

A Hirzebruch proportionality principle in Arakelov geometry

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Abstract

We show that a conjectural extension of a fixed point formula in Arakelov geometry implies results about a tautological subring in the arithmetic Chow ring of bases of abelian schemes. Among the results are an Arakelov version of the Hirzebruch proportionality principle and a formula for a critical power of \hat{c}_1 of the Hodge bundle.

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1 Introduction

The purpose of this note is to exploit some implications of a conjectural fixed point formula in Arakelov geometry when applied to the action of the (-1) involution on abelian schemes of relative dimension d . It is shown that the fixed point formula's statement in this case is equivalent to giving the values of arithmetic Pontrjagin classes of the Hodge bundle, where these Pontrjagin classes are defined as polynomials in the arithmetic Chern classes defined by Gillet and Soulé. When combined with the statement of the non-equivariant arithmetic Grothendieck-Riemann-Roch formula, one obtains a formula for the class $\hat{c}_1^{1+d(d-1)/2}$ of the Hodge bundle in terms of topological classes and a certain special differential form γ . Finally we derive an Arakelov version of the Hirzebruch proportionality principle, namely a ring homomorphism from the Arakelov Chow ring of Lagrangian Grassmannians to the arithmetic Chow ring of bases of abelian schemes.

A fixed point formula for maps from arithmetic varieties to $\text{Spec } D$ has been proven by Roessler and the author in [KR1], where D is a regular arithmetic ring. In [KR2, Appendix] we described the generalization to flat

equivariantly projective maps between arithmetic varieties over D . The still missing ingredient to the proof of this conjecture is the equivariant version of Bismut's formula for the behavior of analytic torsion forms under the composition of immersions and fibrations [B4], i.e. a merge of [B3] and [B4].

We work only with regular schemes as bases; extending these results to moduli stacks and their compactifications remains an open problem, as Arakelov geometry for such situations is not yet developed. In particular one could search an analogue of the full Hirzebruch-Mumford proportionality principle in Arakelov geometry. When this article was almost finished, we learned about related work by van der Geer concerning the classical Chow ring of the moduli stack of abelian varieties and its compactifications [G]. The approach there to determine the tautological subring uses the non-equivariant Grothendieck-Riemann-Roch theorem applied to the line bundle associated to theta divisor. Thus it might be possible to avoid the use of the fixed point formula in our situation by mimicking this method, possibly by extending the methods of Yoshikawa [Y]; but computing the occurring objects related to the theta divisor is presumably not easy.

According to a conjecture by Oort, there are no complete subvarieties of codimension d in the complex moduli space for $d \geq 3$. Thus a possible application of our formula for $\hat{c}_1^{1+d(d-1)/2}$ of the Hodge bundle could be a proof of this conjecture by showing that the height of potential subvarieties would be lower than the known lower bounds for heights. Van der Geer [G, Cor. 7.2] used the degree with respect to the Hodge bundle to show that complete subvarieties have codimension $\geq d$.

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2 Torsion forms

Let $\pi : E^{1,0} \rightarrow B$ denote a d -dimensional holomorphic vector bundle over a complex manifold. Let Λ be a lattice subbundle of the underlying real vector bundle $E_{\mathbf{R}}^{1,0}$ of rank $2d$. Thus the quotient bundle $M := E^{1,0}/\Lambda \rightarrow B$ is a holomorphic fibration by tori Z . Let

$$\Lambda^* := \{\mu \in (E_{\mathbf{R}}^{1,0})^* \mid \mu(\lambda) \in 2\pi\mathbf{Z} \forall \lambda \in \Lambda\}$$

denote the dual lattice bundle. Assume that $E^{1,0}$ is equipped with an Hermitian metric such that the volume of the fibres is constant. Such a metric is induced by a polarization.

Let N_V be the number operator acting on $\Gamma(Z, \Lambda^q T^{*0,1} Z)$ by multiplication with q . Let Tr_s denote the supertrace with respect to the $\mathbf{Z}/2\mathbf{Z}$ -grading on $\Lambda T^* B \otimes \text{End}(\Lambda T^{*0,1} Z)$. Let ϕ denote the map acting on $\Lambda^{2p} T^* B$ as multiplication by $(2\pi i)^{-p}$. We write $\tilde{\mathcal{A}}(B)$ for $\tilde{\mathcal{A}}(B) := \bigoplus_{p \geq 0} (\mathcal{A}^{p,p}(B) / (\text{Im } \partial + \text{Im } \bar{\partial}))$, where $\mathcal{A}^{p,p}(B)$ denotes the C^∞ differential forms of type (p, p) on B .

In [K, Section 3], a superconnection A_t acting on the infinite-dimensional vector bundle $\Gamma(Z, \Lambda T^{*0,1} Z)$ over B had been introduced, depending on $t \in \mathbf{R}^+$. For a fibrewise acting holomorphic isometry g the limit $\lim_{t \rightarrow \infty} \phi \text{Tr}_s g^* N_H e^{-A_t^2} =: \omega_\infty$ exists and is given by the respective trace restricted to the cohomology of the fibres. The equivariant analytic torsion form $T_g(\pi, \mathcal{O}_M) \in \tilde{\mathcal{A}}(B)$ was defined there as the derivative at zero of the zeta function with values in differential forms on B given by

$$-\frac{1}{\Gamma(s)} \int_0^\infty (\phi \text{Tr}_s g^* N_H e^{-A_t^2} - \omega_\infty) t^{s-1} dt$$

for $\text{Re } s > d$.

Theorem 2.1 *Let an isometry g act fibrewise with isolated fixed points on the fibration by tori $\pi : M \rightarrow B$. Then the equivariant torsion form $T_g(\pi, \mathcal{O}_M)$ vanishes.*

Proof: Let $f_\mu : M \rightarrow \mathbf{C}$ denote the function $e^{i\mu}$ for $\mu \in \Lambda^*$. As is shown in [K, §5] the operator A_t^2 acts diagonally with respect to the Hilbert space decomposition

$$\Gamma(Z, \Lambda T^{*0,1} Z) = \bigoplus_{\mu \in \Lambda^*} \Lambda E^{*0,1} \otimes \{f_\mu\}.$$

As in [KR4, Lemma 4.1] the induced action by g maps a function f_μ to a multiple of itself if only if $\mu = 0$ because g acts fixed point free on $E^{1,0}$ outside the zero section. In that case, f_μ represent an element in the cohomology. Thus the zeta function defining the torsion vanishes. **Q.E.D.**

Remark: As in [KR4, Lemma 4.1], the same proof shows the vanishing of the equivariant torsion form $T_g(\pi, L)$ for coefficients in a g -equivariant line bundle L with vanishing first Chern class.

We shall also need the following result of [K] for the non-equivariant torsion form $T(\pi, \mathcal{O}_M) := T_{\text{id}}(\pi, \mathcal{O}_M)$: Assume for simplicity that π is Kähler. Consider for $\text{Re } s < 0$ the zeta function with values in $(d-1, d-1)$ -forms on B

$$Z(s) := \frac{\Gamma(2n - s - 1)}{\Gamma(s)(n-1)!} \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{\bar{\partial} \partial}{4\pi i} \|\lambda^{1,0}\|^2 \right)^{\wedge(d-1)} (\|\lambda^{1,0}\|^2)^{s+1-2n}$$

where $\lambda^{1,0}$ denotes a lattice section in $E^{1,0}$. Then the limit $\gamma := \lim_{s \rightarrow 0^-} Z'(0)$ exists and

$$\frac{\bar{\partial}\partial}{2\pi i}\gamma = c_n(\overline{E^{0,1}}) .$$

In [K, Th. 4.1] the torsion form is shown to equal

$$T(\pi, \mathcal{O}_M) = \frac{\gamma}{\mathrm{Td}(\overline{E^{0,1}})}$$

in $\tilde{\mathcal{A}}(B)$. The differential form γ was intensively studied in [K], in particular its behavior under the action of Hecke operators was determined.

We shall denote a vector bundle F together with an Hermitian metric h by \overline{F} . Then $\mathrm{ch}_g(\overline{F})$ shall denote the Chern-Weil representative of the equivariant Chern character associated to the restriction of (F, h) to the fixed point subvariety. Recall also that $\mathrm{Td}_g(\overline{F})$ is the differential form

$$\frac{\mathrm{Td}(\overline{F}^g)}{\sum_{i \geq 0} (-1)^i \mathrm{ch}_g(\Lambda^i \overline{F})} .$$

3 Abelian schemes and the fixed point formula

We shall use the Arakelov geometric concepts and notation of [SABK] and [KR1]. In this article we shall only give a brief introduction to Arakelov geometry, and we refer to [SABK] for details. Let D be a regular arithmetic ring, i.e. a regular, excellent, Noetherian integral ring, together with a finite set \mathcal{S} of ring monomorphism of $D \rightarrow \mathbf{C}$, invariant under complex conjugation. We shall denote by μ_n the diagonalizable group scheme over D associated to $\mathbf{Z}/n\mathbf{Z}$. We choose once and for all a primitive n -th root of unity ζ_n . Let $f : Y \rightarrow \mathrm{Spec} D$ be an equivariant arithmetic variety, i.e. a regular integral scheme, endowed with a μ_n -projective action over $\mathrm{Spec} D$. The groups of n -th roots of unity acts on $Y(\mathbf{C})$ by holomorphic automorphisms and we shall write g for the automorphism corresponding to ζ_n .

We write f^{μ_n} for the map $Y_{\mu_n} \rightarrow \mathrm{Spec} D$ induced by f on the fixed point subvariety. Complex conjugation induces an antiholomorphic automorphism of $Y(\mathbf{C})$ and $Y_{\mu_n, \mathbf{C}}$, both of which we denote by F_∞ . The space $\tilde{\mathcal{A}}(Y)$ is the subspace of $\tilde{\mathcal{A}}(Y(\mathbf{C}))$ of classes of differential forms ω such that $F_\infty^* \omega = (-1)^p \omega$.

Let $\widehat{\mathrm{CH}}^*(Y)$ denote the Gillet-Soulé arithmetic Chow ring, consisting of arithmetic cycles and suitable Green currents on $Y(\mathbf{C})$. Let $\mathrm{CH}^*(Y)$ denote

the classical Chow ring. Then there is an exact sequence in any degree p

$$\mathrm{CH}^{p,p-1}(Y) \xrightarrow{\rho} \widetilde{\mathcal{A}}^{p-1,p-1}(Y) \xrightarrow{a} \widehat{\mathrm{CH}}^p(Y) \xrightarrow{\zeta} \mathrm{CH}^p(Y) \rightarrow 0. \quad (1)$$

For Hermitian vector bundles \overline{E} on Y Gillet and Soulé defined arithmetic Chern classes $\widehat{c}_p(\overline{E}) \in \widehat{\mathrm{CH}}^*(Y)_{\mathbf{Q}}$.

By "product of Chern classes", we shall understand in this article any product of at least two equal or non-equal Chern classes of degree larger than 0 of a given vector bundle.

Lemma 3.1 *Let*

$$\widehat{\phi} = \sum_{j=0}^{\infty} a_j \widehat{c}_j + \text{products of Chern classes}$$

denote an arithmetic characteristic class with $a_j \in \mathbf{Q}$, $a_j \neq 0$ for $j > 0$. Assume that for a vector bundle \overline{F} on an arithmetic variety Y , $\widehat{\phi}(\overline{F}) = m + a(\beta)$ where β is a differential form on $Y(\mathbf{C})$ with $\partial\bar{\partial}\beta = 0$ and $m \in \widehat{\mathrm{CH}}^0(Y)_{\mathbf{Q}}$. Then

$$\sum_{j=0}^{\infty} a_j \widehat{c}_j(\overline{F}) = m + a(\beta).$$

Proof: By induction: for the term in $\widehat{\mathrm{CH}}^0(Y)_{\mathbf{Q}}$, the formula is clear. Assume now for $k \in \mathbf{N}_0$ that

$$\sum_{j=0}^k a_j \widehat{c}_j(\overline{F}) = m + \sum_{j=0}^k a(\beta)^{|j|}.$$

Then $\widehat{c}_j(\overline{F}) \in a(\ker \partial\bar{\partial})$ for $1 \leq j \leq k$, thus products of these \widehat{c}_j 's vanish by [SABK, Remark III.2.3.1]. Thus the term of degree $k+1$ of $\widehat{\phi}(\overline{F})$ equals $a_{k+1} \widehat{c}_{k+1}(\overline{F})$. **Q.E.D.**

We define **arithmetic Pontrjagin classes** $\widehat{p}_j \in \widehat{\mathrm{CH}}^{2j}$ of arithmetic vector bundles by the relation

$$\sum_{j=0}^{\infty} (-z^2)^j \widehat{p}_j := \left(\sum_{j=0}^{\infty} z^j \widehat{c}_j \right) \left(\sum_{j=0}^{\infty} (-z)^j \widehat{c}_j \right).$$

Thus,

$$\widehat{p}_j(\overline{F}) = (-1)^j \widehat{c}_{2j}(\overline{F} \oplus \overline{F}^*) = \widehat{c}_j^2(\overline{F}) + 2 \sum_{l=0}^{j-1} (-1)^{j+l} \widehat{c}_l(\overline{F}) \widehat{c}_{2k-l}(\overline{F})$$

for an arithmetic vector bundle \overline{F} (compare [MiS, §15]). Lemma 3.1 holds with Chern classes replaced by Pontrjagin classes.

Let $f : Y \rightarrow \text{Spec } D$ denote a quasi-projective arithmetic variety and let $\pi : X \rightarrow Y$ denote a principally polarized abelian scheme of relative dimension d . Set $\overline{E} := (R^1\pi_*\mathcal{O}, \|\cdot\|_{L^2})^*$. This bundle $E = \mathbf{Lie}(X/Y)^*$ is the Hodge bundle. Then by [BBM, Prop. 2.5.2], the full direct image of \mathcal{O} under π is given by

$$\overline{R^\bullet\pi_*\mathcal{O}} = \Lambda^\bullet\overline{E}^* \quad (2)$$

and the relative tangent bundle is given by

$$\overline{T\pi} = \pi^*\overline{E}^* . \quad (3)$$

See also [FC, Th. VI.1.1], where these properties are extended to toroidal compactifications. The underlying real vector bundle of $E_{\mathbf{C}}$ is flat, as the period lattice determines a flat structure. Thus the topological Pontrjagin classes $p_j(E_{\mathbf{C}})$ vanish. For an action of $G = \mu_N$ on X [KR2, conjecture 3.2] states

Conjecture 3.2

$$\widehat{\text{ch}}_G(\overline{R^\bullet\pi_*\mathcal{O}}) - a(T_g(\pi_{\mathbf{C}}, \overline{\mathcal{O}})) = \pi_*^G(\widehat{\text{Td}}_G(\overline{T\pi})(1 - a(R_g(T\pi_{\mathbf{C}}))))$$

where R_g denotes Bismut's equivariant R -class. We shall assume henceforth that this conjecture holds. Thus we obtain the equation

$$\widehat{\text{ch}}_G(\Lambda^\bullet\overline{E}^*) - a(T_g(\pi_{\mathbf{C}}, \overline{\mathcal{O}})) = \pi_*^G(\widehat{\text{Td}}_G(\pi^*\overline{E}^*)(1 - a(R_g(\pi^*E_{\mathbf{C}}^*)))) . \quad (4)$$

Using the equation

$$\widehat{\text{ch}}_G(\Lambda^\bullet\overline{E}^*) = \frac{\widehat{c}_{\text{top}}(\overline{E}^G)}{\widehat{\text{Td}}_G(\overline{E})}$$

(4) simplifies to

$$\frac{\widehat{c}_{\text{top}}(\overline{E}^G)}{\widehat{\text{Td}}_G(\overline{E})} - a(T_g(\pi_{\mathbf{C}}, \overline{\mathcal{O}})) = \widehat{\text{Td}}_G(\overline{E}^*)(1 - a(R_g(E_{\mathbf{C}}^*)))\pi_*^G\pi^*1$$

or, using that $a(\ker \bar{\partial})$ is an ideal of square zero,

$$\widehat{c}_{\text{top}}(\overline{E}^G)(1 + a(R_g(E_{\mathbf{C}}^*))) - a(T_g(\pi_{\mathbf{C}}, \overline{\mathcal{O}}))\text{Td}_g(\overline{E}_{\mathbf{C}}) = \widehat{\text{Td}}_G(\overline{E})\widehat{\text{Td}}_G(\overline{E}^*)\pi_*^G\pi^*1 . \quad (5)$$

If G acts fibrewise with isolated fixed points (over \mathbf{C}), the left hand side of equation (5) is an element of $\widehat{CH}^0(Y)_{\mathbf{Q}(\zeta_n)} + a(\ker \bar{\partial})$. If G does not act with

isolated fixed points, then the right hand side vanishes, $c_{\text{top}}(E^G)$ vanishes and we find

$$\widehat{c}_{\text{top}}(\overline{E}^G) = a(T_g(\pi_{\mathbf{C}}, \overline{\mathcal{O}})\text{Td}_g(\overline{E}_{\mathbf{C}})) . \quad (6)$$

As was mentioned in [K, eq. (7.8)], one finds in particular

$$\widehat{c}_d(\overline{E}) = a(\gamma) .$$

Now we restrict ourself to the action of the automorphism (-1) . We need to assume that this automorphism corresponds to a μ_2 -action. This condition can always be satisfied by changing the base $\text{Spec } D$ to $\text{Spec } D[\frac{1}{2}]$ ([KR1, Introduction] or [KR4, section 2]).

Theorem 3.3 *Let $\pi : X \rightarrow Y$ denote a principally polarized abelian scheme of relative dimension d over an arithmetic variety Y . Set $\overline{E} := (R^1\pi_*\mathcal{O}, \|\cdot\|_{L^2})^*$. Assume [KR2, conjecture 3.2]. Then the Pontrjagin classes of \overline{E} are given by*

$$\widehat{p}_k(\overline{E}) = (-1)^k \left(\frac{2\zeta'(1-2k)}{\zeta(1-2k)} + \sum_{j=1}^{2k-1} \frac{1}{j} - \frac{2 \log 2}{1-4^{-k}} \right) (2k-1)! a(\text{ch}(E)^{[2k-1]}) . \quad (7)$$

The log 2-term actually vanishes in the arithmetic Chow ring over $\text{Spec } D[1/2]$.

Proof: Let $Q(z)$ denote the power series in z given by the Taylor expansion of

$$4(1+e^{-z})^{-1}(1+e^z)^{-1} = \frac{1}{\cosh^2 \frac{z}{2}}$$

at $z = 0$. Let \widehat{Q} denote the associated multiplicative arithmetic characteristic class. Thus by definition for $G = \mu_2$

$$4^d \widehat{\text{Td}}_G(\overline{E}) \widehat{\text{Td}}_G(\overline{E}^*) = \widehat{Q}(\overline{E})$$

and \widehat{Q} can be represented by Pontrjagin classes, as the power series Q is even. Now we can apply Lemma 3.1 for Pontrjagin classes to equation (5). By a formula by Cauchy [Hi3, §1, eq. (10)], the summand of \widehat{Q} consisting only of single Pontrjagin classes is given by taking the Taylor series in z at $z = 0$ of

$$Q(\sqrt{-z}) \frac{d}{dz} \frac{z}{Q(\sqrt{-z})} = \frac{\frac{d}{dz}(z \cosh^2 \frac{\sqrt{-z}}{2})}{\cosh^2 \frac{\sqrt{-z}}{2}} = 1 + \frac{\sqrt{-z}}{2} \tanh \frac{\sqrt{-z}}{2}$$

and replacing every power z^j by \widehat{p}_j . The bundle \overline{E} is trivial, hence $\widehat{c}_{\text{top}}(\overline{E}^G) = 1$. Thus we obtain by equation (5) with $\pi_*^G \pi^* 1 = 4^d$

$$\sum_{k=1}^{\infty} \frac{(4^k - 1)(-1)^{k+1}}{(2k-1)!} \zeta(1-2k) \widehat{p}_k(\overline{E}) = -a(R_g(E_{\mathbf{C}})) .$$

Consider the zeta function $L(\alpha, s) = \sum_{k=1}^{\infty} k^{-s} \alpha^k$ for $\operatorname{Re} s > 1$, $|\alpha| = 1$. It has a meromorphic continuation to $s \in \mathbf{C}$ which shall be denoted by L , too. Then $L(-1, s) = (2^{1-s} - 1)\zeta(s)$ and the function

$$\tilde{R}(\alpha, x) := \sum_{k=0}^{\infty} \left(\frac{\partial L}{\partial s}(\alpha, -k) + L(\alpha, -k) \sum_{j=1}^k \frac{1}{2j} \right) \frac{x^k}{k!}$$

defining the Bismut equivariant R -class in [KR1, Def. 3.6] verifies for $\alpha = -1$

$$\begin{aligned} \tilde{R}(-1, x) - \tilde{R}(-1, -x) &= \sum_{k=1}^{\infty} \left[(4^k - 1)(2\zeta'(1 - 2k) + \zeta(1 - 2k) \sum_{j=1}^{2k-1} \frac{1}{j}) \right. \\ &\quad \left. - 2 \log 2 \cdot 4^k \zeta(1 - 2k) \right] \frac{x^{2k-1}}{(2k-1)!}. \end{aligned} \quad (8)$$

Thus we finally obtain the desired result. **Q.E.D.**

The first Pontrjagin classes are given by

$$\hat{p}_1 = -2\hat{c}_2 + \hat{c}_1^2, \quad \hat{p}_2 = 2\hat{c}_4 - 2\hat{c}_3\hat{c}_1 + \hat{c}_2^2, \quad \hat{p}_3 = -2\hat{c}_6 + 2\hat{c}_5\hat{c}_1 - 2\hat{c}_4\hat{c}_2 + \hat{c}_3^2.$$

In general, $\hat{p}_k = (-1)^k 2\hat{c}_{2k} + \text{products of Chern classes}$. Thus knowing the Pontrjagin classes allows us to express the Chern classes of even degree by the Chern classes of odd degree.

4 A Hirzebruch proportionality principle and other applications

Let U denote the additive characteristic class associated to the power series

$$\sum_{k=1}^{\infty} \left(\frac{\zeta'(1 - 2k)}{\zeta(1 - 2k)} + \sum_{j=1}^{2k-1} \frac{1}{2j} - \frac{\log 2}{1 - 4^{-k}} \right) \frac{x^{2k-1}}{(2k-1)!}.$$

Corollary 4.1 *The part of $\widehat{\operatorname{ch}}(\overline{E})$ in $\widehat{CH}^{\operatorname{even}}(Y)_{\mathbf{Q}}$ is given by the formula*

$$\widehat{\operatorname{ch}}(\overline{E})^{[\operatorname{even}]} = d - a(U(E)).$$

Proof: The part of $\widehat{\operatorname{ch}}(\overline{E})$ of even degree equals

$$\widehat{\operatorname{ch}}(\overline{E})^{[\operatorname{even}]} = \frac{1}{2} \widehat{\operatorname{ch}}(\overline{E} \oplus \overline{E}^*),$$

thus it can be expressed by Pontrjagin classes. More precisely by Newton's formulae ([Hi3, §10.1]),

$$(2k)! \widehat{\text{ch}}^{[2k]} - \widehat{p}_1 \cdot (2k-2)! \widehat{\text{ch}}^{[2k-2]} + \cdots + (-1)^{k-1} \widehat{p}_{k-1} 2! \widehat{\text{ch}}^{[2]} = (-1)^{k+1} k \widehat{p}_k$$

for $k \in \mathbf{N}$. As products of the arithmetic Pontrjagin classes vanish in $\widehat{CH}(Y)_{\mathbf{Q}}$ by Lemma 3.3, we thus observe that the part of $\widehat{\text{ch}}(\overline{E})$ in $\widehat{CH}^{\text{even}}(Y)_{\mathbf{Q}}$ is given by

$$\widehat{\text{ch}}(\overline{E})^{[\text{even}]} = d + \sum_{k>0} \frac{(-1)^{k+1} \widehat{p}_k(\overline{E})}{2(2k-1)!}.$$

Thus the result follows from Lemma 3.3. **Q.E.D.**

Theorem 4.2 *Assume [KR2, conjecture 3.2]. There is a real number $r_d \in \mathbf{R}$ and a Chern-Weil form $\phi(\overline{E})$ on $Y_{\mathbf{C}}$ of degree $(d-1)(d-2)/2$ such that*

$$\widehat{c}_1^{1+d(d-1)/2}(\overline{E}) = a(r_d \cdot c_1^{d(d-1)/2}(E) + \phi(\overline{E})\gamma).$$

The form $\phi(\overline{E})$ is actually a polynomial with integral coefficients in the Chern forms of \overline{E} .

Proof: Consider the graded ring R_d given by $\mathbf{Q}[u_1, \dots, u_d]$ divided by the relations

$$\left(1 + \sum_{j=1}^{d-1} u_j\right) \left(1 + \sum_{j=1}^{d-1} (-1)^j u_j\right) = 1, \quad u_d = 0 \quad (9)$$

where u_j shall have degree j ($1 \leq j \leq d$). This ring is finite dimensional as a vector space over \mathbf{Q} with basis

$$u_{j_1} \cdots u_{j_m}, \quad 1 \leq j_1 < \cdots < j_m < d, 1 \leq m < d.$$

In particular, any element of R_d has degree $\leq \frac{d(d-1)}{2}$. As the relation (9) is verified for $u_j = \widehat{c}_j(\overline{E})$ up to multiples of the Pontrjagin classes and $\widehat{c}_d(\overline{E})$, any polynomial in the $\widehat{c}_j(\overline{E})$'s can be expressed in terms of the $\widehat{p}_j(\overline{E})$'s and $\widehat{c}_d(\overline{E})$ if the corresponding polynomial in the u_j 's vanishes in R_d .

Thus we can express $\widehat{c}_1^{1+d(d-1)/2}(\overline{E})$ as the image under a of a topological characteristic class of degree $\frac{d(d-1)}{2}$ plus γ times a Chern-Weil form of degree $\frac{(d-1)(d-2)}{2}$. As any element of degree $\frac{d(d-1)}{2}$ in R_d is proportional to $u_1^{d(d-1)/2}$, the theorem follows. **Q.E.D.**

Any other arithmetic characteristic class of \overline{E} vanishing in R_d can be expressed in a similar way.

Example: We shall compute $\widehat{c}_1^{1+d(d-1)/2}(\overline{E})$ explicitly for small d . Define topological cohomology classes r_j by $\widehat{p}_j(\overline{E}) = a(r_j)$ via Lemma 3.3. For $d = 1$, clearly

$$\widehat{c}_1(\overline{E}) = a(\gamma) .$$

In the case $d = 2$ we find by the formula for \widehat{p}_1

$$\widehat{c}_1^2(\overline{E}) = a(r_1 + 2\gamma) = a\left[(-1 + \frac{8}{3} \log 2 + 24\zeta'(-1))c_1(E) + 2\gamma\right] .$$

Combining the formulae for the first two Pontrjagin classes we get

$$\widehat{p}_2 = 2\widehat{c}_4 - 2\widehat{c}_3\widehat{c}_1 + \frac{1}{4}\widehat{c}_1^4 - \frac{1}{2}\widehat{c}_1^2\widehat{p}_1 + \frac{1}{4}\widehat{p}_1^2 .$$

Thus for $d = 3$ we find, using $c_3(E) = 0$ and $c_1^2(E) = 2c_2(E)$,

$$\begin{aligned} \widehat{c}_1^4(\overline{E}) &= a(2c_1^2(E)r_1 + 4r_2 + 8c_1(E)\gamma) \\ &= a\left[\left(-\frac{17}{3} + \frac{48}{5} \log 2 + 48\zeta'(-1) - 480\zeta'(-3)\right)c_1^3(E) + 8c_1(\overline{E})\gamma\right] . \end{aligned}$$

For $d = 4$ one obtains

$$\begin{aligned} \widehat{c}_1^7(\overline{E}) &= a\left[64c_2(E)c_3(E)r_1 - (8c_1(E)c_2(E) + 32c_3(E))r_2 + 64c_1(E)r_3 \right. \\ &\quad \left. + 16(7c_1(\overline{E})c_2(\overline{E}) - 4c_3(\overline{E}))\gamma\right] . \end{aligned}$$

As in this case $\text{ch}(E)^{[1]} = c_1(E)$, $3!\text{ch}(E)^{[3]} = -c_1^3(E)/2 + 3c_3(E)$ and $5!\text{ch}(E)^{[5]} = c_1^5(E)/16$, we find

$$\begin{aligned} \widehat{c}_1^7(\overline{E}) &= a\left[\left(-\frac{1063}{60} + \frac{1520}{63} \log 2 + 96\zeta'(-1) - 600\zeta'(-3) + 2016\zeta'(-5)\right)c_1^5(E) \right. \\ &\quad \left. + 16(7c_1(\overline{E})c_2(\overline{E}) - 4c_3(\overline{E}))\gamma\right] . \end{aligned}$$

Now we are going to formulate an Arakelov version of Hirzebruch's proportionality principle. In [Hi2, p. 773] it is stated as follows: Let G/K be a non-compact irreducible symmetric space with non-compact dual G'/K and let $\Gamma \subset G$ be a cocompact subgroup. Then there is a ring monomorphism

$$h : H^*(G'/K, \mathbf{Q}) \rightarrow H^*(\Gamma \backslash G/K, \mathbf{Q})$$

such that $h(c(TG'/K)) = c(TG/K)$ (and similar for other bundles F' , F corresponding to K -representation V' , V dual to each other). This implies in particular that Chern numbers on G'/K and $\Gamma \backslash G/K$ are proportional [Hi1,

p. 345]. Now in our case think about Y as the moduli space of principally polarized abelian varieties of dimension d . Its projective dual is the Lagrangian Grassmannian B_d over $\text{Spec } \mathbf{Z}$ parametrizing isotropic subspaces in symplectic vector spaces of dimension $2d$ over any field, $B_d(\mathbf{C}) = \mathbf{Sp}(d)/\mathbf{U}(d)$. But as the moduli space is a non-compact quotient, the proportionality principle must be altered slightly by considering Chow rings modulo certain ideals corresponding to boundary components in a suitable compactification. For that reason we consider the Arakelov Chow group $\text{CH}^*(\overline{B}_{d-1})$, which is the quotient of $\text{CH}^*(\overline{B}_d)$ modulo the ideal $(\widehat{c}_d(\overline{S}), a(c_d(\overline{S})))$ with \overline{S} being the tautological bundle on B_d , and we map it to $\widehat{\text{CH}}^*(Y)/(a(\gamma))$. Here B_{d-1} shall be equipped with the canonical symmetric metric. For the Hermitian symmetric space B_{d-1} , the Arakelov Chow ring is a subring of the arithmetic Chow ring $\widehat{\text{CH}}(B_{d-1})$ ([GS, 5.1.5]) such that the quotient abelian group depends only on $B_{d-1}(\mathbf{C})$.

Theorem 4.3 *Assume [KR2, conjecture 3.2]. There is a homomorphism of graded rings*

$$h : \text{CH}^*(\overline{B}_{d-1})_{\mathbf{Q}} \rightarrow \widehat{\text{CH}}^*(Y)/(a(\gamma))_{\mathbf{Q}}$$

with

$$h(\widehat{c}(\overline{S})) = \widehat{c}(\overline{E}) \left(1 + a \left(\sum_{k=1}^{d-1} \left(\frac{\zeta'(1-2k)}{\zeta(1-2k)} - \frac{\log 2}{1-4^{-k}} \right) (2k-1)! \text{ch}^{[2k-1]}(E) \right) \right)$$

and

$$h(a(c(\overline{S}))) = a(c(\overline{E})) .$$

Note that S^* and E are ample. One could as well map $a(c(\overline{S}^*))$ to $a(c(E))$, but the correction factor for the arithmetic characteristic classes would have additional harmonic number terms.

Proof: The Arakelov Chow ring $\text{CH}^*(\overline{B}_{d-1})$ has been investigated by Tamvakis in [T]. Consider the graded commutative ring

$$\mathbf{Z}[\widehat{u}_1, \dots, \widehat{u}_{d-1}] \oplus \mathbf{R}[u_1, \dots, u_{d-1}]$$

where the ring structure is such that $\mathbf{R}[u_1, \dots, u_{d-1}]$ is an ideal of square zero. Let \widehat{R}_d denote the quotient of this ring by the relations

$$\left(1 + \sum_{j=1}^{d-1} u_j \right) \left(1 + \sum_{j=1}^{d-1} (-1)^j u_j \right) = 1$$

and

$$(1 + \sum_{k=1}^{d-1} \widehat{u}_k)(1 + \sum_{k=1}^{d-1} (-1)^k \widehat{u}_k) = 1 - \sum_{k=1}^{d-1} \left(\sum_{j=1}^{2k-1} \frac{1}{j} \right) (2k-1)! \text{ch}^{[2k-1]}(u_1, \dots, u_{d-1}) \quad (10)$$

where $\text{ch}(u_1, \dots, u_{d-1})$ denotes the Chern character polynomial in the Chern classes, taken of u_1, \dots, u_{d-1} . Then by [T, Th. 1], there is a ring isomorphism $\Phi : \widehat{R}_d \rightarrow \text{CH}^*(\overline{B}_{d-1})$ with $\Phi(\widehat{u}_k) = \widehat{c}_k(\overline{S}^*)$, $\Phi(u_k) = a(c_k(\overline{S}^*))$. The Chern character term in (10), which could be written more carefully as $(0, \text{ch}^{[2k-1]}(u_1, \dots, u_{d-1}))$, is thus mapped to $a(\text{ch}^{[2k-1]}(c_1(\overline{S}^*), \dots, c_{d-1}(\overline{S}^*)))$. When writing the relation (10) as

$$\widehat{c}(\overline{S})\widehat{c}(\overline{S}^*) = 1 + a(\epsilon_1)$$

and the relation in theorem 3.3 as

$$\widehat{c}(\overline{E})\widehat{c}(\overline{E}^*) = 1 + a(\epsilon_2)$$

we see that a ring homomorphism h is given by

$$h(\widehat{c}_k(\overline{S})) = \sqrt{\frac{1 + a(\epsilon_1)}{1 + a(\epsilon_2)}} \widehat{c}_k(\overline{E}) = (1 + \frac{1}{2}a(\epsilon_1 - \epsilon_2)) \widehat{c}_k(\overline{E}) .$$

Here the factor $1 + \frac{1}{2}a(\epsilon_1 - \epsilon_2)$ has even degree, and thus

$$h(\widehat{c}_k(\overline{S}^*)) = \sqrt{\frac{1 + a(\epsilon_1)}{1 + a(\epsilon_2)}} \widehat{c}_k(\overline{E}^*)$$

which provides the compatibility with the cited relations. **Q.E.D.**

Note that this proof does not make use of the remarkable fact that $h(\epsilon_1^{(k)})$ and $\epsilon_2^{(k)}$ are proportional forms for any degree k .

In particular Tamvakis' height formula [T, Th. 3] provides a combinatorial formula for the real number r_d occurring in theorem 4.2. Replace each term \mathcal{H}_{2k-1} occurring in [T, Th. 3] by

$$-\frac{2\zeta'(1-2k)}{\zeta(1-2k)} - \sum_{j=1}^{2k-1} \frac{1}{j} + \frac{2 \log 2}{1-4^{-k}}$$

and divide the resulting value by half of the degree of B_{d-1} . Using Hirzebruch's formula

$$\deg B_{d-1} = \frac{(d(d-1)/2)!}{\prod_{k=1}^{d-1} (2k-1)!!}$$

for the degree of B_{d-1} (see [Hi1, p. 364]) and the \mathbf{Z}_+ -valued function $g^{[a,b]_{d-1}}$ from [T] counting involved combinatorial diagrams, we obtain

Corollary 4.4 *The real number r_d occurring in theorem 4.2 is given by*

$$r_d = \frac{2^{1+(d-1)(d-2)/2} \prod_{k=1}^{d-1} (2k-1)!!}{(d(d-1)/2)!} \cdot \sum_{k=0}^{d-2} \left(-\frac{2\zeta'(-2k-1)}{\zeta(-2k-1)} - \sum_{j=1}^{2k+1} \frac{1}{j} + \frac{2 \log 2}{1-4^{-k-1}} \right) \cdot \sum_{b=0}^{\min\{k, d-2-k\}} (-1)^b 2^{-\delta_{b,k}} g^{[k-b, b]_{d-1}}$$

where $\delta_{b,k}$ is Kronecker's δ .

In [G, Th. 2.5] van der Geer shows that R_d embeds into the (classical) Chow ring $\text{CH}^*(\mathcal{A}_d)_{\mathbf{Q}}$ of the moduli stack \mathcal{A}_d of principally polarized abelian varieties. Using this result one finds

Lemma 4.5 *Let Y be a regular finite covering of the moduli space \mathcal{A}_d of principally polarized abelian varieties of dimension d . Then for any non-vanishing polynomial $p(u_1, \dots, u_{d-1})$ in R_d ,*

$$h(p(\widehat{c}_1(\overline{S}), \dots, \widehat{c}_{d-1}(\overline{S}))) \notin \text{im } a .$$

In particular, h is non-trivial in all degrees. Furthermore, h is injective iff $a(c_1(E)^{d(d-1)/2}) \neq 0$ in $\widehat{\text{CH}}^{d(d-1)/2+1}(Y)_{\mathbf{Q}}/(a(\gamma))$.

Proof: Consider the canonical map $\zeta : \widehat{\text{CH}}^*(Y)_{\mathbf{Q}}/(a(\gamma)) \rightarrow \text{CH}^*(Y)_{\mathbf{Q}}$. Then

$$\zeta(h(p(\widehat{c}_1(\overline{S}), \dots, \widehat{c}_{d-1}(\overline{S})))) = p(c_1(E), \dots, c_{d-1}(E)) ,$$

and the latter is non-vanishing according to [G, Th. 1.5]. This shows the first part.

If $a(c_1(E)^{d(d-1)/2}) \neq 0$ in $\widehat{\text{CH}}^{d(d-1)/2+1}(Y)_{\mathbf{Q}}/(a(\gamma))$, then by the same induction argument as in the proof of [G, Th. 2.5] R_d embeds in $a(\ker \bar{\partial})$. Finally, by [T, th. 2] any element z of \widehat{R}_d can be written in a unique way as a linear combination of

$$\widehat{u}_{j_1} \cdots \widehat{u}_{j_m} \quad \text{and} \quad u_{j_1} \cdots u_{j_m}, \quad 1 \leq j_1 < \cdots < j_m < d, 1 \leq m < d .$$

Thus if $z \notin \text{im } a$, then $h(z) \neq 0$ follows by van der Geer's result, and if $z \in \text{im } a \setminus \{0\}$, then $h(z) \neq 0$ follows by embedding $R_d \otimes \mathbf{R}$. **Q.E.D.**

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