Math. Subj. Class. (1991): 14G22, 14H10, 14H37

Equivariant deformation of Mumford curves and of ordinary curves in positive characteristic

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Introduction.

Equivariant deformation theory is the correct framework for formulating and answering questions such as "given a curve X of genus g over a field k and a finite group of automorphisms $\rho : G \hookrightarrow \mathrm{Aut}(X)$ of X, in how many ways can X be deformed into another curve of the same genus on which the same group of automorphisms still acts?" The precise meaning of this question is related to the deformation functor $D_{X,\rho}$ of the pair (X, ρ), which associates to any element A of the category $\mathcal{C}_k$ of local Artinian k-algebras the set of isomorphism classes of liftings $(X^{\sim}, \rho^{\sim}, \phi^{\sim})$, where $X^{\sim}$ is a smooth scheme of finite type over A, $\phi^{\sim}$ is an isomorphism of $X^{\sim} \otimes k$ with X, and where $\rho^{\sim} : G \to \mathrm{Aut}_A(X)$ lifts ρ via $\phi^{\sim}$. In general, $D_{X,\rho}$ has a pro-representable hull $H_{X,\rho}$ in the sense of Schlessinger ([18]). This means that there is a smooth map of functors $\mathrm{Hom}(H_{X,\rho}, -) \to D_{X,\rho}$ that induces an isomorphism on the level of tangent spaces, where $H_{X,\rho}$ is a pro-object of $\mathcal{C}_k$, viz., a complete local k-algebra all of whose non-trivial quotients are in $\mathcal{C}_k$. The above question can then be reformulated as the computation of the Krull-dimension of $H_{X,\rho}$. Two remarks are in order: if $g \geq 2$, then the functor is even pro-representable by $H_{X,\rho}$, so Spf $H_{X,\rho}$ is a formal scheme which is, so to speak, the universal basis of a family of curves which have the same automorphism group as X. Secondly, $H_{X,\rho}$ is even algebrizable (cf. Grothendieck [11], §3), and the underlying algebraic scheme over k might be considered as the genuine "universal basis" scheme.

If the characteristic of the ground field k is zero, the dimension of $H_{X,\rho}$ is easy to compute. All obstructions and group cohomology (cf. infra) disappear, and we find that

$$\dim H_{X,\rho} = 3g_Y - 3 + n, \tag{1}$$

where $Y := G\backslash X$ is the quotient of X, g_Y its genus and n is the number of branch points on Y. This result can be found in any classical text on Riemann surfaces (*e.g.*, [7], V.2.2); the moral is that ramification data of $X \to Y$ provide all the necessary information. Note that $3g_Y - 3$ is the degree of freedom of varying the moduli of Y, and one extra degree of freedom comes in for every branch point.

In this work, we are interested in the corresponding question in positive characteristic. Let us first present the motivating example for our studies: moduli schemes for rank two Drinfeld modules with principal level structure (see, e.g., Gekeler & Reversat [8]).

Example. Let $q = p^t$, $F = \mathbf{F}_q(T)$, and $A = \mathbf{F}_q[T]$; let $F_\infty = \mathbf{F}_q((T^{-1}))$ be the completion of F and C a completion of the algebraic closure of F_∞. On Drinfeld's "upper half plane" $\Omega := \mathbf{P}^1_C - \mathbf{P}^1_{F_\infty}$ (which is a rigid analytic space over C), the group $GL(2, A)$ acts by fractional transformations. Let $\mathcal{Z} \cong \mathbf{F}_q^*$ be its center. For $\mathfrak{n} \in A$, the quotients of Ω by congruence subgroups $\Gamma(\mathfrak{n}) = \{\gamma \in GL(2, A) : \gamma = \mathbf{1} \bmod \mathfrak{n}\}$ are open analytic curves which can be compactified to projective curves $X(\mathfrak{n})$ by adding finitely many cusps. These curves are analogues in the function field setting of classical modular curves $X(n)$ for $n \in \mathbf{Z}$. Clearly, elements from $G(\mathfrak{n}) := \Gamma(1)/\Gamma(\mathfrak{n})\mathcal{Z}$ induce automorphisms of $X(\mathfrak{n})$. It is even known that $G(\mathfrak{n})$ is the full automorphism group of $X(\mathfrak{n})$ if $p \neq 2, q \neq 3$ (cf. [5], prop. 4). It follows from (1) that a "classical" modular curve does not admit equivariant deformations, since $X(n) \to X(1) = \mathbf{P}^1$ is ramified above three points (the most famous such curve is probably $X(7)$, which is isomorphic to Klein's quartic of genus 3 with $PSL(2, 7)$ as automorphism group). What is the analogous result for the Drinfeld modular curves $X(\mathfrak{n})$? Note that $X(\mathfrak{n}) \to X(1) = \mathbf{P}^1$ is ramified above two points with ramification groups $\mathbf{Z}/(q+1)\mathbf{Z}$ and $(\mathbf{Z}/p\mathbf{Z})^{td} \rtimes \mathbf{Z}/(q-1)\mathbf{Z}$ where $d = \deg(\mathfrak{n})$ (cf. loc. cit.), which already shows that something goes wrong when dully applying (1).

The correct framework for carrying out such computations is that of equivariant cohomology (Grothendieck [10]). It predicts that in positive characteristic, the group cohomology of the ramification groups with values in the tangent sheaf at branch points contribute to the deformation space. Bertin and Mézard have considered the case of a cyclic group of prime order $G = \mathbf{Z}/p\mathbf{Z}$ in [1], mainly concentrating on mixed characteristic lifting. In this paper, we do not want to impose direct restrictions on the group G, but rather on the curve X, which we require to be *ordinary*. This means that the p-rank of its Jacobian satisfies

$$\dim_{\mathbf{F}_p} \mathrm{Jac}(X)[p] = g.$$

The property of being ordinary is open and dense in the moduli space of curves of genus g, so our calculations do apply to a "large portion" of that moduli space (there is some cheating here, since curves without automorphisms are also dense; but notice that the analytic constructions in part B of the paper at least show the existence of lots of ordinary curves with automorphisms). The main advantage of working with ordinary curves is that their ramification groups are of a very specific form, so the Galois cohomological computation becomes feasible (cf. prop. 1.4): if P_i is a point on Y branched in $X \to Y$, then

$$G_{P_i} \cong (\mathbf{Z}/p\mathbf{Z})^{t_i} \rtimes \mathbf{Z}/n_i\mathbf{Z} \tag{2}$$

for some integers (t_i, n_i) satisfying $n_i | p^{t_i} - 1$. If $t_i > 0$, we say that P_i is wildly branched. Our main algebraic result is the following (which we state here in a form which excludes a few anomalous cases); one can think of it giving the "error term" to the Riemann surface computation in (1).

Main Algebraic Theorem (cf. 5.1). *Assume that $p \neq 2, 3$, X is an ordinary curve of genus $g \geq 2$ over a field of characteristic $p > 0$, G is a finite group acting via $\rho : G \to Aut(X)$ on X, such that $X \to Y := G\backslash X$ is branched above n points, of which the first s are wildly branched. Then the deformation functor $D_{X,\rho}$ is pro-representable by a ring $H_{X,\rho}$ whose Krull-dimension is given by*

$$\dim H_{X,\rho} = 3g_Y - 3 + n + \sum_{i=1}^{s} \frac{t_i}{s(n_i)},$$

where $s(n_i) := [\mathbf{F}_p(\zeta_{n_i}) : \mathbf{F}_p] = \min\{s' > 0 : n_i | p^{s'} - 1\}$, g_Y is the genus of Y and (t_i, n_i) are the data corresponding to the wild ramification points via (2).

The proof goes by first computing the first-order local deformation functors, then studying their liftings to all of $\mathcal{C}_k$, and putting the results together via the localization theorem of [1]. In the end, we can even describe the ring $H_{X,\rho}$ explicitly. It turns out to be a formal polydisc modulo some nilpotents:

$$H_{X,\rho} = k[[x_1, ..., x_m]]/(x_1^{\frac{p-1}{2}}, ..., x_r^{\frac{p-1}{2}}),$$

where the nilpotent relations come from lifting obstructions in the presence of ramification groups with $n_i \leq 2$.

Example (continued). For the Drinfeld modular curve $X(\mathfrak{n})$, we find a $(d-1)$-dimensional reduced deformation space.

The second part of this paper is concerned with analytic equivariant deformation theory. Let $(K, |\cdot|)$ be a non-archimedean valued field of positive characteristic $p > 0$ with residue field k. Recall that a *Mumford curve* over K is a curve X whose stable reduction is isomorphic to a union of rational curves intersecting in k-rational points. Mumford has shown that this is equivalent to its analytification X^{an} being isomorphic to an analytic space of the form $\Gamma\backslash(\mathbf{P}^{1,an}_K - \mathcal{L}_\Gamma)$, where Γ is a discontinuous group in $PGL(2, K)$ with $\mathcal{L}_\Gamma$ as set of limit points. It is known that Mumford curves are ordinary, so the above algebraic theory applies. But what interests us most in this second part is to find out where these deformations "live" in the realm of discrete groups.

The set-up is as follows: let X be a Mumford curve, and let N be a group contained in the normalizer $N(\Gamma)$ of the corresponding "Schottky" group Γ such that $\Gamma \subseteq N$. Then there is an injection $\rho : G := N/\Gamma \hookrightarrow \mathrm{Aut}(X)$ (isomorphism if $N = N(\Gamma)$). By rigidity, two Mumford curves are isomorphic if and only if their Schottky groups are conjugate in $PGL(2, K)$. Hence it is natural to consider the analytic deformation functor $D_{N,\phi} : \mathcal{C}_K \to \mathrm{Set}$ that associates to $A \in \mathcal{C}_K$ the set of homomorphisms $N \to PGL(2, A)$ that lift the given morphism ϕ in the obvious sense. This functor comes with a natural conjugation action by the formal completion $PGL(2)^\wedge$ at the neutral element of $PGL(2)$, and we denote by $D^{\sim}_{N,\phi}$ the quotient functor $D^{\sim}_{N,\phi} = PGL(2)^\wedge \backslash D_{N,\phi}$. We now set out to compute the hulls corresponding to these functors.

For this, we use the fact that N can be described by the theorem of Bass-Serre, which states that there is a graph of groups T_N such that N is a semidirect product of a free group (of rank the cyclotomic number of T_N) and the tree ("amalgamation") product associated to a lifting of stabilizer groups from T_N to the universal covering of T_N as as graph. In a technical proposition, we show that this decomposition of the group N induces a decomposition of functors

$$D_{N,\phi} \cong \varprojlim_{T_N} D_{N_\bullet,\phi|_{N_\bullet}} \times D_{F_c,\phi\circ s},$$

where s is a section of the semi-direct product (note that this is a *direct product* of functors, and the inverse limit is over T_N, *not* over its universal covering, and that we are using $D_{N,\phi}$, not $D^{\sim}_{N,\phi}$). This reduces everything to the computation of $D_{N,\phi}$ for free N (which is easy) or finite N acting on $\mathbf{P}^1$. The latter can be done by using the classification of finite subgroups of $\mathbf{P}^1$ and the algebraic results from the first part in the particular case of $\mathbf{P}^1$. Modulo a few anomalous cases, the result is the following:

Main Analytic Theorem (cf. 8.4). *If X is a Mumford curve of genus $g \geq 2$ over a non-archimedean field K of characteristic $p > 3$ with Schottky group Γ, then for a given discrete group N contained in the normalizer of Γ with corresponding graph of groups T_N, the analytic deformation functor $D^{\sim}_{N,\phi}$ is pro-representable by a ring $H^{\sim}_{N,\phi}$ whose dimension satisfies*

$$\dim H^{\sim}_{N,\phi} = 3c(T_N) - 3 + \sum_{\substack{v=\text{vertex}\\ \text{of } T_N}} d(v) - \sum_{\substack{e=\text{edge}\\ \text{of } T_N}} d(e),$$

where $c(*)$ denotes the chromatic number of a graph $*$ and

$$d(*) = \begin{cases} 2 & \text{if} \quad * \text{ is cyclic of prime-to-}p \text{ order}, \\ 3 & \text{if} \quad * = A_4, S_4, A_5, D_n, PGL(2,p^t), PSL(2,p^t) \\ t & \text{if} \quad * = (\mathbf{Z}/p\mathbf{Z})^t, \\ t/s(n)+2 & \text{if} \quad * = (\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/n\mathbf{Z} \end{cases}$$

Example (continued). The Drinfeld modular curve $X(\mathfrak{n})$ is known to be a Mumford curve, and the normalizer of its Schottky group is isomorphic to an amalgam (cf. [5])

$$N(\mathfrak{n}) = PGL(2,q) *_{(\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/(q-1)\mathbf{Z}} (\mathbf{Z}/p\mathbf{Z})^{td} \rtimes \mathbf{Z}/(q-1)\mathbf{Z}. \tag{3}$$

The above formula gives again a $(d-1)$-dimensional deformation space. We can see these deformations explicitly as follows: by conjugation with $PGL(2,K)$, we can assume that $PGL(2,p^t)$ in (3) is induced by the standard $\mathbf{F}_q \subseteq C$. Then a little matrix computation shows that the freedom of choice left is that of a d-dimensional $\mathbf{F}_q$-vector space V of dimension d in C which contains the standard $\mathbf{F}_q$ in order to embed the order-p elements from the second group involved in the amalgam as $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ for $x \in V$.

In the final section of the paper, we compare algebraic and analytic deformation functors. What comes out is an isomorphism of functors

$$D^{\sim}_{N,\phi} \cong D_{X,\rho}.$$

To prove this, we remark that the whole construction of Mumford can be carried out in the category $\mathcal{C}_K$ (by fixing a lifting of N to $PGL(2,A)$) to produce a scheme X_Γ over Spec A whose central fiber is isomorphic to X, and such that X_Γ carries an action of N/Γ which reduces to the given action on X.

Remark. We did not take the more “global” road of “equivariant Teichmuller space” to analytic deformation, as Herrlich does in [12] and [13]: one can consider the space

$$\mathcal{M}(N,\Gamma) = \mathrm{PGL}(2,K)\backslash \mathrm{Hom}^*(N, PGL(2,K))/\mathrm{Aut}_\Gamma(N),$$

where Hom* means the space of injective morphisms with discrete image, the action of $PGL(2,K)$ from the left is by conjugation, and the action of the group of automorphisms of N which fix Γ is on the right. The relations in N impose a natural structure on $\mathcal{M}(N,\Gamma)$ as an analytic space, but in positive characteristic, this structure might be non-reduced due to the presence of parabolic elements, e.g., if N contains $\mathbf{Z}/p\mathbf{Z}, p > 3$, then the condition that a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be of order p leads to

$$(\mathrm{tr}^2(\gamma) - 4\det(\gamma))^{\frac{p-1}{2}} \cdot b = 0.$$

Therefore in [13], global considerations are restricted to the case where N does not contain parabolic elements (pp. 148), whereas our local calculations are independent of such restrictions.

Let us note that if K has characteristic 0, then [13] gives a formula for the dimension of $\mathcal{M}(N,\Gamma)$ (which turns out to be an equidimensional space) compatible with our main analytical theorem.

Convention. Throughout this paper, if R is a local ring, we let $\dim R$ denote its Krull-dimension, and if V is a k-vector space, we let $\dim_k V$ denote its dimension as a k-vector space.

PART A: ALGEBRAIC THEORY

1. Deformation of Galois covers

1.1 The global deformation functor. We will start by recalling the definition of equivariant deformation of a curve with a given group of automorphisms (cf. [1] for an excellent survey).

Let k be a field, and X a smooth projective curve over k. We fix a finite subgroup G in the automorphism group $\mathrm{Aut}_k(X)$ of X, and denote by ρ the inclusion

$$\rho : G \hookrightarrow \mathrm{Aut}_k(X) : \sigma \to \rho_\sigma.$$

Let $Y = G\backslash X$ be the quotient curve, and denote by π the quotient map $X \to Y$.

Let $\mathcal{C}_k$ be the category of Artinian local k-algebras with the residue field k. A *lifting* of (X, ρ) to A is a triple $(X^\sim, \rho^\sim, \phi^\sim)$ consisting of a scheme $X^\sim$ which is smooth of finite type over A, an injective group homomorphism

$$\rho^\sim : G \hookrightarrow \mathrm{Aut}_A(X^\sim) : \sigma \to \rho^\sim_\sigma,$$

and an isomorphism

$$\phi^\sim : X^\sim \otimes_{\mathrm{Spec}\, A} \mathrm{Spec}\; k \to X$$

of schemes over k such that $\overline{\rho^\sim} = \rho$. Here, $\overline{\rho^\sim}$ denotes the composite of $\rho^\sim$ and restriction onto the central fiber, identified with X by $\phi^\sim$.

Two liftings $(X^\sim, \rho^\sim, \phi^\sim)$ and $(X^\approx, \rho^\approx, \phi^\approx)$ are said to be *isomorphic* if there exists an isomorphism $\psi : X^\approx \to X^\sim$ of schemes over A such that $\phi^\sim \circ (\psi \otimes_A k) = \phi^\approx$ and, for any $\sigma \in G$, $\psi \circ \rho^\approx_\sigma = \rho^\sim_\sigma \circ \psi$.

We arrive at the *deformation functor*

$$D_{X,\rho} : \mathcal{C}_k \to \mathrm{Set}$$

that assigns to any $A \in \mathcal{C}_k$ the set of isomorphism classes of liftings of (X, ρ).

1.2 The functor π^G_* on tangents. In order to compute the tangent space to this deformation functor, we need to recall the equivariant cohomology theory of Grothendieck.

The morphism π induces a tangent map $\mathcal{T}_X \to \pi^*\mathcal{T}_Y$, which is a monomorphism since π is generically étale. Note that both $\mathcal{T}_X$ and $\pi^*\mathcal{T}_Y$ are G-$\mathcal{O}_X$-modules (cf. [10, 5.1]), and the tangent map $\mathcal{T}_X \to \pi^*\mathcal{T}_Y$ is a morphism of G-$\mathcal{O}_X$-modules; the G-structure of $\pi^*\mathcal{T}_Y = \mathcal{O}_X \bigotimes_{\mathcal{O}_Y} \pi^{-1}\mathcal{T}_Y$ is the tensor product of the usual G-structure on $\mathcal{O}_X$ and the trivial G-structure on $\pi^{-1}\mathcal{T}_Y$ (cf. [loc. cit.]). We apply the functor π^G_* to the tangent map; recall that by definition, for a G-sheaf $\mathcal{F}$ on X and an open U of Y, $\pi^G_*(\mathcal{F})(U)$ is the set of sections of $\mathcal{F}$ on $\pi^{-1}(U)$ invariant under G. Since $\pi^G_*\pi^*\mathcal{T}_Y \cong \mathcal{T}_Y$ (cf. [10, (5.1.1)]), we get

$$(1.2.1) \qquad 0 \to \pi^G_*\mathcal{T}_X \longrightarrow \mathcal{T}_Y,$$

The fact that this map is a monomorphism is due to the left exactness of π^G_*. Note that $\pi^G_*\mathcal{T}_X$ is an invertible sheaf on Y: Indeed, it is coherent due to the finiteness of π and the classical invariant theory; since π is étale outside ramification points, it is not a zero sheaf; now, since $\mathcal{O}_{Y,y}$ $(y \in Y)$ is a PID, every stalk of $\pi^G_*\mathcal{T}_X$ is a free $\mathcal{O}_{Y,y}$-module of rank one, thereby the assertion.

1.3 Ordinary curves. From now on, we will assume that the field k is of characteristic $p > 0$ and that the curve X is *ordinary* and of genus $g > 0$. This means that

$$\dim_{\mathbf{F}_p} \mathrm{Jac}(X)[p] = g.$$

We will consider $\mathbf{P}^1$ to be ordinary. The property of being ordinary is open and dense on the moduli space of curves of genus g ([17], §4, [14]), since it is essentially a maximal rank condition on the Hasse-Witt matrix of the curve (hence open), and ordinary curves exist for all g.

1.4 Proposition. *If X is an ordinary curve and G a finite group of automorphisms of X, then the ramification groups in the cover $X \to G\backslash X$ are of the form $(\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/n\mathbf{Z}$ for $n|p^t - 1$.*

Proof. Let P be a ramified point in that cover and let $G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots$ be the standard filtration on the ramification group G_0 at P. In [17] (Theorem 2(i)), S. Nakajima has shown that for an ordinary curve, $G_2 = 1$. It is also known that G_i/G_{i+1} for $i \geq 1$ are elementary abelian p-groups and that G_0 is the semi-direct product of a group of order prime to p and G_1 ([19], IV.2 cor. 1-4). The result follows from this. □

1.5 Notations. Recall that a branch point in $X \to Y$ is called *tame* if $t = 0$ in the representation of (1.4), and *wild* otherwise. Let $b_1, \dots, b_r \in Y$ (resp. $w_1, \dots, w_s \in Y$) be tame (resp. wild) branch points of π. We define the divisor Δ on Y by

$$\Delta = \begin{cases} \sum_{i=1}^r b_i + 2\sum_{j=1}^s w_j & (p \neq 2) \\ \sum_{i=1}^r b_i + \sum_{j=1}^s w_j & (p = 2) \end{cases} \quad \text{where } \delta := \deg(\Delta) = \begin{cases} r + 2s & (p \neq 2) \\ r + s & (p = 2) \end{cases}.$$

Let $\mathbf{Z}_p^{t_i} \rtimes \mathbf{Z}_{n_i}$ denote the ramification groups of the wild branch points w_i.

These data allow us to compute $\pi_*^G \mathcal{T}_X$.

1.6 Proposition. *We have $\pi_*^G \mathcal{T}_X \cong \mathcal{T}_Y(-\Delta)$.*

Proof. First note that the morphism (1.2.1) is an isomorphism outside branch points. Hence the question becomes local around each branch point.

First we look at a tame ramification point $\eta \in Y$. The morphism is described locally by the Kummer equation $y = x^n$, where x and y are local regular parameters on X and Y, respectively, and n is co-prime to p. Then,

$$ny\frac{d}{dy} = x\frac{d}{dx}$$

gives a basis for invariant derivations in the stalk $(\pi_*^G \mathcal{T}_X)_\eta$, and hence $(\pi_*^G \mathcal{T}_X)_\eta$ is identified with $y\mathcal{T}_{Y,\eta}$.

Next we deal with the case of a wild ramification point with a pure p-ramification group $(\mathbf{Z}/p\mathbf{Z})^t$. If $t = 1$, then (1.2.1) is described by an Artin-Schreier equation (centered at 0) $y = x^p/(1 - x^{p-1})$. Differentiating both sides, we get

$$y^2\frac{d}{dy} = -x^2\frac{d}{dx},$$

which gives a basis in $(\pi_*^G \mathcal{T}_X)_\eta$ if $p > 2$. If $p = 2$, one can divide both sides further by y, which leaves them being regular, i.e., $y\frac{d}{dy} = (x-1)\frac{d}{dx}$. Hence $(\pi_*^G \mathcal{T}_X)_\eta$ is identified with $y^2\mathcal{T}_{Y,\eta}$ if $p > 2$ or with $y\mathcal{T}_{Y,\eta}$ if $p = 2$. For $t > 1$, one can "pile up" these calculations to find that the results are the same. In $p=2$, one can easily see the result directly: the equation is $y = x^{2^t}/(1+x)^{2^t-1}$, from which it follows $y\frac{d}{dy} = (x+1)\frac{d}{dx}$, as desired.

The general case of a ramification group of the form $(\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/n\mathbf{Z}$ can be dealt with in the same spirit. The morphism in question is the composite of a pile of Artin-Schreiers $y = x^p/(1-x^{p-1})$ followed by $z = y^n$. One finds that

$$nz^2\frac{d}{dz} = (-1)^t y^{n-1} x^2 \frac{d}{dx}$$

if $p > 2$. In $p=2$, again a direct calculation immediately gives $nz\frac{d}{dz} = (x+1)\frac{d}{dx}$. □

1.7 Cohomology of π_*^G. We now recall the two spectral sequences from [10, 5.2]. Let $\mathcal{F}$ be a coherent G-$\mathcal{O}_X$-module, and write

$$\begin{array}{lcl} \mathrm{H}^i(X;G,\mathcal{F}) & = & \mathrm{R}^i\Gamma_X^G\mathcal{F} \\ \mathcal{H}^i(G,\mathcal{F}) & = & \mathrm{R}^i\pi_*^G\mathcal{F}, \end{array}$$

where $\Gamma_X^G = \Gamma_Y \circ \pi_*^G$, i.e., $\Gamma_X^G\mathcal{F} = (\Gamma_X\mathcal{F})^G$. There are two cohomological spectral sequences

$$\begin{array}{rcl} {}^IE_2^{p,q} = \mathrm{H}^p(Y,\mathcal{H}^q(G,\mathcal{F})) & \Longrightarrow & \mathrm{H}^{p+q}(X;G,\mathcal{F}) \\ {}^{II}E_2^{p,q} = \mathrm{H}^p(G,\mathrm{H}^q(X,\mathcal{F})) & \Longrightarrow & \mathrm{H}^{p+q}(X;G,\mathcal{F}). \end{array}$$

The first one gives rise to the edge sequence

$$(1.7.1) \qquad 0 \longrightarrow \mathrm{H}^1(Y,\pi_*^G\mathcal{F}) \longrightarrow \mathrm{H}^1(X;G,\mathcal{F}) \longrightarrow \mathrm{H}^0(Y,\mathcal{H}^1(G,\mathcal{F})) \longrightarrow 0$$

as we are on a curve Y.

1.8 Tangent space to the global deformation functor. We can use this equivariant cohomology to compute the tangent space to our deformation functor. Recall that the ring of dual numbers is defined as $k[\epsilon] = k[E]/(E^2)$; clearly, it belongs to $\mathcal{C}_k$. The tangent space to the deformation functor $D_{X,\rho}$ is by definition its value on the ring of dual numbers with its natural k-linear structure (cf. [18]).

1.9 Proposition ([1], (3.2.1), (3.3.1)) *We have* $D_{X,\rho}(k[\epsilon]) \cong \mathrm{H}^1(X;G,\mathcal{T}_X)$. □

1.10 Localization. We will now describe localization for our deformation functor. Let $v_j \in X$ be a point lying over w_j for $1 \le j \le s$, and G_j the stabilizer of v_j, which acts on the local ring $\mathcal{O}_{X,v_j}$ by k-algebra automorphisms; we denote it by $\rho_j\colon G_j \to \mathrm{Aut}_k(\mathcal{O}_{X,v_j})$. Changing the choice of v_j does not affect, up to a suitable equivalence, on the action ρ_j. Every lifting $(X^\sim,\rho^\sim,\phi^\sim)$ induces a *lifting of the local representation* ρ_j for any $1 \le j \le s$ in the following sense:

1.11 The local deformation functor. Let R be a k-algebra, and G a group acting via $\rho: G \to \mathrm{Aut}_k(R)$ on R by k-algebra automorphisms. Let $A \in \mathcal{C}_k$ and $R_A = R \otimes_k A$. Let $\bar{\cdot}$ denote the composite the reduction map $R_A \to R$ modulo the maximal ideal m_A of A.

We define a lifting of ρ to A as a group homomorphism $\rho^\sim\colon G \to \mathrm{Aut}_A(R_A)$ such that $\overline{\rho^\sim} = \rho$. Two liftings $\rho^\sim$ and $\rho^\approx$ are said to be *isomorphic* (over ρ) if there exists $\psi \in \mathrm{Aut}_A(R_A)$ with $\overline{\psi} = \mathrm{Id}_R$ such that, for any $\sigma \in G$, $\psi \circ \rho_\sigma^{\approx} = \rho_\sigma^{\sim} \circ \psi$. We arrive at a *local deformation functor*

$$D_\rho \colon \mathcal{C}_k \longrightarrow \mathrm{Set}$$

such that $D_\rho(A)$ is the set of all isomorphism classes of liftings of ρ to A. It is but a formal check using Schlessingers criterion to see that D_ρ has a pro-representable hull if G is a finite group, which we will denote by H_ρ (compare [1], 2.2).

In our situation, we get a transformation of functors

$$D_{X,\rho} \longrightarrow D_{\rho_1} \times \cdots \times D_{\rho_s} \tag{1.10.1}$$

It turns out that this morphism is formally smooth ([1], 3.3.4). Hence we get the following localization result:

1.12 Proposition ([1], 3.3.1, 3.3.5). *The functor $D_{X,\rho}$ has a pro-representable hull $H_{X,\rho}$; in fact*

$$H_{X,\rho} \cong H_{\rho_1} \hat{\otimes} \cdots \hat{\otimes} H_{\rho_s}[[u_1, \ldots, u_N]],$$

where $N = \dim_k \mathrm{H}^1(Y, \pi_*^G \mathcal{T}_X)$. □

1.13 Remark. As soon as $g \geq 2$, the curve X does not have infinitesimal automorphisms, so the functor $D_{X,\rho}$ is in fact *pro-representable* by $H_{X,\rho}$ (cf. [1], 2.1).

2. Lifting of group actions and group cohomology

The aim of the next three sections is to compute the tangent space and the pro-representable hull of the local deformation functors for the representations of the branch groups in quotients of ordinary curves as automorphisms of the local stalks of the tangent sheaf.

2.1 Action on derivations. Let k be a field and R a k-algebra. We denote by $\mathcal{T}_R$ the R-module of k-derivations, i.e., the set of all maps $\delta \colon R \to R$ such that $\delta(xy) = \delta(x)y + x\delta(y)$ for $x, y \in R$ and that $\delta(a) = 0$ for $a \in k$. Let $\varphi \in \mathrm{Aut}_k(R)$ be an automorphism of R over k. Then it induces a k-module automorphism of $\mathcal{T}_R$, denoted by Ad_φ, by

$$\mathrm{Ad}_\varphi(\delta) = \varphi \circ \delta \circ \varphi^{-1}.$$

Note that Ad_φ is not an R-module automorphism, but rather equivariant: for $x \in R$ and $\delta \in \mathcal{T}_R$, we have $\mathrm{Ad}_\varphi(x\delta) = \varphi(x)\mathrm{Ad}_\varphi(\delta)$. Let G be a finite group and suppose it acts on R by k-algebra automorphisms; i.e., suppose a group homomorphism

$$\rho \colon G \to \mathrm{Aut}_k(R) \ : \ \sigma \mapsto \rho_\sigma$$

is given. The induced action of G on $\mathcal{T}_R$ by k-module automorphisms is denoted by

$$\mathrm{Ad}_\rho \ : \ \sigma \mapsto \mathrm{Ad}_{\rho,\sigma}.$$

2.2 Tangent space to the local deformation functor. Let $k[\epsilon]$ be the ring of dual numbers. Then $R[\epsilon] = R \bigotimes_k k[\epsilon]$ is the $k[\epsilon]$-algebra of elements $x + y\varepsilon$ with

$x, y \in R$. Let $\overline{\cdot}\ : R[\epsilon] \to R$ denote the reduction map modulo-ϵ. In the following proposition, we make an explicit identification between the tangent space $D_\rho(k[\epsilon])$ to the deformation functor D_ρ and the first group cohomology with values in the derivations.

2.3 Proposition. *There exists a bijection (depending on the deformation parameter ϵ) $d : D_\rho(k[\epsilon]) \xrightarrow{\sim} \mathrm{H}^1(G, \mathfrak{T}_R)$ described as follows: The 1-cocycle $d_{\rho^\sim}$ associated to a lifting $\rho^\sim$ is given by the formula*

$$d_{\rho^\sim}\sigma = \frac{\rho^\sim_\sigma \circ \rho_\sigma^{-1} - \mathrm{Id}}{\epsilon} \quad (= \frac{d}{d\epsilon}(\rho^\sim_\sigma \circ \rho_\sigma^{-1})|_{\epsilon=0}),$$

for any $\sigma \in G$.

Proof. For $\sigma \in G$ we set $\rho^\sim_\sigma(x) = \rho_\sigma(x) + \rho'_\sigma(x)\epsilon$ ($x \in R$). Then for $x + y\epsilon \in R[\epsilon]$, we have

$$\rho^\sim_\sigma(x + y\epsilon) = \rho^\sim_\sigma(x) + \rho^\sim_\sigma(y)\epsilon = \rho_\sigma(x) + [\rho'_\sigma(x) + \rho_\sigma(y)]\epsilon,$$

i.e., ρ'_σ determines the lifting $\rho^\sim$. The 1-cocycle $d_{\rho^\sim}$ is given by $d_{\rho^\sim}\sigma = \rho'_\sigma \circ \rho_\sigma^{-1}$. The following two formulas are straightforward:

(i) $\rho'_\sigma(xy) = \rho_\sigma(x)\rho'_\sigma(y) + \rho'_\sigma(x)\rho_\sigma(y)$ for $x, y \in R$.
(ii) $\rho'_{\sigma\tau} = \rho'_\sigma \circ \rho_\tau + \rho_\sigma \circ \rho'_\tau$ for $\sigma, \tau \in G$.

From (i) it follows that $d_{\rho^\sim}\sigma \in \mathfrak{T}_R$, and from (ii) that $d_{\rho^\sim}$ is a cocycle, i.e., $d_{\rho^\sim}\sigma\tau = d_{\rho^\sim}\sigma + \mathrm{Ad}_{\rho,\sigma}(d_{\rho^\sim}\tau)$.

Suppose two liftings $\rho^\sim$ and $\rho^\approx$ are isomorphic by $\psi \in \mathrm{Aut}_{k[\epsilon]}(R[\epsilon])$. By the condition $\overline{\psi} = \mathrm{Id}_R$ we can write $\psi(x) = x + \delta(x)\epsilon$ ($x \in R$), where $\delta \in \mathfrak{T}_R$. The equality $\psi \circ \rho^\approx_\sigma = \rho^\sim_\sigma \circ \psi$ implies that

$$\rho''_\sigma - \rho'_\sigma = \rho_\sigma \circ \delta - \delta \circ \rho_\sigma, \tag{2.3.1}$$

where ρ''_σ is the ϵ-part in $\rho^\approx$ (as ρ'_σ was the ϵ-part of $\rho^\sim$). Hence

$$d_{\rho^\approx}\sigma - d_{\rho^\sim}\sigma = \mathrm{Ad}_{\rho,\sigma}(\delta) - \delta, \tag{2.3.2}$$

which implies that $(d_{\rho^\approx}) - (d_{\rho^\sim})$ is a coboundary. Conversely, if we have (2.3.2) for some $\delta \in \mathfrak{T}_R$, then one can define $\psi \in \mathrm{Aut}_{k[\epsilon]}(R[\epsilon])$ by the obvious formula $\psi(x) = x + \delta(x)\epsilon$ (or, equivalently, $\psi(x + y\epsilon) = x + [y + \delta(x)]\epsilon$), which gives an isomorphism between the liftings $\rho^\sim$ and $\rho^\approx$. Therefore, the map d is well-defined, and is injective.

We now show surjectivity. A given cocycle $d : G \to \mathfrak{T}_R$ induces an automorphism $\rho^\sim_\sigma$ for any $\sigma \in G$ by the formula: $\rho^\sim_\sigma(x) = \rho_\sigma(x) + (d\sigma) \circ \rho_\sigma(x)\epsilon$ for $x \in R$. One can easily check that $\sigma \mapsto \rho^\sim_\sigma$ gives a lifting of ρ whose associated 1-cocycle is exactly d. $\square$

3. Computation of the group cohomology

3.1 In this section, k denotes a field of characteristic $p > 0$, and $\mathcal{O}$ a discrete valuation ring over k with the residue field k. We fix a regular parameter x for $\mathcal{O}$. The $\mathcal{O}$-module $\mathfrak{T}_\mathcal{O}$ (defined in (2.1)) is free of rank 1 since every $\delta \in \mathfrak{T}_\mathcal{O}$ is determined by $\delta(x)$. Let $\frac{d}{dx}$ be the unique k-derivation such that $\frac{d}{dx}(x) = 1$. Then $\mathcal{T}_{\mathcal{O}} = \mathcal{O}\frac{d}{dx}$. Let a group G act on $\mathcal{O}$ by k-algebra automorphisms. In this section we write the action of ρ exponentially ($f \to f^{\sigma}$ for $f \in \mathcal{O}$ and $\sigma \in G$), omitting ρ from the notation – we do the same for the induced action Ad on the tangent space $\mathcal{T}_{\mathcal{O}}$. The G-equivariancy condition now becomes $(f\delta)^{\sigma} = f^{\sigma}\delta^{\sigma}$ for $f \in \mathcal{O}$. In this section, we will compute the group cohomology $\mathrm{H}^1(G, \mathcal{T}_{\mathcal{O}})$ in the following three situations, which are exactly the ones that arise for the local action of the ramification group of a branch point on an ordinary curve:

(3.1.1) $G = \langle\tau\rangle \cong \mathbf{Z}/n\mathbf{Z}$ with $(n,p) = 1$ acting on $\mathcal{O}$ by $x^{\tau} = \zeta x$, where ζ is a primitive n-th root of unity;

(3.1.2) $G = \prod_{i=1}^{t}\langle\sigma_i\rangle \cong (\mathbf{Z}/p\mathbf{Z})^t$ acting on $\mathcal{O}$ by $x^{\sigma_i} = x/(1 - u_i x)$ where $u_1, \ldots, u_t \in \mathcal{O}$ are linearly independent over $\mathbf{F}_p$;

(3.1.3) $G = (\prod_{i=1}^{t}\langle\sigma_i\rangle) \rtimes \langle\tau\rangle \cong (\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/n\mathbf{Z}$ with $n > 1$ and $n|p^t - 1$ whose action on $\mathcal{O}$ is the combination of the above ones.

3.2 The case (3.1.1). Since the G-module $\mathcal{T}_{\mathcal{O}}$ is killed by p, but p is prime to the order of G, all higher group cohomology vanishes: $\mathrm{H}^n(G, \mathcal{T}_{\mathcal{O}}) = 0$ for $n > 0$ ([22], 3.1.8).

3.3 The case (3.1.2). Let V be the t-dimensional $\mathbf{F}_p$-vector subspace in k spanned by $u_1, \ldots, u_t$. The group G and its action are isomorphic to the vector group V, acting on $\mathcal{O}$ by $x^u = x/(1 - ux)$ for $u \in V$; by this, we can pretend that $G = V$. Since $(\frac{d}{dx})^u = (1 - ux)^2\frac{d}{dx}$, the G-module $\mathcal{T}_{\mathcal{O}}$ is isomorphic to $\mathcal{O}$ with the G-action given by

$$(3.3.1) \qquad f^u(x) = f\left(\frac{x}{1 - ux}\right)(1 - ux)^2$$

for $f \in \mathcal{O}, u \in V$.

3.4 Lemma. *The G-module $\mathcal{O}$ with G-action (3.3.1) is isomorphic to $M \bigoplus x\mathcal{O}$, where $M = k \bigoplus kx \bigoplus kx^2$ such that:*

(1) The G-module structure of M is given by

$$(a_0 + a_1 x + a_2 x^2)^u = a_0 + (a_1 - 2ua_0)x + (a_2 - ua_1 + u^2 a_0)x^2,$$

i.e., w.r.t the basis $\{1, x, x^2\}$,

$$u \longleftrightarrow \Phi(u) = \begin{pmatrix} 1 & 0 & 0 \\ -2u & 1 & 0 \\ u^2 & -u & 1 \end{pmatrix}.$$

(2) The G-module structure on the second factor $x\mathcal{O}$ is the original G-action on $\mathcal{O}$, i.e., $f^u(x) = f(\frac{x}{1-ux})$ for $f(x) \in x\mathcal{O}$.

Proof. Note that $\mathcal{O} = M \oplus x^3\mathcal{O}$ is a G-stable direct decomposition. The action of G on the first factor is as stated in (1). For $x^2 f(x) \in x^3\mathcal{O}$ (i.e., $f(x) \in x\mathcal{O}$), the action (3.3.1) gives $(x^2 f(x))^u = x^2 f(x/(1 - ux))$. $\square$

3.5 Let G act on $x\mathcal{O}$ as in 3.4 (2), and let $F = \mathrm{Frac}(\mathcal{O})$ be the fraction field of $\mathcal{O}$. Since this action comes from that on $\mathcal{O}$ by ring automorphisms, it extends to F, respecting the decomposition $F = x\mathcal{O} \oplus k[x^{-1}]$. We have

$$\mathrm{H}^1(G, x\mathcal{O}) \oplus \mathrm{H}^1(G, k[x^{-1}]) = \mathrm{H}^1(G, F) = 0$$

by Hilbert 90, and hence $\mathrm{H}^1(G, x\mathcal{O}) = 0$. Thus $\mathrm{H}^1(G, \mathcal{T}_{\mathcal{O}}) \cong \mathrm{H}^1(G, M)$.

We will now compute $\mathrm{H}^1(G, M)$. Since G is a commutative p-group, the condition on a map $d : V \to M$ to be a CO cycle ($d(u+v) = du + (dv)^u$) implies

$$\begin{cases} d(pu) = 0 \iff du + (du)^u + (du)^{2u} + \cdots + (du)^{(p-1)u} = 0 & \text{— (i)} \\ d(u+v) = d(v+u) \iff du + dv^u = dv + du^v & \text{— (ii)} \end{cases}$$

Let us write a cocycle d as $du = a_0(u) + a_1(u)x + a_2(u)x^2$ for $u \in V$.

(3.5.1) Calculating the matrix $1 + \Phi(u) + \Phi(2u) + \cdots + \Phi((p-1)u)$, we deduce that condition (i) is

(1-a) empty unless $p = 2$ or 3,
(1-b) equivalent to $a_0(u) = 0$ if $p = 3$,
(1-c) equivalent to $ua_0(u) + a_1(u) = 0$ if $p = 2$.

(3.5.2) Condition (ii) is equivalent to $2ua_0(u) = 2va_0(v)$ and $u^2a_0(v) - ua_1(v) = v^2a_0(u) - va_1(u)$. Hence

(2-a) if $p \neq 2$, (ii) is equivalent to $a_0(u) = ua_0$ and $a_1(u) = u(a_1 - ua_0)$, where a_0 and a_1 are constants independent of u,
(2-b) if $p = 2$, (ii) together with (i) is equivalent to $a_0(u) = ua_0$ and $a_1(u) = u^2a_0$, where a_0 is a constant independent of u.

Thus we get

$$\mathrm{Z}^1(G, M) = \begin{cases} \{du = ua_0 + u(a_1 - ua_0)x + a_2(u)x^2\} & (p \neq 2, 3) \\ \{du = ua_1x + a_2(u)x^2\} & (p = 3) \\ \{du = ua_0 + u^2a_0x + a_2(u)x^2\} & (p = 2) \end{cases}$$

In each case, the cocycle condition is equivalent to the fact that the function a_2 satisfies

$$a_2(u+v) = a_2(u) + a_2(v) + uv[(u+v)a_0 - a_1]$$

(with $a_0 = 0$ if $p = 3$ and $a_1 = 0$ if $p = 2$). In particular, whenever $a_0 = a_1 = 0$ the function a_2 is $\mathbf{F}_p$-linear. Hence $\mathrm{Z}^1(G, M)$ contains the t-dimensional k-subspace $\mathrm{Hom}_{\mathbf{F}_p}(V, k) = V^* \bigotimes k$. The dimension over k of $\mathrm{Z}^1(G, M)$ is therefore $t + 2$ if $p \neq 2, 3$ or is $t + 1$ otherwise.

3.6 Let $g = b_0 + b_1x + b_2x^2$. Then a coboundary is of the form

$$g^u - g = -2ub_0x + (-ub_1 + u^2b_0)x^2.$$

We can then compute a k-basis for $\mathrm{H}^1(G, T_{\mathcal{O}})$ as follows:

(3.6.1) If $p \neq 2, 3$, a non-trivial such cohomology class $[d_0]$ is given by the cocycle d_0 with

$$d_0 u = u - u^2 x.$$

The other cohomology classes come from the subspace $\{a_0 = a_1 = 0\} \cong \mathrm{Hom}_{\mathbf{F}_p}(V, k)$ in $\mathrm{Z}^1(G, M)$, in which the coboundary classes are spanned by $du = ux^2$. Hence the part of the cohomology coming from $\{a_0 = a_1 = 0\}$ is isomorphic to

$$\mathrm{Hom}_{\mathbf{F}_p}(V, k)/k \cdot \iota,$$

where $\iota \colon V \hookrightarrow k$ ($\iota = \mathbf{1}^*$) is the natural inclusion. Hence

$$\mathrm{H}^1(G, T_{\mathcal{O}}) \cong k \cdot [d_0] \bigoplus \mathrm{Hom}_{\mathbf{F}_p}(V, k)/k \cdot \iota.$$

In particular, $\dim_k \mathrm{H}^1(G, T_{\mathcal{O}}) = t$.

(3.6.2) If $p = 3$, then only the cocycles coming from $\{a_0 = a_1 = 0\}$ survive. Hence

$$\mathrm{H}^1(G, T_{\mathcal{O}}) \cong \mathrm{Hom}_{\mathbf{F}_p}(V, k)/k \cdot \iota,$$

and we have $\dim_k \mathrm{H}^1(G, T_{\mathcal{O}}) = t - 1$.

(3.6.3) If $p = 2$, the cocycle d_0 gives a non-zero cohomology class, and the remaining part $\{a_0 = a_1 = 0\} \cong \mathrm{Hom}_{\mathbf{F}_p}(V, k)$ contains 2-dimensional coboundaries spanned by ι and the Frobenius embedding

$$\mathrm{Frob} \colon V \hookrightarrow k \;:\; u \mapsto u^2.$$

Observe that $\mathrm{Frob} = \iota$ exactly if $t = 1$. Hence

$$\mathrm{H}^1(G, T_{\mathcal{O}}) \cong k \cdot [d_0] \bigoplus \mathrm{Hom}_{\mathbf{F}_p}(V, k)/(k \cdot \iota + k \cdot \mathrm{Frob}).$$

Therefore, $\dim_k \mathrm{H}^1(G, T_{\mathcal{O}})$ is $t - 1$ if $t > 1$ or is 1 of $t = 1$.

3.7 Explicit liftings. We can make the liftings corresponding to these cocycles explicit: First we consider the lifting

$$x^u = \frac{x + a_0 u \epsilon}{1 - u(x + a_1 \epsilon)} \tag{3.7.1}$$

If $p \neq 2, 3$, this always defines a lifting. If $p = 3$ (resp. $p = 2$) however, this is a lifting if and only if $a_0 = 0$ (resp. $a_1 = 0$). The corresponding cocycle d is by the formula in Proposition 2.3 given as

$$du = \frac{d}{d\epsilon}\left(\frac{x^u}{1 + ux^u}\right)\Big|_{\epsilon=0} = ua_0 + u(a_1 - ua_0)x.$$

In particular, if $a_1 = 0$, then it is $a_0 d_0$. The other liftings corresponding to the cocycles coming from $\mathrm{Hom}_{\mathbf{F}_p}(V, k)$ are

$$x^u = \frac{x}{1 - (u + a_2(u)\epsilon)x} \tag{3.7.2}$$

for $a_2 \in \mathrm{Hom}_{\mathbf{F}_p}(V,k)$.

3.8 The case (3.1.3). Let us write $N = \prod_{i=1}^t \langle \sigma_i \rangle$ and $H = \langle \tau \rangle$; $G = N \rtimes H$. As in the previous paragraphs, we can view N as an $\mathbf{F}_p$-vector subgroup V in k. The group H acts on N by inner automorphisms in G, and since

$$x^{\tau \cdot u \cdot \tau^{-1}} = \frac{x}{1 - \zeta u x},$$

the $\mathbf{F}_p$-vector space V is stable under multiplication by ζ. Let $s := [\mathbf{F}_p(\zeta) : \mathbf{F}_p]$ be the degree of ζ over $\mathbf{F}_p$; then

$$s = \min\{s' \in \mathbf{Z}_{>0} : n | p^{s'} - 1\}$$

([15], 2.47(ii)). In this way, V becomes a vector space over $\mathbf{F}_p(\zeta) = \mathbf{F}_{p^s}$; in particular, we have $s|t$. Since $|H|$ is coprime to the characteristic of k, we have

$$\mathrm{H}^1(G, M) = \mathrm{H}^1(N, M)^H$$

(see, e.g., [4], III.10.4), so we only have to describe the action of H on the cohomology $\mathrm{H}^1(N, M)$ (which has already been computed).

First recall that the action of H on $\mathcal{T}_{\mathcal{O}}$ is given by

$$(x^r \frac{d}{dx})^\tau = \zeta^{r-1} x^r \frac{d}{dx}.$$

It stabilizes M, on which is acts by

$$(a_0 + a_1 x + a_2 x^2)^\tau = \zeta^{-1} a_0 + a_1 x + \zeta a_2 x^2.$$

The action of H on the cohomology $\mathrm{H}^1(N, M)$ is induced from the action on the space of cocycles given by $d^\tau u = (du^\tau)^{\tau^{-1}}$ (cf. loc. cit.). Hence $d_0^\tau u = (d_0 \zeta u)^{\tau^{-1}} = (\zeta u - \zeta^2 u^2 x)^{\tau^{-1}} = \zeta^2 (u - u^2 x) = \zeta^2 d_0 u$; thus

$$d_0^\tau = \zeta^2 d_0.$$

This implies that the class d_0 is not H-invariant as long as $n \neq 2$. Next we look at the remaining part $\{a_0 = a_1 = 0\} \cong \mathrm{Hom}_{\mathbf{F}_p}(V, k)$. An element $a_2 \in \mathrm{Hom}_{\mathbf{F}_p}(V, k)$ (corresponding to the cocycle $du = a_2(u) x^2$) is H-invariant if and only if $d^\tau u = \zeta^{-1} a_2(\zeta u) x^2 = a_2(u) x^2$, or equivalently, $a_2(\zeta u) = \zeta a_2(u)$. Since ζ generates $\mathbf{F}_{p^s}$, it is equivalent to the $\mathbf{F}_p$-linear function a_2 being actually $\mathbf{F}_{p^s}$-linear. Summing up, we have the following:

(3.8.1) Suppose $p \neq 2, 3$. If $n \neq 2$, then

$$\mathrm{H}^1(G, \mathcal{T}_{\mathcal{O}}) \cong \mathrm{H}^1(N, M)^H \cong \mathrm{Hom}_{\mathbf{F}_{p^s}}(V, k)/k \cdot \iota,$$

and hence is of dimension $t/s - 1$. If $n = 2$ (which implies $s = 1$), it is of dimension t, since d_0 is also H-invariant.

(3.8.2) If $p=3$, then we always have

$$\mathrm{H}^1(G,\mathcal{T}_{\mathcal{O}}) \cong \mathrm{H}^1(N,M)^H \cong \mathrm{Hom}_{\mathbf{F}_{p^s}}(V,k)/k\cdot\iota,$$

and $\dim_k \mathrm{H}^1(G,\mathcal{T}_{\mathcal{O}}) = t/s-1$.

(3.8.3) If $p=2$, then $n\neq 2$ and $s>1$ (since $1\neq n|2^s-1$). The subspace $\mathrm{Hom}_{\mathbf{F}_{p^s}}(V,k)$ in $\mathrm{Hom}_{\mathbf{F}_p}(V,k)$ has a trivial intersection with the one dimensional subspace spanned by the Frobenius embedding $u\mapsto u^2$. Hence

$$\mathrm{H}^1(G,\mathcal{T}_{\mathcal{O}}) \cong \mathrm{H}^1(N,M)^H \cong \mathrm{Hom}_{\mathbf{F}_{p^s}}(V,k)/k\cdot\iota,$$

and $\dim_k \mathrm{H}^1(G,\mathcal{T}_{\mathcal{O}}) = t/s-1$.

4. The local pro-representable hull

4.1 We are going to take a closer look at the hull H_ρ of D_ρ, where the group G and its action on $R=\mathcal{O}$ are as in (3.1.1), (3.1.2), and (3.1.3). The first one is a trivial case, i.e., the tame cyclic action is rigid (compare [1], 4.3).

To analyze the case (3.1.2), we have to look at the lifting obstructions. First we remark that a lifting of the form

$$x^u = \frac{x}{1-(u+\alpha(u))x}, \tag{4.1.1}$$

(which is in the direction of (3.7.2)), can always be taken for any $\mathbf{F}_p$-linear map $\alpha\colon V\to\mathfrak{m}_A$. In other words, the liftings to the directions of $\mathrm{Hom}_{\mathbf{F}_p}(V,k)/k\cdot\iota$ (or $\mathrm{Hom}_{\mathbf{F}_p}(V,k)/(k{\cdot}\iota{+}k{\cdot}\mathrm{Frob})$) in the tangent space $\mathrm{H}^1(G,\mathcal{T}_{\mathcal{O}})$ are always unobstructed.

So the question is when the lifting in the direction of a_0d_0 is unobstructed (which only occurs for $p\neq 3$). Such a lifting would be given by $x^u=(x+\alpha(u))/(1-ux)$ for a function $\alpha\colon V\to\mathfrak{m}_A$. We get that $\alpha(u)/u$ is constant due to the compatibility condition $(x^u)^v=(x^v)^u$, hence the lifting is of the form

$$x^u = \frac{x+\alpha u}{1-ux} \tag{4.1.2}$$

for $\alpha\in\mathfrak{m}_A$. Let Φ be the corresponding matrix:

$$\Phi = \begin{pmatrix} 1 & \alpha u \\ -u & 1 \end{pmatrix}$$

The question is when it is of order p. Now assume $p>2$. Since Φ has eigenvalues $1\pm\sqrt{\alpha}u$, the action (4.1.2) is of order p if and only if

$$\alpha^{\frac{p-1}{2}} = 0, \tag{4.1.3}$$

which gives a genuine obstruction. If $p=2$ on the contrary, Φ^2 is diagonal, so the lifting is unobstructed.

4.2 Summing up the results obtained so far, we have the following description of the hull:

(4.2.1) If $p \neq 2, 3$, then

$$H_\rho \cong k[[x_0, x_1, \ldots, x_t]]/(x_0^{\frac{p-1}{2}}, x_1 + \cdots + x_t).$$

The Zariski tangent space ($\cong \mathrm{H}^1(G, \mathcal{T}_\mathcal{O})$) is of dimension t, while the Krull-dimension of H_ρ is $t-1$ (because of the obstruction).

(4.2.2) If $p = 3$, then

$$H_\rho \cong k[[x_1, \ldots, x_t]]/(x_1 + \cdots + x_t).$$

Both the dimension of the Zariski tangent space and the Krull-dimension are $t-1$.

(4.2.3) If $p = 2$, then

$$H_\rho \cong k[[x_0, x_1, \ldots, x_t]]/(x_1 + \cdots + x_t, u_1 x_1 + \cdots + u_t x_t),$$

where $\{u_1, \ldots, u_t\}$ is an $\mathbf{F}_p$-basis of V. The dimension of the Zariski tangent space and the Krull-dimension are equal, and are $t-1$ if $t > 1$ or 1 if $t = 1$.

4.3 Finally, in the general case (3.1.3), the arguments are completely similar; we only have to check that the action of $H = \langle \tau \rangle$ is liftable. The results are as follows:

(4.3.1) If $n \neq 2$ or $p = 2, 3$, then

$$H_\rho \cong k[[x_1, \ldots, x_{t/s}]]/(x_1 + \cdots + x_{t/s}).$$

If $p \neq 2, 3$ and $n = 2$ (hence $s = 1$), then the action of H extends to the lifting (4.1.2) (just replacing u by ζu), and the obstructions do not change. Hence, as in (4.2.1), we have

$$H_\rho \cong k[[x_0, x_1, \ldots, x_t]]/(x_0^{\frac{p-1}{2}}, x_1 + \cdots + x_t).$$

4.4 Theorem. *Let $\rho : G \to Aut(\mathcal{T}_\mathcal{O})$ be a local representation of a finite group G, where $\mathcal{O}$ is of characteristic p. Let n be an integer coprime to p, define $s := \min\{s' : n | p^{s'} - 1\}$, and let $[d_0]$ be the cohomology class defined in* (3.6)*. The following table lists the dimension of the group cohomology $H^1(G, \mathcal{T}_\mathcal{O})$, the fact whether $[d_0]$ is trivial (—), unobstructed* (unobs.) *or obstructed* (obs.)*, and the Krull-dimension* $\dim H_\rho$ *of the pro-representable hull H_ρ of the local deformation functor D_ρ:*

G	(p, t, n)	$h^1(G, \mathcal{T}_\mathcal{O})$	$[d_0]$	$\dim H_\rho$
$\mathbf{Z}/n$		0	—	0
$(\mathbf{Z}/p)^t$	$p \neq 2, 3$	t	obs.	$t-1$
	$p = 3$	$t-1$	—	$t-1$
	$p = 2, t > 1$	$t-1$	unobs.	$t-1$
	$p = 2, t = 1$	1	unobs.	1
$(\mathbf{Z}/p)^t \rtimes \mathbf{Z}/n$	$n \neq 2$ or $p = 2, 3$	$t/s - 1$	—	$t/s - 1$
	$n = 2$	t	obs.	$t-1$

5. Main theorem on algebraic equivariant deformation

We have now collected all information needed to prove our main algebraic theorem:

5.1 Theorem. *Let X be an ordinary curve over a field of characteristic $p > 0$ and let G be a finite group of acting on X via $\rho : G \to Aut(X)$.*

(a) *The Krull-dimension of the pro-representable hull of the deformation functor $D_{X,\rho}$ is given by*

$$\dim H_{X,\rho} = 3g_Y - 3 + \delta + \sum_{i=1}^{s} \dim H_{\rho_i},$$

where g_Y is the genus of $Y := G\backslash X$, δ is given in (1.5) *and H_{ρ_i} is the pro-representable hull of the local deformation functor associated to the representation of the ramification group G_i at the branch point w_i in $X \to Y$, whose dimension was given in* (4.4), unless *in the following four cases:*

1. *when $p = 2, Y = \mathbf{P}^1$ and $X \to Y$ is branched above 2 points; then $\dim H_{X,\rho} = \dim H_{\rho_1} + \dim H_{\rho_2}$;*
2. *when $X = \mathbf{P}^1 \to Y = \mathbf{P}^1$ is tamely branched above two points, then $\dim H_{X,\rho} = 0$;*
3. *when $X = \mathbf{P}^1 \to Y = \mathbf{P}^1$ is wildly branched above a unique point; then $\dim H_{X,\rho} = \dim H_{\rho_1}$, (in this case, G is a pure-p group and $X \to Y$ is an Artin-Schreier cover);*
4. *when $X \to Y$ is an unramified cover of elliptic curves; then $\dim H_{X,\rho} = 1$.*

Furthermore, if the genus g of X is ≥ 2, then $D_{X,\rho}$ is pro-representable by $H_{X,\rho}$.

(b) *The dimension of the tangent space to the functor $D_{X,\rho}$ as a k-vector space satisfies*

$$\dim_k D_{X,\rho}(k[\epsilon]) = \dim H_{X,\rho} + \begin{cases} 0 & \text{if } p = 2, 3, \\ \#\{i : n_i \leq 2\} & \text{otherwise.} \end{cases}$$

Proof. Let Y be the quotient $G\backslash X$. From proposition (1.12), we find that

$$\dim H_{X,\rho} = \sum_{i=1}^{s} \dim H_{\rho_i} + h^1(Y, \pi_*^G \mathcal{T}_X),$$

where H_{ρ_i} are the hulls of the local deformation functors associated to the action of the ramification groups G_i at wild ramifications points $w_1, ..., w_s$ on the space of local derivations, which was computed in (4.4).

To compute the h^1-term, recall from (1.6) that $\pi_*^G \mathcal{T}_X = \mathcal{T}_Y(-\Delta)$, where Δ is defined in (1.5). By Riemann-Roch, we find

$$h^1(\pi_*^G \mathcal{T}_X) = 3g_Y - 3 + \delta + h^0(\mathcal{T}_Y(-\Delta)),$$

where g_Y is the genus of Y.

Since $\deg(\mathcal{T}_Y(-\Delta)) = 2 - 2g_Y - \delta$, the last term vanishes if $g_Y > 1$ or $g_Y = 1$ and $\delta > 0$ or $g_Y = 0$ and $\delta > 2$.

If $g_Y = 1$ and $\delta = 0$, X is an unramified cover of an elliptic curve, hence is an elliptic curve itself.

Assume that $g_Y = 0$, that there are at least two branch points on Y and that $\delta = 2$. If $p \neq 2$, then these branch points have to be tame, so $\delta = 2$, and the Hurwitz formula implies that $g_X = 0$ too. If $p = 2$, they can be wild, but still $\delta = 2$; in both cases, $h^0(\mathcal{T}_Y(-\delta)) = h^0(\mathcal{O}_{\mathbf{P}^1}) = 1$.

If $g_Y = 0$ and only one point on Y is branched, then it follows from Hurwitz's formula (using the fact that second ramification groups vanish in the ordinary case, cf. [17]) that $g_X = 0$ too, and the ramification has to be wild at this point. So if $p \neq 2$, we find $\delta + h^0(\mathcal{T}_Y(-\delta)) = 2 + h^0(\mathcal{O}_{\mathbf{P}^1}) = 3$. On the other hand, if $p = 2$, then $\delta + h^0(\mathcal{T}_Y(-\delta)) = 1 + h^0(\mathcal{O}_{\mathbf{P}^1}(1)) = 3$. Let np^t be the order of the ramification group at that unique point, where n is coprime to p. Hurwitz's formula gives in particular that $(n-1)p^t + 2$ divides $2np^t$, and this (together with $n | p^t - 1$) implies $n = 1$. Hence we do get a $\mathbf{Z}/p\mathbf{Z}$-cover. This finishes the proof of part (a).

For part (b), we apply the formula from (1.9) in combination with (1.7.1). It thus suffices to compute $h^0(Y, \mathcal{H}^1(G, \mathcal{T}_X))$, but $\mathcal{H}^1(G, \mathcal{T}_X)$ is concentrated in the branch points w_i, where it equals the group cohomology $H^1(G_i, \mathcal{T}_{\mathcal{O}_{w_i}})$ ([1], 3.3), so $\dim H_{X,\rho}$ and $\dim_k D_{X,\rho}(k[\epsilon])$ differ only at places where $[d_0]$ is obstructed in table (4.4), which is if $n_i \leq 2$ and $p \neq 2, 3$. $\square$

5.2 Corollary. *Assume that $p \neq 2, 3$, X is an ordinary non-elliptic curve over a field of characteristic $p > 0$, G is a finite group acting via $\rho : G \to Aut(X)$ on X, such that $X \to Y := G\backslash X$ is branched above n points, of which s are wildly branched with ramification groups $(\mathbf{Z}/p\mathbf{Z})^{t_i} \rtimes (\mathbf{Z}/n_i\mathbf{Z})$ $(i = 1, ..., s)$. Then the Krull dimension of the pro-representable hull of the equivariant deformation functor $D_{X,\rho}$ is given by*

$$\dim H_{X,\rho} = 3g_Y - 3 + n + \sum_{i=1}^{s} \frac{t_i}{s_i},$$

where g_Y is the genus of Y and $s_i := \min\{s' > 0 : n_i | p^{s'} - 1\}$. $\square$

5.3.1 Example: Artin-Schreier curves. The Artin-Schreier curve whose affine equation is given by $(y^{p^t} - y)(x^{p^t} - x) = c$ for some constant $c \in k^*$ has automorphism group

$$G = (\mathbf{Z}/p\mathbf{Z})^{2t} \rtimes D_{p^t-1},$$

where D_* denotes a dihedral group of order $2*$. The quotient $Y = G\backslash X$ is a projective line, and the branching groups are $\mathbf{Z}/2\mathbf{Z}$ (twice if $p \neq 2$ and once if $p = 2$) and $(\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/(p^t - 1)\mathbf{Z}$ (once). This curve with its full automorphism groups hence allows for a one-dimensional deformation space (for different reasons if $p \neq 2$ and $p = 2$). This deformation is exactly given by varying c.

5.3.2 Example: Drinfeld modular curves. The Drinfeld modular curves $X(n)$ from the introduction have automorphism group $G := \Gamma(1)/\Gamma(n)\mathbf{F}_q^*$ for $d :=$

$\deg(n) > 1$. The quotient $Y := G\backslash X(n)$ is a projective line, over which X is branched at 2 points with ramification groups $\mathbf{Z}/(p+1)\mathbf{Z}$ and $\mathbf{F}_q^d \rtimes \mathbf{F}_q^*$ respectively. Hence $X(n)$ can be deformed in $d-1$ ways (regardless of p, but for different reasons if $p=2$ – then we are in exceptional case (1) from (5.1)).

PART B: ANALYTIC THEORY

6. Equivariant deformation of Mumford curves

6.1 Mumford curves. Let $(K, |\cdot|)$ be a complete discrete valuation field with valuation ring $\mathcal{O}_K$ and residue field $\mathcal{O}_K/\mathfrak{m}_{\mathcal{O}_K} = k$. Recall that a projective curve X over K is called a *Mumford curve* if it is "uniformized over K by a Schottky group". This means that there exists a free subgroup Γ in $PGL(2,K)$ of rank g, acting on $\mathbf{P}^1_K$ with limit set $\mathcal{L}_\Gamma$ such that X satisfies $X^{\mathrm{an}} \cong \Gamma\backslash(\mathbf{P}^{1,\mathrm{an}}_K - \mathcal{L}_\Gamma)$ as rigid analytic spaces. Mumford ([16]) has shown that these conditions are equivalent to the existence of a stable model of X over $\mathcal{O}_K$ whose special fiber consists only of rational components with k-rational double points. Because of the "GAGA"-correspondence for one-dimensional rigid analytic spaces, we do not have to (and will not) distinguish between analytic and algebraic curves. It is well-known that Mumford curves are ordinary (this is basically because their Jacobian is uniformized by $(\mathbf{G}^{\mathrm{an}}_{m,K})^g/\Gamma^{\mathrm{ab}}$ where g is the genus of X, cf. [5], 1.2). Thus, the results from the previous section essentially solve the equivariant deformation problem for Mumford curves in a cohomological way. In this part however, we want to develop an independent theory of analytic deformation of Mumford curves based on the groups that uniformize them. This will make the liftings and obstructions whose cohomological existence was proven in the previous part more "visible" as actual deformations of 2×2-matrices over K.

6.2 Automorphisms. It is well-known ([5], 1.3) that for a Mumford curve X of genus $g \geq 2$ with Schottky group Γ, $\mathrm{Aut}(X) = N(\Gamma)/\Gamma$, where $N(\Gamma)$ is the normalizer of Γ in $PGL(2,K)$. Conversely, if N is a discrete subgroup of $PGL(2,K)$ containing Γ and contained in $N(\Gamma)$, then it induces a group of automorphisms

$$\rho : N/\Gamma \hookrightarrow \mathrm{Aut}(X).$$

6.3 Notation. If a finitely generated discrete subgroup N of $PGL(2,K)$ is given, let $\mathrm{Hom}^*(N, PGL(2,K))$ denote the set of *injective* homomorphisms $\phi : N \to PGL(2,K)$ *with discrete image*. Then such N contains a finite index normal free subgroup of finite rank Γ ([9], I.3), and if Γ is non-trivial, the pair (N,Γ) gives rise to a Mumford curve with an action of N/Γ as is being considered here.

6.4 Rigidity. Two Mumford curves X, X' with Schottky groups Γ, Γ' are isomorphic if and only if Γ and Γ' are conjugate in $PGL(2,K)$ ([16], 4.11).

6.4.1 Remark. Note that this is very different from the situation in the uniformization theory of Riemann surfaces S, where in a representation $S = \Gamma\backslash\Omega$ with Γ a Schottky subgroup of $PGL(2,\mathbf{C})$, the domain of discontinuity Ω of Γ is not the universal topological covering space of S, whereas this does hold for Mumford curves.

6.5 Analytic deformation functors. Recall that $\mathcal{C}_K$ is the category of local Artinian K-algebras. If N and $\phi \in \mathrm{Hom}^*(N, PGL(2,K))$ are given, we consider the *analytic deformation functor*

$$D_{N,\phi} : \mathcal{C}_K \to \mathrm{Sets}$$

of the pair (N, ϕ), which sends $A \in \mathcal{C}_K$ to the set of liftings of (N, ϕ) to A. Here, a lifting is a morphism $\phi^\sim \in \mathrm{Hom}(N, PGL(2,A))$ which, when composed with reduction modulo the maximal ideal $\mathfrak{m}_A$ of A, equals the original embedding ϕ (in particular, $\phi^\sim$ is injective). Note that we do not consider classes of liftings modulo conjugacy by $PGL(2)$ — this implies that $D_{N,\phi}$ is naturally equipped with an action of the formal completion $PGL(2)^\wedge$ of $PGL(2)$ at the neutral element, and we denote the quotient by

$$D^\sim_{N,\phi} := PGL(2)^\wedge \backslash D_{N,\phi}.$$

Since N is finitely generated, it is not difficult to show that the functors $D_{N,\phi}$ and $D^\sim_{N,\phi}$ have pro-representable hulls $H_{N,\phi}$ and $H^\sim_{N,\phi}$. We want to compute the dimension of the tangent spaces to these functors, and the Krull-dimension of their hulls, and this will be done by "decomposing" N using its structure as a group acting on a tree to give a decomposition of $D_{N,\phi}$; note that such a decomposition is *not* given on the level of $D^\sim_{N,\phi}$.

7. Structure of N as a group acting on a tree.

We fix a finitely generated discrete subgroup N of $PGL(2,K)$, and we will now recall how the structure of N can be seen from its action on the Bruhat-Tits tree (cf. section 2 of [5]).

7.1 The Bruhat-Tits tree. Let $\mathfrak{T}$ denote the Bruhat-Tits tree of $PGL(2,K)$ (*i.e.*, its vertices are similarity classes Λ of rank two $\mathcal{O}_K$-lattices in K^2, and two vertices are connected by an edge if the corresponding quotient module has length one – *see* Serre [20], Gerritzen & van der Put [9]). We assume K to be large enough so that all fixed points of N are defined over K; then N acts without inversion on $\mathfrak{T}$. It is a regular tree in which the edges emanating from a given vertex are in one-to-one correspondence with $\mathbf{P}^1(k)$. The tree $\mathfrak{T}$ admits a left action by $PGL(2,K)$.

7.2 Notations on trees. For any subtree T of $\mathfrak{T}$, let $\mathrm{Ends}(T)$ denote its set of ends (*i.e.*, equivalence classes of half-lines differing by a finite segment). There is a natural correspondence between $\mathbf{P}^1(K)$ and $\mathrm{Ends}(\mathfrak{T})$. Let $V(T)$ and $E(T)$ denote the set of vertices and edges of T respectively. For $\sigma \in E(T)$, let $o(\sigma)$ (respectively $t(\sigma)$) denote the origin (respectively, terminal) vertex of σ. Let N_x denote the stabilizer of a vertex or edge x of T for the action of N. The maps o, t induces maps $N_\sigma \to N_\Lambda$ for $\Lambda = o(\sigma)$ or $\Lambda = t(\sigma)$, which will be denoted by the same letter. For any $u, v \in \mathbf{P}^1(K)$, let $]u, v[$ denote the apartment in $\mathfrak{T}$ connecting u and v (seen as ends of $\mathfrak{T}$).

7.3 The trees associated to N. We can construct a locally finite tree $\mathcal{T}(\mathcal{L})$ (possibly empty) from any compact subset $\mathcal{L}$ of $\mathbf{P}^1(K)$: it is the minimal subtree of $\mathcal{T}$ whose set of ends coincides with $\mathcal{L}$, or equivalently, the minimal subtree of $\mathcal{T}$ containing $\bigcup_{u,v\in\mathcal{L}}]u,v[$.

We define $\mathcal{T}_N$ to be the tree associated to the subset $\mathcal{L}_N$ consisting of the limit points of N in $\mathbf{P}^1(K)$. Since N is a finitely generated discrete group, $\mathcal{T}_N$ coincides with the tree of N as it is defined in Gerritzen & van der Put [9].

7.4 The graph associated to N. $\mathcal{T}_N$ admits a natural action of N, and we denote the quotient graph by $T_N := N\backslash\mathcal{T}_N$; the corresponding quotient map will be denoted by π_N. The graph T_N is finite and connected.

We turn T_N into a graph of groups as follows: let T be a spanning tree (maximal subtree) of T_N, which we can see as a subtree of $\mathcal{T}_N$ by a fixed section $\iota : T \to \mathcal{T}_N$ of π_N. Let $c = c(T_N)$ denote the cyclomatic number of T_N (= number of edges outside T), and fix $2c$ lifts $e_i^{\pm}$ of these edges outside T to $\mathcal{T}_N$ which satisfy: $t(e_i^+) \in V(\iota(T)), o(e_i^-) \in V(\iota(T))$. Fix c hyperbolic elements $\{\gamma_i\}_{i=1}^c$ in N such that $\gamma_i e_i^+ = e_i^-$. Then $\iota(T) \cup \{e_i^+\}_{i=1}^c$ is a fundamental domain for the action of N on $\mathcal{T}_N$.

For any vertex $\Lambda \in V(\mathcal{T}_N)$ and edge $\sigma = [\Lambda, M] \in E(\mathcal{T}_N)$ we denote by N_Λ and $N_\sigma = N_\Lambda \cap N_M$ their respective stabilizers for the action of N. Note that these groups are finite since N is discrete.

For a vertex $v \in V(T_N) = V(T)$, we let $N_v = N_{\iota(v)}$. For edges $e \in E(T_N)$, either $e \in E(T)$, and then we let $N_e = N_{\iota(e)}$, or else, there is a unique i such that $\pi_N(e_i^{\pm}) = e$, and we let $N_e = N_{e_i^+}$.

The morphisms between these groups are defined as follows: if $e \in E(T)$, then $N_e \hookrightarrow N_{t(e)}$ and $N_e \hookrightarrow N_{o(e)}$ are the natural inclusions; if, on the other hand, $e = \pi_N(e_i^{\pm})$, then $N_e \hookrightarrow N_{t(e)}$ is the natural inclusion, but $N_e \hookrightarrow N_{o(e)}$ is given by $s \to \gamma_i^{-1} s \gamma_i$.

We then have the following description of the group N:

7.5 Theorem. (Bass-Serre [6], 4.1, 4.4, [20]) *For any spanning tree T of T_N, N equals the fundamental group of the graph of groups T_N at T. This means that N is generated by the amalgam of N_v over N_e for all $e \in E(T_N), v \in V(T_N)$ together with the fundamental group of T_N at T as a plain graph, viz., the free group F_c on c generators $\{n_i\}_{i=1}^c$, where $c = c(T_N)$ is the cyclomatic number of T_N. The further relations in N are of the form $n_i t(\gamma) n_i^{-1} = o(\gamma)$ for every $i = 1, ..., c$ and for every $\gamma \in N_e, e \in T_N - T$. In particular, there is a split exact sequence of groups*

$$0 \to \varinjlim_{T_N^{\sim}} N_\bullet^{\sim} \to N \to F_{c(T_N)} \to 0,$$

where $\pi : T_N^{\sim} \to T_N$ is the universal covering of T_N as a plain graph, which has been made into a graph of groups by setting $N_\bullet^{\sim}$ for $\bullet \in V(T_N^{\sim}) \cup E(T_N^{\sim})$ equal to $N_{\pi(\bullet)}$. $\square$

8. Decomposition of the functor $D_{N,\phi}$.

8.1 Proposition. *Let $s : F_{c(T_N)} \to N$ be a splitting of the sequence in (7.5).*

Then there is an isomorphism of functors

$$D_{N,\phi} \cong \varprojlim_{T_N} D_{N_\bullet,\phi|_{N_\bullet}} \times D_{F_c,\phi\circ s},$$

where the inverse limit is in the category of functors (note that morphisms between $N_\bullet$ naturally induce morphisms of functors between $D_{N_\bullet}$).

8.1.1 Remark. Note that we get a direct product of functors, but a limit of functors over T_N (instead of the obvious semi-direct product and limit over $T_N^\sim$). We also note that there is no such decomposition on the level of the functors $D^\sim_{N,\phi}$.

Proof. Let $A \in \mathcal{C}_K$. By restriction, a deformation of N to A trivially gives rise to deformations of $N_\bullet$ and F_c.

For the rest of the proof, we will imitate the construction of T_N as a graph of groups, but we will lift to $T_N^\sim$ instead of $\mathfrak{T}_N$. So choose a fixed maximal spanning tree $\iota : T \hookrightarrow T_N^\sim$ and a basis $\{\gamma_1, ..., \gamma_c\}$ of $s(F_c)$, where $c = c(T_N)$. Take, as before, $2c$ edges $e_i^\pm \in E(T_N^\sim)$ such that $t(e_i^+) \in V(T), o(e_i^-) \in V(T), \gamma_i e_i^+ = e_i^-$. Thus, $T \cup \{e_i^+\}$ is a "fundamental domain" for $T_N^\sim \to T_N$.

To give elements in $\varprojlim_{T_N} D_{N_\bullet,\phi|_{N_\bullet}}(A)$ and $D_{F_c,\phi\circ s}(A)$ means precisely to give a compatible collection of $\phi_v : N_v \hookrightarrow PGL(2, A)$ and $\phi_c : F_c \hookrightarrow PGL(2, A)$. Compatibility means that for $e \in E(T_N)$, the following diagram is commutative:

$$\begin{array}{ccc} N_e & \longrightarrow & N_{o(e)} \\ \Big\downarrow & & \Big\downarrow \phi_{o(e)} \\ N_{t(e)} & \underset{\phi_{t(e)}}{\longrightarrow} & PGL(2, A). \end{array}$$

We want to extend this to an embedding of N.

By the construction of the fundamental domain, there exists for any $v \in V(T_N^\sim)$, a unique $\gamma \in s(F_c)$ such that $v \in \gamma T$, and this allows us to define $N_v \to PGL(2, A)$ to be $\sigma \mapsto \phi_{\gamma^{-1}v}(\gamma^{-1}\sigma\gamma)$. For edges e, we similarly get γ such that $e \in \gamma \cdot (T \cup \{e_i^+\}_{i=1}^c)$, and the same works.

By the compatibility, we thus get an embedding of $\varinjlim_{T_N^\sim} N_\bullet$ into $PGL(2, A)$, which by construction is compatible with the conjugation action of F_c, so that we finally get an embedding $N \hookrightarrow PGL(2, A)$, viz., an element of $D_{N,\phi}(A)$. Since this construction is functorial in A, we get the desired inverse map of functors. $\square$

8.2 Computing the functor $D_{F_c,\phi\circ s}$. The set of morphisms

$$\mathrm{Hom}(F_c, PGL(2, K))$$

is a smooth algebraic variety over K; by choosing a basis of F_c, it is isomorphic to $PGL(2, K)^c$ over K. We can take its formal completion at the K-rational point $\phi \circ s$, and

$$D_{F_c,\phi\circ s} \cong \mathrm{Hom}(F_c, PGL(2, K))^\wedge_{\phi\circ s}$$

as formal functors. In particular,

$$\dim_K D_{F_c,\phi\circ s}(K[\epsilon]) = \dim H_{F_c,\phi\circ s} = 3c,$$

where the first one is the dimension of the tangent space and the second one the Krull-dimension of the pro-representable hull of the functor.

8.3 Computing the functor $D_{N,\phi}$ for finite N. Here, the argument is based on the simple observation that an injective element ϕ of $\mathrm{Hom}(N, PGL(2,K))$ corresponds to a cover $\mathbf{P}^1 \to \mathbf{P}^1$ with Galois group N; and hence, it is related to the algebraic deformation functor (1.1) of the pair $(\mathbf{P}^1, \phi)$ (regarding ϕ as a representation of N into $\mathrm{Aut}(\mathbf{P}^1)$). The functor $D_{\mathbf{P}^1,\phi}$ is defined modulo conjugation by $PGL(2)$, whereas the analytic deformation functor $D_{N,\phi}$ carries a natural action of $PGL(2)^\wedge$. However, it is easy to see that

$$D^{\sim}_{N,\phi} = D_{\mathbf{P}^1,\phi}.$$

From this formula, we get in particular that

$$\dim H_{N,\phi} = \dim H_{\mathbf{P}^1,\phi} + 3 - \nu(\phi(N)), \tag{8.3.1}$$

where for a finite subgroup $G \subseteq PGL(2,K)$,

$$\nu(G) = \dim \mathrm{Nor}_{PGL(2,K)}(G)$$

is the dimension of the normalizer of G in $PGL(2,K)$ as an algebraic group. Formula (8.3.1) continues to hold when hulls are replaced by tangent spaces.

By Dickson's *Hauptsatz*, the finite groups N acting on $\mathbf{P}^1$ in positive characteristic are known. Let us first set up the notation:

8.3.2 Notation. We let D_n denote the dihedral group of order $2n$. We will write $P(2,q)$ to denote either $PGL(2,q)$ or $PSL(2,q)$ by slight abuse of notation, with the convention that any related numerical quantities that appear between set delimiters $\{\}$ are only to be considered for $PSL(2,q)$.

We now recall this classification in the version as it is given in Valentini-Madan, as this more geometrical form immediately allows us to compute $D_{N,\phi}$ using the results from section 5:

8.3.3 Theorem (Dickson, cf. [21]). *Any finite subgroup of $PGL(2,K)$ is isomorphic to a finite subgroup of $PGL(2,p^m)$ for some $m > 0$. The group $PGL(2,p^m)$ has the following finite subgroups G, such that π_G is branched over d points with ramification groups isomorphic to $G_1, ..., G_d$:*

(i) $G = \mathbf{Z}/n\mathbf{Z}$ for $(n;p) = 1$, $d = 2$, $G_1 = G_2 = \mathbf{Z}/n\mathbf{Z}$;

(ii) $G = D_n$ with $p \neq 2$, $n|p^m \pm 1$, $d = 3$, $G_1 = G_2 = \mathbf{Z}/2\mathbf{Z}, G_3 = \mathbf{Z}/n\mathbf{Z}$ or also, $p = 2$, $(n;2) = 1$, $d = 2$ and $G_1 = \mathbf{Z}/2\mathbf{Z}, G_2 = \mathbf{Z}/n\mathbf{Z}$;

(iii) $G = (\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/n\mathbf{Z}$ for $t \leq m$ and $n|p^m - 1, n|p^t - 1$ with $d = 2$ and $G_1 = G, G_2 = \mathbf{Z}/n\mathbf{Z}$ if $n > 1$ and $d = 1, G_1 = G$ otherwise;

(iv) $G = P(2,p^t)$ *with* $d = 2$ *and* $G_1 = (\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/\{\frac{1}{2}\}(p^t-1)\mathbf{Z}, G_2 = \mathbf{Z}/\{\frac{1}{2}\}(p^t+1)\mathbf{Z}$;

(v) A_4 *of* $p \neq 2,3$, $d = 3$, $G_1 = \mathbf{Z}/2\mathbf{Z}, G_2 = G_3 = \mathbf{Z}/3\mathbf{Z}$;

(vi) S_4 *if* $p \neq 2,3$, $d = 3$, $G_1 = \mathbf{Z}/2\mathbf{Z}, G_2 = G_3 = \mathbf{Z}/4\mathbf{Z}$;

(vii) A_5 *if* $5|p^{2m}-1$ *and* $p \neq 2,3,5$ *with* $d = 3$ *and* $G_1 = \mathbf{Z}/2\mathbf{Z}, G_2 = \mathbf{Z}/3\mathbf{Z}, G_3 = \mathbf{Z}/5\mathbf{Z}$ *or* $p = 3$, $d = 2$ *and* $G_1 = \mathbf{Z}/3\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}, G_2 = \mathbf{Z}/5\mathbf{Z}$. □

8.3.4 Lemma. *The normalizer of a finite subgroup* N *of* $PGL(2,K)$ *has dimension* $\nu(N) = 0$, *unless if* N *is cyclic of order prime-to-p; then* $\nu(N) = 1$, *or if* N *is a pure p-group; then* $\nu(N) = 2$.

Proof. Any group from the above list which does not belong to the mentioned exceptions has at least three fixed points on $\mathbf{P}^1$, the set of which should also remain stable under the action of the normalizer of N, which hence is finite.

A cyclic subgroup N of order prime-to-p has a diagonalizable generator, and by a direct computation, this is seen to be exactly stabilized by the one-dimensional group D generated by the center of $PGL(2,K)$ and the involution $\binom{0\,1}{1\,0}$.

A p-group N can be put into upper diagonal form by conjugation, and a little computation shows that the stabilizer of such a group consists precisely of the 2-dimensional group of upper trigonal matrices. □

8.4 Theorem. *Let* N *be as in section 7, and suppose a Bass-Serre representation of* N *is given as in 7.5. Then*

$$\dim H^{\sim}_{N,\phi} = 3c(T_N) - 3 + \sum_{v \in V(T_N)} h(N_v) - \sum_{e \in E(T_N)} h(N_e),$$

and

$$\dim_K D^{\sim}_{N,\phi}(K[\epsilon]) = 3c(T_N) - 3 + \sum_{v \in V(T_N)} t(N_v) - \sum_{e \in E(T_N)} t(N_e),$$

where for a finite group $G \subset PGL(2,K)$, *the numbers* $h(G)$ *and* $t(G)$ *are given in the table below:*

G	(p,t,n)	$h(G)$	$t(G)$
$\mathbf{Z}/n\mathbf{Z}$	$(n;p) = 1$	2	2
D_n	$p \neq 2$	3	3
	$p = 2$	4	4
$(\mathbf{Z}/p\mathbf{Z})^t$	$p \neq 2,3$	t	$t+1$
	$p = 3$ or $p = 2, t > 1$	t	t
	$p = 2, t = 1$	2	2
$(\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/n\mathbf{Z}$	$p \neq 2$ & $n \neq 2$ or $p = 2,3$	$t/s+2$	$t/s+2$
	$n = 2$	$t+2$	$t+3$
$P(2,p^t)$	$\{p^t \neq 5\}$	3	3
A_4, S_4		3	3
A_5	$p \neq 3$	3	3
	$p = 3$	3	4

Proof. Since N is infinite, the action of $PGL(2)^\wedge$ is of dimension 3. By 8.1 and 8.2, we are reduced to computing $D_{N,\phi}$ for finite N occuring in T_N. We know which different G can occur on the edges and vertices of T_N by Dickson's theorem 8.3.3. For each of these, using 8.3.1 we are reduced to the computation of the algebraic data, for which we appeal to 5.1, and to the computation of $\nu(G)$, which is in lemma 8.3.4. If we let $h^{\mathrm{alg}}(G)$ and $t^{\mathrm{alg}}(G)$ denote the Krull-dimension of the pro-representable hull and vector space dimension of the tangent space to $D_{\mathbf{P}^1,\phi|G}$ respectively, we find

G	(p,t,n)	$h^{\mathrm{alg}}(G)$	$t^{\mathrm{alg}}(G)$	$3-\nu(G)$
$\mathbf{Z}/n\mathbf{Z}$	$(n;p)=1$	0	0	2
D_n	$p\neq 2$	0	0	3
	$p=2$	1	1	3
$(\mathbf{Z}/p\mathbf{Z})^t$	$p\neq 2,3$	$t-1$	t	1
	$p=3$ or $p=2, t>1$	$t-1$	$t-1$	1
	$p=2, t=1$	1	1	1
$(\mathbf{Z}/p\mathbf{Z})^t\rtimes\mathbf{Z}/n\mathbf{Z}$	$p\neq 2$ & $n\neq 2$ or $p=2,3$	$t/s-1$	$t/s-1$	3
	$n=2$	$t-1$	t	3
$P(2,p^t)$	$\{p^t\neq 5\}^*$	0	0	3
A_4, S_4		0	0	3
A_5	$p\neq 3$	0	0	3
	$p=3$	0	1	3

In this computation, note at * that $PSL(2,5)=A_5$ does not occur in Dickson's list if $p=5$. □

9. Compatibility between algebraic and analytic deformation

9.1 Deformation of Mumford uniformization. We have already seen how to compare analytic and algebraic deformation functors for finite groups acting on $\mathbf{P}^1$. We now want to compare these functors in general, in particular to achieve equality between the apparently different results from the main algebraic and analytic theorem (5.1 and 8.3.5).

Let X be a Mumford curve over K uniformized by a Schottky group Γ and $\phi: N \hookrightarrow PGL(2,k)$ a discrete group between Γ and its normalizer in $PGL(2,K)$. Let $\rho: N/\Gamma \hookrightarrow \mathrm{Aut}(X)$. To be able to compare the functors $D_{N,\phi}$ and $D_{X,\rho}$, it will be necessary to develop a theory of algebraic deformation of Mumford uniformization; this means just to translate the formalism of Mumford ([16]) from fields K to elements in the category $\mathcal{C}_K$. For lack of a reference, we sketch it here; the reader who wants to follow the details in encouraged to take a copy of [16] at hand.

9.2 Analytic objects in $\mathcal{C}_K$. The maximal ideal of an object A in $\mathcal{C}_K$ is denoted by $\mathfrak{m}_A$. Each object A in $\mathcal{C}_K$ can be made into a K-affinoid algebra in a unique way by a suitable surjective homomorphism $K\langle X_1,\dots,X_n\rangle \to A$ over K (cf. [2] (6.1)). We denote by $\mathcal{O}_A$ (resp. $\mathfrak{m}_{\mathcal{O}_A}$) the subring (resp. the ideal in $\mathcal{O}_A$) consisting of power-bounded (resp. topologically nilpotent) elements in A (cf. loc. cit. (6.2.3)). Since every element in $\mathfrak{m}_A$ is nilpotent, we have $\mathcal{O}_A\bigcap\mathfrak{m}_A\subseteq\mathfrak{m}_{\mathcal{O}_A}$.

By this, it is easily seen that $\mathcal{O}_A$ is a local ring with the maximal ideal $\mathfrak{m}_{\mathcal{O}_A}$, that $\mathcal{O}_A/\mathfrak{m}_{\mathcal{O}_A} \cong k$, and that $\pi_A^{-1}(\mathcal{O}_K) = \mathcal{O}_A$ and $\pi_A^{-1}(\mathfrak{m}_{\mathcal{O}_K}) = \mathfrak{m}_{\mathcal{O}_A}$, where $\pi_A: A \to K$ is the reduction map.

Example. In the ring of dual numbers $K[\epsilon]$, the ring of power-bounded elements is $\mathcal{O}_K + K\epsilon$, whereas the ideal of topologically nilpotent elements is $\mathfrak{m}_{\mathcal{O}_K} + K\epsilon$.

9.3 Lattices. Let A be an object in $\mathcal{C}_K$. By a lattice in A^2 we mean an $\mathcal{O}_A$-submodule M in A^2 that is free of rank 2. By elementary commutative algebra, this is equivalent to $M \subset A^2$ being an $\mathcal{O}_A$-submodule such that the image $\overline{M}$ in K^2 by the reduction map $A^2 \to K^2$ is a lattice in the usual sense. We consider the set $\Delta_A^{(0)}$ of similarity classes of lattices up to multiplication by A^*. Then $\Delta_A^{(0)}$ can be naturally identified with the set of equivalence classes of couples $(\mathbf{P}, \phi)$, where $\mathbf{P}$ is an $\mathcal{O}_A$-scheme isomorphic to $\mathbf{P}^1_{\mathcal{O}_A}$ and ϕ is an isomorphism between $\mathbf{P} \otimes A$ and $\mathbf{P}^1_A$, and two couples $(\mathbf{P}, \phi)$ and $(\mathbf{P}', \phi')$ are equivalent if there exists an $\mathcal{O}_A$-isomorphism $\psi: \mathbf{P} \to \mathbf{P}'$ such that $\phi' \circ \psi = \phi$. The identification between $\Delta_A^{(0)}$ and the space of such couples is given by

$$M \mapsto \mathbf{P}(M) = \mathrm{Proj}(\mathrm{Sym}_{\mathcal{O}_A} M)$$

where ϕ is induced from $M \otimes A \cong A^2$.

9.4 Trees. We take a subgroup $N \subset \mathrm{PGL}(2, A)$ such that its image $\overline{N}$ in $\mathrm{PGL}(2, K)$ by the reduction map is finitely generated, discrete and isomorphic to N. Such a subgroup N contains a normal free subgroup Γ of finite index, since $\overline{N}$ does and N and $\overline{N}$ are isomorphic as groups. This Γ satisfies a "flatness" condition analogous to [16, (1.4)] (or, equivalently, property $*$ in loc. cit. pp.139) in the folowing sense: if Σ is the set of all sections Spec $A \to \mathbf{P}^1_A$ fixed by non-trivial elements $\gamma \in \Gamma$, then for any $P_1, P_2, P_3, P_4 \in \Sigma$, the cross-ratio $R := R(P_1, P_2; P_3, P_4)$ or its inverse R^{-1} lie in $\mathcal{O}_A$ (note: Σ does not depend on the choice of Γ in N). The proof is easy from the fact $\pi_A^{-1}(\mathcal{O}_K) = \mathcal{O}_A$. Given P_1, P_2, P_3 with homogeneous coordinates w_1, w_2, w_3, respectively, let $M = \mathcal{O}_A a_1 w_1 + \mathcal{O}_A a_2 w_2 + \mathcal{O}_A a_3 w_3$, where the a_i satisfy a non-trivial linear relation $a_1 w_1 + a_2 w_2 + a_3 w_3 = 0$. The class $v(P_1, P_2, P_3)$ of M in $\Delta_A^{(0)}$ depends only on P_1, P_2, P_3. We let $\Delta_\Gamma^{(0)}$ be the set of all such $v(P_1, P_2, P_3)$. The set $\Delta_\Gamma^{(0)}$ is "linked" in the sense of loc. cit. (1.11), and the thus obtained tree is obviously the usual tree with respect to $\overline{\Gamma}$.

9.5 The construction of the formal scheme also parallels the original one. For M_1 and M_2 in $\Delta_\Gamma^{(0)}$ one defines the join $\mathbf{P}(M_1) \vee \mathbf{P}(M_2)$ to be the closure of the graph of the birational map $\mathbf{P}(M_1) \cdots \to \mathbf{P}(M_2)$ induced from $\phi_2^{-1} \circ \phi_1$, where $(\mathbf{P}(M_i), \phi_i)$ corresponds to M_i $(i = 1, 2)$ by the correspondence from (9.3). The formal scheme $\mathcal{P}_\Gamma$ over Spf $\mathcal{O}_A$ is then constructed as in loc. cit. pp.156 using these joins. Obviously, its fiber over Spf $\mathcal{O}_K$ is isomorphic to the usual formal scheme; in particular, their underlying topological spaces are isomorphic. It is clear that the associated rigid space Ω_Γ of $\mathcal{P}_\Gamma$ in the sense of [3] §5 is the complement in $\mathbf{P}^{1,\mathrm{an}}_A$ of the closure of the set of fixed rig-points (corresponding to the fixed sections). The quotient and the algebraization can equally well be taken by a r reasoning similar to the usual case. What finally comes out is a scheme X_Γ over A with special fiber over K the Mumford curve corresponding to $\overline{\Gamma}$, and hence one can further take a finite quotient by N/Γ.

9.6 The above construction of infinitesimal deformation of Mumford uniformization induces a morphism of functors

$$\Phi : D^{\sim}_{N,\phi} \longrightarrow D_{X,\rho},$$

by associating to a deformation of $N \hookrightarrow PGL(2,K)$ to $\bar{N} \hookrightarrow PGL(2,A)$ the corresponding "Mumford" curve over $\operatorname{Spec} A$. By an argument parallel to [16] §4, it is not difficult to see the following:

9.7 Proposition. *The morphism Φ is an isomorphism.* □

9.8 Remark. If X is a Mumford curve uniformized by a Schottky group Γ and N is between Γ and its normalizer in $PGL(2,K)$, let $\rho \colon N\backslash\Gamma \to \operatorname{Aut}(X)$ the corresponding representation. Then the (a priori very different looking) results from the algebraic computation 5.1 for $D_{X,\rho}$ and the analytic computation 8.3.5 for $PGL(2)^{\wedge}\backslash D_{N,\phi}$ agree. There might be a more direct combinatorial proof of this formula.

9.9.1 Example (Artin-Schreier-Mumford curves). If, for the Artin-Schreier curves of 5.3.1 over a non-archimedean field K the value of c satisfies $|c| < 1$, then $X_{t,c}$ is a Mumford curve, and the corresponding normalizer of its Schottky group is

$$N_t = ((\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/(p^t-1)\mathbf{Z}) *_{\mathbf{Z}/(p^t-1)\mathbf{Z}} D_{p^t-1}.$$

We compute from this that the analytic infinitesimal deformation space is 1-dimensional, in concordance with the algebraic result.

9.9.2 Example (Drinfeld modular curves). The Drinfeld modular curve $X(\mathfrak{n})$ is known to be a Mumford curve (cf. [8]), and the normalizer of its Schottky group is isomorphic to an amalgam (cf. [5])

$$N(\mathfrak{n}) = PGL(2,p^t) *_{(\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/(p^t-1)\mathbf{Z}} (\mathbf{Z}/p\mathbf{Z})^{td} \rtimes \mathbf{Z}/(p^t-1)\mathbf{Z}$$

(at least if $p \neq 2, q \neq 3$). The above formula gives a $(d-1)$-dimensional infinitesimal analytic deformation space, and this agrees with the algebraic result.

Acknowledgments. The first author is post-doctoral fellow of the Fund for Scientific Research - Flanders (FWO -Vlaanderen). This work was done while the first author was visiting Kyoto University and the MPIM in Bonn, and while the second author was visiting the University of Paris-VI. Thanks to Frans Oort for pointing us to [1].

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