 4.14:

**Conjecture 4.15.** *There exists a positive absolute constant $C$ with the following property. Let $\alpha_1,\ldots,\alpha_n$ be nonzero algebraic numbers and $\log\alpha_1,\ldots,\log\alpha_n$ logarithms of $\alpha_1,\ldots,\alpha_n$ respectively. Assume that the numbers $\log\alpha_1,\ldots,\log\alpha_n$ are $\mathbb{Q}$-linearly independent. Let $\beta_0,\beta_1,\ldots,\beta_n$ be algebraic numbers, not all of which are zero. Denote by $D$ the degree of the number field*

$$\mathbb{Q}(\alpha_1,\ldots,\alpha_n,\beta_0,\beta_1,\ldots,\beta_n)$$

*over $\mathbb{Q}$. Further, let $A_1,\ldots,A_n$ and $B$ be positive real numbers, each $\ge e$, such that*

$$\log A_j\ge\max\left\{\mathsf{h}(\alpha_j),\ \frac{|\log\alpha_j|}{D},\ \frac{1}{D}\right\}\quad(1\le j\le n),$$

$$B\ge\max_{1\le j\le n-1}\mathsf{h}(\beta_j).$$

*Then the number*

$$\Lambda = \beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

*satisfies*

$$|\Lambda| > \exp\{-C^n D^{n+2} (\log A_1) \cdots (\log A_n)(\log B + \log D)(\log D)\}.$$

One is rather close to such an estimate (see [W 2001] §§ 5 and 6, as well as [Matv 2000]). The result is proved now in the so-called rational case, where

$$\beta_0 = 0 \quad \text{and} \quad \beta_i \in \mathbb{Q} \quad \text{for} \quad 1 \le i \le n.$$

In the general case, one needs a further condition, namely

$$B \ge \max_{1 \le i \le n} \log A_i.$$

Removing this extra condition would enable one to prove that numbers like $e^\pi$ or $2^{\sqrt{2}}$ are not Liouville numbers.

These questions are the first and simplest ones concerning transcendence measures,measures of Diophantine approximation, measures of linear independence and measures of algebraic independence. One may ask many further questions on this topic, including an effective version of Schanuel's conjecture. It is interesting to notice that in this case a "technical condition" cannot be omitted ([W 1999b] Conjecture 1.4).

Recall that the rank of a prime ideal $\mathfrak{P} \subset \mathbb{Q}[T_1, \ldots, T_m]$ is the largest integer $r \ge 0$ such that there exists an increasing chain of prime ideals

$$(0) = \mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \cdots \subset \mathfrak{P}_r = \mathfrak{P}.$$

The rank of an ideal $\mathfrak{I} \subset \mathbb{Q}[T_1, \ldots, T_m]$ is the minimum rank of a prime ideal containing $\mathfrak{I}$.

**Conjecture 4.16** (Quantitative Refinement of Schanuel's Conjecture). *Let $x_1, \ldots, x_n$ be $\mathbb{Q}$-linearly independent complex numbers. Assume that for any $\epsilon > 0$, there exists a positive number $H_0$ such that, for any $H \ge H_0$ and $n$-tuple $(h_1, \ldots, h_n)$ of rational integers satisfying $0 < \max\{|h_1|, \ldots, |h_n|\} \le H$, the inequality*

$$|h_1 x_1 + \cdots + h_n x_n| \ge \exp\{-H^\epsilon\}$$

*holds. Let $d$ be a positive integer. Then there exists a positive number $C = C(x_1, \ldots, x_n, d)$ with the following property: for any integer $H \ge 2$ and any $n+1$ tuple $P_1, \ldots, P_{n+1}$ of polynomials in $\mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ with degrees $\le d$ and usual heights $\le H$, which generate an ideal of $\mathbb{Q}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ of rank $n+1$, we have*

$$\sum_{j=1}^{n+1} \bigl|P_j(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})\bigr| \ge H^{-C}.$$

A consequence of Conjecture 4.16 is a quantitative refinement to Conjecture 3.3 on algebraic independence of logarithms of algebraic numbers [W 1999b]:

(?) *If $\log \alpha_1, \ldots, \log \alpha_n$ are $\mathbb{Q}$-linearly independent logarithms of algebraic numbers and $d$ a positive integers, there exists a constant $C > 0$ such that, for any nonzero polynomial $P \in \mathbb{Z}[X_1, \ldots, X_n]$ of degree $\le d$ and height $\le H$ with $H \ge 2$,*

$$|P(\log \alpha_1, \ldots, \log \alpha_n)| \ge H^{-C}.$$

### 4.4. Density: Mazur's Problem

Let $K$ be a number field with a given real embedding. Let $V$ be a smooth variety over $K$. Denote by $Z$ the closure, for the real topology, of $V(K)$ in $V(\mathbb{R})$. In his paper [Maz 1992] on the topology of rational points, Mazur asks:

**Question 4.17** (Mazur). *Assume that $K = \mathbb{Q}$ and that $V(\mathbb{Q})$ is Zariski dense; is $Z$ a union of connected components of $V(\mathbb{R})$?*

An interesting fact is that Mazur asks this question in connection with the rational version of Hilbert's tenth problem (see [Maz 1994] and [Maz 1995]).

The answer to question 4.17 is negative: an example is given in [CSS 1997] by Colliot-Thélène, Skorobogatov and Swinnerton-Dyer of a smooth surface $V$ over $\mathbb{Q}$, whose $\mathbb{Q}$-rational points are Zariski-dense, but such that the closure $Z$ in $V(\mathbb{R})$ of the set of $\mathbb{Q}$-points is not a union of connected components.

However for the special case of abelian varieties, there are good reasons to believe that the answer to question 4.17 is positive. Indeed for this special case a reformulation of question 4.17 is the following:

(?) *Let $A$ be a simple abelian variety over $\mathbb{Q}$. Assume the Mordell-Weil group $A(\mathbb{Q})$ has rank $\geq 1$. Then $A(\mathbb{Q}) \cap A(\mathbb{R})^0$ is dense in the neutral component $A(\mathbb{R})^0$ of $A(\mathbb{R})$.*

This statement is equivalent to the next one:

**Conjecture 4.18.** *Let $A$ be a simple abelian variety over $\mathbb{Q}$, $\exp_A : \mathbb{R}^g \rightarrow A(\mathbb{R})^0$ the exponential map of the Lie group $A(\mathbb{R})^0$ and $\Omega = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_g$ its kernel. Let $u = u_1\omega_1 + \cdots + u_g\omega_g \in \mathbb{R}^g$ satisfy $\exp_A(u) \in A(\mathbb{Q})$. Then $1, u_1, \ldots, u_g$ are linearly independent over $\mathbb{Q}$.*

The following quantitative refinement of Conjecture 4.18 is suggested in [W 1999a] Conjecture 1.1. For $\zeta = (\zeta_0 : \cdots : \zeta_N)$ and $\xi = (\xi_0 : \cdots : \xi_N)$ in $\mathbb{P}_N(\mathbb{R})$, write

$$\mathrm{dist}(\zeta, \xi) = \frac{\displaystyle\max_{0 \leq i,j \leq N} |\zeta_i \xi_j - \zeta_j \xi_i|}{\displaystyle\max_{0 \leq i \leq N} |\zeta_i| \cdot \max_{0 \leq j \leq N} |\xi_j|}.$$

**Conjecture 4.19.** *Let $A$ be a simple Abelian variety of dimension $g$ over a number field $K$ embedded in $\mathbb{R}$. Denote by $\ell$ the rank over $\mathbb{Z}$ of the Mordell-Weil group $A(K)$. For any $\epsilon > 0$, there exists $h_0 > 0$ (which depends only on the Abelian variety $A$, the real number field $K$ and $\epsilon$) such that, for any $h \geq h_0$ and any $\zeta \in A(\mathbb{R})^0$, there is a point $\gamma \in A(K)$ with Néron-Tate height $\leq h$ such that*

$$\mathrm{dist}(\zeta, \gamma) \leq h^{-(\ell/2g)+\epsilon}.$$

Similar problems arise for commutative algebraic groups. Let us consider the easiest case of a torus $\mathbb{G}_{\mathrm{m}}^n$ over the field of real algebraic numbers. We replace the simple abelian variety $A$ of dimension $g$ by the torus $\mathbb{G}_{\mathrm{m}}^n$ of dimension $n$, the Mordell-Weil group $A(K)$ by a finitely generated multiplicative subgroup of $\left(\overline{\mathbb{Q}}^{\times}\right)^n$ and the connected component $A(\mathbb{R})^0$ of the origin in $A(\mathbb{R})$ by $\left(\mathbb{R}_+^{\times}\right)^n$. The corresponding problem is then: given positive algebraic numbers $\gamma_{ij}$ $(1 \leq i \leq n,\ 1 \leq j \leq m)$, consider the approximation of a tuple $(\zeta_1, \ldots, \zeta_n) \in \left(\mathbb{R}_+^{\times}\right)^n$ by tuples of algebraic numbers of the form

$$\left(\gamma_{11}^{s_1} \cdots \gamma_{1m}^{s_m}, \ldots, \gamma_{n1}^{s_1} \cdots \gamma_{nm}^{s_m}\right)$$

with $\underline{s} = (s_1, \ldots, s_m) \in \mathbb{Z}^m$.

The qualitative density question is solved by the following statement, which is a consequence of Conjecture 3.3.

**Conjecture 4.20.** *Let $m$, $n$, $k$ be positive integers and $a_{ij\kappa}$ rational numbers ($1 \le i \le n$, $1 \le j \le m$, $1 \le \kappa \le k$). For $\underline{x} = (x_1, \dots, x_k) \in \left(\mathbb{R}_+^\times\right)^k$ denote by $\Gamma(\underline{x})$ the following finitely generated subgroup of $\left(\mathbb{R}_+^\times\right)^n$:*

$$\Gamma(\underline{x}) = \left\{ \left( \prod_{j=1}^m \prod_{\kappa=1}^k x_k^{a_{ij\kappa}s_j} \right)_{1 \le i \le n} \; ; \; \underline{s} = (s_1, \dots, s_m) \in \mathbb{Z}^m \right\}.$$

*Assume there exists $\underline{x} \in \left(\mathbb{R}_+^\times\right)^k$ such that $\Gamma(\underline{x})$ is dense in $\left(\mathbb{R}_+^\times\right)^n$. Then for any $\underline{\gamma} = (\gamma_1, \dots, \gamma_k)$ in $\left(\mathbb{R}_+^\times\right)^k$ with $\gamma_1, \dots, \gamma_k$ algebraic and multiplicatively independent, the subgroup $\Gamma(\underline{\gamma})$ is dense in $\left(\mathbb{R}_+^\times\right)^n$.*

If there is a $\underline{x}$ in $\left(\mathbb{R}_+^\times\right)^k$ such that $\Gamma(\underline{x})$ is dense in $\left(\mathbb{R}_+^\times\right)^n$, then the set of such $\underline{x}$ is dense in $\left(\mathbb{R}_+^\times\right)^k$. Hence again, loosely speaking, Conjecture 4.20 means that logarithms of algebraic numbers should behave like almost all numbers (see also [L 1991] Chap. IX, § 7 p. 235).

Conjecture 4.20 would provide an effective solution to the question raised by Colliot-Thélène and Sansuc and solved by D. Roy (see [Ro 1992]):

> *Let $k$ be a number field of degree $d = r_1 + 2r_2$, where $r_1$ is the number of real embeddings and $r_2$ the number of pairwise non-conjugate embeddings of $k$. Then there exists a finitely generated subgroup $\Gamma$ of $k^\times$, with rank $r_1 + r_2 + 1$, whose image in $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ is dense.*

The existence of $\Gamma$ is known, but the proof of D. Roy does not yield an explicit example.

Density questions are closely related to transcendence questions. For instance the multiplicative subgroup of $\mathbb{R}_+^\times$ generated by $e$ and $\pi$ is dense if and only if $\log \pi$ is irrational (which is an open question).

The simplest case of Conjecture 4.20 is obtained with $n = 2$ and $m = 3$. It reads as follows.

(?) *Let $\alpha_1, \alpha_2, \alpha_3$, $\beta_1, \beta_2, \beta_3$ be nonzero positive algebraic numbers. Assume that for any $(a, b) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, two at least of the three numbers*

$$\alpha_1^a \beta_1^b, \; \alpha_2^a \beta_2^b, \; \alpha_3^a \beta_3^b$$

*are multiplicatively independent. Then the subgroup*

$$\Gamma = \left\{ \left( \alpha_1^{s_1} \alpha_2^{s_2} \alpha_3^{s_3}, \alpha_1^{s_1} \alpha_2^{s_2} \alpha_3^{s_3} \right) \; ; \; (s_1, s_2, s_3) \in \mathbb{Z}^3 \right\}$$

*of $\left(\mathbb{R}_+^\times\right)^2$ is dense.*

It is easy to deduce this statement from the four exponentials Conjecture 3.7.

The next question is to consider a quantitative refinement. Let $\Gamma$ be a finitely generated subgroup of $(\overline{\mathbb{Q}} \cap \mathbb{R}_+^\times)^n$ which is dense in $\left(\mathbb{R}_+^\times\right)^n$. Fix a set of generators $\underline{\gamma}_1, \dots, \underline{\gamma}_m$ of $\Gamma$. For $\underline{s} = (s_1, \dots, s_m) \in \mathbb{Z}^m$ and $1 \le i \le n$ define

$$\gamma_i(\underline{s}) = \prod_{j=1}^m \gamma_{ij}^{s_j} \in \overline{\mathbb{Q}}^\times.$$

The density assumption means that for any $\underline{\zeta} = (\zeta_1, \dots, \zeta_n) \in \left(\mathbb{R}_+^\times\right)^n$ and any $\epsilon > 0$, there exists $\underline{s} \in \mathbb{Z}^m$ such that

$$\max_{1 \le i \le n} |\gamma_i(\underline{s}) - \zeta_i| \le \epsilon.$$

We wish to bound $|\underline{s}| = \max_{1 \le j \le m} |s_j|$ in terms of $\epsilon$.

We fix a compact neighborhood $\mathcal{K}$ of the origin $(1, \dots, 1)$ in $\left(\mathbb{R}_+^\times\right)^n$. For instance

$$\mathcal{K} = \{ \underline{\zeta} \in \left(\mathbb{R}_+^\times\right)^n \; ; \; 1/2 \le |\zeta_i| \le 2 \; (1 \le i \le n) \}$$

would do.

**Conjecture 4.21.** *For any $\epsilon > 0$ there exists $S_0 > 0$ (depending on $\epsilon$, $\gamma_1, \ldots, \gamma_m$ and $\mathcal{K}$) such that, for any $S \geq S_0$ and any $\underline{\zeta} \in \mathcal{K}$, there exists $\underline{s} \in \mathbb{Z}^m$ with $|\underline{s}| \leq S$ and*

$$\max_{1 \leq i \leq n} |\gamma_i(\underline{s}) - \zeta_i| \leq S^{-1-(1/n)+\epsilon}.$$

These questions raise a new kind of Diophantine approximation problem.

## 5. Further Topics

### 5.1. Metric Problems

Among the motivations to study metric problems in Diophantine analysis (not to mention secular perturbations in astronomy and the statistical mechanics of a gas – see [Ha 1998]), one expects to guess the behaviour of certain classes of numbers (like algebraic numbers, logarithms of algebraic numbers, and numbers given as values of classical functions, suitably normalized [L 1971] p. 658 and 664).

A first example is related with Wirsing-Schmidt's Conjecture: in 1965, V.G. Sprindzuk showed that the conjecture 4.11 is true for almost all $\theta$ (for Lebesgue measure).

A second example is the question of refining Roth Theorem: Conjecture 2.12 is motivated by Khinchine' Theorem ([Sp 1979] Chap. I, § 1, Th. 1 p. 1) which answers the question of rational Diophantine approximation for almost all real numbers. In 1926 A. Khinchine himself extended his result to simultaneous Diophantine rational approximation

$$\max_{1 \leq i \leq n} |q\alpha_i - p_i|$$

([Sp 1979] Chap. I, § 4, Th. 8 p. 28), and in 1938 A.V. Groshev proved the first very general theorem of Khinchine type for systems of linear forms

$$\max_{1 \leq i \leq n} |q_1\alpha_{i1} + \cdots + q_m\alpha_{im} - p_i|$$

([Sp 1979] Chap. I, § 5, Th. 12 p. 33). Using the same heuristic, one may extend Conjecture 2.12 in the context of simultaneous linear combinations of algebraic numbers.

In Conjecture 2.12 (as well as in Khinchine's result for almost all real numbers) the function $q\psi(q)$ is assumed to be nonincreasing. A conjecture of Duffin and Schaeffer (see [Sp 1979] Chap. 1, § 2, p. 17 and [Ha 1998]) would enable one to work without such a restriction. Denote by $\varphi(n)$ Euler's function

$$\varphi(n) = \sum_{\substack{1 \leq k \leq n \\ \gcd(k,n)=1}} 1.$$

**Conjecture 5.1** (Duffin and Schaeffer). *Let $\psi$ be a positive real valued function. Then, for almost all $\theta \in \mathbb{R}$, inequality (2.11) has an infinite number of solutions in integers $p$ and $q$ with $q > 0$ and* $\gcd(p, q) = 1$ *if and only if the series*

$$\sum_{q=1}^{\infty} \frac{1}{q}\psi(q)\varphi(q)$$

*diverges.*

The Khinchine-Goshev Theorem has been extended to certain manifolds (see [Sp 1979], [BeD 2000], as well as more recent papers by V. Bernick, M. Dodson, D Kleinbock and G. Margulis). Further, connections between the metrical theory of Diophantine approximation on one hand, hyperbolic geometry, ergodic theory and dynamics of flows on homogeneous spaces of Lie groups on the other, have been studied my several mathematicians, including D. Sullivan, S.J. Dani, G. Margulis and D. Kleinbock. Also F. Paulin recently investigated the Diophantine approximation properties of geodesic lines on the Heisenberg group, which gives rise to new open questions, for instance to study

$$\max_{1\le i\le n} |q\alpha_i - p_i|^{\kappa_i}$$

when $\kappa_1,\dots,\kappa_n$ are positive real numbers.

The set of real numbers with bounded partial quotients is countable. This is the set of real numbers which are badly approximable by rational numbers. Y. Bugeaud asks a similar question for numbers which are badly approximable by algebraic numbers of bounded degree.

**Question 5.2** (Y. Bugeaud). *Let $n\ge 2$. Denote by $\mathcal{K}_n$ the set of real numbers $\xi$ with the following property: there exists $c_1(\xi)>0$ and $c_2(\xi)>0$ such that for algebraic number $\alpha$ of degree $\le n$,*

$$|\xi-\alpha| \ge c_2(\xi)\mathrm{H}(\alpha)^{-n-1},$$

*and such that there are infinitely many algebraic numbers $\alpha$ of degree $\le n$ with*

$$|\xi-\alpha| \le c_1(\xi)\mathrm{H}(\alpha)^{-n-1},$$

*Do the set $\mathcal{K}_n$ strictly contain the set of algebraic numbers of degree $n+1$?*

In connection with the algebraic independence problems of § 3.1, one would like to understand better the behavior of real (or complex) numbers with respect to Diophantine approximation by algebraic numbers of large degree (see Conjecture 4.12). A natural question is to consider this question from a metrical point of view. Roughly speaking, what is expected is that for almost all real numbers $\xi$, the quality of approximation by algebraic numbers of degree $\le d$ and measure $\le t$ is $e^{-dt}$. Here is a precise suggestion of Y. Bugeaud [Bu 2000b].

For a real number $\kappa>0$, denote by $\mathcal{F}_\kappa$ the set of real numbers $\xi$ with the following property: for any $\kappa'$ with $0<\kappa'<\kappa$ and any $d_0\ge 1$, there exists a real number $h_0\ge 1$ such that, for any $d\ge d_0$ and any $t\ge h_0 d$, the inequality

$$|\xi-\gamma|\le e^{-\kappa' dt}$$

has a solution $\gamma\in\overline{\mathbb{Q}}$ with $[\mathbb{Q}(\gamma):\mathbb{Q}]\le d$ and $\mu(\gamma)\le t$.

Also, denote by $\mathcal{F}'_\kappa$ the set of real numbers $\xi$ with the following property: for any $\kappa'>\kappa$ there exists $d_0\ge 1$ and $h_0\ge 1$ such that, for any $d\ge d_0$ and any $t\ge h_0 d$, the inequality

$$|\xi-\gamma|\le e^{-\kappa' dt}$$

has no solution $\gamma\in\overline{\mathbb{Q}}$ with $[\mathbb{Q}(\gamma):\mathbb{Q}]\le d$ and $\mu(\gamma)\le t$.

These definition are given more concisely in [Bu 2000b]: for $t\ge d\ge 1$ denote by $\overline{\mathbb{Q}}(d,t)$ the set of real algebraic numbers $\gamma$ of degree $\le d$ and measure $\le t$. Then

$$\mathcal{F}_\kappa=\bigcap_{\kappa'<\kappa}\ \bigcup_{d_0\ge 1}\ \bigcup_{h_0\ge 1}\ \bigcap_{d\ge d_0}\ \bigcap_{t\ge h_0 d}\ \bigcup_{\gamma\in\overline{\mathbb{Q}}(d,t)\cap\mathbb{R}} \left]\gamma-e^{-\kappa' dt},\gamma+e^{-\kappa' dt}\right[,$$

$$\mathcal{F}'_\kappa = \bigcap_{\kappa'>\kappa} \bigcup_{d_0\geq 1} \bigcup_{h_0\geq 1} \bigcap_{d\geq d_0} \bigcap_{t\geq h_0 d} \bigcap_{\gamma\in\overline{\mathbb{Q}}(d,t)\cap\mathbb{R}} ]\gamma - e^{-\kappa' dt}, \gamma + e^{-\kappa' dt}[^c,$$

According to Theorem 4 of [Bu 2000b], there exists two positive constants $\widetilde{\kappa}$ and $\widetilde{\kappa}'$ such that, for almost all $\xi \in \mathbb{R}$,

$$\max\{\kappa > 0 \,;\, \xi \in \mathcal{F}_\kappa\} = \widetilde{\kappa} \quad \text{and} \quad \min\{\kappa > 0 \,;\, \xi \in \mathcal{F}'_\kappa\} = \widetilde{\kappa}'.$$

Further,

$$\frac{1}{850} \leq \widetilde{\kappa} \leq \widetilde{\kappa}' \leq 1.$$

Bugeaud's conjecture is $\widetilde{\kappa} = \widetilde{\kappa}' = 1$.

It is an important open question to investigate the simultaneous approximation of almost all tuples in $\mathbb{R}^n$ by algebraic tuples $\underline{\gamma} = (\gamma_1, \ldots, \gamma_n)$ in terms of the degree $[\mathbb{Q}(\underline{\gamma}) : \mathbb{Q}]$ and the measure $\mu(\underline{\gamma})$. Most often previous authors devoted much attention to the dependence on the height, but now it is necessary to investigate more thoroughly the behaviour of the approximation for large degree.

Further problems which we considered in the previous sections deserve to be investigated under the metrical point of view. Our next example is a strong quantitative form of Schanuel's Conjecture for almost all tuples ([W 2000a] Conjecture 4).

**Conjecture 5.3.** *Let $n$ be a positive integer. For almost all $n$-tuples $(x_1, \ldots, x_n)$, there are positive constants $c$ and $D_0$ (depending on $n$, $x_1, \ldots, x_n$ and $\epsilon$), with the following property. For any integer $D \geq D_0$, any real number $\mu \geq D$ and any $2n$-tuple $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ of algebraic numbers satisfying*

$$[\mathbb{Q}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) : \mathbb{Q}] \leq D$$

*and*

$$[\mathbb{Q}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) : \mathbb{Q}] \max_{1\leq i\leq n} \max\{\mathrm{h}(\alpha_i), \mathrm{h}(\beta_i)\} \leq \mu,$$

*we have*

$$\max_{1\leq i\leq n} \max\{|x_i - \beta_i| \,;\, |e^{x_i} - \alpha_i|\} \geq \exp\{-cD^{1/(2n)}\mu\}.$$

One may also expect that $c$ does not depend on $x_1, \ldots, x_n$.

An open metrical problem of uniform distribution has been suggested by P. Erdős to R.C. Baker in 1973 (see [Ha 1998] Chap. 5, p. 163). It is a variant of a conjecture of Khinchine which was disproved by J.M. Marstrand in 1970.

(?) *Let $f$ be a bounded measurable function with period 1. Is-it true that*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} f(n\alpha) = \int_0^1 f(x)dx$$

*for almost all $\alpha \in \mathbb{R}$?*

### 5.2. Function Fields

Let $K$ be a field and $\mathcal{C} = K((T^{-1}))$ the field of Laurent series on $K$. The field $\mathcal{C}$ has similar properties to the real number field, where $\mathbb{Z}$ is replaced by $K[T]$ and $\mathbb{Q}$ by $K(T)$. An absolute value on $\mathcal{C}$ is defined by selecting $|T| > 1$: we set $|\alpha| = |T|^k$ if $\alpha = \sum_{n\in\mathbb{Z}} a_n T^{-n}$ is a nonzero element of $\mathcal{C}$, where $k = \deg(\alpha)$ denotes the least rational integer such that $a_k \neq 0$. Hence $\mathcal{C}$ is the completion of $K(T)$ for this absolute value.

A theory of Diophantine approximation has been developed on $\mathcal{C}$ in analogy with the classical one. If $K$ has zero characteristic, the result are very similar to the classical ones. But if $K$ has finite characteristic, the situation is completely different (see [DML 1999] and [Sch 2000]). It is not yet even clear how to describe the situation from a conjectural point of view. Some algebraic elements satisfy a Roth type inequality, while for some others, Liouville's estimate is optimal. However from a certain point of view much more is known in the function field case, since the exact approximation exponent is known for several classes of algebraic numbers. Even a conjectural description of the set of algebraic numbers for which a Roth type inequality is valid is still missing.

There is also a transcendence theory over function fields. The starting point is a work of Carlitz in the 40's. He defines functions on $\mathcal{C}$ which behave like an analog of the exponential function (*Carlitz module*). A generalization is due to Drinfeld (*Drinfeld modules*), and a number of results on the transcendence of numbers related to these objects are known, going much further than their classical (complex) counterpart. However the theory is far from being complete. An analog of Schanuel's Conjecture for Drinfeld modules is proposed by W.D. Brownawell in [Bro 1998], together with many further related problems, including large transcendence degree, Diophantine geometry, values of Carlitz-Bessel functions and values of Gamma functions.

For the study of Diophantine approximation, an important tool (which is not available in the classical number theoretic case) is the derivation $d/dT$. In the transcendence theory this gives rise to new questions which started to be investigated by L. Denis. Also in the function field case interesting new questions arise by considering several characteristics. So Diophantine analysis for function fields involve different aspects, some which are reminiscent of the classical theory, and some which have no counterpart.

For the related transcendence theory involving automata theory, we refer to the paper by D. Thakur [T 1998], and especially on p.389–390 for the state of the art concerning the following open problem:

**Conjecture 5.4** (J.H. Loxton and A.J. van der Poorten). *Let $(n_i)_{i\geq 0}$ be an increasing sequence of positive integers. Assume there is a prime number $p$ such that the power series*

$$\sum_{i\geq 0} z^{n_i} \in \mathbb{F}_p[[z]]$$

*is algebraic over $\mathbb{F}_p(z)$ and irrational (not in $\mathbb{F}_p(z)$). Then the real number*

$$\sum_{i\geq 0} 10^{-n_i}$$

*is transcendental.*

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